Finite Rank, Relatively Bounded Perturbations of Semigroups Generators.

PART II: Spectrum and Riesz Basis Assignment with Applications to Feedback Systems (*).

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Summary. – *This paper is motivated by, and ultimately directed to, boundary feedback partial differential equations of both parabolic and hyperbolic type, defined on a bounded domain. It is written, however, in abstract form. It centers on the (feedback) operator* $A_F = A + P$; *A the infinitesimal generator of a s.c. semigroup on H; P an A-bounded, one dimensional range operator (typically non-dissipative), so that* $P = (A\cdot, a)b$, *for* $a, b \in H$. *While Part I studied the question of generation of a s.c. semigroup on H by A_F* and *lack thereof, the present Part II focuses on the following topics:* (i) *spectrum assignment of A_F, given A and $a \in H$, via a suitable vector $b \in H$; alternatively, given A, via a suitable pair of vectors $a, b \in H$;* (ii) *spectrality of A_F—and lack thereof—when A is assumed spectral (constructive counter-examples include the case where P is bounded but the eigenvalues of A have zero gap, as well as the case where P is genuinely A-bounded). The main result gives a set of sufficient conditions on the eigenvalues $\{\lambda_n\}$ of A, the given vector $a \in H$ and a given suitable sequence $\{\varepsilon_n\}$ of nonzero complex numbers, which guarantee the existence of a suitable vector $b \in H$ such that A_F possesses the following two desirable properties:* (i) *the eigenvalues of A_F are precisely equal to* $\lambda_n + \varepsilon_n$; (ii) *the corresponding eigenvectors of A_F form a Riesz basis (a fortiori, A_F is spectral). While finitely many ε_n's can be preassigned arbitrarily, it must be however that* $\varepsilon_n \to 0$ *« sufficiently fast ». Applications include various types of boundary feedback stabilization problems for both parabolic and hyperbolic partial differential equations. An illustration to the damped wave equation is also included.*

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0. – Introduction, summary of main results, comparison with the literature.

Let Y be a (separable) Hilbert space with inner product $(\,,\,)$ and let $A: Y \supset \mathscr{D}(A) \to Y$ be the infinitesimal generator of a C_0-semigroup or group on Y, conveniently denoted by $\exp[At]$. We shall assume $0 \in \varrho(A)$. Finally, let $P: Y \supset \mathscr{D}(P) \to Y$ be a finite rank (or range), A-bounded (or relatively bounded) operator so that $\mathscr{D}(A) \subset \mathscr{D}(P)$. Typically, P will be unbounded and hence unclosable [K.1, p. 166]. For the purposes of the present paper, it will suffice to take P of one-dimensional range. We then note that P is necessarily of the general form

$$(0.1)\qquad Py = (Ay, a)b\,, \quad \mathscr{D}(P) = \{y \in Y:\ (Ay, a) = \text{well defined}\} \supset \mathscr{D}(A)$$

for some vectors $a, b \in Y$.

In fact, if P is A-bounded, then—equivalently—the operator PA^{-1} is bounded and, by assumption, it has one dimensional range. Thus, $PA^{-1}h = (h, a)b$ for some vectors a and b in Y, and we then set $y = A^{-1}h$ to verify our claim.

The object of our interest is the perturbed operator

$$(0.2)\qquad A_F = A + P = A + (A\,\cdot, a)b\,, \quad \mathscr{D}(A_F) \equiv \mathscr{D}(A)\,,$$

which we shall regard as arising either in the first order dynamics

$$(0.3)\qquad \dot{y} = A_F y = Ay + (Ay, a)b$$

or in the second order dynamics

$$(0.4)\qquad \ddot{y} = A_F y = Ay + (Ay, a)b\,.$$

These two abstract differential models include a large variety of boundary feedback parabolic and hyperbolic partial differential equations (but not all, see [L-T.6]). These form—in fact—the original motivation, as well as the ultimate goal, of our study. Section 4 will in fact illustrate our results to these more « concrete » equations, which will be defined on a bounded open domain Ω of R^n with boundary Γ. Accordingly, we shall henceforth make the standing *assumption* that

(0.5) (*K*) *the* (*original*) *operator* A *has compact resolvent* $R(\lambda, A)$ *on* Y and actually, for convenience in some points, we shall also assume that the eigenvalues of A are simple, except perhaps for finitely many.

It then follows that, as expected, A_F in (0.2) also has compact resolvent (Lemma 1.1). Thus, the spectrum of A_F is only point spectrum and consists at most of a countable sequence $\{\alpha_n\}_{n=1}^{\infty}$ of isolated points, with $|\alpha_n| \to \infty$, which—for A given once and for all—depends on the pair a, b of vectors in Y. With these preliminaries, we can now introduce the problems, which we have investigated in this paper and the corresponding results which we have obtained. (In Part I, we studied semigroup generation by A_F, or lack thereof, or well posedness of (0.3)—with application to boundary feedback hyperbolic equations.)

Section 1. - To begin with, we examine, in Section 1, the question of pre-assigning (or allocating) the spectrum of the perturbed operator A_F. Our results take two forms. In Theorem 1.2, we assume that (the operator A and) the vector $a \in Y$ be given; we then establish the existence of a vector $b \in Y$, such that the spectrum of the corresponding operator A_F be arbitrarily preassigned, subject to the following condition: that the distance $|\varepsilon_n| = |\alpha_n - \lambda_n|$ between the « new » (desired) eigenvalue α_n of A_F and the « old » eigenvalue λ_n of A be non zero ($|\varepsilon_n| \neq 0$) and asymptotically small. This means that any finite number of new eigenvalues of A_F can be arbitrarily preassigned, while the remaining eigenvalues of A_F will have to remain sufficiently close to the corresponding eigenvalues of A (but $|\varepsilon_n| \neq 0$).

Instead, in Theorem 1.5, the same conclusion is achieved of synthesizing pressigned eigenvalues $\{\alpha_n\}$ of A_F,—once A is given—this time through the *simultaneous* search of the pair a, b of vectors in Y: however, in contrast with the preceeding Theorem 1.2, *all* $\{\alpha_n\}$ will now have to remain sufficiently close to the corresponding eigenvalues $\{\lambda_n\}$ of A, with no freedom now allowed for pre-assigning wholly arbitrarily finitely many α_n's.

Applications of the first result (Thm. 1.2) to boundary control problems include dynamics (0.3) or (0.4) where one wishes to shift on the left hand side of the complex plane C (or at the left of any vertical line in C, for that matter) finitely many unstable eigenvalues of the original A. Applications of the second result (Thm. 1.5) include boundary feedback byperbolic equations, where the spectrum $\{\lambda_n\}$ of the original operator A in model (0.3) lies, say, on the imaginary axis and the feedback perturbation operator shifts it to the open left hand side of C (see Application 4.3, Section 4).

Sections 2 *and* 3. – In the subsequent two sections (2 and 3), we investigate, first of all, whether or not the perturbed operator A_F is, or can be made, *spectral* (in the sense of DUNFORD [D.1] or DUNFORD-SCHWARTZ [D-S.1, Vol. III]). Counterexamples are provided in section 2 and a particularly strong and desirable version of a positive result is then given in Theorem 3.1 (and Corollary 3.2) of section 3. We then combine the spectrality of A_F of Theorem 3.1 with the spectrum allocation result of Theorem 1.2 and thus arrive at the *main positive result of our paper*, Theorem 3.3. This states that: for an unperturbed operator spectral of scalar type, hence similar to an operator A with an orthonormal basis of eigenvectors $\{\Phi_m\}$, given the simple eigenvalues $\{\lambda_m\}$ of A, the coordinates $\{a_m\}$ of a vector $a \in Y$ and a sequence $\{\varepsilon_m\}$ of non zero distinct complex numbers, all subject to certain verifiable assumptions, there exists (constructively) a vector $b \in Y$ such that the corresponding perturbed operator A_F in (0.2) has two major and desirable properties: (i) (spectral assignment) its eigenvalues $\{\alpha_m\}$ are precisely given by $\alpha_m = \lambda_m + \varepsilon_m$; (ii) (preservation of basis properties of eigenvectors) its eigenprojections $\{Q_m\}$ are related to the eigenprojections $\{P_m\}$ of A by the similarity relation $Q_m = W^{-1} P_m W$, where W is a suitable bounded, boundedly invertible operator on Y (i.e. the corresponding normalized eigenvectors $\{\Psi_m\}$ of A_F form a Riesz basis on Y) and A_F is a spectral operator of scalar type.

Significance of main results. – To put these results in proper perspective within the context of dynamical systems, we recall that in the typical parabolic case or in the typical undamped hyperbolic case (written as a first order equation), the generator A in model (0.3) is self-adjoint or normal, skew-adjoint, respectively while in the typical damped hyperbolic equation A is of scalar type. Thus, A is in any case an operator of *scalar type*, special case of a *spectral operator*, in the sense of DUNFORD [D.1] and DUNFORD-SCHWARTZ [D-S.1, Vol. III]. With the perturbation operator P as in (0.1), it is then natural to ask whether or not the perturbed operator A_F is also spectral for some or possibly all choices of the vector pairs a, b in Y. [Note that the assumption of commutativity between A and the perturbation P, which is made in established theory [D-S.1] does not generally hold in our present case.] The importance of producing a spectral feedback operator A_F is self-evident: in fact, in our present case, where the spectrum of A_F is countable, spectrality of A_F amounts to the assertion that A_F posseses the most desirable property of unconditional convergence of its eigenvalue expansion. More precisely, if A_F with spectrum $\{\alpha_n\}_{n=1}^{\infty}$ is spectral, then—because of the countable additivity property of the resolution of the identity associated to spectral operators—every $y \in Y$ has an unconditionally convergent expansion of the type $y = \sum_n y_n = \sum_n E(\alpha_n) y$ where the spectrum of y_n consists of « generalized eigenvectors » associated to α_n: $(A_F - \alpha_n I)^{m_n} y_n = 0$, $n = 1, 2, \ldots$. If A_F is, in particular, a spectral operator of scalar type, then the generalized eigenvectors are simply eigenvectors in the ordinary sense. If A_F is a spectral operator of type m, then the generalized eigenvectors y_n

satisfy the equations $(A_F - \alpha_n I)^{m+1} y_n = 0$, $n = 1, 2, \dots$. Thus, the basis results for A_F in Theorem 3.1 and 3.3 (and Corollary 3.2) produce, a fortiori, operators A_F which are spectral, indeed of scalar type. On the other hand, the counterexamples to spectrality of A_F given in section 2 should be contrasted with the following two positive results:

(i) first, the property for A_F as in (0.2) that $\overline{\text{span}}$ {generalized eigenvectors of $A_F\} = Y$, slightly weaker than spectrality, but closely related to it, does indeed always hold true, for *any* operator A (self-adjoint or skew-adjoint as in the canonical parabolic or hyperbolic cases) and for *any* choice of vectors a, b in Y (see section 2, below (2.1));

(ii) in the special case of bounded perturbations, say P in (0.1) with $a \in \mathscr{D}(A^*)$, the feedback operator $A + (\cdot, a')b$, ($a' = A^* a \in Y$ for instance) is always spectral, provided that A is spectral with simple eigenvalues except finitely many and, among other things, that the distance d_n from λ_n to the rest of the spectrum of A satisfies the condition: $\{1/d_n\} \in l_1$ or, in Hilbert space, $\{1/d_n\} \in l_2$ (see [S.1, Thm. 1, p. 419 and Corol. 1b', p. 424] and, in a more general form which allows P to be A^ν-bounded with $0 \leqslant \nu < 1$, see [D-S, Vol. III, Thm. 7, p. 2296]). For $A = S$ with $-S$ a self adjoint elliptic operator as in the canonical parabolic case, the above condition on $\{1/d_n\}$ in Hilbert space does not hold true, however, for dim $\Omega > 1$. The canonical hyperbolic case $A = \begin{vmatrix} 0 & I \\ S & 0 \end{vmatrix}$ has its eigenvalues reducible to the eigenvalues of S.

Indeed, our counterexamples in section 2 to spectrality of the operator A_F defined by (0.2) refer to two cases: (1) the case where P is bounded but the eigenvalues of A have zero gap ($\inf d_n = 0$); (ii) the case where P is genuinely A-bounded.

Section 4. – The final goal of section 4 is to apply the abstract main result (Theorem 3.3) of spectral allocation and simultaneous preservation of the basis properties for the feedback operator A_F (in (0 2), with A and $a \in Y$ given through a suitable choice of $b \in Y$(to parabolic and hyperbolic boundary feedback dynamics, (see section 3 or Part I). To this end, in subsection 4.1 we first specialize Theorem 3.3 to two corollaries. Corollary 4.1 (i) refers to the first order model (0.3) for parabolic equations, while Corollaries 4.1 (ii) and 4.2 refer to hyperbolic equations in second order form (0.4) or first order form (0.3), respectively. However, before applying these abstract results in Hilbert space Y to boundary feedback parabolic and hyperbolic equations, we shall need to carry out one final step: as in Part I, we shall need to deduce the counterpart of these results for the adjoint operators A_F^*. This is done in Theorem 4.3 (counterpart of Theorem 3.3 for A_F^*) and Corollaries 4.1*-4.2* (counterparts of Corollaries 4.1-4.2 for A_F^*). In the applications to partial differential equations of subsection 4.2, we shall appeal, in fact, not directly to A_F, but to its adjoint A_F^*. The reason for this is not only limited to the considerations of section 3, Part I: that the closed loop feedback dynamics are more satisfactorily modelled by the abstract equation $\dot{z} = A_F^* z$ (or $\ddot{z} = A_F^* z$) in Z, rather than by

$y = A_F y$ (or $\ddot{y} = A_F y$) in Y, since the space Z is much more desirable than the space Y. Actually, there is a more substantial and more critical reason which we shall explain at the end (4.2, Final Comments). This paper is another effort in the general area of « stabilization » or « eigenvalues allocation » for parabolic and hyperbolic equations, by means of a boundary feedback of a simple form (presumably, « easy to implement in practical applications ») [L-T.1-L-T.5], [S.2], [R.1]-[R.2]. However, it presents a spectral analysis approach of the feedback operator, rather than an approach based on the closed loop dynamics as in [L-T.1]-[L-T.3] or a difference delay system as a « canonical » form for two dimensional, first order hyperbolic systems in one space dimension [R.1]-[R.2]: (These last two references address themselves to eigenvalues allocation only, without considering the basis properties of the eigenvectors of the feedback system in general). Our treatment here is simpler than the one followed in [L-T.1]-[L-T.3], however it solves a simpler version of the spectral allocation problem, in that we seek here a synthesizing vector in the interior Ω, and not synthesizing vectors on the boundary Γ as before.

Paper [S.2] by SUN SHUN-HUA, is also based on the spectral analysis of the closed loop operator (in an abstract setting). Our positive results in section 1 go, however, far beyond [S.2] (see comments in Remark 1.4), in that (i) [S.2] considers only bounded perturbations and (ii) [S.2] assumes that the original operator A (corresponding to the original system) has a *nonzero gap* of its eigenvalues (i.e. the distance d_n between two consecutive eigenvalues is uniformly bounded below by a positive constant: $\inf d_n > 0$), which essentially reduces the applications to one-dimensional equations. To overcome these two assumptions, we employ a different machinery that is crucially based on a Lemma of KATO [K.1] concerning the similarity between the orthogonal eigenprojections of the unperturbed operator and the nonnecessarily orthogonal eigenprojections of the perturbed operator.

1. – The question of spectral assignment for the operator $A_F = A + P$, P finite rank and A-bounded.

We return to the perturbed operator A_F

$$A_F = A + P = A + (A\cdot, a)b\,, \qquad a, b \in Y \tag{1.1}$$

on Y, where A has compact resolvent. Let $\{\lambda_n\}$ be the sequence of eigenvalues of A, $|\lambda_n| \to \infty$, with $\{\Phi_n\}$ being the corresponding normalized eigenvectors. It will be proved below (Lemma 1.1) that, as expected, A_F also has compact resolvent, for any a, b in Y. Accordingly, we shall indicate with $\{\alpha_n\}$ and $\{\Psi_n\}$ the eigenvalues and corresponding normalized eigenvectors of A_F (which, of course, depend on the choice of a and b).

Two main results are proved in this section, both aiming at the same final goal which is motivated by stability considerations: assign (or preassign) the eigenvalues

$\{\alpha_n\}$ of A_F in « more desirable » locations of the complex plane than those of the original eigenvalues $\{\lambda_n\}$ of A. This will be achieved by either seeking a suitable vector b, given the vector a in advance (Theorem 1.2); or else by seeking simultaneously a suitable pair of vectors a and b (Theorem 1.5). As described below, these two Theorems have sharply different implications yet both have relevant applications, to say, boundary feedback hyperbolic equations (see Introduction). In the first case, Theorem 1.2, it will be possible to preassign, in a completely arbitrary manner, finitely many of the « new » eigenvalues $\{\alpha_n\}$; in particular, it will be possible to replace or shift all of the finitely many « original » eigenvalues of A situated in the closed right hand side of C with « new » eigenvalues $\{\alpha_n\}$ of A_F which will be all contained in the open left hand side of C. In contrast, this finite degree of freedom will not be allowed in the second case, Theorem 1.5 an application of which will instead be to the stability of hyperbolic equations via boundary feedback. It will consist in removing all the eigenvalues $\{\lambda_n\}$ of A, which are originally located on the imaginary axis (typical case of the free wave equation), and in shifting them (« slightly ») to the open left hand side of C (and asymptotically approaching the imaginary axis). In both theorems, (verifiable) sufficient conditions will be imposed, involving the eigenvalues $\{\lambda_n\}$ of the original A, the distance $|\alpha_n - \lambda_n| \neq 0$ between λ_n and the (desired) « new »-eigenvalues α_n of A_F, and—in the case of Thm. 1.2—the coordinates $a_n = (a, \varphi_n)$ of $a \in Y$ as well.

1.1. *The resolvent operator $R(\lambda, A_F)$ of A_F and preliminaries.*

If $y \in Y$ is given, we seek to solve $(\lambda I - A_F)x = (\lambda I - A)x - (Ax, a)b = y$, for suitable $\lambda \in C$ and $x \in \mathscr{D}(A) = \mathscr{D}(A_F)$. Thus, $x = R(\lambda, A)y + R(\lambda, A)b(Ax, a)$ for $\lambda \in \varrho(A)$ and

$$(Ax, a)[1 - (AR(\lambda, A)b, a)] = (AR(\lambda, A)y, a) .$$

Hence, the sought after vector x is given by

$$x = R(\lambda, A_F)y = R(\lambda, A)y + R(\lambda, A)b\,\frac{(AR(\lambda, A)y, a)}{1 - (AR(\lambda, A)b, a)} \tag{1.2}$$

for all $\lambda \in \varrho(A)$ for which the analytic denominator in (1.2) does not vanish. With $AR(\lambda, A) = [\lambda R(\lambda, A) - 1]$, we have

$$\lim\, (AR(\lambda, A)b, a) = \lim\, [(\lambda R(\lambda, A)b, a) - (b, a)] = (b, a) - (b, a) = 0 ,$$

say for $\lambda = \text{real} \to +\infty$ [B-B.1]. Thus, the denominator cannot vanish identically. Eq. (1.2) gives $R(\lambda, A_F)$ for each such fixed λ as the sum of two obviously compact operators acting on y (recall (K) in (0.5)). Thus $R(\lambda, A_F)$ is compact, $\lambda \in \varrho(A_F)$. Moreover, let $\alpha \in \varrho(A)$; if the denominator in (1.2) vanishes for $\lambda = \alpha$, then α is a pole of $R(\lambda, A_F)$ and so [T-L.1], α is an eigenvalue of A_F; and conversely. We collect all this in a formal statement.

LEMMA 1.1. – (i) For any vectors $a, b \in Y$ and under assumption (K) in (0.5) (A has compact resolvent), the perturbed operator A_F in (1.1) has compact resolvent $R(\lambda, A_F)$ in Y, which is given by (1.2);

(ii) Any eigenvalue α of A_F which is not also an eigenvalue of A (i.e. $\alpha \in \sigma_p(A_F)$ but $\alpha \notin \sigma_p(A) \equiv \{\lambda_n\}$) is characterized by

$$(1.3) \qquad \begin{cases} (a) & (AR(\alpha, A)b, a) = 1, \quad \text{i.e. in particular by} \\ (b) & \displaystyle\sum_{k=1}^{\infty} \frac{\lambda_k a_k b_k}{\alpha - \lambda_k} = 1, \end{cases}$$

$a_k = (a, \Phi_k)$; $b_k = (b, \Phi_k)$, $\alpha \notin \{\lambda_k\} \equiv \sigma_p(A)$, if the $\{\Phi_n\}$ form an orthonormal basis (i.e. A normal operator) [S.4, p. 250]. □

REMARK 1.1. – (i) The adjoint operator A_F^* of A_F is given by

$$(x, A_F^* y) = (A_F x, y) = \big(Ax + (Ax, a)b, y\big) = \big(Ax, y + a(b, y)\big) =$$
$$= \big(x, A^*[y + a(b, y)]\big), \qquad x \in \mathscr{D}(A_F),\ y \in \mathscr{D}(A_F^*)$$

i.e. by

$$(1.4) \qquad \begin{cases} (a) & A_F^* y = A^*[y + a(b, y)] \\ (b) & \mathscr{D}(A_F^*) = \{h \in Y : h + a(b, h) \in \mathscr{D}(A^*)\} \end{cases}$$

(ii) In section 3, we shall invoke the following considerations.

Compute $A_F \Phi_m = A\Phi_m + (A\Phi_m, a)b = \lambda_m \Phi_m + \lambda_m a_m b$.

Thus, if $a_{m_1} = 0$ for some m_1, then $\lambda_{m_1} = \alpha_{m_1}$ and Φ_{m_1} is an eigenvector of A_F as well. Similar conclusions hold, if $b_{m_1} = 0$ for some m_1, in which case by (1.4a) and normality of A: $A_F^* \Phi_{m_1} = A^* \Phi_{m_1} = \bar{\lambda}_{m_1} \Phi_{m_1}$ and $\lambda_{m_1} = \alpha_{m_1}$. We thus conclude, for future use below (3.40): *if the differences $\varepsilon_m = \alpha_m - \lambda_m \neq 0$ for all m, then the products $a_m b_m \neq 0$ as well.*

1.2. *Given A and $a \in Y$, sufficient conditions for spectral assignment of A_F via a vector $b \in Y$. Finite degree of freedom.*

Henceforth we shall introduce

$$(1.4) \qquad \varepsilon_n = \alpha_n - \lambda_n,$$

the difference between the n-th «new» eigenvalue of A_F and the n-th «original» (simple) eigenvalue of A. We shall always be concerned with the case $|\varepsilon_n| > 0$ for all n, where Remark 1.1 (ii) then applies. Our main result in this subsection is that: the spectrum of A_F can be arbitrarily preassigned, subject to the condition that the distance $|\varepsilon_n| \neq 0$ between «new» and «original» eigenvalue is asymptotically small; i.e. a finite number of eigenvalues of A_F can be arbitrarily preassigned, while the remaining eigenvalues of A_F will have to remain sufficiently close to the corresponding eigenvalues of A (but $|\varepsilon_n| > 0$). This will be achieved by having the operator A and the vector $a \in Y$ given in advance, and seeking only a suitable synthesizing vector b in Y (1). Applications of this result to boundary control problems will include those situations, modelled by either dynamics (0.3) or by dynamics (0.4), where one wishes to shift on the left hand side of C finitely many unstable eigenvalues of A. See Section 4.

THEOREM 1.2 (*Spectral assignment of A_F via a vector b*). – (i) Let the normal generator A of the Introduction be given [satisfying assumption (K) in (0.5)] with simple eigenvalues $\{\lambda_m\}$ and an orthonormal basis of eigenvectors $\{\Phi_m\}$ in Y. Let a vector $a \in Y$ be given, with coordinates $a_m = (a, \Phi_m) \neq 0$ for all m (2). Let $\{\varepsilon_m\}$ be a sequence of non zero distinct complex numbers, which are assumed to satisfy the following four conditions involving only $\{\varepsilon_m\}$, $\{\lambda_m\}$, and $\{a_m\}$:

$$(1.5)\qquad (1)\qquad \begin{aligned} &0 < |\varepsilon_m| < d_m/2, \text{ asymptotically, where } \varepsilon_m \to 0 \text{ as } m \to \infty,\\ &d_m \equiv \min\{|\lambda_m - \lambda_{m-1}|, |\lambda_m - \lambda_{m+1}|\}\end{aligned}$$

$$(1.6)\qquad (2)\qquad \sum_{m=1}^{\infty} \frac{|\varepsilon_m|^2}{|\lambda_m a_m|^2} < \infty$$

$$(1.7)\qquad (3)\qquad \sum_{m=1}^{\infty} \frac{|\varepsilon_m|^2 \delta_m}{|\lambda_m a_m|^2} < \infty,$$

where

$$(1.8)\qquad \delta_m \equiv \sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{|a_k \lambda_k|^2}{|\lambda_k - \lambda_m|^2} > 0$$

$$(1.9)\qquad (4)\qquad \sum_{m=1}^{\infty} |\varepsilon_m| \gamma_m < \infty,$$

where

$$(1.10)\qquad \gamma_m \equiv \sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_m|^2}.$$

Then, there exists a suitable vector $b \in Y$ such that the corresponding operator $A_F = A + (A\cdot, a)b$ has its eigenvalues [Lemmas 1.1] $\{\alpha_m\}$ given precisely by

$$(1.11)\qquad \alpha_m \equiv \beta_m + \varepsilon_m, \qquad m = 1, 2, \ldots .$$

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(1) The roles of a and b in Theorem 1.2 are interchangeable: one may assign b in advance and seek a as well.

(2) This is an approximate controllability type of condition.

Moreover, if the space Y split into the direct sum $Y = Y_1 \oplus Y_2$ and thus $b = [b^1, b^2]$, $a = [a^1, a^2]$, $\Phi_m = [\Phi_m^1, \Phi_m^2]$, $(b, \Phi_m)_Y = (b^1, \Phi_m^1)_{Y_1} + (b^2, \Phi_m^2)_{Y_2}$, etc. we may require that the suitable vector $b \in Y$ claimed above satisfies $b^1 = 0$ (i.e. $b \in Y_2$) or else $b^2 = 0$ (i.e. $b \in Y_1$), provided that $\{\Phi_m^2\}$ (resp. $\{\Phi_m^1\}$) are linearly independent in Y_2 (resp. in Y_1).

(ii) The case where the original operator A is spectral of scalar type with compact resolvent and simple eigenvalues can be reduced to the normal case (i). Indeed, by a similarity transformation Π: $\Pi^{-1}A\Pi = A_N$, A_N normal, $\Pi, \Pi^{-1} \in \mathscr{L}(Y)$, and Π can be taken self-adjoint [Werner's, Lemma XV 6.2, p. 1947, D-S III]. Then, $\Pi^{-1}A_F\Pi = A_N + (A_N, \Pi a)_Y \Pi^{-1} b$, the eigenvalues of $\Pi^{-1}A_F\Pi$ and A_F coincide, and we can apply part (i) to $\Pi^{-1}A_F\Pi$.

REMARK 1.2. – (i) For each fixed m, the positive constant δ_m in (1.8) is well defined and finite, since $\{a_n\}_{n=1}^{\infty} \in l_2$.

(ii) For each fixed m, the positive constant γ_m in (1.10) is also well defined and finite, as a consequence of assumption (2). This is so, in view of:

$$\sum_{k=1}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k|^2} = \sum_{k=1}^{\infty} \frac{|\varepsilon_k a_k|}{|\lambda_k a_k \lambda_k|} \leqslant \left(\sum_{k=1}^{\infty} \frac{|\varepsilon_k|^2}{|\lambda_k a_k|^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \frac{|a_k|^2}{|\lambda_k|^2}\right)^{\frac{1}{2}} < \infty . \tag{1.12}$$

(iii) Under assumptions $\{\varepsilon_m\} \in l_1$ and

$$\sum_{k=1}^{\infty} |\varepsilon_k| \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{|\lambda_j - \lambda_k|^p} < \infty \qquad \text{for some } 1 \leqslant p < \infty$$

it will follow from the argument in section 3.1 (see Lemma 3.5) that we have

$$\frac{|a_m b_m \lambda_m|}{|\varepsilon_m|} \sim 1 , \qquad \text{hence } \frac{|\varepsilon_m|}{|a_m \lambda_m|} \sim |b_m|$$

and condition (1.6) is also *necessary* for spectral assignment. □

REMARK 1.3. – The following will be used repeatedly in the sequel. Under assumption $|\varepsilon_m| < d_m/2$ as in (1.5), then the sequence $\alpha_m = \lambda_m + \varepsilon_m$ satisfies

$$\tfrac{1}{2}|\lambda_k - \lambda_j| \leqslant |\lambda_k - \alpha_j| \leqslant \tfrac{3}{2}|\lambda_k - \lambda_j| , \qquad \text{for all } j \neq k . \tag{1.13}$$

In fact

$$|\lambda_j - \alpha_j| = |\varepsilon_j| < \frac{d_j}{2} = \frac{1}{2}\min\left\{|\lambda_j - \lambda_{j-1}|, |\lambda_j - \lambda_{j+1}|\right\} \leqslant \frac{1}{2}|\lambda_j - \lambda_k| , \qquad k \neq j . \tag{$*$}$$

Thus, for $k \neq j$, adding and substracting λ_j:

$$|\lambda_k - \alpha_j| \leqslant |\lambda_k - \lambda_j| + |\varepsilon_j| \leqslant \tfrac{3}{2}|\lambda_k - \lambda_j| \tag{1.13a}$$

by (*). Moreover, again by (*)

$$|\lambda_k - \alpha_j| \geqslant \big||\lambda_k - \lambda_j| - |\varepsilon_j|\big| \geqslant (1 - \tfrac{1}{2})|\lambda_k - \lambda_j| \tag{1.13b}$$

and claim (1.13) follows. □

Proof of Theorem 1.2. - We impose that the constants $\{\alpha_m\}$ as in (1.11) be precisely the eigenvalues of an operator A_F as in (1.1) for the assigned vector $a \in Y$ and for a suitable choice of a vector $b \in Y$. Thus, by Lemma 1.1 (ii), we impose the identities (see (1.3b)) $\big(AR(\alpha_i, A)b, a\big) \equiv 1$, i.e.

$$\left\{\begin{array}{ll} (a) & \dfrac{\lambda_i a_i b_i}{\alpha_i - \lambda_i} + \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{\infty} \dfrac{\lambda_j a_j}{\alpha_i - \lambda_j}\, b_j \equiv 1\,, \quad \text{i.e.} \\ (b) & b_i + \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{\infty} \dfrac{\alpha_i - \lambda_i}{\alpha_i - \lambda_j}\,\dfrac{\lambda_j a_j}{\lambda_i a_i}\, b_j \equiv \dfrac{\alpha_i - \lambda_i}{\lambda_i a_i}\,, \\ \text{or} & \\ (c) & b_i + \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{\infty} \dfrac{\varepsilon_i}{\alpha_j - \lambda_j}\,\dfrac{\lambda_j a_j}{\lambda_i a_i}\, b_j \equiv \dfrac{\varepsilon_i}{\lambda_i a_i}\,, \qquad i = 1, 2, \ldots \end{array}\right. \tag{1.14}$$

which can be more coincisely re-written as

$$(I + T)b = v \tag{1.15}$$

for a sought after vector $b \in Y$, (if $Y = Y_1 \oplus Y_2$, we may also require b to be of the form $[b^1, 0]$, or $[0, b^2]$, in which case $b_i = (b, \Phi_i)_Y$ are $(b^1, \Phi_i^1)_{Y_1}$ or $(b^2, \Phi_i^2)_{Y_2}$, respectively). Here, I is the infinite identity matrix, and T is the infinite matrix

$$T = [(t_{ij})]\,, \qquad t_{ij} = \left\{\begin{array}{ll} \dfrac{\varepsilon_i}{\alpha_i - \lambda_j}\,\dfrac{\lambda_j a_j}{\lambda_i a_i} & i \neq j \\ 0 & i = j \end{array}\right. \qquad i, j = 1, 2, \ldots \tag{1.16}$$

$$b = [b_1, b_2, \ldots]\,, \qquad v = [v_1, v_2, \ldots]\,, \qquad v_i = \frac{\varepsilon_i}{\lambda_i a_i}\,, \qquad i = 1, 2, \ldots\,. \tag{1.17}$$

Of course, b must be in l_2: Moreover, by assumption (2) in Theorem 1.2 (see (1.6)), we have the known $v \in l_2$ as well. We next prove:

LEMMA 1.3. – Under assumptions (1) (i.e. (1.5)) and (3) (i.e. (1.7)) of Theorem 1.2, the infinite matrix T in (1.16) defines a Hilbert-Schmidt (hence compact) operator $l_2 \to l_2$. □

PROOF OF LEMMA 1.3. – We want to check that the double norm of T is finite, i.e. that

$$\sum_{i,j=1}^{\infty} |t_{ij}|^2 < \infty \tag{1.18}$$

We compute via (1.16)

$$\sum_{i,j=1}^{\infty} |t_{ij}|^2 = \sum_{\substack{i,j\\ i\neq j}} \frac{|\varepsilon_i \lambda_j a_j|^2}{|\alpha_i - \lambda_j| |\lambda_i a_i|^2} = \sum_{i=1}^{I} \frac{|\varepsilon_i|^2}{|\lambda_i a_i|^2} \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{|\lambda_j a_j|^2}{|\alpha_i - \lambda_j|^2} \equiv ① + \sum_{i>I} \frac{|\varepsilon_i|^2}{|\lambda_i a_i|^2} \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{|\lambda_j a_j|^2}{|\alpha_i - \lambda_j|^2} \equiv ② \tag{1.19}$$

where I is a finite index. The first term ① in (1.19) is obviously finite as, for fixed i, $i \neq j$, the infinite sum in j is finite $(\{a_j\} \in l_2)$. As to the second term 2 in (1.19), by assumption (1.5), we invoke (1.13) in Remark 1.2 and replace α_i with λ_i, to obtain:

$$② \leqslant 2 \sum_{i>I} \left|\frac{\varepsilon_i}{\lambda_i a_i}\right|^2 \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{|\lambda_j a_j|^2}{|\lambda_i - \lambda_j|^2} < \infty$$

as the infinite sum in j is precisely δ_i in (1.8), and so assumption (1.7) applies Eq. (1.18) is proved. □

Having ascertained that Eq. (1.15) is well defined as an operator equation in l_2 with unknown $b \in l_2$, we next seek l_2-inversion of $(I + T)$.

LEMMA 1.4. – Under assumptions (1) (i.e. (1.5)), (3) (i.e. (1.7)), and (4) (i.e. (1.9)) of Theorem 1.2, the operator $(I + T)$ on l_2 admits a bounded inverse $(I + T)^{-1}$ defined on all of l_2; in other words, $-1 \in \varrho(T)$. □

PROOF OF LEMMA 1.4. – By Lemma 1.3, it suffices to show that: $-1 \notin \sigma_p(T)$, i.e. $\lambda = -1$ is not an eigenvalue of T. This will be achieved via perturbation theory. In fact, if $\{T_n\}$ is any sequence of bounded operators on l_2, with the two properties that:

(1.20) (i) $\|T_n - T\|_{l_2 \to l_2} \to 0$, as $n \to \infty$;

(1.21) (ii) $B(-1, r) \subset \varrho(T_n) =$ resolvent set of T_n, for some $r > 0$, and all n sufficiently large (where $B(-1, r)$ denotes the closed ball (disk) of the complex plane, centered at the point $\lambda = -1$ and of positive radius r), then it follows by virtue of the standard upper semicontinuity of the spectrum [KATO, Theorem 3.1, and Remark 3.3, p. 208]

$$B(-1, r) \subset \varrho(T) \tag{1.22}$$

as well; in particular $-1 \in \varrho(T)$. As a sequence $\{T_n\}$ we take the natural restriction of T to its first $n\times n$ entries, while imposing all other entries equal to zero; i.e.

$$T_n \equiv \begin{bmatrix} t_{ij} & \text{as in (1.16)}\,, & \text{for } i, j = 1, \ldots, n; \\ t_{ij} = 0 & \text{otherwise}\,. & \end{bmatrix}$$

PROOF OF (1.20). – For $x \in l_2$, we compute easily

$$\|(T_n - T)x\|_{l_2}^2 \equiv \sum_{i=1}^{n}\left|\sum_{j=n+1}^{\infty} t_{ij}x_j\right|^2 + \sum_{i=n+1}^{\infty}\left|\sum_{j=1}^{\infty} t_{ij}x_j\right|^2 \leqslant \leqslant \left\{\sum_{j=n+1}^{\infty}\sum_{i=1}^{n}|t_{ij}|^2 + \sum_{i=n+1}^{\infty}\sum_{j=1}^{\infty}|t_{ij}|^2\right\}\|x\|_{l_2}^2 \tag{1.23}$$

and, by (1.18), each of the two terms in { } of (1.23) goes to zero as $n \to \infty$.

PROOF OF (1.21). – It is enough to prove that ($I_n = n\times n$ identity matrix)

$$\lim_{n\to\infty} \det(I_n + T_n) \neq 0\,. \tag{1.24}$$

The computation of the determinant is a somewhat tedius task, which is carried out in [S.2]. The result is:

$$\det(I_n + T_n) = \prod_{j=1}^{n-1}\prod_{k=j+1}^{n}\left(1 + \frac{\varepsilon_j\varepsilon_k}{(\lambda_k - \alpha_j)(\lambda_j - \alpha_k)}\right). \tag{1.25}$$

Thus, to achieve (1.24), it suffices to show

$$\sigma \equiv \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\frac{|\varepsilon_j\varepsilon_k|}{|\lambda_k - \alpha_j||\lambda_j - \alpha_k|} < \infty\,. \tag{1.26}$$

In view of assumption (1.5), we again invoke the left hand side of the double inequality (1.13) in replacing α_j with λ_j and α_k with λ_k. We obtain

$$\sigma \leqslant 4\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\frac{|\varepsilon_j\varepsilon_k|}{|\lambda_k - \lambda_j|^2} = 4\sum_{j=1}^{\infty}|\varepsilon_j|\sum_{k=j+1}^{\infty}\frac{|\varepsilon_k|^{\frac12}}{|\lambda_k - \lambda_j|}\frac{|\varepsilon_k|^{\frac12}}{|\lambda_k - \lambda_j|} \leqslant$$

$$\leqslant 4\sum_{j=1}^{\infty}|\varepsilon_j|\left\{\sum_{k=j+1}^{\infty}\frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|^2}\right\}^{\frac12}\left\{\sum_{k=j+1}^{\infty}\frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|^2}\right\}^{\frac12} =$$

$$= 4\sum_{j=1}^{\infty}|\varepsilon_j|\sum_{k=j+1}^{\infty}\frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|^2} \leqslant 4\sum_{j=1}^{\infty}|\varepsilon_j|\gamma_j < \infty$$

by assumption (1.9), and (1.26) follows as desired. Thus, (1.24) is proved, and so is (1.21). □

Continuing with the proof of Theorem 1.2, we now invoke Lemma 1.4 and solve the vector equation in (1.15) in the unknown $b \in l_2$. Then, the operator A_F corresponding to the assigned vector a and the vector b just found, has spectrum $\{\alpha_m\}$ satisfying (1.11). The proof of Theorem 1.2 is complete. □

REMARK 1.4. – Theorem 1.2 is of similar nature as one in SHUN-HUE [S.2]. There are, however, two major differences, in that (i) [S.2] considers only bounded perturbations (i.e. $a \in \mathscr{D}(A^*)$ and, moreover, (ii) [S.2] assumes a *non zero* gap: $\inf\{|\lambda_{n+1}-\lambda_n|, |\lambda_n-\lambda_{n-1}|\} = c > 0$ of the eigenvalues of A). Assumption (ii) is also very restrictive, as it rules out say, selfadjoint elliptic operators in space dimension strictly greater than one. □

REMARK 1.5. – An analysis of the proof of Theorem 1.2 shows that its assumptions: (1) $\Leftrightarrow$ (1.5); (2) $\Leftrightarrow$ (1.6); (3) $\Leftrightarrow$ (1.7); (4) $\Leftrightarrow$ (1.9) were employed as follows.

(i) Assumption (1) $\Rightarrow$ inequality (1.13) in replacing α with λ;

(ii) Assumption (1), Assumption (3) $\Rightarrow$ T is Hilbert-Schmidt on l_2 $\Downarrow$ (1.20): $\|T_n - T\|_{l_2\to l_2} \to 0$ in Lemma 1.4

Assumption (4) $\Downarrow$ (1.24): $\lim \det(I_n + T_n) \neq 0$ in Lemma 1.4

(both) $\Rightarrow$ Lemma 1.4: $(I+T)^{-1}$ bounded on all of l_2

(iii) Assumption (2) $\Rightarrow$ γ_m (see (1.10) in the definition of Assumption (4)) is finite (Remark 1.2 (i)).

Some variations are possible, of course. In particular,

CLAIM. – Let

$(V1)$ $\{\varepsilon_m\} \in l_1$;

$(V2)$ $$\begin{cases} \displaystyle\sum_{m=1}^{\infty} |\varepsilon_m| \Bigg[\sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k-\lambda_m|^{2p}}\Bigg]^{1/p} < \infty: \quad \text{this reduces to (1.9) for } p=1 \\ \text{for some} \quad 1 \leqslant p < \infty . \end{cases}$$

Then, assumption (4) $\Leftrightarrow$ (1.9) is well defined and holds true. □

In fact, with $1/p + 1/p' = 1$, we compute

$$\sum_{m=1}^{\infty} |\varepsilon_m| \sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_m|^2} = \sum_{m=1}^{\infty} |\varepsilon_m| \sum_{\substack{k=1\\k\neq m}}^{\infty} |\varepsilon_k|^{1/p'} \frac{|\varepsilon_k|^{1/p}}{|\lambda_k - \lambda_m|^2} \leqslant$$

$$\leqslant \sum_{m=1}^{\infty} |\varepsilon_m| \left[\sum_{k=1}^{\infty} |\varepsilon_k|\right]^{1/p'} \left[\sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_m|^{2p}}\right]^{1/p} =$$

$$= \left\{\sum_{k=1}^{\infty} |\varepsilon_k|\right\}^{1/p'} \left\{\sum_{m=1}^{\infty} |\varepsilon_m| \sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_m|^{2p}}\right\}^{1/p}$$

and (1.9) follows via ($V1$)-($V2$).

We remark that in the important situations where A is self-adjoint (canonical parabolic case) or skew-adjoint (canonical hyperbolic case), for the first order model (0.3) it is *always true* that [see [C-H, Ch. V1, § 3.3-§ 3.4], [T.1, p. 392-5]):

$$(*) \qquad \sum_{k=1}^{\infty} \frac{1}{|\lambda_n|^{2p}} < \infty \quad \text{for some } 1 \leqslant p < \infty .$$

Thus, in these cases, for each m fixed, the infinite sum in k in ($V2$) is always finite. Thus, ($V1$) along with the condition

$$(V2'): \qquad \sum_{\substack{k=1\\k\neq m}} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_m|^{2p}} \leqslant \text{const} < \infty , \qquad m = 1, 2, \ldots \qquad \text{some } 1 \leqslant p < \infty$$

a fortiori guarantee ($V2$). An upperbound for ($V2'$) is (recall (1.5)):

$$\sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_m|^{2p}} \leqslant \sum_{k=1}^{\infty} \frac{|\varepsilon_k|}{d_k^{2p}}, \qquad \text{some } 1 \leqslant p < \infty .$$

This shows that finiteness of the right hand side can always be achieved by imposing that the distance $|\varepsilon_k|$ between new and original k-th eigenvalues be chosen as to compensate the natural gap d_k of the original eigenvalues, raised to the power $2p$, no matter how large p is. □

1.3. *Given A, sufficient conditions for spectral assignment of A_F, via a suitable pair of vectors a, b in Y.*

Given A, a, and $\{\varepsilon_n\}$, Theorem 1.2 produces a vector b which synthesizes the desired eigenvalues $\alpha_n = \lambda_n + \varepsilon_n$ for A_F. A variation of this consists in synthesizing the eigenvalues of A_F through the *simultaneous* search of suitable vectors a, b in Y, given only A and $\{\varepsilon_n\}$. (This last situation does *not* mean, of course, that for all $a \in Y$, we can find a suitable vector $b \in Y$, such that we can synthesize the eigenvalues of A_F). This is achieved in Theorem 1.5 below, whose proof is similar to (and simpler than) the one in section 1.2. A major difference with Theorem 1.2, will be that Theorem 1.5 will not allow to assign wholly arbitrarily finitely many eigenvalues of A_F. Rather, all eigenvalues of A_F will have to remain sufficiently close to the corresponding eigenvalues of A. Despite this limitation, Theorem 1.5 will have a physically significant application to boundary feedback hyperbolic equations, where the spectrum of the original wave operator A is on the imaginary axis and the feedback (perturbation) operator shifts it to the open left hand side of C, while keeping it asymptotically approaching the imaginary axis. See Application 4.3 of Section 4.

THEOREM 1.5 (spectral assignment of A_F via vectors a and b). – (i) Let the generator A of the Introduction be given satisfying (K) of (0.5) with simple eigenvalues $\{\lambda_m\}$ and a basis of normalized eigenvectors $\{\Phi_m\}$ in Y. Let $\{\varepsilon_m\}$ be a sequence of non zero distinct complex numbers, which are assumed to satisfy the following conditions involving only $\{\varepsilon_m\}$ and $\{\lambda_m\}$:

(1) $0 < |\varepsilon_m| < d_m/2$, d_m as in (1.5), asymptotically [same as assumption (1) of Theorem 1.2]

$$\text{(1.27)}\quad (2)\quad \sum_{m=1}^{\infty} \frac{|\varepsilon_m|}{|\lambda_m|} \equiv C_1 < \infty .$$

$$\text{(1.28)}\quad (3)\quad \sup_j \sum_{\substack{i=1\\ j\neq i}}^{\infty} \frac{|\varepsilon_i|}{|\lambda_j - \lambda_i|} \equiv C_2 < \infty .$$

$$\text{(1.29)}\quad (4)\quad C_1 + C_2 < \tfrac{1}{2} .$$

Then, there exists suitable vectors $a, b \in Y$ such that the corresponding operator $A_F = A + (A\cdot, a)b$ has its eigenvalues [Lemma 1.1] $\{\alpha_m\}$ given precisely by

$$\alpha_m = \lambda_m + \varepsilon_m , \quad m = 1, 2, \dots . \tag{1.30}$$

Moreover, if Y splits into the direct sum $Y = Y_1 \oplus Y_2$, see Theorem 1.2 (i) below (1.10), then we may further require that the suitable pair $a, b \in Y$ claimed above satisfies the additional condition that either a, or b, or both be only in one component space, as before.

(ii) The case where the original operator A is spectral of scalar type with compact resolvent and simple eigenvalues can be reduced to the normal case (i), as in Theorem 1.2 (ii). □

REMARK 1.6. – Various sets of conditions can be given to insure (2)-(4). For instance, C_2 in (1.28) is upper bounded by (recall (1.5))

$$C_2 \leqslant \sum_{i=1}^{\infty} \frac{|\varepsilon_i|}{|d_i|} \equiv C_2' \tag{1.31}$$

and the infinite sum in (1.31) will be finite, if the ε_i's are chosen as to « compensate » the natural gap d_i of the original eigenvalues $\{\lambda_m\}$. □

PROOF OF THEOREM 1.5. – This time we impose that the constants $\{\alpha_m\}$ in (1.30) be precisely the eigenvalues of an operator A_F as in (1.1) for a suitable pair of vectors a and b in Y. Since we require that no α_m be also an eigenvalue of A, we invoke Lemma 1.1 (ii) and impose the identities (see (1.3) or (1.14a)):

$$\frac{\lambda_i}{\alpha_i-\lambda_i}a_ib_i+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\frac{\lambda_j}{\alpha_i-\lambda_j}a_jb_j\equiv 1\,,\qquad i=1,2,\dots$$

which, however, we rewrite now as

$$a_ib_i+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\frac{a_i-\lambda_i}{\alpha_i-\lambda_j}\frac{\lambda_j}{\lambda_i}a_jb_j\equiv\frac{\alpha_i-\lambda_i}{\lambda_i}$$

or

$$a_ib_i+\sum_{\substack{i=1\\ j\neq i}}^{\infty}\frac{\varepsilon_i}{\lambda_i}\frac{\lambda_j}{\alpha_i-\lambda_j}a_jb_j\equiv\frac{\varepsilon_i}{\lambda_i},\qquad i=1,2,\dots\,. \tag{1.32}$$

Eq. (1.32) can be more coincisely re-written as

$$(I+V)x=f \tag{1.33}$$

where I is the infinite identity matrix, and

$$V=[(v_{ij})]\,,\qquad v_{ij}=\begin{cases}\dfrac{\lambda_j\varepsilon_i}{\lambda_i(\alpha_i-\lambda_j)} & i\neq j\\ 0 & i=j\end{cases} \tag{1.34}$$

while

$$f=[f_1,f_2,\dots]\,,\qquad f_i=\varepsilon_i/\lambda_i \tag{1.35}$$

and the unknown x is

$$x=[x_1,x_2,\dots]\,,\qquad x_i=a_ib_i\,. \tag{1.36}$$

We seek x in l_1. By assumption (2) (see (1.27)), the given vector $f\in l_1$. We next prove

LEMMA 1.6. – Under assumptions (1) and (2) (i.e. (1.27)) and (3) (see (1.28) of Theorem 1.5, the infinite matrix V in (1.34) defines a bounded operator $l_1\to l_1$. Moreover, if assumption (4) (i.e. (1.29)) of the Theorem also holds, then $\|V\|_{l_1\to l_1}<1$.

PROOF OF LEMMA 1.6. – By a standard result [T-L.1] we compute from (1.34)

$$\|V\|_{l_1\to l_1} \equiv \sup_j \sum_{\substack{i=1\\ i\neq j}}^{\infty} |v_{ij}| = \sup_j \left\{ \sum_{\substack{i=1\\ 1\neq j}}^{\infty} \frac{|\lambda_j \varepsilon_i|}{|\lambda_i||\alpha_i - \lambda_j|} \right\}$$

(invoking the left hand side of (1.13) via assumption (1))

$$\leqslant 2 \sup_j \left\{ \sum_{\substack{i=1\\ i\neq j}}^{\infty} \frac{|\lambda_j \varepsilon_i|}{|\lambda_i||\lambda_i - \lambda_j|} \right\}$$

(adding and subtracting λ_i inside $|\ |$ in the numerator)

$$\leqslant 2 \sup_j \left\{ \sum_{i=1}^{\infty} \left|\frac{\varepsilon_i}{\lambda_i}\right| + \sum_{\substack{i=1\\ i\neq j}}^{\infty} \frac{|\varepsilon_i|}{|\lambda_i - \lambda_j|} \right\} = 2C_1 + 2C_2 < \infty \tag{1.37}$$

and where the right hand side is indeed < 1, if (4) is assumed

COROLLARY 1.7. – Under the assumptions of Lemma 1.6, the operator equation (1.33) in l_1 has a unique solution $x \in l_1$.

Continuing with the proof of Theorem 1.5, the solution $x \in l_1$ provided by Corollary 1.7 can be synthesized (in infinitely many ways, in fact) by two l_2 vectors $\{a_i\}$, $\{b_i\}$ such that $x_i = a_i\, b_i$. Theorem 1.5 is proved. □

2. – Spectrality of A_F. Counterexamples in two cases: (i) P is A-bounded. (ii) P is bounded, but the eigenvalues of A have zero gap.

We shall make use of the following characterization of spectrality [D-S, III. p. 2257]: The (perturbed) operator A_F on Y with compact resolvent (see Lemma 1.1) and eigenvalues $\{\alpha_k\}$, $|\alpha_k| \to \infty$ is spectral if and only if the following two conditions hold:

(i) the family of sums of finite collections of projections $E(\alpha_k, A_F)$ corresponding to single points α_k in the spectrum $\sigma(A_F)$ is uniformly bounded; i.e.

$$\left\| \sum_{k=1}^{K} E(\alpha_k, A_F) \right\|_{Y\to Y} \leqslant \text{const}, \quad \text{for all } K; \tag{2.1}$$

(ii) no nonzero y in Y satisfies all of the equations $E(\alpha_i, A_F)y = 0$, $i \geqslant 1$.

Condition (ii) means that ([D-S, III Definit., p. 2295 and Lemma 5, p. 2355], [S.1, p. 4451])

$$\overline{\text{span}}\,\{\text{generalized eigenvectors of } A_F^*\} = Y$$

and it always hold for $A_F = A + (A\cdot, a)b$, whose adjoint is $A_F^* = A^*[\cdot + a(b,\cdot)]$ (see (1.4)) in our cases of interest, when A is self-adjoint (canonical parabolic case) or skew-adjoint (canonical hyperbolic case): See the theorem in [D-S, III, p. 2374] as applied to A_F^*, which we can invoke, since, in these cases, A^{-p} is Hilbert-Schmidt for some positive integer p. This follows from the well known estimates $\lambda_m \sim m^{2/\dim \Omega}$ [C-H.1; T-1] for selfadjoint elliptic differential operators. Thus, we shall attempt in this section to violate condition (2.1) for suitable vectors $a, b \in Y$ and a suitable skewadjoint operator A.

We recall (1.2)

$$R(\lambda, A_F)y_0 = R(\lambda, A)y_0 + R(\lambda, A)b\,\frac{(AR(\lambda, A)y_0, a)}{1-(AR(\lambda, A)b, a)}$$

valid for all $\lambda \in \varrho(A)$ for which the denominator does not vanish. Also

$$E(\alpha_k, A_F) = \int_{\Gamma_k} R(\lambda, A_F)\, d\lambda \tag{2.2}$$

where Γ_k is any smooth close curve surrounding the point α_k (say a circle centered at α_k) and containing no other points of $\sigma(A_F)$.

2.1. *Constructive counterexamples.*

(i) *The case where P is A-bounded, but not bounded.* Consider the skew-adjoint operator A (corresponding to the canonical hyperbolic equation) defined by $A\Phi_n = -ir_n\Phi_n$, where $\{\Phi_n\}$ are an orthonormal basis of eigenvectors in Y with eigenvalues $\lambda_n = -ir_n$, $r_n > 0$, r_n strictly increasing to ∞. Thus, $A = i$ (a selfadjoint operator) and $\exp[At]$ is a unitary operator on Y, $t \in R$. We have, with $y_n = (y, \Phi_n)$:

$$\exp[At]y = \sum_{n=1}^{\infty} \exp[-ir_n t]y_n\Phi_n; \qquad R(\lambda, A)y = \sum_{n=1}^{\infty} \frac{y_n\Phi_n}{\lambda + ir_n}.$$

We shall next define suitable vectors a, b, and y_0 in Y, so that the corresponding operator A_F violates condition (2.1) of the first paragraph. To this end, we define these vectors by means of their coordinates through the following steps

(1) Let $\{s_n\}$ be an l_1-sequence of positive numbers such that

$$\begin{cases} r_{n_k}s_{n_k} \to +\infty\,, & \text{as } k\to\infty\,, \quad \text{for } n = \text{subsequence } n_k\,, \quad k = 1, 2, \ldots \\ s_n \equiv 0\,, & \text{for } n \neq n_k \end{cases} \tag{2.3}$$

where the subsequence $\{n_k\}$ and its translate by one $\{1 + n_k\}$ have no common elements: $\{n_k\} \cap \{1 + n_k\} = 0$; i.e. no $[1 + n_k]$ is a member of the sequence $\{n_k\}$.

(2) Next we impose that

$$s_n \equiv y_{0,n}a_n\,, \qquad n = 1, 2, \ldots \tag{2.4}$$

for the two l_2-sequences $\{y_{0,n}\}$ and $\{a_n\}$, coordinates of y_0 and a, say of positive numbers. We shall further require that $\|y_0\| \leqslant 1$. [All this can always be achieved by imposing $y_{0,n} \equiv a_n \equiv \sqrt{s_n}$, with $\sum_n s_n \leqslant 1$.]

Thus, by (2.4), to satisfy (2.3) we require

$$(2.5) \qquad \begin{cases} a_n \equiv 0\,, & \text{for } n \neq n_k\,, \quad k = 1, 2, \ldots \\ a_{n_k} = \text{committed by the requirement that} \\ r_{n_k}(y_{0,n_k} a_{n_k}) \to +\infty \\ \text{and further specified below.} \end{cases}$$

[Note that (2.5) implies that *neither* y_0, *nor* a *belong* to $\mathscr{D}(A)$.]

Thus, *P is A-bounded, but not bounded.*

(3) As to the vector b, we impose that

$$(2.6) \qquad \begin{cases} b_n \quad \equiv 0\,, & \text{for } n = n_k\,, \quad k = 1, 2, \ldots \\ b_{1+n_k} = \text{to be specified below} \\ \text{while the other co-ordinates are arbitrary.} \end{cases}$$

(4) The sequences $\{a_{n_k}\}_{k=1}^{\infty}$ and $\{b_{1+n_k}\}_{k=1}^{\infty}$ left uncommitted in (2.5)-(2.6) are chosen as to satisfy

$$(2.7) \qquad r_{n_k} s_{n_k} b_{1+n_k} \equiv r_{n_k}(y_{0,n_k} a_{n_k}) b_{1+n_k} \to \infty\,, \qquad \text{as } \to \infty$$

[note that $r_{n_k} s_{n_k} \to +\infty$, while $b_{1+n_k} \to 0$].

Specific examples satisfying (1)-(4).

Let $r_n \equiv n$ (one dimensional wave equation) and let

$$(2.8) \qquad s_n = \begin{cases} c/\ln n & n = n_k = [\exp[k^2]]\,, \quad k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where $[\exp[k^2]]$ = smallest integer larger or equal to $\exp[k^2]$.

Thus, $s_{n_k} \sim c/k^2$, $|\{s_n\}|_{l_1} \leqslant c \sum_k 1/k^2 = 1$ for $c = 6/\pi^2$.

$$r_{n_k} s_{n_k} \equiv n_k s_{n_k} \sim c\,\frac{[\exp[k^2]]}{k^2} \to \infty$$

and $\{n_k\} \cap \{1 + n_k\} = \emptyset$. Requirement (1) is checked. Then, take $a_n \equiv y_{0,n} = \sqrt{s_n}$

to satisfy (2). Finally define a sequence $\{b_n\} \in l_2$ by

$$\begin{cases} b_{1+n_k} = 1/k & k = 1, 2, \ldots \\ b_n = 0 & \text{otherwise}. \end{cases} \tag{2.9}$$

Then, with $r_n \equiv n$,

$$r_{n_k} s_{n_k} b_{1+n_k} \sim \frac{c[\exp[k^2]]}{k^k} \to \infty \tag{2.10}$$

and requirements (3)-(4) are checked as well.

Variations of these examples are immediate.

Continuation of analysis. As a result of $a_n \equiv 0$ for $n \neq n_k$ and $b_n \equiv 0$ for $n = n_k$, $k = 1, 2, \ldots$ we obtain $a_n b_n \equiv 0$ for all n and thus,

$$(AR(\lambda, A)b, a) = -i \sum_{n=1}^{\infty} \frac{r_n a_n b_n}{\lambda + ir_n} \equiv 0 .$$

Also, the Y-valued function

$$R(\lambda, A)b = \sum_{k=1}^{\infty} \frac{b_{1+n_k} \Phi_{1+n_k}}{\lambda + ir_{1+n_k}} \tag{2.12}$$

has simple poles at $\{-ir_{1+n_k}\}_{k=1}^{\infty}$, while the scalar function

$$(AR(\lambda, A)y_0, a) = -i \sum_{k=1}^{\infty} \frac{r_{n_k} y_{0,n_k} a_{n_k}}{\lambda + ir_{n_k}} \tag{2.13}$$

has simple poles at $\{-ir_{n_k}\}_{k=1}^{\infty}$. Then, by (1.2) recalled above (2.2) and (2.11), we obtain for $\lambda \in \varrho(A)$:

$$R(\lambda, A_F)y_0 = R(\lambda, A)y_0 + G(\lambda) \tag{2.14}$$

where

$$G(\lambda) \equiv G(\lambda; a; b; y_0) = R(\lambda, A)b(AR(\lambda, A)y_0, a) . \tag{2.15}$$

Since the integer $1 + n_k$ is strictly smaller than the integer n_{1+k}, we have $r_{1+n_k} < r_{n_{1+k}}$ for all k. Thus, if Γ_k is a circle centered at $-ir_{n_k}$ of radius, say $< \frac{1}{3} d_{n_k}$, where $d_{n_k} = \min\{|r_{1+n_k} - r_{n_k}|, |r_{n_k} - r_{n_k - 1}|\}$, it follows from the assertions below (2.12) and (2.13) that

$$\text{inside } \Gamma_k : \begin{cases} R(\lambda, A)b & \text{is analytic} \\ (AR(\lambda, A)y_0, a) & \text{has a simple pole at } -ir_{n_k} . \end{cases} \tag{2.16}$$

Now, from (2.13)

$$(2.17)\qquad (AR(\lambda, A)y_0, a)(\lambda + ir_{n_k}) = -ir_{n_k}y_{0,n_k}a_{n_k} - (\lambda + ir_{n_k})\sum_{\substack{l=1\\ l\neq k}}^{\infty}\frac{r_{n_l}y_{0,n_l}a_{n_l}}{\lambda + ir_{n_l}}$$

so that

$$(2.18)\qquad \frac{1}{2\pi i}\int_{\Gamma_k} G(\lambda)\,d\lambda = [\text{residue of } G(\lambda) \text{ at } \lambda = -in_k] = \lim_{\lambda\to -ir_{n_k}} G(\lambda)(\lambda + ir_{n_k}) =$$

(from (2.15)-(2.17))

$$= -ir_{n_k}y_{0,n_k}a_{n_k}R(-ir_{n_k}, A)b =$$

from (2.12)

$$= -ir_{n_k}y_{0,n_k}a_{n_k}\sum_{l=1}^{\infty}\frac{b_{1+n_l}\Phi_{1+n_l}}{-ir_{n_k} + ir_{1+n_l}} = \frac{-r_{n_k}y_{0,n_k}a_{n_k}b_{1+n_k}}{r_{1+n_k} - r_{n_k}}\Phi_{1+n_k} + v_k$$

where the vector v_k is given by

$$v_k = -r_{n_k}y_{0,n_k}a_{n_k}\sum_{\substack{l=1\\ l\neq k}}^{\infty}\frac{b_{1+n_l}\Phi_{1+n_l}}{r_{1+n_l} - r_{n_k}}$$

is orthogonal to Φ_{1+n_k}. Thus, by (2.18)

$$(2.19)\qquad \left|\frac{1}{2\pi i}\int_{\Gamma_k} G(\lambda)\,d\lambda\right|^2 \geqslant \frac{|r_{n_k}y_{0,n_k}a_{n_k}b_{1+n_k}|^2}{|r_{1+n_k} - r_{n_k}|^2}$$

while

$$(2.20)\qquad \left|\frac{1}{2\pi i}\int_{\Gamma_k} R(\lambda, A)y_0\,d\lambda\right| = \|y_{0,n_k}\Phi_{n_k}\| = |[\text{residue of } R(\lambda, A)y_0 \text{ at } \lambda = ir_{n_k}]| = y_{0,n_k}\downarrow 0\,, \quad \text{as } k\to\infty\,.$$

Thus, recalling (2.2) and (2.14) and using (2.19)-(2.20), we get for $\{-ir_n\} = \sigma(A)\subset\sigma(A_F)$ and $\|y_0\|\leqslant 1$:

$$(2.21)\qquad \|E(-ir_{n_k}, A_F)\| \geqslant \|E(-ir_{n_k}, A_F)y_0\| = \left\|\frac{1}{2\pi i}\int_{\Gamma_k}[R(\lambda, A)y_0 + G(\lambda)]\,d\lambda\right\| \geqslant \left|y_{0,n_k} - \frac{r_{n_k}y_{0,n_k}a_{n_k}b_{1+n_k}}{r_{1+n_k} - r_{n_k}}\right| \to\infty\,, \quad \text{as } k\to\infty$$

where the right hand side blows up to infinity in view of (2.7). Thus, condition (2.1) is violated, as desired.

(ii) *The case where P is bounded, but A has zero gap.* We make two observations:

(1) for the class of counterexamples considered above, we had that $a\notin\mathscr{D}(A)$,

as noted below (2.5). Thus, the perturbation $P = (A\cdot, a)b$ is A-bounded, but not bounded on Y;

(2) the right hand side of (2.21) would still blow up to infinity, even with the numerator *failing* to satisfy (2.7), provided that the denominator $[r_{1+n_k} - r_{n_k}]$ goes to zero faster than the numerator. This, in particular requires that the operator A defined before has eigenvalues $\{\lambda_n = -ir_n\}$ with *zero gap*, where the gap of the eigenvalues $\{\lambda_n\}$ is defined by

$$\text{gap} \equiv \inf_n \{|\lambda_n - \lambda_{n-1}|, \ |\lambda_n - \lambda_{n+1}|\} .$$

This way the arguments leading to Eq. (2.21) apply also to the case where the perturbation $P = (A\cdot, a)b$ is a *bounded operator*, i.e. $a \in \mathscr{D}(A)$ [in which case $r_n y_{0,n} a_n \to 0$, and $r_{n_k} y_{0,n_k} a_{n_k} b_{1+n_k} \to 0$], provided that (i) the eigenvalues $\{\lambda_n = -ir_n\}$ of A have zero gap and (ii)

$$\lim_{k\to\infty} \frac{r_{n_k} y_{0,n_k} a_{n_k} b_{1+n_k}}{r_{1+n_k} - r_{n_k}} = \lim_{k\to\infty} \frac{y_{0,n_k} a_{n_k} b_{1+n_k}}{r_{1+n_k}/r_n - 1} = \infty .$$

Since the $\{r_n\}$ are arbitrary, this can always be achieved.

2.2. *Indirect proof that $A_F = A + (A\cdot, a)b$ cannot be spectral for all $a, b \in Y$ (in fact, $(a, b) \in Y_a \times Y_b$).*

(i) *The case P is A-bounded, but not bounded.* Let A have an orthonormal basis $\{\Phi_n\}$ of eigenvectors on Y. A may either be self-adjoint (canonical parabolic case) or else skew adjoint (canonical hyperbolic case).

Consider the subspace $Y_a \times Y_b$ of $Y \times Y$ defined by

$$Y_a \times Y_b = \left\{(a, b): \begin{array}{l} a_n = (a, \Phi_n) \equiv 0, \ n = 1, 3, 5, \dots \\ b_n = (b, \Phi_n) \equiv 0, \ n = 2, 4, 6, \dots \end{array}\right\} .$$

Hence $\sigma(A) \subset \sigma(A_F)$, see Remark 1.1 (ii). If A_F were a spectral operator for vectors a and b in $Y_a \times Y_b$, then condition (2.1) would imply a fortiori that the eigenprojections $E(\alpha_k, A_F)$—which we now write as $E(\alpha_k; a, b)$ to emphasize the dependence on a and b—satisfy

$$\|E(\alpha_k; a, b)\|_{Y\to Y} \leq C_{a,b} , \quad \text{for } k = 1, 2, \dots, \quad \{\alpha_k\} = \sigma(A_F) \tag{2.23}$$

with $C_{a,b}$ a constant depending on a and b. Next, for $(a, b) \in Y_a \times Y_b$, we always have $(AR(\lambda, A)b, a) \equiv 0$ and hence (2.14) holds

$$R(\lambda, A_F) y_0 = R(\lambda, A) y_0 + G(\lambda; a; b; y_0)$$

with $G(\ )$ defined by (2.15). Thus, if (2.23) were to hold, then as before

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_k} G(\lambda, a; b; y_0)\, d\lambda \right\|_{Y \to Y} \leqslant C_{a,b,y_0}, \quad k = 1, 2, \ldots, \quad (a, b) \in Y_a \times Y_b \tag{2.24}$$

would follow, where Γ_k is a small circle centered at α_k and surrounding no other point of $\sigma(A_F)$. But the map $(a, b) \to G(\cdot, a; b; y_0)$ from $Y_a \times Y_b$ to Y is linear in a for fixed b and linear in b for fixed a. Applying twice the Principle of Uniform Boundedness on (2.24) for $(a, b) \in Y_a \times Y_b$ yields

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_k} G(\lambda; a; b; y_0)\, d\lambda \right\|_{Y \to Y} \leqslant C_{y_0}, \quad k = 1, 2, \ldots, \quad (a, b) \in \text{unit sphere of } Y_a \times Y_b \tag{2.25}$$

which is then a necessary condition for A_F to be spectral, for all vectors a and b in $Y_a \times Y_b$.

To prove our point, we shall now contradict the statement in (2.25), thereby showing indirectly that the operator A_F is *not spectral for some* $(a, b) \in Y_a \times Y_b$. For fixed k, let us define vectors a^k, b^k, y_0^k depending on k by

$$b^k \equiv \underbrace{[0, \ldots, 0, 1, 0, \ldots]}_{k\text{-th coordinate}} \tag{2.26}$$

$$y_0^k \equiv a^k \equiv \underbrace{[0, \ldots, 0, 1, 0, \ldots]}_{(k+1)\text{-th coordinate}} \tag{2.27}$$

so that $(a^k, b^k) \in$ sphere of radius 2 of $Y_a \times Y_b^1$, and $\|y_0^k\| \equiv 1$. Then

$$R(\lambda, A) b^k = \frac{1}{\lambda - \lambda_k}; \quad (AR(\lambda, A) y_0, a) = \frac{\lambda_{k+1}}{\lambda - \lambda_{k+1}}. \tag{2.28}$$

Thus, if the circle Γ_k is centered at λ_k and excludes any other eigenvalue, then by (2.15) and (2.28)

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_k} G(\lambda; a^k; b^k; y_0^k)\, d\lambda \right\|_{Y \to Y} = \left\| \frac{1}{2\pi i} \int_{\Gamma_k} \frac{\lambda_{k+1}\, d\lambda}{(\lambda - \lambda_k)(\lambda - \lambda_{k+1})} \right\| = \tag{2.29}$$
$$= |[\text{Residue of integrand] at } \lambda = \lambda_k| = \frac{|\lambda_{k+1}|}{|\lambda_k - \lambda_{k+1}|} \to \infty \quad \text{as } k \to \infty$$

thereby contradicting (2.25), as desired.

(ii) *The case P is bounded, but A has zero gap.* The two remarks made below (2.21) in the constructive proof have counterparts in the present indirect approach.

(1′) The presence of λ_{k+1} in the numerator of (2.29) reflects that P is A-bounded but is not bounded, for, in this latter case, $a \in \mathscr{D}(A)$, and the identity at the right of (2.28) would not contribute λ_{k+1} in the numerator.

(2′) In the case where P is bounded (λ_{k+1} does not appear in the numerator of (2.29)), the blowing up is still possible, and this occurs precisely when the eigenvalues of A have *zero gap*.

3. – A desirable case of spectrality: forcing the eigenvectors of A_F to form a Riesz basis with preassigned spectrum.

3.1. *Main results*: (i) *preservation of basis properties for* A_F (*Theorem* 3.1) *and* (ii) *spectral and Riesz basis assignement for* A_F *via a vector* $b \in Y$ (*Theorem* 3.3).

Our major result of this section is Theorem 3.3, which refers to the case where the unperturbed operator is spectral of scalar type, and hence similar to a normal operator A with an orthonormal basis of eigenvectors $\{\Phi_m\}$. Then, given the simple eigenvalues $\{\lambda_m\}$ of A, the co-ordinates $\{a_m\}$ of a vector $a \in Y$, and a sequence $\{\varepsilon_m\}$ of non zero distinct complex numbers, all subject to certain assumptions, we assert the existence (constructively) of a vector $b \in Y$, such that the corresponding perturbed operator $A_F = A + (A\cdot, a)b$ has two major and desirable properties: (i) (spectral assignement) its eigenvalues $\{\alpha_m\}$ are precisely given by $\alpha_m = \lambda_m + \varepsilon_m$; (ii) (preservation of basis properties of eigenvectors) its eigenprojections $\{Q_m\}$ are related to the complete, orthogonal eigenprojections $\{P_m\}$ of A by the similarity relation, $Q_m = W^{-1}P_m W$, m large enough, for a suitable bounded, boundedly invertible operator W on Y. (Following established terminology, we shall then say that the normalized eigenvectors Ψ_m's of A_F form a Riesz basis on Y.) Theorem 3.3 is the culmination, and combination, of two results: the spectral assignment result, Theorem 1.2, of section 1 and the basis preservation result at the beginning of this section, Theorem 3.1—Corollary 3.2. In Theorem 1.2, A and $a \in Y$ were given, and we then synthesized a suitable $b \in Y$, which forced A_F to have a preassigned spectrum $\{\alpha_m\}$. In Theorem 3.1, we start instead with a given spectrum $\{\alpha_m\}$ of A_F, $\{\lambda_m\}$ of A, and a vector $a \in Y$, and we deduce, under suitable assumptions, that A_F has in fact, a Riesz basis of eigenvectors. (Corollary 3.2 puts the assumptions on the given $\{\alpha_m\}$, $\{\lambda_m\}$ and a in more easily verifiable, yet less general, form.) Finally, we then combine these two results, to provide the desired synthesis of $b \in Y$ (via Theorem 1.2), which generates eigenvalues $\{\alpha_m\}$ for A_F, which in turn satisfy the assumptions of Corollary 3.2. This is Theorem 3.3. Some obvious implications of Theorem 3.3 to stability related questions of parabolic and hyperbolic generators A will be given next as corollaries in subsection 4.1, while subsection 4.2 will apply these corollaries to parabolic and hyperbolic boundary feedback systems.

THEOREM 3.1 (*Preservation of «basis» properties*). – Let the generator A of the Introduction be given [satisfying (K) in (0.5)], with simple eigenvalues $\{\lambda_m\}$ and eigenvectors $\{\Phi_m\}$ forming an orthonormal basis in Y. With $A_F = A + (A\cdot, a)b$, let $\{\alpha_m\}_{m=1}^{\infty}$ be the distinct eigenvalues [see Lemma 1.1 (i)] of A_F, all different from all $\{\lambda_m\}$. Set $\varepsilon_m = \alpha_m - \lambda_m$ and assume the following hypothesis (H1)-(H4), involving only $\{\lambda_m\}$, $\{a_m\}$ and $\{\varepsilon_m\}$:

(3.1) (H1) $0 < |\varepsilon_m| < d_m/2$, $\varepsilon_m \to 0$ as $m \to \infty$, where

$$d_1 = |\lambda_1 - \lambda_2| , \qquad d_m = \min\{|\lambda_m - \lambda_{m-1}|, |\lambda_m - \lambda_{m+1}|\} , \qquad m = 2, 3, \ldots$$

[(H1) coincides with assumption (1) of Theorem 1.2];

$$\text{(3.2)} \quad (H2) \qquad \sup_{j\geq K} \sum_{\substack{k=K\\k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|} \leq \frac{1}{2}(1-\delta) , \qquad 1 > \delta > 0$$

for $K = K_\delta^1$, henceforth kept fixed, and

$$\text{(3.3)} \quad (H2') \qquad \lim_{j\to\infty} \sum_{\substack{k=K\\k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|} = 0 \ (K \text{ fixed}) ;$$

$$\text{(3.4)} \quad (H3) \qquad \sum_{j=1}^{\infty} \frac{|\varepsilon_j|^2}{|\lambda_j a_j|^2} \sum_{\substack{k=1\\k\neq j}}^{\infty} \frac{|\lambda_k a_k|^2}{|\lambda_k - \lambda_j|^2} \leq \text{const} < \infty$$

[(H3) coincides with assumption (3) of Theorem 1.2];

(H4) (see (1.10))

$$\text{(3.5)} \qquad |\varepsilon_j|\gamma_j \equiv |\varepsilon_j| \sum_{\substack{k=1\\k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|^2} \to 0 \qquad \text{as } j \to \infty$$

[(H4) is implied a fortiori by assumption (4) of Theorem 1.2].

Then, the orthogonal complete eigenprojections $\{P_m\}_{m=1}^{\infty}$ of A and the eigenprojections $\{Q_m\}_{m=1}^{\infty}$ of A_F are related to each other by the similarity transformation in the following sense

$$\text{(3.6a)} \qquad Q_m = W^{-1} P_m W , \quad m = J , \quad J+1, \ldots; \quad Q_{0,J'} = W^{-1} P_{0,J} W;$$

J sufficiently large

$$Q_{0,J'} = Q_1 + \ldots + Q_{J'-1}; \qquad P_{0,J} = P_1 + \ldots P_{J-1};$$

J' possibly less then J in which case the projections Q_j, $J' \leq j \leq J-1$ are redundant, where $W = W_J^1$ is a bounded and boundedly invertible transformation on Y (which is defined below in (3.17)). As a consequence

$$\text{(3.6b)} \qquad \sum_{m=1}^{\infty} Q_m = I , \qquad \text{strongly and unconditionally on } Y$$

(even excluding the values of m: $J' \leqslant m \leqslant J-1$) and there is a constant $C = C_J = (\|W^{-1}\| \|W\|)^2 > 0$ such that

$$(3.6c) \qquad \frac{1}{C}\sum_{m=J}^{\infty} \|Q_m y\|_Y^2 + \|Q_{0,J'} y\|_Y^2 \leqslant \|y\|_Y^2 \leqslant C \sum_{m=J}^{\infty} \|Q_m y\|_Y^2 + \|Q_{0,J'} y\|_Y^2; \qquad y \in Y .$$

In terms of the normalized eigenvectors $\{\Psi_m\}$ of A_F: $A_F \Psi_m = \alpha_m \Psi_m$, $m = 1, 2,$ except $J' \leqslant m \leqslant J-1$ we have that: $Q_m y = c_m(y)\Psi_m$ for bounded linear functionals $\{c_m\}$ on Y, which form a biorthogonal sequence with the eigenvectors $\{\Psi_m\}$; i e : $c_m(\Psi_n) =$ (Kroneker) δ_{mn}. Indeed, from (3.6a)

$$c_1(y)\Psi_1 + \ldots + C_{J'-1}(y)\Psi_{J'-1} = (Wy, \Phi_1) W^{-1}\Phi_1 + \ldots + (Wy, \Phi_{J-1}) W^{-1}\Phi_{J-1}$$

$$c_m(y)\Psi_m = (Wy, \Phi_m) W^{-1}\varphi_m; \qquad |c_m(y)| \leqslant \|W\| \|W^{-1}\| \|y\| , \qquad m = J,\ J+1, \ldots$$

and the (normalized eigenvectors $\{\Psi_m\}_{m=J}^{\infty}$ of A_F and $\{\Phi_m\}_{=mJ}^{\infty}$ of A are related by

$$\Psi_m = (W\Psi_m, \Phi_m) W^{-1}\varphi_m , \qquad m = J , \quad J+1, \ldots$$

$$\left[\text{Also, } \frac{Q_m \Phi_m}{(W\Phi_m, \Phi_m)} = W^{-1}\Phi_m, \qquad m = J , \quad J+1, \ldots\right.$$

are non-normalized eigenvectors of A_F, written as the image of Φ_m under the bounded, boundedly invertible operator W^{-1}]. Thus, the following expansions hold (unconditionally):

$$(a) \qquad y = \sum_{m=1}^{\infty} c_m(y)\Psi_m , \qquad y \in Y$$

(even exluding $J' \leqslant m \leqslant J-1$: this will not be repeated below), and the $\{c_m\}$ are eigenvectors of A_F^* corresponding to the eigenvalues $\{\bar{\alpha}_m\}$, the complex conjugates of $\{\alpha_m\}$. From (3.6b), and hence

$$(3.7) \begin{cases} (b) \quad A_F y = \displaystyle\sum_{m=1}^{\infty} \alpha_m c_m(y)\Psi_m , \qquad y \in \mathscr{D}(A_F) = \mathscr{D}(A) \\ (c) \quad \exp[A_F t] y = \displaystyle\sum_{m=1}^{\infty} c_m(y) \exp[\alpha_m t]\, \Psi_m , \qquad y \in Y,\ t \geqslant 0 . \end{cases}$$

Moreover, by virtue of (3.6c), we have for any $y \in Y$ as in (3.7a):

$$(3.8a) \qquad C_{1J} \sum_m |c_m(y)|^2 \leqslant \|y\|_Y^2 \leqslant C_{2J} \sum_m |c_m(y)|^2 , \qquad 0 < C_{1J},\ C_{2J}$$

from $m = 1, 2, \ldots$ excluding $J' \leqslant m \leqslant J-1$.

This, applied to (3.7c) gives, in particular for $y \in Y$ and $t \geqslant 0$

$$(3.8b) \qquad C_{1J} \sum_m |c_m(y) \exp[\alpha_m t]|^2 \leqslant \|\exp[A_F t] y\|_Y^2 \leqslant C_{2J} \sum_m |c_m(y) \exp[\alpha_m t]|^2 .$$

A fortiori, A_F is a spectral operator. □

REMARK 3.1. – A more checkable set of conditions which imply assumptions $(H2)$ and $(H2')$ is given by the following variation

$$
(3.9a)\quad \begin{cases} (W2) \quad \{\varepsilon_m\} \in l_1 \\ \text{and} \\ \displaystyle\sum_{k=1}^{\infty} |\varepsilon_k| \sum_{\substack{j=1 \\ j\neq k}}^{\infty} \frac{1}{|\lambda_j - \lambda_k|^p} \equiv M < \infty \quad \text{for some } 1 \leqslant p < \infty . \end{cases}
$$

In fact, in this case, given $1 > \delta > 0$, it is always possible to find $K = K_\delta$ such that

$$
(3.9b)\qquad 2^P \sum_{k=K}^{\infty} |\varepsilon_k|^{p-1} M \leqslant (1-\delta)^p .
$$

Next, if we now set

$$
\sigma_j(K) \equiv \sum_{\substack{k=K \\ k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|},
$$

we see that both assumptions $(H2) = (3.2)$ and $(H2') = (3.3)$ are a fortiori simultaneously implied by the condition

$$
(3.9c)\qquad \sum_{j=K}^{\infty} \sigma_j^P(K) \leqslant \frac{(1-\delta)^P}{2^P} .
$$

To achieve (3.9c), we compute with $1/p' + 1/p = 1$:

$$
\sum_{j=K}^{\infty} \sigma_j^P(K) = \sum_{j=K}^{\infty} \left\{ \sum_{\substack{k=K \\ k\neq j}}^{\infty} |\varepsilon_k|^{1/p'} \frac{|\varepsilon_k|^{1/p}}{|\lambda_k - \lambda_j|} \right\}^p \leqslant \sum_{j=K}^{\infty} \left\{ \left[\sum_{\substack{k=K \\ k\neq j}}^{\infty} |\varepsilon_k| \right]^{1/p'} \left[\sum_{\substack{k=K \\ k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|^P} \right]^{1/p} \right\}^P \leqslant
$$

$$
\leqslant \left[\sum_{k=K}^{\infty} |\varepsilon_k| \right]^{p/p'} \sum_{j=K}^{\infty} \sum_{\substack{k=K \\ k\neq j}} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|^P} = \left[\sum_{k=K}^{\infty} |\varepsilon_k| \right]^{P-1} \sum_{k=K}^{\infty} |\varepsilon_k| \sum_{\substack{j=K \\ j\neq k}}^{\infty} \frac{1}{|\lambda_k - \lambda_j|^P} \leqslant (1-\delta)^P / 2^P
$$

as desired, where in the last step we have used (3.9b).

We summarize the above argument by the implications:

$$
(W2) \equiv (3.9a) \to (3.9b) \to (3.9c) \to \begin{cases} (H2) \\ (H2') \end{cases}
$$

and our claim is proved.

Note, moreover that

$$
\sum_{m=1}^{\infty} \frac{1}{|\lambda_m|^p} < \infty, \quad \text{for some } 1 \leqslant p < \infty
$$

holds *always* true either in the case of A selfadjoint (canonical parabolic case) or of A skewadjoint (canonical hyperbolic case): see Remark 1.5. Thus, in these cases, the series in j which occurs in $(W2) = (3.9a)$ is automatically convergent for each k fixed. □

We state formally this result, which is less general, but in more easily verifiable form.

COROLLARY 3.2. – The same conclusions of Theorem 3.1 hold under assumptions $(H1)$, $(W2)$, $(H3)$, $(H4)$ on the sequences $\{\lambda_m\}$, $\{a_m\}$ and $\{\varepsilon_m\}$ which are described in Theorem 3.1. □

We now combine Corollary 3.2 with Theorem 1.2 for the purpose of *synthesizing* the properties of the eigenvalues $\{\alpha_m = \lambda_m + \varepsilon_m\}$ of A_F required in Corollary 3.2 via *a suitable vector* $b \in Y$, *once* A *and* $a \in Y$ *are given.* To this end, we recall what already noted:

$(H1)$ in Theorem 3.1 *coincides* with (1) in Theorem 1.2;

$(H3)$ in Theorem 3.1 *coincides* with (3) in Theorem 1.2;

$(H4)$ in Theorem 3.1 *is implied by* (4) in Theorem 1.2.

Moreover, as we have seen in Remark 3.1 above that

$$(W2) \text{ in Corollary 3.2} \Rightarrow \begin{cases} (H2) \\ (H2') \end{cases}$$

while we shall see in Lemma 3.5 below and Remark 3.3 that

$$\left.\begin{array}{l} (H2) \xrightarrow{(1.13)} (3.30a) \\ \\ (H2') \xrightarrow{(1.13)} (3.32) \end{array}\right\} \rightarrow \begin{array}{l} (3.31) \\ \\ (3.32) \end{array} \rightarrow \text{assumption (2) of Thm. 1.2, i.e. (1.6)}\,.$$

Thus, given A and $a \in Y$, we can synthesize (claim the existence of) a vector $b \in Y$, such that the corresponding operator $A_F = A + (A\cdot, a)b$ satisfies the conclusions of Theorem 3.1 under the sole assumptions:

(1) of Theorem 1.2; i.e. (1.5);

$(W2)$ of Corollary 3.2; i.e. (3.9a);

(3) of Theorem 1.2; i.e. (1.7)-(1.8);

(4) of Theorem 1.2; (1.9)-(1.10).

[We also note that $(W2)$ for $p = 2$ implies (4).]
We collect all this in the *main* result of the present section.

THEOREM 3.3 (*Spectral and Riesz basis assignment for A_F via a vector b*). – (i) Let the generator A of the Introduction be given [satisfying (K) of (0.5)] with simple eigenvalues $\{\lambda_m\}$ and eigenvectors $\{\Phi_m\}$ forming an orthonormal basis in Y, $m = 1, 2, \ldots$. Let $a = \{a_m = (a, \Phi_m)\}$, $a_m \neq 0$ be given in Y. Let $\{\varepsilon_m\}$ be a sequence of non-zero distinct complex numbers, which are assumed to satisfy the following conditions, involving only $\{\varepsilon_m\}$, $\{\lambda_m\}$ and $\{a_m\}$:

Condition (1) of Theorem 1.2; i.e.

$$0 < |\varepsilon_m| < d_m/2\,, \quad d_m \text{ defined by (1.5); asymptotically} \tag{3.10}$$

Condition ($W2$) of Corollary 3.2; i.e.

$$\begin{cases} \{\varepsilon_m\} \in l_1 \\ \text{and} \\ \displaystyle\sum_{k=1}^{\infty} |\varepsilon_k| \sum_{\substack{j=1\\ j\neq k}}^{\infty} \frac{1}{|\lambda_j - \lambda_k|^p} \equiv M < \infty\,, \quad \text{for some } 1 < p < \infty\,; \end{cases} \tag{3.11}$$

Condition (3) of Theorem 1.2; i.e.

$$\sum_{k=1}^{\infty} \frac{|\varepsilon_k|^2}{|\lambda_k a_k|^2} \sum_{\substack{j=1\\ j\neq k}}^{\infty} \frac{|\lambda_j a_j|^2}{|\lambda_j - \lambda_k|^2} < \infty\,; \tag{3.12}$$

Condition (4) of Theorem 1.2; i.e.

$$\sum_{k=1}^{\infty} |\varepsilon_k| \sum_{\substack{j=1\\ j\neq k}}^{\infty} \frac{|\varepsilon_j|}{|\lambda_j - \lambda_k|^2} < \infty\,. \tag{3.13}$$

Then, there exists a suitable vector $b \in Y$—obtained as in Theorem 1.2—which is such that the corresponding operator $A_F = A + (A\cdot, a)b$ has the following properties:

(i) the eigenvalues of A_F are precisely

$$\alpha_m = \lambda_m + \varepsilon_m\,, \quad m = 1, 2, \ldots;$$

(ii) the co-ordinates $b_m = (b, \Phi_m)$ of b satisfy

$$b_m \sim \frac{\varepsilon_m}{\lambda_m a_m} \tag{3.14}$$

(see more precisely (3.31) and (3.33));

(iii) the corresponding eigenprojections $\{Q_m\}_{m=1}^{\infty}$ of A_F are related to the complete orthogonal family of eigenprojections $\{P_m\}_{m=1}^{\infty}$ of A by the same similarity relations as in (3.6a)

[so that the sequence $\{\Psi_m\}_{m=J}^{\infty}$ of normalized eigenvectors of A_F forms a Riesz basis in $Y \setminus P_{0,J} Y$] and all the results in the conclusion of Theorem 3.1 following (3.6a) hold, in particular the expansions (3.7) and the double inequalities (3.8). Thus A_F is a fortiori a spectral operator.

(ii) The case where the original operator A is spectral of scalar type with compact resolvent and simple eigenvalues can be reduced to the normal case (i), by a similarity transformation Π: $\Pi^{-1} A \Pi = A_N$, A_N normal, $\Pi, \Pi^{-1} \in \mathscr{L}(Y)$, Π self adjoint [see Theorem 1.2 (ii)]. We can apply part (i) to the operator $\Pi^{-1} A_F \Pi = A_N + (A_N, \Pi a) \Pi^{-1} b$ whose eigenvalues are the same as those of A_F, whose eigenvectors are the image under Π of those of A_F, and whose semigroup $\exp[\Pi^{-1} A_F \Pi t]$ has the desirable expansions of part (i). But, then

$$\exp[A_F t] = \Pi \exp[\Pi^{-1} A_F \Pi t] \Pi^{-1} . \quad \square$$

3.2. *Proof of Theorem* 3.1.

STEP 1. – We shall invoke Lemma 4.17 a, p. 294 in KATO [K.1] where a sufficient condition is given, which guarantees that the eigenprojections $\{Q_m\}$ of A_F satisfy (3.6a), so that its eigenvectors $\{\Psi_m\}$ form a Riesz basis. Following Kato's notation, we shall let P_j and Q_j denote in this section the projections onto the finite-dimensional eigenspaces [Lemma 1.1] corresponding to the eigenvalues λ_j of A and α_j of A_F, $j = 1, 2, \ldots$ respectively, i.e.

$$P_j = \frac{1}{2\pi i} \int_{\mathscr{C}_j} R(\lambda, A)\, d\lambda \, ; \qquad Q_j = \frac{1}{2\pi i} \int_{C_j} R(\lambda, A_F)\, d\lambda \tag{3.15}$$

where $\mathscr{C}_j$ (resp. C_j) is a small circle centered at λ_j (resp. at α_j) and surrounding no other eigenvalue of A (resp. of A_F). According to Kato's Lemma, our major task is to show that under the assumptions of Theorem 3.1, we have

$$\sum_{j=J'}^{\infty} \|P_j(Q_j - P_j) y\|_Y^2 \leq c_{J'} \|y\|_Y^2 , \qquad y \in Y,\ c_{J'} < 1 , \qquad J' \text{ sufficiently large} . \tag{3.16a}$$

After this, we define $Q_{0,J'} = Q_1 \cdots + Q_{J'-1}$, and take J, possibly larger than J' such that with $P_{0,J} = P_1 + \ldots + P_{J-1}$ we have $J - 1 = \dim P_{0,J} = \dim Q_{0,J'}$. Thus, (i) the family $[P_{0,J}, \{P_j\}_{j=J}^{\infty}]$ is a complete family of orthogonal projections on Y. (ii) Eq. (3.16a) holds a fortiori with J' replaced by the possibly larger J and we re-write it for further reference

$$\sum_{j=J}^{\infty} \|P_j(Q_j - P_j) y\|_Y^2 \leq C_J \|y\|_Y^2 , \qquad y \in Y,\ C_J < 1 , \qquad J \text{ sufficiently large} . \tag{3.16b}$$

Thus, we can apply Kato's Lemma with respect to the family of eigenprojections $[Q_{0,J'}, \{Q_j\}_{j=J}^{\infty}]$ and conclude that $Q_j = W^{-1}P_jW$, $j = J, J+1, \dots$, etc. as in (3.6*a*) with the transformation $W = W_J$ given by (see [K 1, Eq. (4.32)])

$$W = W_J = P_{0,J}Q_{0,J'} + \sum_{j=J}^{\infty} P_jQ_j\,, \qquad W, W^{-1} \in \mathscr{L}(Y)\,. \tag{3.17}$$

Also, $P_jx = (x, \Phi_j)\Phi_j$ and $Q_jx = c_j(x)\Psi_j$ for some unit vector Ψ_j, $c_j(\Psi_k) = \delta_{jk}$. By commutativity of Q_j and A_F, then $Q_j\Psi_j = \Psi_j$ yields that Ψ_j is an eigenvector of A_F with corresponding eigenvalue $\alpha_j = c_j(A_F\Psi_j)$. From this and $\sum_{j=1} Q_j = I$, strongly and unconditionally, we can derive the expansions (3.7).

Also, the double inequality (3.6*c*) follows from (3.6*a*): in one direction we have

$$\|Q_{0,J'}y\|^2 + \sum_{m=J}^{\infty} \|Q_my\|^2 \leqslant \|W^{-1}\|^2\Big[\|P_{0,J}Wy\|^2 + \sum_{m=J}^{\infty} \|P_mWy\|^2\Big] = \|W^{-1}\|^2\|Wy\|^2$$

by orthonormality of the complete family $P_{0,J}$, $\{P_m\}_{m=J}^{\infty}$ and the left inequality in (3.6*c*) follows. In the other direction, we have by (3.6*a*)

$$\|x\|^2 = \|P_{0,J}x\|^2 + \sum_{m=J}^{\infty} \|P_mx\|^2 \leqslant \|W\|^2\|Q_{0,J'}W^{-1}x\| + \sum_{m=J}^{\infty} \|Q_mW^{-1}x\|^2$$

and the right inequality on (3.6*c*) follows by setting

$$y = W^{-1}x\,, \qquad \|y\| \leqslant \|W^{-1}\|\,\|x\|\,.$$

Thus, Theorem 3.1 will be proved, as soon as (3.16) is established.

REMARK 3.2. – In Kato's formulation, the constant c appearing in his Eq. (4.31), p. 294 is required to be strictly less than one; i.e. in our corresponding Eq. (3.16) above, C_J must be < 1. This will be achieved at the expenses of incorporating in P_0 and Q_0 in Kato's notation more eigenprojections, as indicated by the definition of our $P_{0,J}$ and $Q_{0,J'}$ in (3.6*a*), thereby having the index j in our Eq. (3.16*b*) run only fron a sufficiently large J on. □

The core of our proof consists in showing (3.16*b*). We begin with computing $(Q_j - P_j)$ for j large enough. Recalling our assumption (3.1) $|\varepsilon_j| < d_j/2$, we can take a circle Γ_j encircling only λ_j and α_j and no other eigenvalue of A or A_F: Invoking (1.2), we obtain from (3.15):

$$2\pi i(Q_j - P_j)y = \int_{\Gamma_j} [R(\lambda, A_F) - R(\lambda, A)]y\, d\lambda = \int_{\Gamma_j} \frac{R(\lambda, A)b\big(AR(\lambda, A)y, a\big)}{1 - \big(AR(\lambda, A)b, a\big)}\, d\lambda$$

(using the identities $(AR(\alpha_j, A)b, a) \equiv 1$, see (1.3a), which define the α_j's all different from all $\{\lambda_m\}$)

$$= \int_{\Gamma_j} \frac{R(\lambda, A) b(AR(\lambda, A)y, a)}{(A[R(\alpha_j, A)b - R(\lambda, A)]b, a)} d\lambda$$

(using the first resolvent equation)

$$(3.18) \qquad = \int_{\Gamma_j} \frac{R(\lambda, A) b(AR(\lambda, A)y, a)}{(\lambda - \alpha_j)(AR(\alpha_j, A)R(\lambda, A)b, a)} d\lambda \equiv \int_{\Gamma_j} u_j(\lambda)\, d\lambda =$$

$$= 2\pi i \left\{ \begin{bmatrix} \text{[Residue of integrand } u_j \\ \text{at } \lambda = \alpha_j] \end{bmatrix} + \begin{bmatrix} \text{[Residue of integrand } u_j \\ \text{at } \lambda = \lambda_j] \end{bmatrix} \right\}.$$

Accordingly, we compute

$$)3.19) \qquad \begin{bmatrix} \text{Residue of } u_j \\ \text{at } \lambda = \alpha_j \end{bmatrix} = \lim_{\lambda \to \alpha_j} u_j(\lambda)(\lambda - \alpha_j) = \frac{R(\alpha_j, A) b(AR(\alpha_j, A)y, a)}{(AR^2(\alpha_j, A)b, a)}.$$

Moreover, since

$$u_j(\lambda) = \frac{\left\{\dfrac{b_j \Phi_j}{\lambda - \lambda_j} + \sum\limits_{\substack{k=1 \\ k \neq j}}^{\infty} \dfrac{b_k \Phi_k}{\lambda - \lambda_k}\right\}\left\{\dfrac{\lambda_j y_j a_j}{\lambda - \lambda_j} + \sum\limits_{\substack{k=1 \\ k \neq j}}^{\infty} \dfrac{\lambda_k y_k a_k}{\lambda - \lambda_k}\right\}}{(\lambda - \alpha_j)\left\{\dfrac{\lambda_j a_j b_j}{(\alpha_j - \lambda_j)(\lambda - \lambda_j)} + \sum\limits_{\substack{k=1 \\ k \neq j}}^{\infty} \dfrac{\lambda_k a_k b_k}{(\alpha_j - \lambda_k)(\lambda - \lambda_k)}\right\}}$$

after multiplying numerator and denominator by $(\lambda - \lambda_j)$, we readily find

$$(3.20) \qquad \begin{bmatrix} \text{Residue of } u_j \\ \text{at } \lambda = \lambda_j \end{bmatrix} = \lim_{\lambda \to \lambda_j} u_j(\lambda)(\lambda - \lambda_j) = -y_j \Phi_j.$$

We combine (3.18)-(3.20) and obtain

$$(3.21) \qquad (Q_j - P_j)y = -y_j \Phi_j + \frac{R(\alpha_j, A) b(AR(\alpha_j, A)y, a)}{(AR^2(\alpha_j, A)b, a)}.$$

From here, we obtain the desired term

$$(3.22) \qquad P_j(Q_j - P_j)y = -y_j \Phi_j + \frac{b_j(AR(\alpha_j, A)y, a)}{(\alpha_j - \lambda_j)(AR^2(\alpha_j, A)b, a)} \Phi_j.$$

Using again the eigenvector expansion in which we isolate the j-th term and standard manupulations, we finally arrive at

$$(3.23) \qquad P_j(Q_j - P_j)y = \Phi_j \left\{ -y_j + \frac{y_j + \mathcal{A}_j(y)}{1 + \mathcal{B}_j} \right\}$$

where

$$\mathcal{A}_j(y) \equiv \mathcal{A}_j(y;a) = \frac{\alpha_j - \lambda_j}{\lambda_j a_j} \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \frac{\lambda_k y_k a_k}{a_j - \lambda_k} \tag{3.24}$$

(we wish to explicitly note the dependence on the vector a)

$$\mathscr{B}_j = \frac{(\alpha_j - \lambda_j)^2}{\lambda_j a_j b_j} \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \frac{\lambda_k a_k b_k}{(\alpha_j - \lambda_k)^2}. \tag{3.25}$$

Thus, Kato's condition (3.16) becomes, via (3.23)-(3.25):

$$\sum_{j=J}^{\infty} \left| -y_j + \frac{y_j + \mathcal{A}_j(y)}{1 + \mathscr{B}_j} \right|^2 \leqslant C_J \|y\|_Y^2, \quad y \in Y, \ C_J < 1, \tag{3.26}$$

for sufficiently large J.

In order to satisfy this condition, it suffices to require

$$\begin{cases} (a) \quad |\mathscr{B}_j| \to 0 \quad \text{as } j \to \infty \\ (b) \quad \sum_{j=J}^{\infty} |\mathscr{B}_j(y)|^2 \leqslant C_J \|y\|_Y^2, \quad y \in Y, \ C_J < 1. \end{cases} \tag{3.27}$$

We formalize what we have so far obtained in the following Lemma, in which we re-write (3.27*a*) and provide an explicit sufficient condition for (3.27*b*) to hold (via Schwarz inequality).

LEMMA 3.4. – Theorem 3.1 holds as soon as we prove that its assumptions (*H*1)-(*H*4) guarantee that the following conditions can be fulfilled:

$$|\mathscr{B}_j| = \frac{|\varepsilon_j|}{|\lambda_j a_j b_j|} |\varepsilon_j| \left| \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \frac{\lambda_k a_k b_k}{(\alpha_j - \lambda_k)^2} \right| \to 0, \quad \text{as } j \to \infty \tag{3.28}$$

$$\sum_{j=J}^{\infty} \frac{|\varepsilon_j|^2}{|\lambda_j a_j|^2} \left\{ \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \frac{|\lambda_k a_k|^2}{|\alpha_j - \lambda_k|^2} \right\} \leqslant C_J < 1, \quad J \text{ sufficiently large.} \tag{3.29}$$

STEP 2. – It remains to show that the above conditions (3.28)-(3.29) can indeed be fulfilled. To this end, we shall start by investigating the ratio $\varepsilon_j/\lambda_j a_j b_j$ which occurs in (3.28) for j large.

In the next Lemma, no conditions at all are imposed on the first $(K-1)$ differences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{K-1}$, a situation which arises when one wishes to replace the original eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{K-1}$ of A by new eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_{K-1}$ of A_F with no constraints at all.

LEMMA 3.5. – (I) Let $\{\lambda_m\}$ and $\{\alpha_m\}$ be the eigenvalues of A and A_F, $m = 1, 2, \ldots$ respectively and set $\varepsilon_m = \alpha_m - \lambda_m$. Assume that $0 < |\varepsilon_k| < d_k/2$ (see (1.5)) for $k \geqslant K$ and, moreover, that

$$\sup_{j\geqslant K} s_j(K) \leqslant 1 - \delta\,, \quad 1 > \delta > 0 \tag{3.30a}$$

where

$$s_j(K) \equiv \sum_{\substack{k=K\\k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \alpha_j|}\,, \quad j \geqslant K \tag{3.30b}$$

for $K = K_\delta$, henceforth kept fixed.

[(3.30) is implied by $(H2) = (3.2)$ of Theorem 3.1, via (1.13).] Then, the corresponding vectors a and b in the definition of A_F have coordinates $a_k b_k \neq 0$, $k = K$, $K+1, \ldots$, which satisfy the asymptotic relations:

$$0 < c_\delta \leqslant \frac{|\varepsilon_j|}{|\lambda_j a_j b_j|} \quad \text{for all } j \geqslant K \tag{3.31}$$

where c_δ is a positive constant depending on δ but not on $j \geqslant K$.

Under the additional assumption that

$$\lim_{j\to\infty} s_j(K) = 0\,, \quad K \text{ fixed} \tag{3.32}$$

[(3.32) is equivalent to $(H2') = (3.3)$ of Theorem 3.1, by (1.13)] then the vectors a and b in A_F satisfy also the following upper bounds

$$\frac{|\varepsilon_j|}{|\lambda_j a_j b_j|} \leqslant \frac{1}{1-2\varepsilon} < \infty \quad \text{for all } j \geqslant J(K, \varepsilon)\,, \quad 0 < \varepsilon < \tfrac{1}{2} \tag{3.33}$$

ε arbitrary positive number given in advance and $J(K, \varepsilon)$ is a positive integer depending on K and ε (but not on $j \geqslant J(K, \varepsilon)$), which is generally larger than K.

(II) A checkable set of conditions which are sufficient for the simultaneous fulfillment of both (3.30) and (3.32) is, as we shall see, that we require ($\{\varepsilon_m\} \in l_1$ and, moreover)

$$\sum_{k=K}^{\infty} |\varepsilon_k| \equiv C_{1,K} < \infty \tag{3.34}$$

$$\sum_{k=K}^{\infty} |\varepsilon_k| \sum_{\substack{j=K\\j\neq k}}^{\infty} \frac{1}{|\lambda_j - \lambda_k|^p} \equiv C_{2,K} < \infty\,, \quad \text{for some } 1 \leqslant p < \infty \tag{3.35}$$

with

$$2^p (C_{1,K})^{p-1} C_{2,K} \leqslant (1-\delta)^p\,, \quad K = K_\delta,\ 0 < \delta < 1\,. \tag{3.36}$$

Note, that (3.34)-(3.36) are, in turn, implied by the more easily verified condition $(W2) = (3.9a)$ of Corollary 3.2. □

REMARK 3.3. – Note, that (3.31) and (3.33) together give $b_m \sim \varepsilon_m/\lambda_m a_m$ and so assumption (2) of Theorem 1.2 (see (1.6)) is automatically fulfilled. Thus, in particular, condition $(W2) = (3.9a)$ implies (1.6), as noted below Corollary 3.2. □

PROOF OF LEMMA 3.5. – Identity (1.3b) which defines the sequence $\{\alpha_k\}$, $k \geqslant K$, is

$$\frac{\lambda_j a_j b_j}{\alpha_j - \lambda_j} + \sum_{\substack{k=K\\k\neq j}}^{\infty} \frac{\alpha_k - \lambda_k}{\alpha_j - \lambda_k}\,\frac{\lambda_k a_k b_k}{\alpha_k - \lambda_k} = 1 - \sum_{k=1}^{K-1} \frac{\lambda_k a_k b_k}{\alpha_j - \lambda_k} \equiv f_{K,j}\,, \quad j = K,\, K+1, \dots \tag{3.36a}$$

which can be coincisely written as

$$v_j + \sum_{\substack{k=K\\k\neq j}}^{\infty} t_{jk} v_k = f_{K,j}\,, \qquad j \geqslant K\,, \tag{3.36b}$$

or

$$v + T_K v = f_K \tag{3.36c}$$

where v is the infinite vector

$$v = [(v_j),\, j \geqslant K]\,, \qquad v_j = \frac{\lambda_j a_j b_j}{\varepsilon_j} \tag{3.37}$$

(we do not indicate for v the dependence on K) and T_K is the infinite matrix

$$T_K = [(t_{jk})]\,, \qquad j,\, k \geqslant K\,, \qquad t_{jk} = \begin{cases} \dfrac{\varepsilon_k}{\alpha_j - \lambda_k}\,, & k \neq j \\ 0\,, & k = j\,. \end{cases} \tag{3.38}$$

The vector $f_K = [(f_{K,j}),\, j \geqslant K]$ defined by the right hand side of (3.36a) plainly belongs to l_∞. Accordingly, our aim is to show that (3.36c) can be rightfully viewed as an operator equation in the unknown vector v in the space l_∞. Thus, we want to establish that:

(i) the infinite matrix T_K in (3.38) defines a bounded (linear) operator $l_\infty \to l_\infty$;

(ii) $(I + T_K)$ has a bounded inverse $(I + T_K)^{-1}$ defined on all of l_∞ (where I = infinite identity matrix).

To achieve (i) and (ii), it suffices to require that [T-L.1, p. 223] (see (3.38))

$$\|T_K\|_\infty = \|T_K\|_{l_\infty \to l_\infty} = \sup_{j \geqslant K} \sum_{k=K}^{\infty} |t_{jk}| = \sup_{j \geqslant K} \sum_{\substack{k=K\\k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \alpha_j|} \equiv \sup_{j \geqslant K} s_j(K) \leqslant 1 - \delta < 1 \tag{3.39}$$

for some $\delta > 0$, and $K = K_\delta$ which is precisely our assumption (3.30). With (3.39) guaranteed, we can then solve uniquely (3.36c) for $v \in l_\infty$: by (3.37), this means that

$$|v_j| = \frac{|\lambda_j a_j b_j|}{|\varepsilon_j|} \leqslant \text{const}_K , \qquad \text{for all } j \geqslant K \tag{3.40a}$$

(in fact, e.g. for $j \geqslant K$)

$$\frac{|\lambda_j a_j b_j|}{|\varepsilon_j|} = |v_j| \leqslant \|v\|_\infty \leqslant \|(I + T_K)^{-1}\|_\infty \|f_K\|_\infty \leqslant \frac{1}{1 - \|T_K\|} \|f_K\|_\infty \leqslant \frac{1}{\delta} \|f_K\|_\infty . \tag{3.40b}$$

By Remark 1.1 (ii), $a_j b_j \neq 0$ (since $|\varepsilon_j| > 0$, $j \geqslant K$) and the desired inequality (3.31) follows. To prove now the inequality (3.33) under the additional assumption (3.32), we return to (3.36b), which we write explicitly (see (3.36a) and (3.38)):

$$v_j = 1 - \sum_{k=1}^{K-1} \frac{\lambda_k a_k b_k}{\alpha_j - \lambda_k} - \sum_{\substack{k=K \\ k \neq j}}^{\infty} \frac{\varepsilon_k}{\alpha_j - \lambda_k} v_k , \qquad j \geqslant K \tag{3.41}$$

with $K = K_\delta$ fixed by (3.30a). Thus, as $j \to \infty$ (with K fixed), the finite sum in (3.41) goes to zero and $\left|\sum_{k=1}^{K-1}\right| < \varepsilon$, for all $j \geqslant J_1(K, \varepsilon)$, ε arbitrary positive constant. As to the infinite sum in (3.41), we have (using that $v \in l_\infty$, say (3.40b))

$$\left| \sum_{\substack{k=K \\ k \neq j}}^{\infty} \frac{\varepsilon_k}{\alpha_j - \lambda_k} v_k \right| \leqslant \|v\|_\infty \sum_{\substack{k=K \\ k \neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \alpha_j|} = \|v\|_\infty s_j(K) < \varepsilon , \qquad \text{for all } j \geqslant J_2(K, \varepsilon)$$

by assumption (3.32). Thus, we conclude from (3.41) that for $0 < \varepsilon < \frac{1}{2}$

$$\frac{|\lambda_j a_j b_j|}{|\varepsilon_j|} = |v_j| \geqslant 1 - 2\varepsilon , \qquad \text{for all } j \geqslant \max [J_1(K, \varepsilon), J_2(K, \varepsilon)] \equiv J(K, \varepsilon)$$

and (3.33) follows as before ($a_j b_j \neq 0$). The first part (I) of the Lemma is proved.

We now show the second part (II) involving the more explicit conditions (3.34)-(3.36) by proceding as in Remark 3.1. A sufficient condition for both (3.39) = (3.30) and (3.32) to hold simultaneously is obtained by imposing that

$$\sum_{j=K}^{\infty} s_j^p(K) \leqslant (1 - \delta)^p , \qquad \text{for some } \infty > p \geqslant 1, \text{ i.e. (with } 1/p' + 1/p = 1\text{):}$$

$$\sum_{j=K}^{\infty} s_j^p(K) \equiv \sum_{j=K}^{\infty} \left\{ \sum_{\substack{k=K \\ k \neq j}}^{\infty} |\varepsilon_k|^{1/p'} \frac{|\varepsilon_k|^{1/p}}{|\lambda_k - \alpha_j|} \right\}^p \leqslant \sum_{j=K}^{\infty} \left\{ \left[\sum_{k=K}^{\infty} |\varepsilon_k| \right]^{1/p'} \left[\sum_{\substack{k=K \\ k \neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \alpha_j|^p} \right]^{1/p} \right\}^p =$$

$$= \left[\sum_{k=K}^{\infty} |\varepsilon_k| \right]^{p/p'} \sum_{j=K}^{\infty} \sum_{\substack{k=K \\ k \neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \alpha_j|^p} = \left[\sum_{k=K} |\varepsilon_k| \right]^{p-1} \sum_{k=K}^{\infty} |\varepsilon_k| \sum_{\substack{j=K \\ j \neq k}}^{\infty} \frac{1}{|\lambda_k - \alpha_j|^p} \leqslant$$

$$\text{(by (1.13))} \qquad \leqslant 2^p \left[\sum_{k=K}^{\infty} |\varepsilon_k| \right]^{p-1} \left[\sum_{k=K}^{\infty} |\varepsilon_k| \sum_{\substack{j=K \\ j \neq k}}^{\infty} \frac{1}{|\lambda_k - \lambda_j|^p} \right] = 2^p (C_{1,K})^{p-1} C_{2,K} \leqslant (1 - \delta)^p$$

where the last step comes from assumptions (3.34)-(3.36). From here, it follows a fortiori that $\lim_{j\to\infty} s_j(K) = 0$, i.e. (3.32)) as well as $s_j^P(K) \leqslant (1-\delta)^P$, $j \geqslant K$, i.e. (3.30*a*). Lemma 3.5 is fully proved.

To complete the proof of Theorem 3.1, we need to show that conditions (3.28) and (3.29) in Lemma 3.4 are fulfilled. But, by (1.13) in Remark 1.3, condition (3.29) is guaranteed by assumption (*H*4) of Theorem 3.1. As to (3.28), we re-write it as

$$(3.42) \qquad |\mathscr{B}_j| = \frac{|\varepsilon_j|}{|\lambda_j a_j b_j|}\,|\varepsilon_j| \sum_{\substack{k=1\\k\neq j}}^{\infty} \frac{\lambda_k a_k b_k}{(\lambda_k - \alpha_j)^2} \leqslant \frac{4}{(1-2\varepsilon)c_\delta}\,|\varepsilon_j| \sum_{\substack{k=1\\k\neq j}}^{\infty} \frac{|\varepsilon_k|}{|\lambda_k - \lambda_j|^2}, \qquad j \geqslant J(K,\varepsilon)$$

where in the last step we have used (3.33) and (3.31) of Lemma 3.5, as well as (1.13) in Remark 1.3. Thus, the right hand side of (3.42) is (see (1.10))

$$\text{const}_{\varepsilon,K}|\varepsilon_j|\gamma_j\,, \qquad j \geqslant J(K,\varepsilon)$$

and $|\varepsilon_j|\gamma_j \to 0$, (keeping ε, K fixed) from our assumption (*H*4) of Theorem 3.1.

Thus, $|\mathscr{B}_j| \to 0$ as $j \to \infty$ and (3.28) holds true, as well. The proof of Theorem 3.1 is complete

REMARK 3.4. – Instead of assuming A normal and obtaining A_F of scalar type as in Theorem 3.1, one could assume only A spectral and then force A_F to be spectral. To this end, instead of using Kato's lemma, one may invoke the somewhat related characterization of spectrality given by (i)-(ii), at the beginning of section 2, as a necessary condition for A and as a sufficient condition for A_F. The computations made at the beginning of the proof of Theorem 3.1 are then needed. We have, however, not yet carried out in detail this program of modifying Theorem 3.1 (and subsequently Theorem 3.3). □

4. – Corollaries and applications to boundary feedback parabolic and hyperbolic equations.

4.1. *Corollaries.*

We now apply the results of section 3 to relevant dynamical systems, in particular to parabolic and hyperbolic boundary feedback equations (see section 3, Part I). For clarity, we distinguish three general situations relevant to the applications that we have in mind, but for conciseness—we refer only to the case of A normal, Theorem 3.3 (i) in the statements. An important hyperbolic case (damped wave equation) which does not fit Theorem 3.3 (i) but Theorem 3.3 (ii) instead [i.e. A is not normal, however, it is of scalar type] is illustrated in Application 4.4.

We begin with an application of the main Theorem 3.3 to the first order dynamics (0.3) [resp. second order dynamics (0.4)] in the case of parabolic [resp. hyperbolic] equations, where a degree of freedom in shifting finitely many eigenvalues of A is allowed.

COROLLARY 4.1 (to Theorem 3.3). – (i) (*Parabolic equations in the form of model* (0.3)). – Let A be a normal operator which satisfies assumption (K) of (0.5) and which is also the generator of a s.c., analytic semigroup on Y for $t > 0$. Let the eigenvalues of A be simple, non zero, and ordered by their real parts:

$$\dots \leqslant \operatorname{Re} \lambda_3 \leqslant \operatorname{Re} \lambda_2 \leqslant \operatorname{Re} \lambda_1 \tag{4.1}$$

(so that if, say, N of them have positive real part, the respective eigensolutions of $\dot{y} = Ay$ blow up in time exponentially).

Then, for any vector $a \in Y$, $a_m \neq 0$ and any positive number δ, there exists a suitable vector $b \in Y$, as described by Theorem 3.3, (i.e. Theorem 1.2) such that the corresponding feedback operator A_F in (0.2) has a Riesz basis of normalized eigenvectors $\{\Psi_m\}_{m=1}^{\infty}$ with corresponding distinct eigenvalues $\{\alpha_m\}_{m=1}^{\infty}$, which satisfy $\sup \operatorname{Re} \sigma(A) = \sup \operatorname{Re} \alpha_m \leqslant -\delta < 0$. Moreover, A_F generates a s.c., analytic semigroup on Y. Expansions (3.7) and inequalities (3.8*b*) hold true with

$$y(t, y_0) = \exp [A_F t] y_0$$

solutions of the abstract equation (0.3). A fortiori

$$\|\exp [A_F t]\| \leqslant C_\delta \exp [-\delta t], \qquad t \geqslant 0; \tag{4.2}$$

(ii) (*Hyperbolic equations in the form of model* (0.4)). In the special case that A is self-adjoint, one may require that the eigenvalues $\{\alpha_m\}$ of A_F, corresponding to the Riesz basis of normalized eigenvectors $\{\Psi_m\}$, be distinct real numbers, indeed negative with $\alpha_m \leqslant -\delta < 0$. In this case, A_F generates a s.c. cosine operator $C_F(t)$ on Y, with corresponding sine operator $S_F(t)$ given by

$$C_F(t) y = \sum_{m=1}^{\infty} c_m(y) \cos \sqrt{-\alpha_m}\, t \Psi_m \tag{4.3}$$

$$S_F(t) y = \int_0^t C_F(\tau) y \, d\tau = \sum_{m=1}^{\infty} \frac{c_m(y)}{\sqrt{-\alpha_m}} \sin \sqrt{-\alpha_m}\, t \Psi_m, \qquad y \in Y, \ t \in R \tag{4.4}$$

and $y(t; y_0, y_1) = C_F(t) y_0 + S_F(t) y_1$ are the solutions of the second order abstract equation (0.4). (Recall that $A_F^* c_m = \bar{\alpha}_m c_m$, and $(c_m, \Psi_n) =$ Kroneker δ_{mn}). □

PROOF. – (i) A has, as required, on orthonormal basis of eigenvectors $\{\Phi_m\}$ with eigenvalues $\{\lambda_m\}$ and A_F generates a s.c., analytic semigroups on Y (by a standard result [K.1, p. 497]). The expansion (3.7) and inequalities (3.8b), in particular the upper bound (4.2), follow from the conclusions of Theorem 3.3, by imposing that the non zero distinct numbers $\varepsilon_i = \alpha_i - \lambda_i$ be chosen so that the following two conditions hold: (i) $\mathrm{Re}\, \alpha_i \leqslant -\delta < 0$ for the finitely many $\lambda_1, \ldots, \lambda_I$ with $\mathrm{Re}\, \lambda_i \geqslant -\delta$; (ii) the remaining $\{\varepsilon_i\}_{i=I+1}^{\infty}$ be sufficiently small in absolute value so that the assumptions (3.10)-(3.13) of Theorem 3.3 are fulfilled yielding a suitable $b \in Y$.

(ii) Part (ii) is a specialization of part (i) to the case where all the eigenvalues $\{\lambda_m\}_{m=1}^{\infty}$ of A are real and indeed only finitely many, say I, of them larger then $-\delta$. Thus, we may force $\alpha_1, \ldots, \alpha_I$ to be negative at the left of $-\delta$, while keeping $\alpha_{I+1}, \alpha_{I+2}, \ldots$ also to be negative and asymptotically close to $\lambda_{I+1}, \lambda_{I+2}, \ldots$ respectively, so that Theorem 3.3 applies. □

We finally apply Theorem 3.3 to the first order dynamics (0.3) in the case of hyperbolic equations.

Here, in the canonical case with no damping, A is then the generator of a s.c. unitary group on Y and thus, by Stone Theorem, $A = iS$, with S self-adjoint. Hence, A has an orthonormal basis of eigenvectors, $\{\Phi_m\}$, as required, with corresponding eigenvalues $\{\lambda_m\}$ lying exactly on the imaginary axis.

The case of damping will be treated in Application 4.4.

COROLLARY 4.2 (to Theorem 3.3). - (i) Let A be the generator of a s.c. unitary group on Y satisfying (K) in (0.5) (so that $\|\exp[At]y_0\|_Y \equiv 1$ for the solution $y(t, y_0) = \exp[At]y_0$ of the free system $\dot{y} = Ay$), with simple eigenvalues on $\mathrm{Re}\, \lambda = 0$.

Then, for any vector $a \in Y$, $a_m \neq 0$ there exists a suitable vector $b \in Y$, as described by Theorem 3.3, (i.e. by Theorem 1.2) such that the corresponding feedback operator A_F in (0.2) has a Riesz basis of normalized eigenvectors $\{\Psi_m\}$ with corresponding eigenvalues $\{\alpha_m\}$ which lie all in the open left hand side C^- of the complex plane (i.e.: $\mathrm{Re}\, \alpha_m < 0$ for all m), with finitely many of them arbitrarily located in C^-, while the sequence $\{\alpha_m\}$ approaches asymptotically the imaginary axis. Moreover, A_F generates a s.c. semigroup on Y given by (see expansions (3.7))

$$\exp[A_F t]y = \sum_m c_m(y) \exp[\alpha_m t]\Psi_m\,, \qquad y \in Y,\ t \geqslant 0 \tag{4.5}$$

and $\exp[A_F t]$ is strongly stable; i.e.

$$\|\exp A_F t]y\|_Y \to 0 \quad \text{as } t \to \infty\,, \quad \text{for all } y \in Y \tag{4.6}$$

with $y(t, y_0) = \exp[A_F t]y_0$ solutions of (0.3).

(ii) If A is a normal group generator and its simple eigenvalues $\{\lambda_m\}$ satisfy, instead, $\mathrm{Re}\, \lambda_m \leqslant -\varepsilon < 0$ and approach asymptotically the line $\mathrm{Re}\, \lambda = -\varepsilon$, the same conclusion as in part (i) holds, with the eigenvalues $\{\alpha_m\}$ of A_F this time approaching, from the left the line $\operatorname{Re}\lambda = -\varepsilon$, and with finitely many of them otherwise arbitrarily located in the half plane $\operatorname{Re}\lambda \leqslant -\varepsilon$. Moreover, we have the uniform decay

$$(4.7)\qquad \|\exp[A_F t]y\|_Y \leqslant c\exp[-\varepsilon t]\|y\|_Y\,, \qquad t\geqslant 0,\ y\in Y\,.$$

PROOF. – (i) Here we select the non zero numbers $\varepsilon_m = \alpha_m - \lambda_m$ of Theorem 3.3 so that $\operatorname{Re}\alpha_m < 0$, the $\{\alpha_m\}$ accumulate asymptotically along the imaginary axis (i.e. $0 \neq |\varepsilon_m| \to 0$) with the $|\varepsilon_m|$ asymptotically so small as to satisfy assumptions (3.10)-(3.13) of Theorem 3.3. From (4.5) we then obtain (4.6):

$$\exp[A_F t]y = \sum_{m=1}^{M} c_m(y)\exp[\alpha_m t]\Psi_m + \sum_{m=M+1}^{\infty} c_m(y)\exp[\alpha_m t]\Psi_m$$

where, with $\exp[\operatorname{Re}\alpha_m t]\leqslant 1$ for $t\geqslant 0$, given ε and y we can select $M = M_{\varepsilon,y}$ so that

$$(4.8)\qquad \Big|\sum_{m=M+1}^{\infty}\Big|_Y < \varepsilon\,, \quad t\geqslant 0 \quad \text{and} \quad \Big|\sum_{m=1}^{M}\Big| \leqslant c_y \exp[-c_M t]\to 0 \quad \text{as } t\to\infty$$

with $-c_M \equiv \sup\{\operatorname{Re}\alpha_m,\ m=1,\dots,M\} < 0$ and part (i) is proved.

As to part (ii) the important exponential decay (4.7) is a consequence of [the $\{\Psi_m\}$ being a Riesz basis, i.e. of] the right hand side inequality in (3.8*b*) with $\sup \operatorname{Re}\alpha_m = -\varepsilon < 0$. □

As explained in the Introduction, in applying our abstract main results (Theorem 3.3 and the above corollaries) to boundary feedback parabolic or hyperbolic equations, we shall appeal to the operator A_F^*, rather than directly to the operator A_F itself.

Thus, we need a version of our main result Theorem 3.3, and corollaries above, for the adjoint operator A_F^*. This is quickly done, by [D-S, Vol. III, Lemma 4, p. 2354]. Thus the definition of the projection Q_j for A_F implies that Q_j^* is the corresponding projection of A_F^*

$$(4.9)\qquad Q_j^* = \frac{1}{2\pi i}\int_{C_j} R(\lambda, A_F^*)\,d\lambda\,.$$

Moreover (3.6*a*) implies

$$(4.10a)\qquad Q_m^* = W^* P_m W^{*-1}\,, \quad m = J,\ J+1\,, \quad Q_{0,J'}^* = W^* P_{0,J} W^{*-1}$$

since $P_m = P_m^*$ by orthogonality assumption [T-L.1, p. 250]. Thus—under the same assumptions as in Theorem 3.3—we have:

$$(4.10b)\qquad \sum_{m=1}^{\infty} Q_m^* = I\,, \quad \text{strongly and unconditionally in } Y$$

(counterpart of (3.6b)). Moreover

$$\frac{1}{C_J}\left\{\sum_{m=1}^{\infty}\|Q_m^* y\|_Y^2 + \|Q_{0,J'}^* y\|^2\right\} \leqslant \|y\|_Y^2 \leqslant C_J\left\{\sum_{m=1}^{\infty}\|Q_m^* y\|_Y^2 + \|Q_{0,J'}^* y\|_Y^2\right\}, \qquad y \in Y \tag{4.10c}$$

(counterpart of (3.6c)). Thus, the counterpart of Theorem 3.3 for the adjoint operator A_F^* is

THEOREM 4.3 (*Spectral and Riesz basis assignment for A_F^* via a vector b*). – Under the same assumptions as in Theorem 3.3, the projection $\{Q_m^*\}$ of A_F^*, defined by (4.9), satisfy the similarity relationship (4.10a), with W as in (3.17), as well as identity (4.10b) and inequalities (4.10c). Thus, if $\{\Psi_*^m\}$ are the normalized eigenvectors of A_F^* (proportional to c_m in Eq. (3.7)) corresponding to the eigenvalues $\{\bar{\alpha}_m\}$, the following counterparts of expansions (3.7) hold (for simplicity of notation we take $J' = J$ in Theorem 3.3)

$$(4.11)\qquad \begin{cases} (a) & y = \sum_{m=1}^{\infty} k_m(y)\Psi_m^*, & y \in Y \\ (b) & A_F^* y = \sum_{m=1}^{\infty} \bar{\alpha}_m k_m(y)\Psi_m^*, & y \in \mathscr{D}(A_F^*) \\ (c) & \exp[A_F^* t]y = \sum_{m=1}^{\infty} k_m(y)\exp[\bar{\alpha}_m t]\Psi_m^*, & y \in Y,\ t \geqslant 0 \end{cases}$$

along with the inequalities (counterparts of (3.8))

$$(4.12)\qquad \begin{cases} (a) & \frac{1}{c}\sum_{m=1}^{\infty}|k_m(y)|^2 \leqslant \|y\|_Y^2 \leqslant C\sum_{m=1}^{\infty}|k_m(y)|^2, \qquad y \in Y \\ (b) & \frac{1}{c}\sum_{m=1}^{\infty}|k_m(y)\exp[\alpha_m t]| \leqslant \|\exp[A_F^* t]y\|_Y \leqslant c\sum_{m=1}^{\infty}|k_m(y)\exp[\alpha_m t]|^2, \\ & \hfill t \geqslant 0,\ y \in Y \end{cases}$$

$k_m(\Psi_n^*) =$ Kroneker δ_{mn}, $k_m(A_F^*\Psi_m^*) = \bar{\alpha}_m$, k_m eigenvector of A_F corresponding to α_m, hence proportional to Φ_m. Thus a fortiori, A_F^* is a spectral operator. □

As a consequence, Corollaries 4.1-4.2 have full counterparts for the adjoint operators A_F^*.

COROLLARY 4.1*. – (i) (*Parabolic equations in the form $\dot{z} = A_F^* z$*). Let $A: Z \supset \supset \mathscr{D}(A) \to Z$ be a normal operator on the Hilbert space Z, which satisfies assumption (K) of (0.5) and which is also the generator of a s.c. analytic semigroup on Y for $t > 0$. Assume that the eigenvalues of A are as in (4.1).

Then, for any vector $a \in Z$, $a_m \neq 0$ and any positive number δ, there exists a suitable vector $b \in Z$ such that the corresponding adjoint operator $A_F^* = A^* + a(\cdot, b)_Z$

(see (1.4)) has a Riesz basis of normalized eigenvectors $\{\Psi^*_m\}_{m=1}^\infty$ with corresponding eigenvalues $\{\bar{\alpha}_m\}_{m=1}^\infty$ satisfying $\operatorname{Re}\alpha_m \leqslant -\delta < 0$. Moreover, A^*_F generates a s.c. semigroup on Z, analytic for $t > 0$.

Expansions (4.11) and inequalities (4.12) hold on Z, so that, a fortiori (see (4.2))

$$\|\exp[A^*_F t]\| = \|(\exp[A_F t])^*\| = \|\exp[A_F t]\| \leqslant c_\delta \exp[-\delta t], \quad t \geqslant 0 \tag{4.13}$$

$\| \ \|$ being the uniform operator topology $Z \to Z$.

(ii) (*Hyperbolic equations in the form* $\ddot{z} = A^*_F z$). In particular, if A is self-adjoint, the same conclusion as in Corollary 4.1 (ii) holds for $C^*_F(t)$, the s.c. cosine operator generated by A^*_F on Z, except that in (4.3)-(4.4), $\{\Psi_m\}$ and $e_m(\cdot)$ are now replaced by $\{\Psi^*_m\}$ and $k_m(\cdot)$: (the α_m are negative):

$$\begin{cases} (a) \quad C^*_F(t)z = \sum_{m=1}^{\infty} k_m(z)\cos\sqrt{-\alpha_m}\,t\Psi^*_m \\ (q) \quad S^*_F(t)z = \sum_{m=1}^{\infty} \dfrac{k_m(z)}{\sqrt{-\alpha_m}}\sin\sqrt{-\alpha_m}\,t\Psi^*_m \end{cases} \qquad z \in Z,\ t \in R \tag{4.14}$$

and $z(t, z_0, z_1) = C^*_F(t)z_0 + S^*_F(t)z_1$ are the solutions of $\ddot{z} = A^*_F z$. □

COROLLARY 4.2*. – Part (i) and (ii) of Corollary 4.2 holds also verbatim for A^*_F, except that the bi-orthogonal sequences $\{\Psi_m\}$ and $e_m(\cdot)$ in (4.5) are now replaced by the bi-orthogonal sequences $\{\Psi^*_m\}$ and $k_m(\cdot)[\{\Psi^*_m\}$ normalized eigenvectors of $A^*_F]$ and conclusion (4.6) is now replaced by

$$\|\exp[A^*_F t]z\|_Z \to 0 \quad \text{as } t \to \infty,\ z \in Z \tag{4.15}$$

in the counterpart of case (i), while conclusion (4.7) is now replaced by

$$\|\exp[A^*_F t]z\|_Z \leqslant c\exp[-\varepsilon t]\|z\|_Z, \quad z \in Z,\ t \geqslant 0 \tag{4.16}$$

in the counterpart of case (ii). □

4.2. *Application to parabolic and hyperbolic dynamics.*

APPLICATION 4.1 [This is Application 3.1, Part I, which we re-write for convenience.] – The boundary feedback hyperbolic equation « with interior observation of the position » in the Dirichlet B.C. is

$$\begin{aligned} &x_{tt}(t,\xi) = \mathcal{A}(\xi,\partial)x(t,\xi) && \text{in } (0,T]\times\mathcal{A} \\ &x(0,\xi) = x_0(\xi); \quad x_t(0,\xi) = x_1(\xi) && \xi \in Q \\ &x(t,\sigma) = \langle x(t,\cdot), w(\cdot)\rangle g(\sigma) && \text{in } (0,T]\times\Gamma \end{aligned} \tag{4.17}$$

in the notation of (3.1), Part I, $\langle\, ,\rangle$ being the $L_2(\Omega)$-inner product, where now $\mathcal{A}(\xi, \partial)$ plus zero Dirichlet B.C. realizes a self-adjoint operator $-A$. The well-posedness Theorems 3.1. Part I applies. It easily follows from (3.6), Part I, with $x = z_1$, $\dot{x} = z_2$ that the second order abstract model in dual form for problem (4.17) is

$$\ddot{x} = -A[x - Dg\langle x, w\rangle] = A_F^* x \quad \text{on } X = L_2(\Omega) \tag{4.18}$$

Then, *Corollary* 4.1* (ii) *applies* with $Z = L_2(\Omega)$ and

$$A_F^* = -A[I - Dg\langle\cdot, w\rangle]\,, \quad a = -Dg,\ b = w \tag{4.19a}$$

and gives, in particular, for appropriate choices of $\{\varepsilon_n\}$ all negative, the following conclusion: *given a vector* $g \in L_2(\Gamma)$, with

$$\langle Dg, \Phi_m\rangle \neq 0\,, \quad \text{equivalently} \left(g, \frac{\partial \Phi_m}{\partial \nu}\Big|_\Gamma\right)_{L_2(\Gamma)} \neq 0\,, \quad m = 1, 2, \ldots \tag{4.19b}$$

(see e.g. [L-T.1] for the equivalence), $\{\Phi_m\}$ orthonormal eigenvectors of A, for any choice of $\{\varepsilon_n\}$, all negative, satisfying the assumptions of Theorem 3.3, *there exists a suitable vector* $w \in L_2(\Omega)$ *such that the expansions* (4.14) *hold*, whereby all (closed loop) *feedback solutions of* (4.17) *can have oscillations with arbitrarily prescribed speed*, i.e. $\alpha_n = \lambda_n + \varepsilon_n$ *negative* and *less than a prescribed negative number*. Moreover, note that with

$$g \text{ real} \in L_2(\Gamma)\,, \quad \{\lambda_m\} \text{ real and } \{\varepsilon_n\} \text{ real}\,, \quad \text{then the vector} \quad b = w \in L_2(\Omega)$$

guaranteed as a solution of (1.15) by Theorem 1.5 can be also taken real. □

REMARK 4.1. – The analogous problem for (4.17), where w is given in $L_2(\Omega)$ and a suitable $g \in L_2(\Gamma)$ is sought—in which case the *synthesis problem* mentioned in the Final Comments below arises—was solved in [L-T.1], through an entirely different approach; in the present paper we study the spectral properties of A_F, or A_F^*, while in [L-T.1] we followed the feedback dynamics of (4.17). In [L-T.1], we spoke of *almost periodic boundary feedback stabilization* (4.17), due to the expansions (4.14). □

APPLICATION 4.2. – The parabolic problem corresponding to (4.17) is

$$\begin{aligned} x_t(t, \xi) &= \mathcal{A}(\xi, \partial)x(t, \xi) && \text{in } (0, T]\times\Omega \\ x(0, \xi) &= x_0(\xi) && \xi \in \Omega \\ x(t, \sigma) &= \langle x(t,\cdot), w(\cdot)\rangle g(\sigma) && \text{in } (0, T]\times\Gamma \end{aligned} \tag{4.20}$$

where we now assume that $\mathcal{A}(\xi, \partial)$ plus zero Dirichlet B.C. realizes a normal generator $-A$ of a s.c. analytic semigroup. Well posedness of (4.20) follows from [L-T.7] or [T.5]. The corresponding first order abstract model in dual form for problem (4.20) is

$$\dot{x} = -A[x - Dg\langle x, w\rangle] = A_F^* x\,, \quad \text{on } X = L_2(\Omega)\,.$$

Then, *Corollary* 4.1* (i) *applies* with $Z = L_2(\Omega)$ and the *same* quantities defined in (4.19). Thus, in particular we obtain: *given a vector* $g \in L_2(\Gamma)$, as in (4.19*b*) *there exists for any choice of the* $\{\varepsilon_m\}$ *as in Theorem* 3.3 *a suitable vector* $w \in L_2(\Omega)$ *such that the expansions* (4.11) *hold*, $\alpha_m = \lambda_m + \varepsilon_m$, *whereby, in particular, by* (4.13), *all feedback solutions of* (4.20) *decay asymptotically to zero with arbitrarily prescribed speed* $-\delta < 0$. Moreover, in the self-adjoint case, if g is real, then w can likewise be taken real. □

REMARK 4.2. – The analogous problem, where w is given in $L_2(\Omega)$ and a suitable $g \in L_2(\Gamma)$ is sought—in which case the *synthesis problem* mentioned in the Final Comments arises—was solved through an entirely different approach in [L-T.2] for problem (4.20), and in [L-T.3] for a more demanding parabolic problem with an *unbounded* observation (canonically the gradient of the solution acting in the Neumann B.C.).

However, the full boundary case with observation of the trace $x(t,)|_\Gamma$ of the solution, i.e. the term $(x(t,)|_\Gamma, w)_\Gamma g$ in the Neumann B.C., $w, g \in L_2(\Gamma)$ escaped the analysis of [L-T.3]: in this case we were only able to obtain in [L-T.4] the weaker result of an arbitrarily preassigned (uniform) decay of all feedback solution (stabilization) as in (4.13) in particular in the canonical cases: the Laplacian translated, defined on spheres and on parallelepipeds. □

APPLICATION 4.3 [This is Application 3.2, Part I, which we rewrite for convenience]. – The boundary feedback hyperbolic equation « with interior observation of the velocity » in the Dirichlet B.C.

$$\text{(4.21)} \qquad \begin{aligned} &x_{tt}(t, \xi) = \Delta x(t, \xi) && \text{in } (0, T]\times\Omega \\ &x(0, \xi) = x_0(\xi)\,, \quad x_t(0, \xi) = x_1(\xi)\,, && \xi \in \Omega \\ &x(t, \sigma) = [\langle x(t,\cdot), w_1(\cdot)\rangle + \langle x_t(t,\cdot), w_2(\cdot)\rangle]g(\sigma) && \text{in } (0, T]\times\Gamma \end{aligned}$$

in the notation of (3.2), Part I $\big(w_i \in L_2(\Omega),\, g \in L_2(\Gamma)\big)$, $\langle\,,\,\rangle$ being the $L_2(\Omega)$-inner product. The well-posedness Theorem 3.2, Part I applies. The abstract first order model in dual form of problem (4.21) is [see Part I]

$$\text{(4.22}a\text{)} \qquad \dot{z} = \mathscr{A}[z + a(b, z)] = A_F^* z$$

with $z_1 = x$, $z_2 = \dot{x}$, $Z = L_2(\Omega) \times [\mathscr{D}(A^{\frac{1}{2}})]'$

$$(4.22b) \qquad \mathscr{A} = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}; \quad a = \begin{vmatrix} -Dg \\ 0 \end{vmatrix} \in Z; \quad b = \begin{vmatrix} w_1 \\ Aw_2 \end{vmatrix} \in Z$$

with $w_1 \in L_2(\Omega)$ and $w_2 \in \mathscr{D}(A^{\frac{1}{2}})$, as required by Theorem 3.2, Part I and $-A$ being the self-adjoint operator given by the Laplacian with zero Dirichlet B.C. The eigenvalues of the skew-adjoint operator $\mathscr{A}$ are: $\lambda_m^{+,-} = \pm i\sqrt{\mu_m}$, $m = 1, 2, \dots$ with the corresponding orthonormal basis of eigenvectors $\Phi_m^{+,-} = [e_m, \pm i\sqrt{\mu_m}e_m]$ on Z, where $\{\mu_m\}$ are the positive eigenvalues of A on $L_2(\Omega)$ (assumed simple) and $\{e_m\}$ the corresponding eigenvectors, $2\|e_m\|^2_{L_2(\Omega)} = 1$. Thus, for the sought after vector b in (4.22b) we have

$$(4.23) \qquad b = \begin{vmatrix} w_1 \\ Aw_2 \end{vmatrix} = \sum_{m=1}^{\infty} [\langle w_1, e_m\rangle - i\sqrt{\mu_m}\langle w_2, e_m\rangle]\Phi_m^+ + \\ + \sum_{m=1}^{\infty} [\langle w_1, e_m\rangle + i\sqrt{\mu_m}\langle w_2, e_m\rangle]\Phi_m^-$$

and the coordinates $(b, \Phi_m^+)_Z$ and $(b, \Phi_m^-)_Z$ are complex conjugate of each other if and only if w_1 and w_2 are real, the case of interest. (We are likewise taking $g \in L_2(\Gamma)$ real.) To handle this extra condition on the co-ordinates of the sought after vector b, we proceed as follows.

We first impose the assumption $(a, \Phi_m^{+,-})_Z \neq 0$, equivalently

$$(4.24) \qquad \langle Dg, e_m\rangle \neq 0 \Leftrightarrow \left(g, \frac{\partial e_m}{\partial \nu}\Big|_{\Gamma}\right)_{L_2(\Gamma)} \neq 0$$

$m = 1, 2, \dots$ as in (4.19b). Next we apply Corollary 4.2* (i) separately on $Z^+ = \overline{\text{sp}}\,\{\Phi_m^+, m = 1, 2, \dots\}$ and on $Z^- = \overline{\text{sp}}\,\{\Phi_m^-, m = 1, 2, \dots\}$. Thus, after assigning $\{\varepsilon_m^+\}$ and $\{\varepsilon_m^-\}$ as prescribed by Corollary 4.2* (i), we obtain suitable vectors $b^+ \in Z^+$ and $b^- \in Z^-$, via Theorem 1.2 (i.e. solution of the corresponding equation (1.14c) or (1.15)). We next impose the extra condition that ε_m^+ be complex conjugate of ε_m^-: $\varepsilon_m^+ = \overline{\varepsilon_m^-}$. This implies, as is easily seen from (1.14c) with λ_m^+ complex conjugate of λ_m^-, that we likewise obtain $(b^+, \Phi_m^+)_{Z^+}$ complex conjugate of $(b^-, \Phi_m^-)_{Z^-}$ for the two solutions. Thus, the vector $b = b^+ + b^- \in Z$ is a solution of (1.14c) on the entire Z corresponding to $\{\varepsilon_m^+, \overline{\varepsilon_m^+}\}$, $m = 1, 2, \dots$ and for such b we impose, by (4.23) that

$$(4.25) \qquad \langle w_1, e_m\rangle - i\sqrt{\mu_m}\langle w_2, e_m\rangle = (b, \Phi_m^+)_Z, \qquad m = 1, 2, \dots$$

which determines uniquely $w_1 \in L_2(\Omega)$ (real) and $w_2 \in \mathscr{D}(A^{\frac{1}{2}})$ (real), as desired.

We conclude that: *given a real vector* $g \in L_2(\Gamma)$ *as in* (4.25), *for any choice of the* $\{\varepsilon_m^{+,-}\}$ *with* $\varepsilon_m^+ = \overline{\varepsilon_m^-}$, $\text{Re}\,\varepsilon_m^{+,-} < 0$ *and otherwise satisfying the assumptions* of Theorem 3.3, *the above procedure yields real vectors* $w_1 \in L_2(\Omega)$ and $w_2 \in \mathscr{D}(A^{\frac{1}{2}})$ [note that

$A^{\frac{1}{2}}$ is invertible] *such that all feedback solutions* of (4.21) *admit the expansion*

$$z(t, z_0, z_1) = \exp[A_F^* t] \begin{vmatrix} z_0 \\ z_1 \end{vmatrix} = \sum_{m=1}^{\infty} k_m \left(\begin{vmatrix} z_0 \\ z_1 \end{vmatrix} \right) \exp[\bar{\alpha}_m t] \Psi_n^* . \tag{4.26}$$

Here, $\{\Psi_m^*\}$ is a Riesz basis in Z of eigenvectors of A_F^*, with preassigned eigenvalues $\{\bar{\alpha}_m\}$ of A_F^* which lie all in C^- (i.e. $\operatorname{Re} \alpha_m < 0$ for all m) and approach asymptotically the imaginary axis, *while an arbitrary finite number of them can be located at will in* C^-. *Moreover, for all such solutions* $z(t, z_0, z_0)$ *we have*

$$z(t, z_0, z_1) = \exp[A_F^* t] \begin{vmatrix} z_0 \\ z_1 \end{vmatrix} \to 0 \qquad \text{as } t \to \infty \tag{4.27}$$

i.e. strong decay to zero.

Note, from (4.22*b*), that we could have also taken $Z = \mathscr{D}(A^{\frac{1}{4}-\delta}) \times [\mathscr{D}(A^{\frac{1}{4}+\delta})]'$, $\delta > 0$. □

REMARK 4.3. – A similar, in a sense a converse, result was proved in [L-T.5] through a different approach: in [L-T.5] we obtained conclusion (4.27) for the case with preassigned vectors $g \in L_2(\Gamma)$, $w_2 \in L_2(\Omega)$ and $w_1 = 0$ in (4.21) satisfying certain easily verifiable assumptions; however, neither did we obtain the Riesz basis expansion (4.26), nor could we of course claim much about the corresponding closed loop eigenvalues, beyond the mere statement that they are in $\operatorname{Re} \lambda < 0$. □

In our next application, the operator A in model (0.3) fails to be normanl but it is of scalar type. Thus, we must appeal to Theorem 3.3 (ii) and its dual version.

APPLICATION 4.4 [Damped wave equation]. – Consider the same canonical situation as in (4.21) except that now damping is present in the right hand side of the equation:

$$\begin{aligned} x_{tt}(t, \xi) &= \Delta x(t, \xi) - 2k x_t(t, \xi) && \text{in } (0, T] \times \Omega \\ x(0, \xi) &= x_0(\xi), \quad x_t(0, \xi) = x_1(\xi) && \text{in } \Omega \\ x(t, \sigma) &= [\langle x(t, \cdot), w_1 \rangle + \langle x_t(t, \cdot), w_2 \rangle] g(\sigma) && \text{in } (0, T] \times \Gamma \end{aligned} \tag{4.28}$$

with damping coefficient $k > 0$, $w_i \in L_2(\Omega)$, $g \in L_2(\Gamma)$, $\langle \, , \rangle$ being the $L_2(\Omega)$—inner product. The first order abstract model in dual form of problem (4.28) is now

$$\dot{z} = \mathscr{A}[z + a(b, z)] = A_F^* z \tag{4.29a}$$

with $z_1 = x$, $z_2 = \dot{x}$, $Z = L_2(\Omega) \times [\mathscr{D}(A^{\frac{1}{2}})]'$ as in Application 4.3, $-A$ being the self-adjoint operator given by the Laplacian with homogeneous Dirichlet B.C.

$$\mathscr{A} = \begin{vmatrix} 0 & I \\ -A & -2kI \end{vmatrix}; \quad a = \begin{vmatrix} -Dg \\ 0 \end{vmatrix} \in Z; \quad b = \begin{vmatrix} w_1 \\ Aw_2 \end{vmatrix} \in Z \tag{4.29b}$$

$w_1 \in L_2(\Omega)$, $w_2 \in \mathscr{D}(A^{\frac{1}{2}})$. In the notation of Application 4.3, $\{\mu_m\}$ are the positive, increasing eigenvalues of A (assumed simple) and $\{e_m\}$ corresponding eigenvectors. The eigenvalues and normalized eigenvectors of $\mathscr{A}$ are, with $2|e_m|^2_{L_2(\Omega)} \equiv 1$:

$$\left\{\begin{array}{l} \lambda_m^{+,-} = -k \pm i\sqrt{\mu_m - k^2}\,; \quad \Phi_m^{+,-} = \begin{vmatrix} e_m \\ \lambda_m^{+,-} \quad e_m \end{vmatrix} \\ \lambda_m^+ \lambda_m^- = \mu_m = |\lambda_m^{+,-}|^2\,, \end{array}\right. \quad m = 1, 2, \ldots \tag{4.30}$$

where, without changing the qualitative analysis, we are assuming « small » damping k i.e. $\mu_1 > k^2$. The solutions of the *free* system (i.e. (4.28) with $g = 0$) satisfy the bound

$$\left\|\begin{vmatrix} x(t) \\ x_t(t) \end{vmatrix}\right\|_E = \left\|\exp[\mathscr{A}t]\begin{vmatrix} x_0 \\ x_1 \end{vmatrix}\right\|_E \leqslant C\exp[-kt]\left\|\begin{vmatrix} x_0 \\ x_1 \end{vmatrix}\right\|_E, \quad t \geqslant 0\,. \tag{4.31}$$

This standard result can be shown in several ways: (i) by the usual change of variable $v(t, \xi) = \exp[kt]x(t, \xi)$ which converts the equation of (4.28) into the easier one $v_{tt} = \Delta v + k^2 v$; (ii) by spectral analysis, using properties of the eigenvectors of $\mathscr{A}$ which will be listed below in ($P1$)-($P5$). The well-known « multipliers » or energy methods also work [C.1] yielding a decay $\exp[-ct]$ for some $c > 0$, but do not identify c as $c = k$. We next note that $\mathscr{A}$ in (4.29*b*) is *not* normal in Z. Indeed

$$\mathscr{A}^*\mathscr{A} = \mathscr{A}\mathscr{A}^* + 4k\begin{vmatrix} 0 & I \\ A & 0 \end{vmatrix}, \quad \mathscr{A}^* = \begin{vmatrix} 0 & -I \\ A & -2kI \end{vmatrix} \tag{4.32}$$

and $\mathscr{A}$ is normal in Z if and only if $k = 0$, the undamped case. We now check that $\mathscr{A}$ is (indeed a rather special case of) a spectral operator of scalar type. To this end, we analyze the spectral properties of $\mathscr{A}$. The following properties can be directly verified in addition to (4.30):

($P1$) $\{\Phi_m^+\}_{m=1}^\infty$ and $\{\Phi_m^-\}_{m=1}^\infty$ are each on orthonormal family on Z.

($P2$) $\left\{\begin{array}{l} \Phi_n^+ \text{ is orthogonal to } \{\Phi_m^-\}_{m=1}^\infty \text{ in } Z,\ m \neq n\,, \\ \Phi_n^- \text{ is orthogonal to } \{\Phi_m^+\}_{m=1}^\infty \text{ in } Z,\ m \neq n\,. \end{array}\right.$

($P3$) $2\lambda_m^-(\Phi_m^+, \Phi_m^-)_Z = 2\lambda_m^+(\Phi_m^-, \Phi_m^+)_Z = \lambda_m^- + \lambda_m^+\,.$

($P4$) Completeness of $\{\Phi_m^+, \Phi_m^-, m = 1, 2, \ldots\}$ on Z.

Thus, setting $Z^+ = \overline{\text{sp}}\,\{\Phi_m^+\}_{m=1}^\infty$ and $Z^- = \overline{\text{sp}}\,\{\Phi_m^-\}_{m=1}^\infty$, $\mathscr{A}$ restricted on $Z^{+,-}$ is normal. Moreover

($P5$) $Z = Z^+ + Z^-$; $\quad Z^+ \cap Z^- = \{0\}$ (direct sum) and the expansions

$$z = z^+ + z^- = \sum_{m=1}^{\infty} (z^+, \Phi_m^+)_Z \Phi_m^+ + \sum_{m=1}^{\infty} (z^-, \Phi_m^-)_Z \Phi_m^- , \qquad z \in Z \tag{4.31}$$

$$\mathscr{A} z = \sum_{m=1}^{\infty} \lambda_m^{+,-} (z^{+,-}, \Phi_m^{+,-})_Z \Phi_m^{+,-} , \qquad z \in \mathscr{D}(\mathscr{A}) \tag{4.32}$$

hold. A fortiori $\mathscr{A}$ is a spectral operator of scalar type.

(*P6*) The eigenvalues of $\mathscr{A}^*$ given by (4.32) are $\overline{\lambda_m^{+,-}} = \lambda_m^{-,+}$ with corresponding eigenvectors, $[e_m, -\lambda_m^{-,+} e_m]$, respectively.

If we set $\nu_m^{-,+} = (\lambda_m^{+,-} - \lambda_m^{-,+})/2$, then the following nonnormalized eigenvectors of $\mathscr{A}^*$ corresponding to its eigenvalues $\overline{\lambda_m^{+,-}}$:

$$\Phi_m^{*-} = \begin{vmatrix} \dfrac{e_m}{\nu_n^-} \\ \dfrac{-\lambda_m^-}{\nu_m^-} e_m \end{vmatrix} ; \qquad \Phi_m^{*+} = \begin{vmatrix} \dfrac{e_m}{\nu_m^+} \\ \dfrac{-\lambda_m^+}{\nu_m^+} e_m \end{vmatrix} \tag{4.33}$$

form a *bi-orthogonal system* with respect to the eigenvectors $\{\Phi_m^{+,-}\}$ of $\mathscr{A}$ corresponding to its eigenvalues $\lambda_m^{+,-}$:

$$\begin{cases} (\Phi_m^{*-}, \Phi_n^+)_Z = (\Phi_m^{*+}, \Phi_n^-)_Z = \text{Kroneker } \delta_{mn} \\ (\Phi_m^{*-}, \Phi_n^-)_Z = (\Phi_m^{*+}, \Phi_m^+)_Z = 0 \qquad \text{all } n, m. \qquad \square \end{cases} \tag{4.34}$$

Taking the inner product in E of z in (4.31) with $\Phi_n^{*+,-}$ and using (4.34) yields $(z, \Phi_n^{*-,+})_Z = (z^{+,-}, \Phi_n^{+,-})_Z$, hence

$$z = \sum_{m=1}^{\infty} (z, \Phi_m^{*-})_Z \Phi_m^+ + \sum_{m=1}^{\infty} (z, \Phi_m^{*+})_Z \Phi_m^- . \tag{4.35}$$

[We made similar considerations for A_F in (3.7)].

We now introduce a transformation that maps $\{\Phi_m^+, \Phi_m^-\}$ in an orthonormal basis on Z. Define on Z the operator R by setting:

$$\begin{cases} 0_m^+ \equiv R\Phi_m^+ = R \begin{vmatrix} e_m \\ \lambda_m^+ \; e_m \end{vmatrix} = \begin{vmatrix} \dfrac{\lambda_m^+}{\sqrt{\mu_m}} e_m \\ -\sqrt{\mu_m}\, e_m \end{vmatrix} = \begin{vmatrix} \dfrac{\lambda_m^+}{\sqrt{\mu_m}} & 0 \\ 0 & -\dfrac{\sqrt{\mu_m}}{\lambda_m^+} \end{vmatrix} \begin{vmatrix} e_m \\ \lambda_m^+ \; e_m \end{vmatrix} , \\ 0_m^- \equiv R\Phi_m^- = \Phi_m^- = \begin{vmatrix} e_m \\ \lambda_m^- \; e_m \end{vmatrix} , \qquad m = 1, 2, \ldots . \end{cases} \tag{4.36}$$

Then, one verifies that:

(*P7*) $\{0_m^+, 0_m^-, m = 1, 2, \ldots\}$ is an orthonormal basis on Z

(orthonormal and complete).

($P8$) R is bounded and boundedly invertible on Z.

Setting

$$\Pi = R^{-1}\,, \quad R = \Pi^{-1} \quad \text{on } Z \tag{4.37}$$

we have, by (4.36) and ($P7$) that

$$\{\Phi_m^{+,-},\ m = 1, 2, \ldots\} = \Pi\{0_m^{+,-},\ m = 1, 2, \ldots\} = \text{a Riesz basis on } Z\,. \tag{4.38}$$

Using the (non selfadjoint) Π as a similarity transformation (guaranteed, in general, by Werner's theorem) transforms (4.29), via the usual change of coordinate $\zeta = \Pi^{-1}z$, into

$$\dot{\zeta} = \mathscr{A}_N[\zeta + \Pi^{-1}a(\zeta, \Pi^* b)] \quad \text{on } Z \tag{4.39}$$

$\mathscr{A}_N = \Pi^{-1}\mathscr{A}\Pi =$ normal operator on Z. Eq. (4.39) is of the form, for which we intend to apply Corollary 4.2* (i) with $\varepsilon = k$ in (4.15). We begin by imposing the required assumption that $(\Pi^{-1}a, 0_m^{+,-})_Z \neq 0$ in terms of $g \in L_2(\Gamma)$. To this end, we recall expansions (4.31) and (4.35) with $z = a = [-Dg, 0]$ (by (4.29b)), apply R across and use (4.33) and (4.36). We find

$$\left(\Pi^{-1}\begin{vmatrix}-Dg\\ 0\end{vmatrix}, 0_m^+\right)_Z = (\Pi^{-1}a, 0_m^+)_Z = (a^+, \Phi_m^+)_Z = (a, \Phi_m^{*-})_Z = = -2\,\frac{1}{\lambda_m^- - \lambda_m^+}\langle Dg, e_m\rangle \tag{4.39}$$

$$\left(\Pi^{-1}\begin{vmatrix}-Dg\\ 0\end{vmatrix}, 0_m^-\right)_Z = (\Pi^{-1}a, 0_m^-)_Z = (a^-, \Phi_m^-)_Z = (a, \Phi^{*+})_Z = = -2\,\frac{1}{\lambda_m^+ - \lambda_m^-}\langle Dg, e_m\rangle\,. \tag{4.40}$$

Thus, as in (4.24), for $m = 1, 2, \ldots$

$$(\Pi^{-1}a, 0_m^{+,-})_Z \neq 0 \Leftrightarrow \langle Dg, e_m\rangle \neq 0 \Leftrightarrow \left(g, \frac{\partial e_m}{\partial \nu}\bigg|_\Gamma\right)_{L_2(\Gamma)} \neq 0\,. \tag{4.41}$$

On the other hand with $\Pi^* = R^{*-1}$ and b as in (4.29b)

$$\Pi^* b = R^{*-1}\begin{vmatrix}w_1\\ Aw_2\end{vmatrix} = \sum_{m=1}^{\infty}(R^{*-1}b, 0_m^{+,-})_Z 0_m^{+,-} = \sum_{m=1}^{\infty}(b, \Pi 0_m^{+,-})_Z 0_m^{+,-} = = \sum_{m=1}^{\infty}\left(\begin{vmatrix}w_1\\ Aw_2\end{vmatrix}, \Phi_m^{+,-}\right)_Z 0_m^{+,-} = \sum_{m=1}^{\infty}[\langle w_1, e_m\rangle + \lambda_m^-\langle w_2, e_m\rangle]0_m^+ + + \sum_{m=1}^{\infty}[\langle w_2, e_m\rangle + \lambda_m^+\langle w_2, e_m\rangle]0_m^- \tag{4.42}$$

and the co-ordinates of $(\Pi^* b, 0_m^+)_Z$ and $(\Pi^* b, 0_m^-)_Z$ are complex conjugate of each other if and only if w_1 and w_2 are real, the case of interest:

$$(4.43)\quad \begin{cases} (a) & (\Pi^* b, 0_m^+)_Z = [\langle w_1, e_m\rangle - k\langle w_2, e_m\rangle] - i\sqrt{\mu_m - k^2}\,\langle w_2, e_m\rangle \\ (b) & (\Pi^* b, 0_m^-)_Z = [\langle w_1, e_m\rangle - k\langle w_2, e_m\rangle] + i\sqrt{\mu_m - k^2}\,\langle w_2, e_m\rangle\,. \end{cases}$$

To obtain this we proceed as in Application 4.3, following (4.24). We apply Corollary 4.2* (i) on $Z^+ = \overline{\text{sp}}\,\{0_m^+\}$ and on $Z^- = \overline{\text{sp}}\,\{0_m^-\}$ separately, with the $\{\varepsilon_m^+\}$ which are taken complex conjugate of the $\{\varepsilon_m^-\}$: $\varepsilon_m^+ = \overline{\varepsilon_m^-}$. As a result, we obtain suitable vectors $\beta^+ \in Z^+$ and $\beta^- \in Z^-$ from (1.14*c*) in Theorem 1.2 such that the $(\beta^+, 0_m^+)_{Z^+}$ are complex conjugate of the $(\beta^-, 0_m^-)_{Z^-}$. (Note from (4.39)-(4.40) that $(\Pi^{-1}a, 0_m^+)_Z$ is complex conjugate of $(\Pi^{-1}a, 0_m^-)_Z$). Then $b = \Pi^{*-1}[\beta^+ + \beta^-]$, the real vector $w_2 \in \mathscr{D}(A^{\frac{1}{2}})$ is identified by

$$(4.44)\qquad \langle w_2, e_m\rangle = -\frac{\text{Im}\,(\beta^+, 0_m^+)_Z}{\sqrt{\mu_m - k^2}}\,, \qquad m = 1, 2, \ldots$$

and the real vector $w_1 \in L_2(\Omega)$ by

$$(4.45)\qquad \langle w_1, e_m\rangle = k\langle w_2, e_m\rangle + \text{Re}\,(\beta^+, 0_m^+)_Z\,.$$

We finally conclude: *given a real vector* $g \in L_2(\Gamma)$ as in (4.41), *for any choice of the* $\{\varepsilon_m^{+,-}\}$ *with* $\varepsilon_m^+ = \overline{\varepsilon_m^-}$, $\text{Re}\,\varepsilon_m^{+,-} < 0$, and *otherwise satisfying the assumptions* of Theorem 3.3, *the above procedure yields real vectors* $w_1 \in L_2(\Omega)$ *and* $w_2 \in \mathscr{D}(A^{\frac{1}{2}})$, *such that all feedback solutions of* (4.28) *admit an expansion as in* (4.26), *with preassigned eigenvalues* $\alpha_m^{+,-} = \varepsilon_m^{+,-} + \lambda_m^{+,-}$ *of* A_F^* *which lie all in the half plane* $\text{Re}\,\lambda < -k$ *and approach asymptotically the vertical line* $\text{Re}\,\lambda = -k$, *while an arbitrary finite number of them can be located at will in* $\text{Re}\,\lambda < -k$.

Moreover, *all such feedback solutions* $z(t) = [x(t), x_t(t)]$ *satisfy the decay* (4.16), *with* $Z = L_2(\Omega) \times [\mathscr{D}(A^{\frac{1}{2}})]'$ *and* $\varepsilon = k$. □

The abstract theory of the present paper covers also, of course, higher order differential operators, say beams or plates equations, and generalization thereof. This is illustrated by the next application.

APPLICATION 4.5. – Consider the elastic system

$$(4.46)\quad \begin{cases} x_{tt}(t, \xi) + \Delta^2 x(t, \xi) = 0 & \text{in } (0, T]\times\Omega \\ x(0, \xi) = x_0(\xi)\,, \quad x_t(0, \xi) = x_1(\xi) & \text{in } \Omega \\ x(t, \sigma) \equiv 0 & \text{in } (0, T]\times\Gamma \\ \Delta x(t, \sigma) = [\langle \Delta x(t, \cdot), w_1(\cdot)\rangle + \langle \Delta x_t(t, \cdot), w_2(\cdot)\rangle]\,g(\sigma) & \text{in } (0, T]\times\Gamma \end{cases}$$

with the same notation as before.

Setting $Af = \Delta^2 f$ on $\mathfrak{D}(A) = H^4(\Omega) \cap H_0^2(\Omega)$, we have that A is a positive, self-adjoint operator on $L_2(\Omega)$ and $A^{\frac{1}{2}} f = \Delta f$ on $\mathfrak{D}(A^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$. If we define $h = Gu$ to mean the solution of the corresponding elliptic problem

$$\left|\begin{array}{l} \Delta^2 h = 0 \\ h|_\Gamma = 0 \\ \Delta h|_\Gamma = 0 \end{array}\right. \quad \text{in } \Omega \qquad \text{i.e. with } \Delta h = p \left\{\begin{array}{ll} \begin{cases} \Delta p = 0 \\ p|_\Gamma = u \end{cases} & \text{in } \Omega \\ \begin{cases} \Delta h = p \\ h|_\Gamma = 0 \end{cases} & \text{in } \Omega \end{array}\right.$$

we see that $p = Du$ (same D as in previous applications) and $A^{\frac{1}{2}} h = p$, i.e. $h = A^{-\frac{1}{2}} Du$. Thus

$$G = A^{-\frac{1}{2}} D \colon L_2(\Gamma) \to H^{\frac{5}{2}}(\Omega) .$$

The abstract version of (4.46) is now

$$\ddot{x} = -A\{x + Gg[\langle A^{\frac{1}{2}} x, w_1\rangle + \langle A^{\frac{1}{2}} \dot{x}, w_2\rangle]\}$$

i.e.

$$\dot{z} = \mathcal{A}\{z + a(b, z)_Z\}$$

on $Z = \mathfrak{D}(A^{\frac{1}{2}}) \times L_2(\Omega)$, where $z = [x, \dot{x}]$, $\mathcal{A}$ is the skew adjoint operato $\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}$ and

$$a = \begin{vmatrix} -Gg \\ 0 \end{vmatrix} = \begin{vmatrix} -A^{-\frac{1}{2}} Dg \\ 0 \end{vmatrix}, \qquad b = \begin{vmatrix} A^{-\frac{1}{2}} w_1 \\ A^{\frac{1}{2}} w_2 \end{vmatrix}$$

so that we must take $w_2 \in \mathfrak{D}(A^{\frac{1}{2}})$. As in, say, Application 4.3, we compute that the condition $a_m \neq 0$ for the Z-coordinates of a is equivalent to

$$\left(g, \frac{\partial e_m}{\partial \nu}\bigg|_\Gamma\right)_{L_2(\Gamma)} \neq 0 , \qquad m = 1, 2, \ldots$$

with $\{e_m\}$ eigenvectors of A. □

REMARK 4.4. – In light of Application 4.4, the special class of two dimensional, first order hyperbolic systems in one space dimension treated in [R.1], [R.2] could also be cast as an application of our Theorem 3.3. This would yield a result close in spirit to the one arrived at in [R.1]-[R.2] through a quite different approach, based on a difference—delay system as a canonical form of the original first order hyperbolic systems). □

FINAL COMMENTS. – These refer to Remark 4.1 and 4.2. For sake of definiteness, we limit our considerations to Application 4.1, Eq. (4.17), with say, the Laplacian. We want to explain why, it was expedient to work with the dual model $\ddot{z} = A_F z$ on $Z = L_2(\Omega)$ as we have done, and not with the original model $\ddot{y} = A_F y$. The latter is, in fact, given by

$$y = -\tilde{A}y + \tilde{A}^{\frac{3}{4}+\varepsilon}A^{\frac{1}{4}-\varepsilon}Dg(\tilde{A}y, \tilde{A}^{\frac{1}{2}}w)_Y = A_F y \tag{4.47}$$

$$b = \tilde{A}^{\frac{1}{4}+\varepsilon}A^{\frac{3}{4}-\varepsilon}Dg \in Y; \quad a = \tilde{A}^{\frac{1}{2}}w \in Y \tag{4.48}$$

on the «undesirable» space $Y = [\mathscr{D}(A^{\frac{3}{4}+\varepsilon})]'$ where $\tilde{A}$ denotes the extension by isomorphism $L_2(\Omega) \to Y$ of the original $A: L_2(\Omega) \supset \mathscr{D}(A) \to L_2(\Omega)$; also $A^{\frac{1}{4}-\varepsilon}Dg \in \in L_2(\Omega)$ for $g \in L_2(\Gamma)$, $\varepsilon > 0$. Choosing g corresponds now to assigning $b \in Y$, not $a \in Y$, but we have seen that Theorem 3.1 is symmetric in the roles played by a and b. It is of course possible, but annoying, to return from the desired result in Y to the desired result in $Z = L_2(\Omega)$; see e.g. the technique in [L-T.7]. We wish, however, to make another point: that if we preassign w in (4.48), i.e. $a \in Y$, and we seek $g \in L_2(\Gamma)$, then an extra nontrivial difficulty arises, which we call «synthesis problem». Indeed once a *suitable* $b \in Y$ is obtained through Theorem 3.3 (or Theorem 3.1), one still has to recover $g \in L_2(\Gamma)$; i.e. one has to prove that such vector b can, indeed, be written (synthesized) as in (4.48) for some $g \in L_2(\Gamma)$. But $b \in Y$ is given in Theorem 3.3 by its co-ordinates and the vector $Dg \in L_2(\Omega)$ is nothing but an harmonic function in $L_2(\Omega)$ (see definition of D in (3.4), Part I). Thus, the *synthesis* problem mentioned in Remark 4.1 amounts to the possibility of being able to *characterize* the totality (or at least identify a suitably large class) of harmonic functions Dg in $L_2(\Omega)$, with $g \in L_2(\Gamma)$, in terms of their coordinates $\langle Dg, \Phi_i \rangle$ with respect to the orthonormal basis $\{\Phi_i\}$ of eigenvectors of the operator A (the Laplacian Δ with zero Dirichlet B.C.). We are not aware of any satisfactory solution to this problem in the literature. However, in the approach taken in [L-T.1], we did succeed in preassigning $w \in L_2(\Omega)$ and in obtaining $g \in L_2(\Gamma)$, thus solving the «synthesis problem». □

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