Finite State Automata Representing Two-Dimensional Subshifts

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Overview

- Background and Motivation
- Automata Representing 2D Sofic Shifts
- Uniform Horizontal Transitivity and Periodicity
- State Merging
- Open Questions

2D Shift of Finite Type

- Σ is a finite alphabet.
- Q is a set of $k \times k$ states: $[0, k-1] \times [0, k-1] \to \Sigma$.
- Shift of finite type defined by Q is $X \subseteq \Sigma^{\mathbb{Z}^2}$ such that

$$\forall x \in X, \{x_{[i,i+k-1]\times[j,j+k-1]} \mid i, j \in \mathbb{Z}\} \in Q.$$

Ex: 2D Golden Mean

a	a
a	a

b	a
a	a

a	**b**
a	a

a	a
b	a

a	a
a	**b**

For $\Sigma = \{a, b\}$, Q is finite set of states defining set X of all possible configurations of the plane having any appearance of b surrounded by a's.

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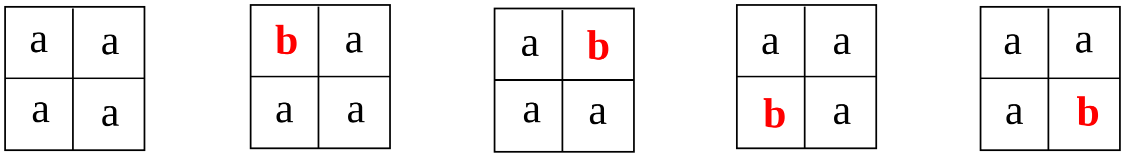

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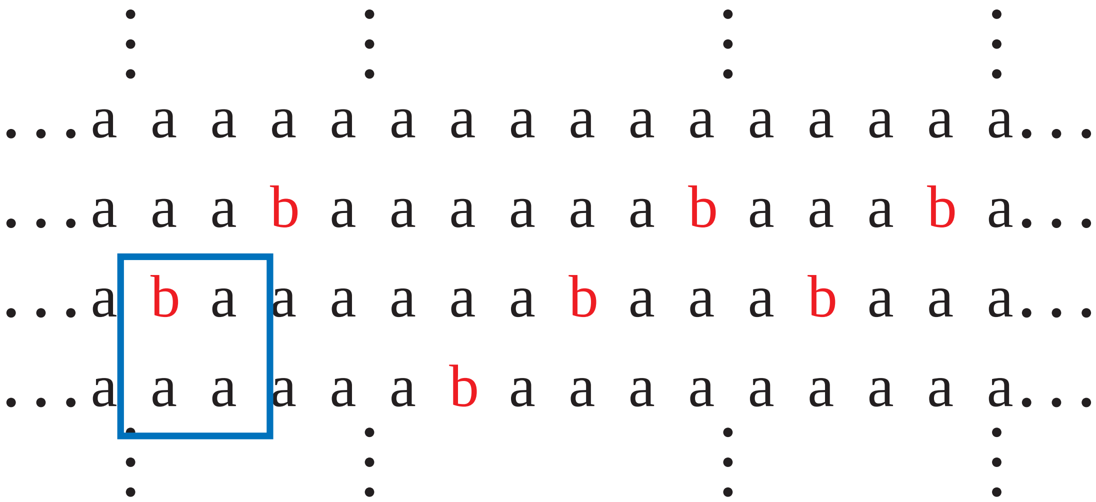

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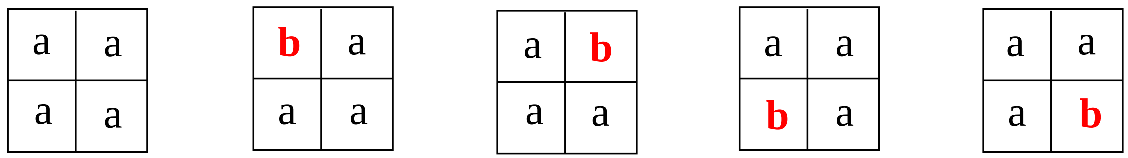

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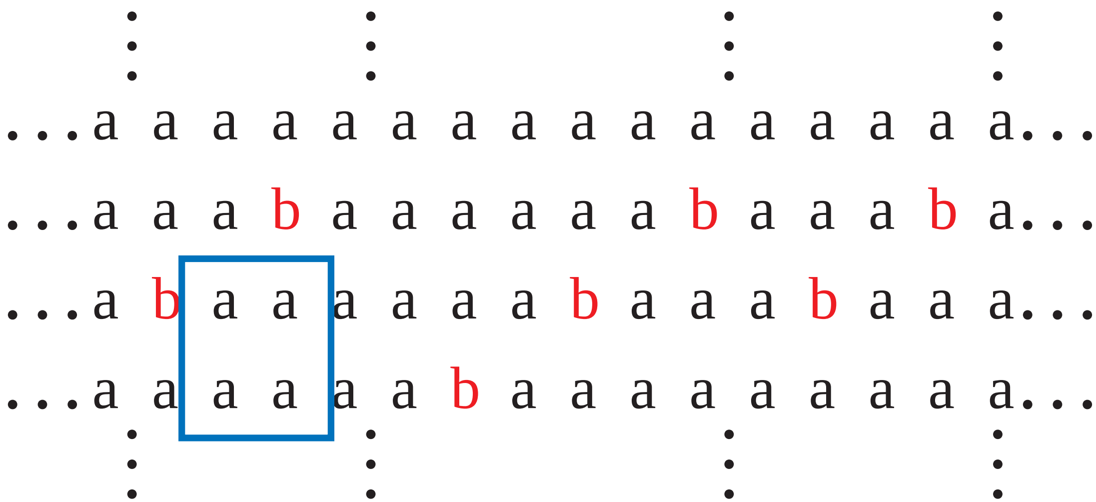

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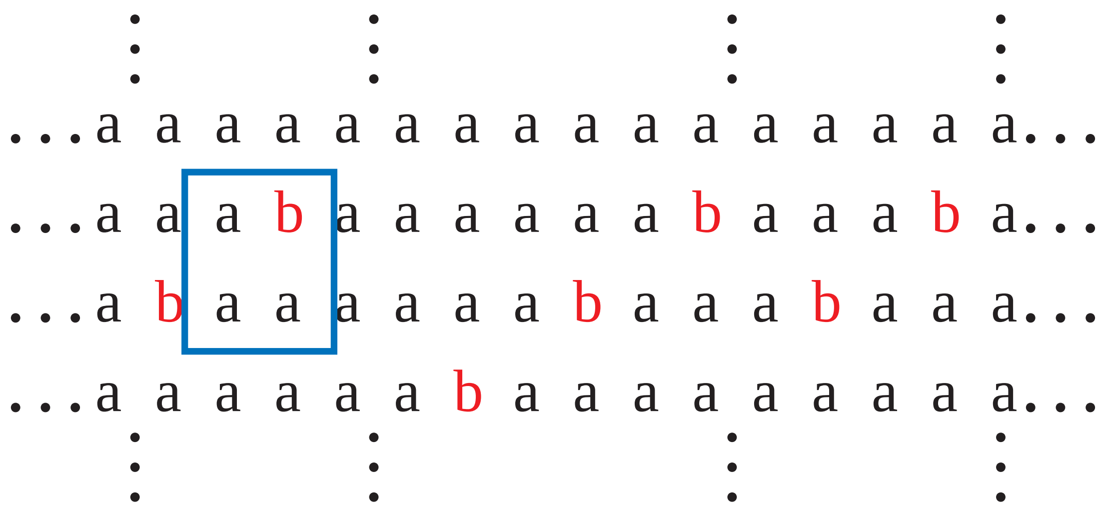

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⋮ ⋮ ⋮ ⋮

...a a a a a a a a a a a a a a a a...

...a a a b a a a a a a b a a a b a...

...a b a a a a a a b a a a b a a a...

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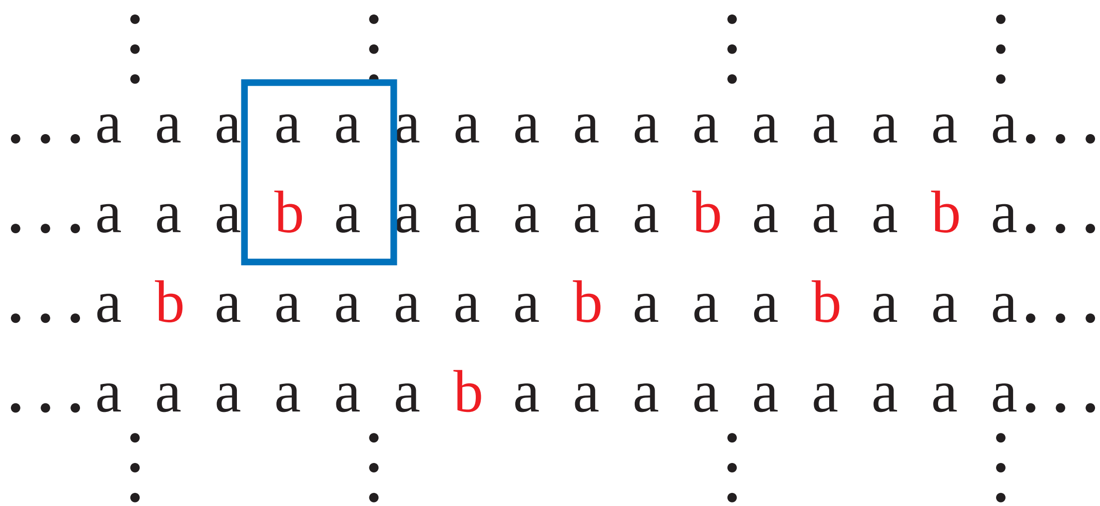

The Emptiness Problem

- Is it possible for any finite set of equal-sized square tiles with colored edges to tile the plane in such a way that contiguous edges have the same color?
(H. Wang, 1961)
- NOTE: A set of Wang tiles that can tile the plane satisfy the definition of a 2D shift of finite type.

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- NOTE: A set of Wang tiles that can tile the plane satisfy the definition of a 2D shift of finite type.
- Incorrect proof of affirmative hinges on assumption that any set of tiles capable of tiling the plane must admit a periodic tiling.
- In 1D, shift of finite type X is nonempty $\Leftrightarrow$ X contains a periodic point.

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- There exists a set of Wang tiles that can only tile the plane aperiodically. (R. Berger, 1966)
- <u>The Emptiness Problem:</u> Wang's question is now known to be undecidable.

Factors vs. Allowed Blocks

- Factors of X: For subshift X, $F(X)$ denotes set of all blocks that appear in some point of the subshift.
- Allowed blocks: $A(X)$ denotes set of all blocks that can be constructed from finite set Q which defines X.

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- Factors of X: For subshift X, $F(X)$ denotes set of all blocks that appear in some point of the subshift.
- Allowed blocks: $A(X)$ denotes set of all blocks that can be constructed from finite set Q which defines X.
- In 1D, $F(X) = A(X)$ for all shifts of finite type.
- In 2D, $F(X) \subseteq A(X)$ for all shifts of finite type, but $F(X) = A(X)$ is undecidable (Emptiness Problem).

Automata for 2D Sofic Subshifts

- Two separate graphs (matrices) have been used to represent horizontal and vertical movement in a 2D shift of finite type X.
- However, sofic subshifts that are the image of X under a block code generally can not be represented by simply relabeling the underlying pair of graphs that represent X.

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- Investigate periodicity in 2D subshifts having property $A(X) = F(X)$
- Initiate state merging to reduce graph size

$\mathcal{M}_{F(X)}$ Recognizing 2D Shifts of Finite Type

Let X be a 2D shift of finite type defined by set of $k \times k$ states Q where X has the property $A(X) = F(X)$.

The finite state automaton $\mathcal{M}_{F(X)} = (Q, E, s, t, \lambda)$ defined by Q is a finite directed graph such that:

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- Edge set is $E = E_h \cup E_v$, where ...

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$e_h : q \longrightarrow r \in E_h$ if and only if

$$\begin{matrix} q_{(1,k-1)} & \cdots & q_{(k-1,k-1)} \\ \vdots & \ddots & \vdots \\ q_{(1,0)} & \cdots & q_{(k-1,0)} \end{matrix} = \begin{matrix} r_{(0,k-1)} & \cdots & r_{(k-2,k-1)} \\ \vdots & \ddots & \vdots \\ r_{(0,0)} & \cdots & r_{(k-2,0)} \end{matrix}$$

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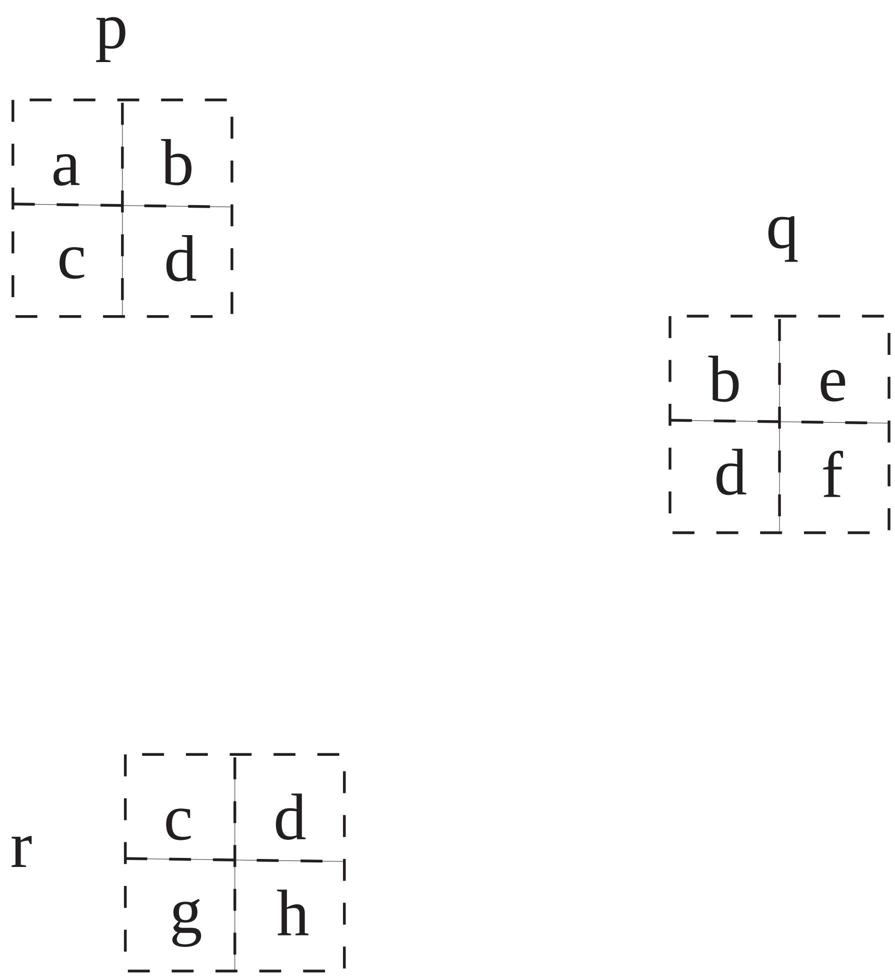
Labeling Function



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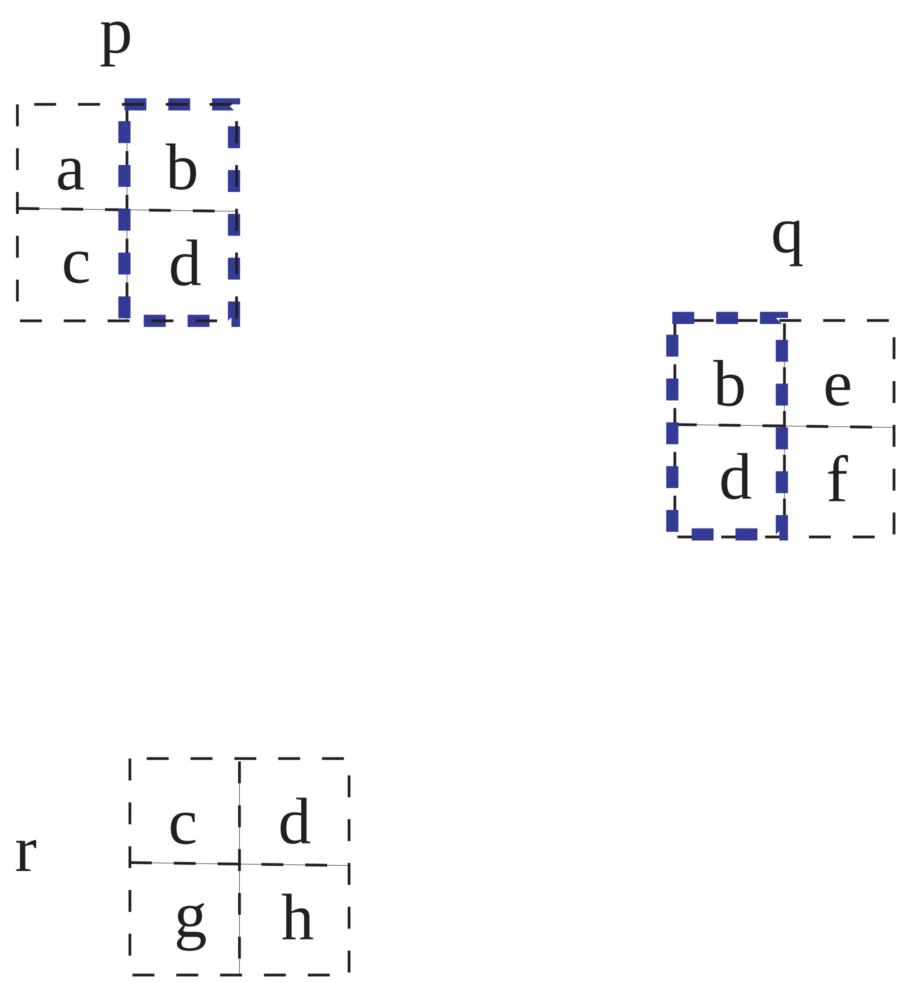

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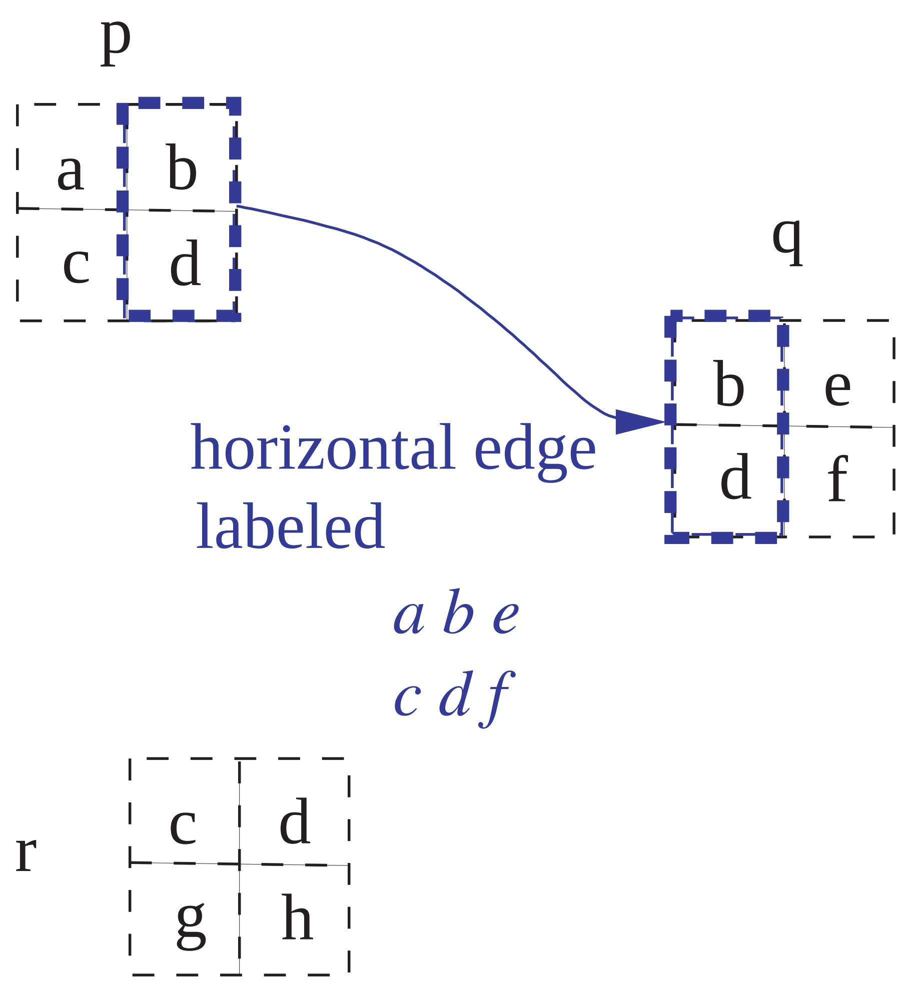

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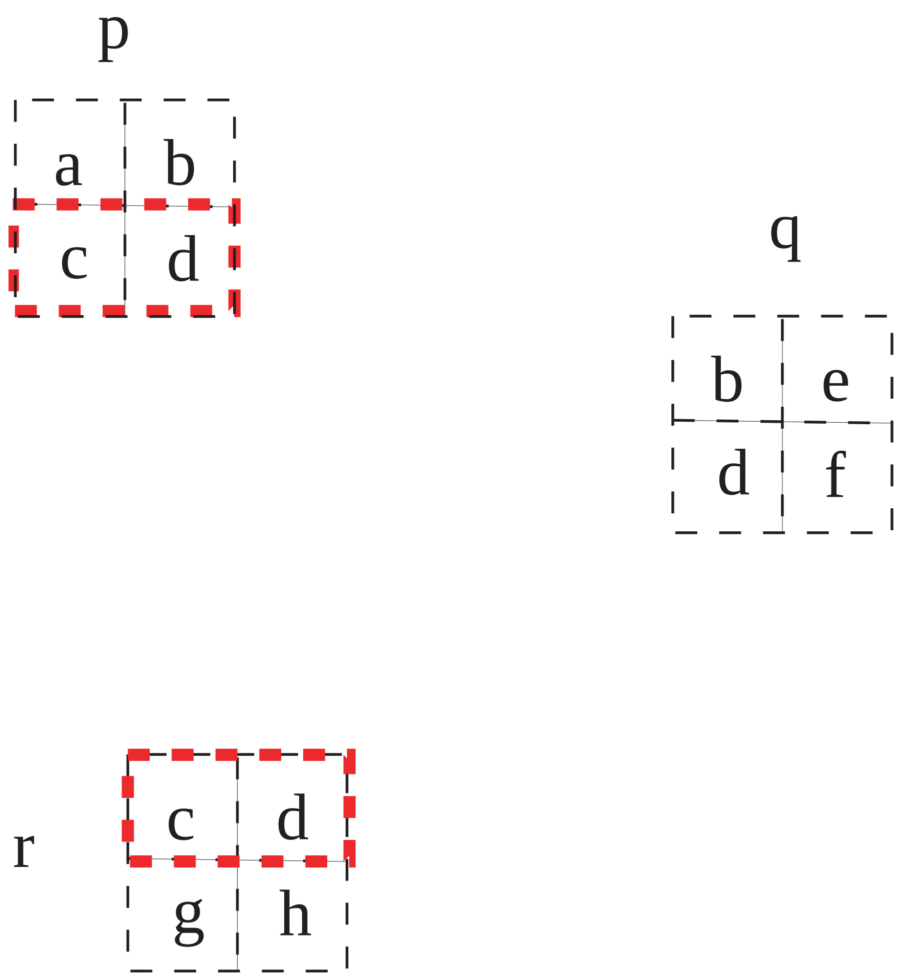

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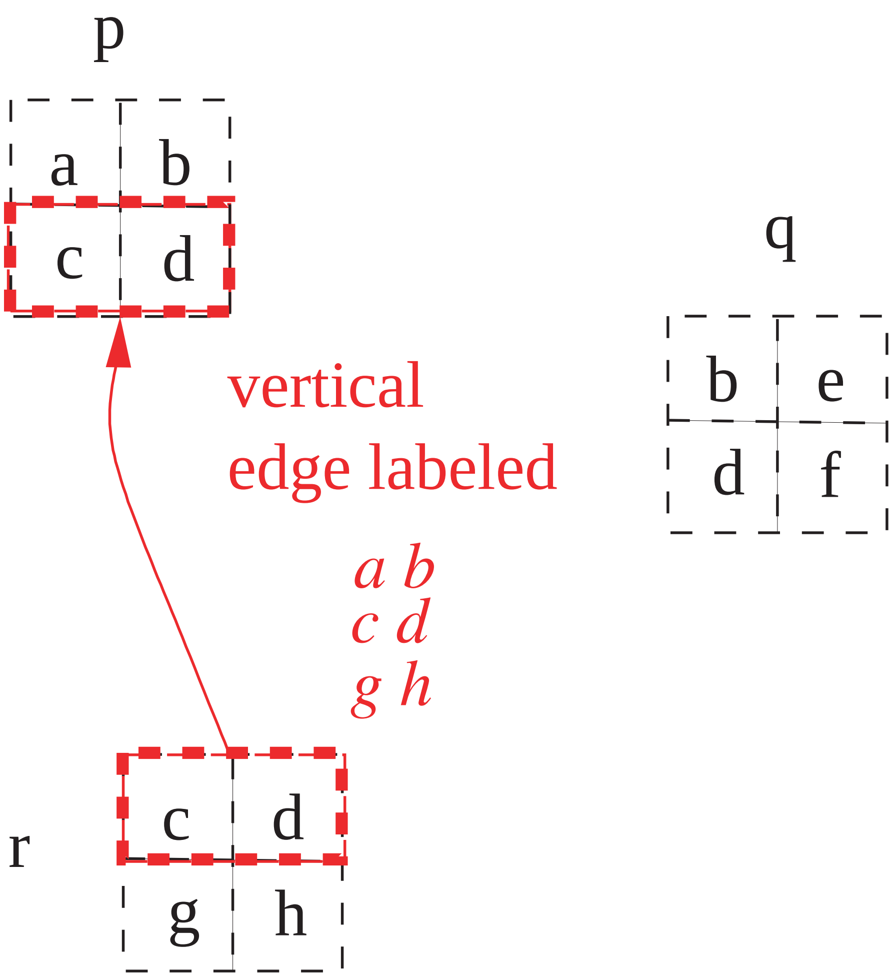

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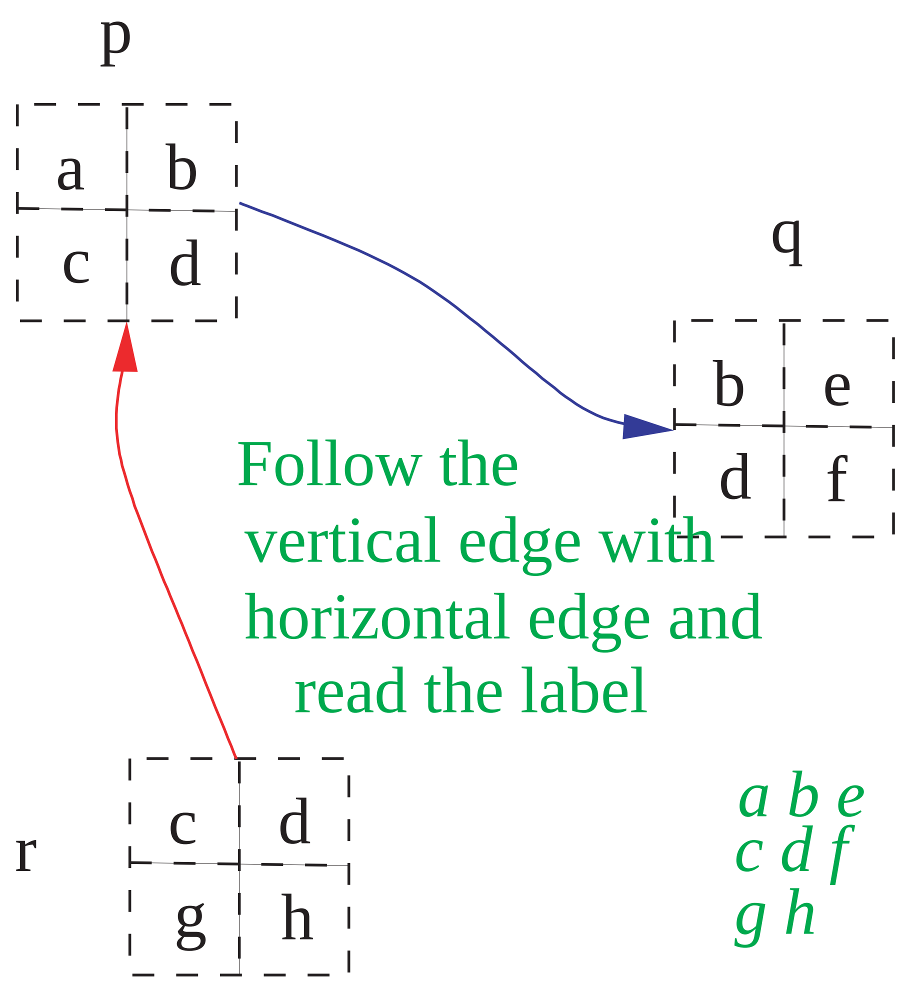

Acceptance of Non-block Factors

If X is given by a set Q of $k \times k$ blocks then a k-phrase is a shape obtained by repeated extension of rows and/or columns of width k.

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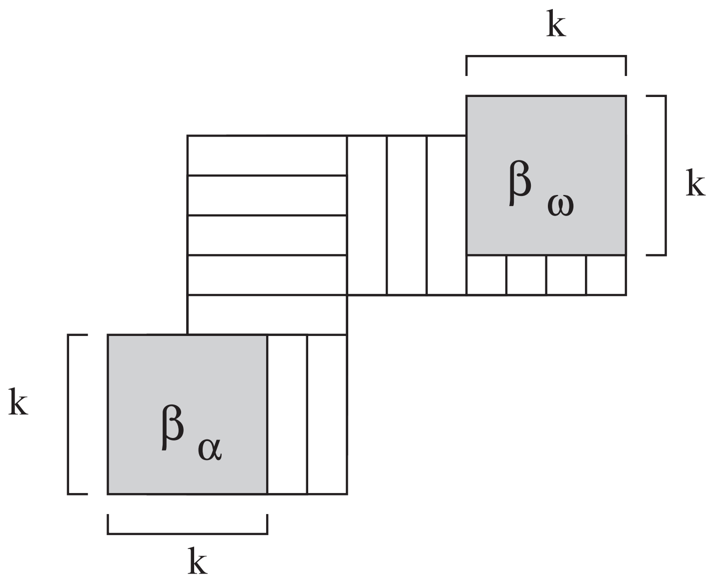

A k-phrase is said to be *accepted* by $\mathcal{M}_{F(X)}$ if there is a path in $\mathcal{M}_{F(X)}$ having P as its label.

Block Acceptance, Shifts of Finite Type

Block $B_{m,n}$ is said to be *accepted* by $\mathcal{M}_{F(X)}$ if all k-phrases of $B_{m,n}$ are accepted.

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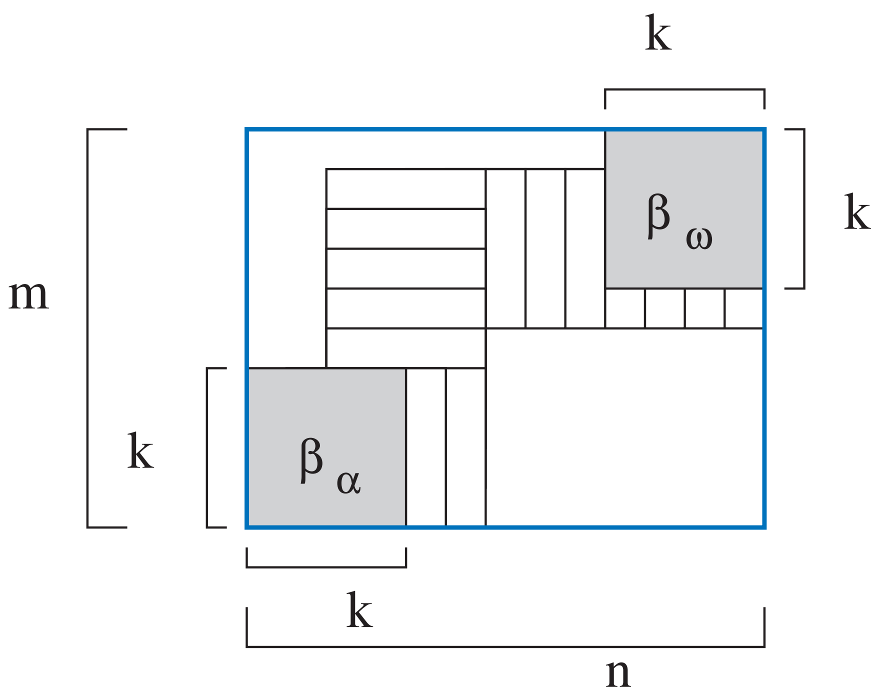
(Check all k-phrases of $B_{m,n}$ that start with β_α and terminate in β_ω after a sequence of $n-k$ horizontal transitions and $m-k$ vertical transitions.)



Proposition

For a 2D shift of finite type X having property $F(X) = A(X)$, automaton $\mathcal{M}_{F(X)}$ is such that

$$F(X) = L(\mathcal{M}_{F(X)}) = \left\{B \;:\; B \in \Sigma^{**}, B \text{ is accepted by } \mathcal{M}_{F(X)}\right\}.$$

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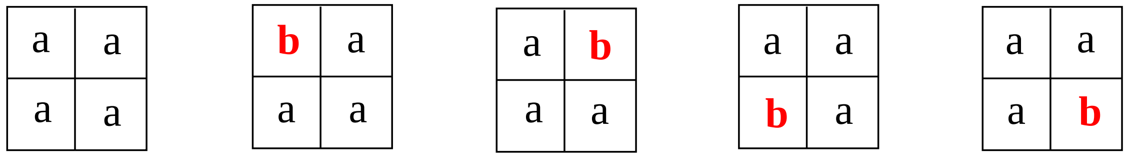

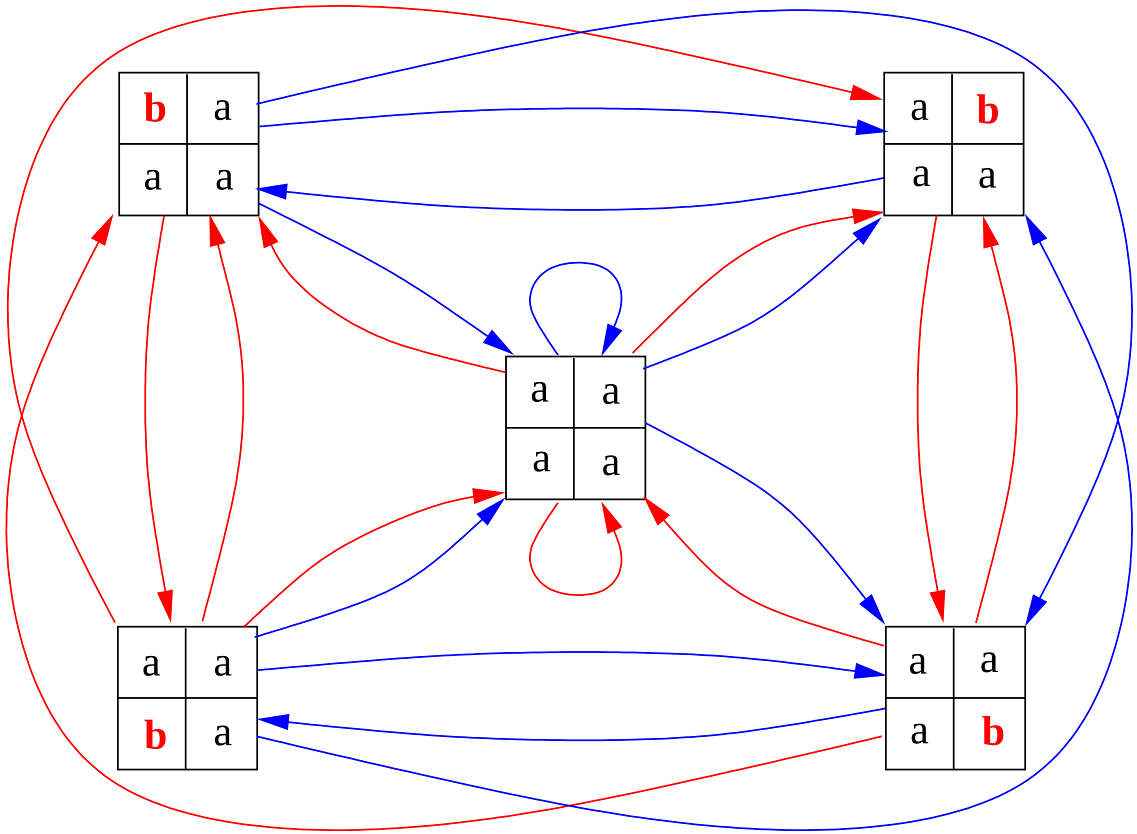

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$$
\begin{array}{ccccccc}
q_{[0,m-k+1]} & \rightharpoonup & q_{[1,m-k+1]} & \rightharpoonup & \cdots & q_{[n-k+1,m-k+1]} \\
\vdots & & \vdots & & & \vdots \\
\upharpoonleft & & \upharpoonleft & & & \upharpoonleft \\
q_{[0,1]} & \rightharpoonup & q_{[1,1]} & \rightharpoonup & \cdots & q_{[n-k+1,1]} \\
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q_{[0,0]} & \rightharpoonup & q_{[1,0]} & \rightharpoonup & \cdots & q_{[n-k+1,0]}
\end{array}
$$

Block Acceptance, Alternate Definition

- An $m \times n$ block is accepted IFF it is represented by a block path comprised of the appropriate number of states and transitions.
- The original definition of block acceptance for shifts of finite type is a special case of this since all states bear distinct labels.

Grid-Infinite Paths

- A configuration of the plane is represented by a <u>grid-infinite path</u>.

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & q_{[0,1]} & \rightharpoonup & q_{[1,1]} & \cdots \\ & \upharpoonleft & & \upharpoonleft & \\ \cdots & q_{[0,0]} & \rightharpoonup & q_{[1,0]} & \cdots \\ & \vdots & & \vdots & \end{array}$$

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- For a 2D shift of finite type X, there is a $1 - 1$ correspondence between points in X and grid-infinite paths in $\mathcal{M}_{F(X)}$.

Proposition

Let X be represented by $\mathcal{M}_{F(X)} = (Q, E, s, t, \lambda)$, and let Y be the image of X under the block map Φ.

If $\mathcal{M}^{\Phi}_{F(X)}$ is the automaton having underlying graph $\mathcal{M}_{F(X)}$ with state set Q' and edge set E' relabeled according to Φ, then $L(\mathcal{M}^{\Phi}_{F(X)}) = F(Y)$.

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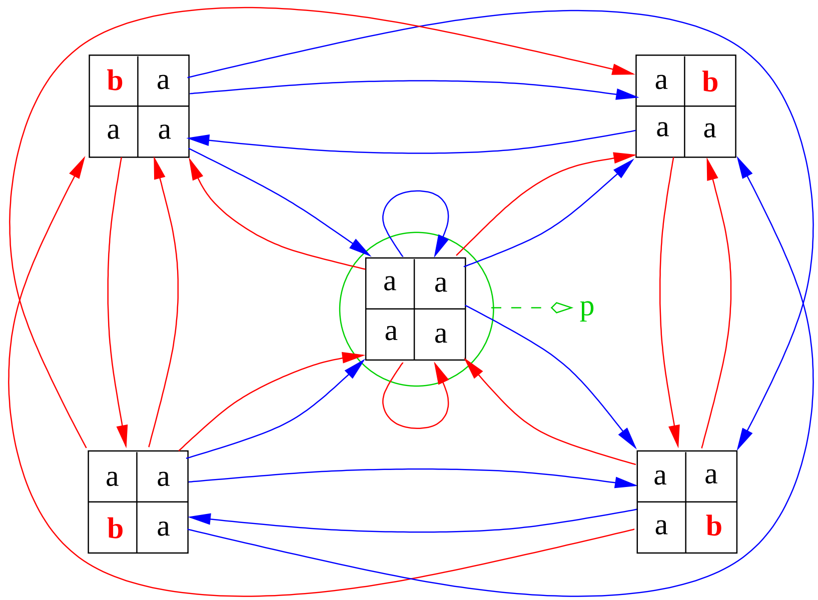
- The sofic shift Y need not be shift of finite type.
- There need no longer exist a $1-1$ correspondence between points in Y and grid-infinite paths in $\mathcal{M}^{\Phi}_{F(X)}$.

Example: Strictly Sofic Subshift



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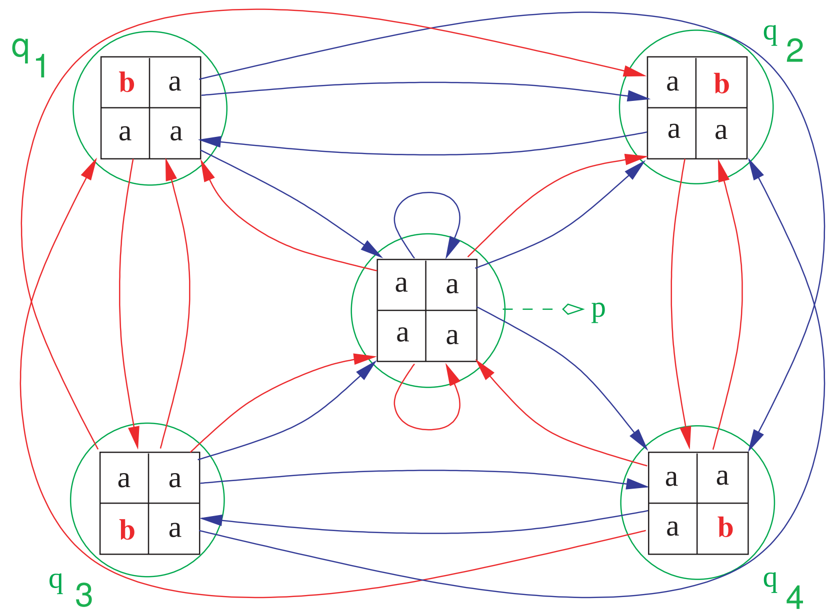
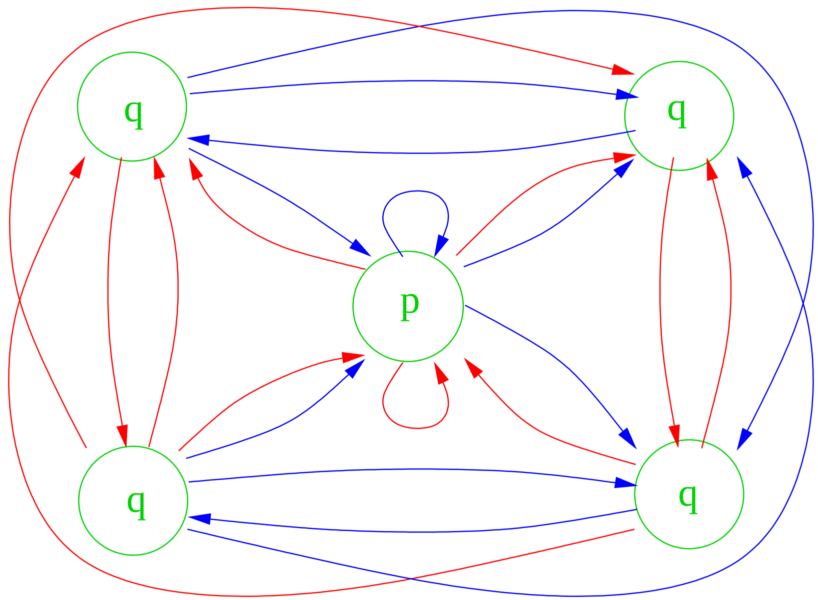


Notice q always appears in 2×2 tiles as $\begin{matrix} q_4 & q_3 \\ q_2 & q_1 \end{matrix}$.

Example: Strictly Sofic Subshift

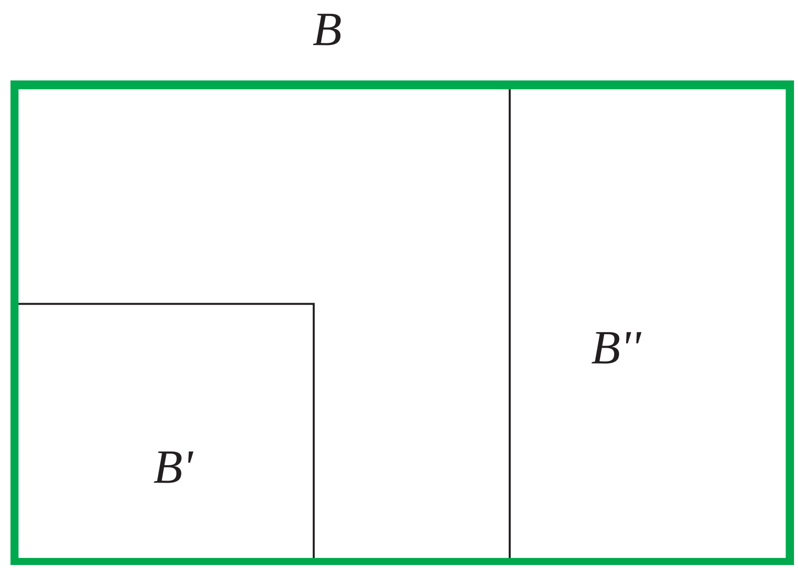


Automaton represents all configurations of the plane that

can be obtained by tiling with p and $\begin{matrix} q & q \\ q & q \end{matrix}$.

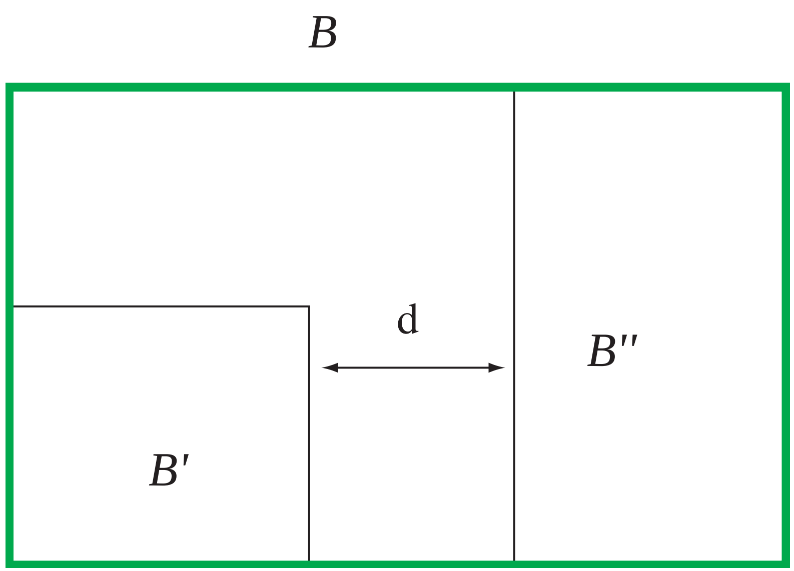
2D Uniform Horizontal Transitivity

For a 2D subshift X, we say the factor language $F(X)$ has horizontal transitivity if for every pair of blocks $B', B'' \in F(X)$ the block B' meets B'' along direction vector $\langle 1, 0 \rangle$ within some larger block $B \in F(X)$.



2D Uniform Horizontal Transitivity

For a 2D subshift X, we say the factor language $F(X)$ has uniform horizontal transitivity if there is a positive integer K such that for every pair of blocks $B', B'' \in F(X)$ that meet along direction vector $\langle 1, 0 \rangle$ there is a block $B \in F(X)$ that encloses B' and B'' in a way that $d(B', B'') < K$.

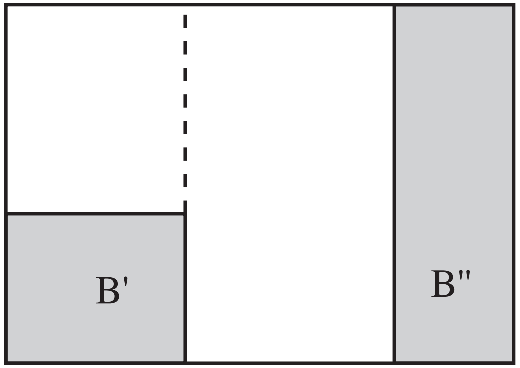


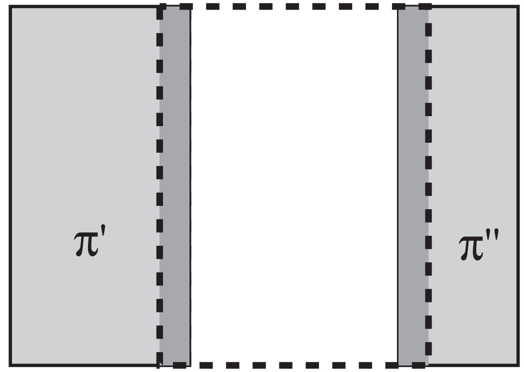
Theorem

Let X be 2D subshift represented by $\mathcal{M}^{\Phi}_{F(X)}$.

Given distance K, there is algorithm which decides whether $F(X)$ has uniform horizontal transitivity at distance K.

Automaton Facilitates Proof





We seek block path β that overlaps final and initial states of block paths representing B' and B'', respectively.

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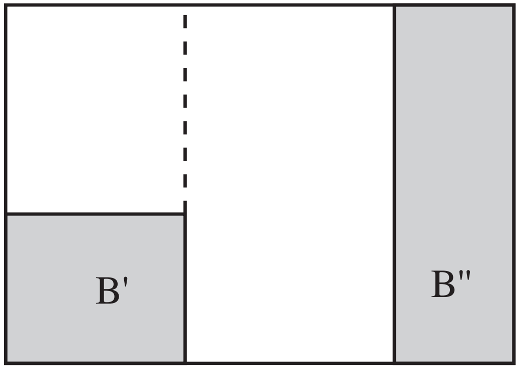

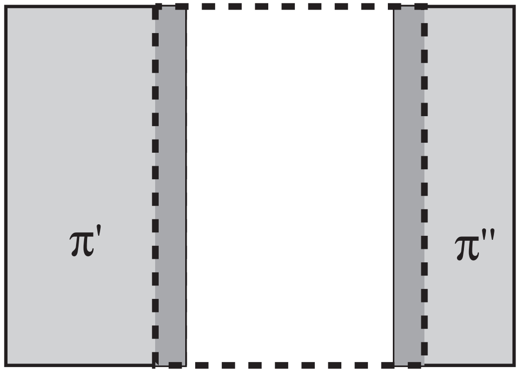

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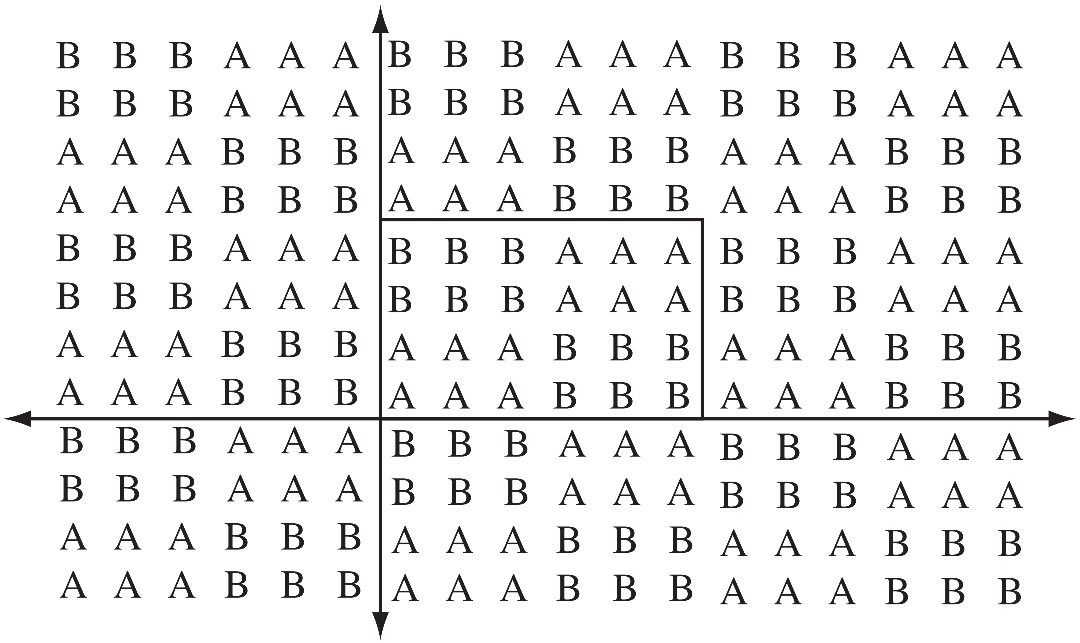
Uniformity condition permits application of well-known results from 1D automata theory.

2D Periodic Points

Given 2D shift space X, $x \in X$ is periodic of period $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ iff $x_{(i,j)} = x_{(i+a,j+b)}$ for every $(i, j) \in \mathbb{Z}^2$.



Theorem

Let X be 2D subshift represented by $\mathcal{M}^{\Phi}_{F(X)}$.

If $F(X)$ exhibits uniform horizontal transitivity at some distance K, then X has a periodic point of period (a, b) for some $a \leq K + k$.

Follower-Separated Graphs

The follower set of state $q_i \in Q$ is the set of all blocks that have bottom-left corner $\beta_\alpha = q_i$.

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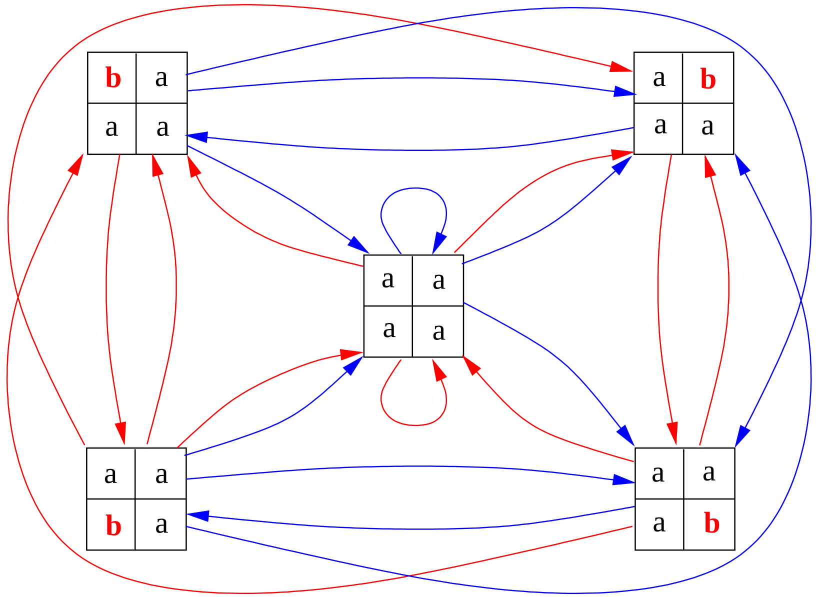
Graphs with distinct follower sets for each state are called follower-separated graphs.

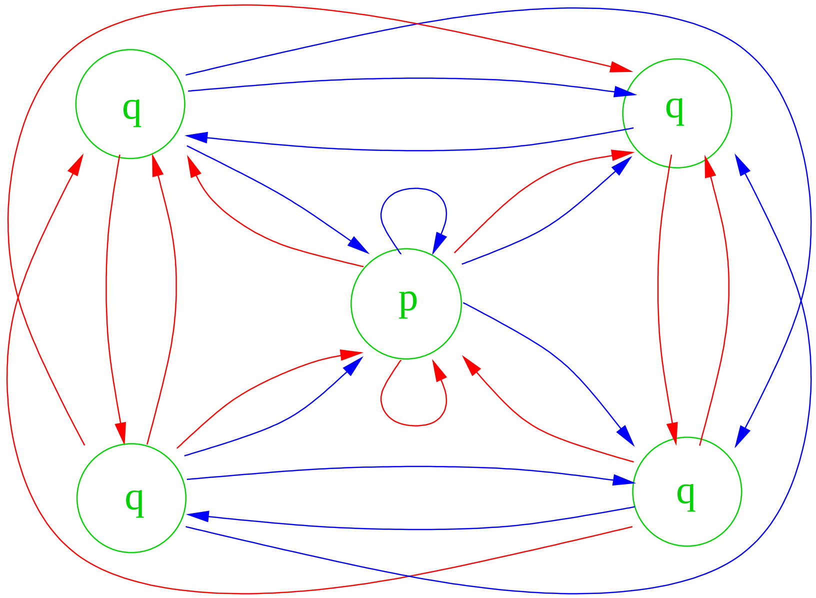
Ex: Follower-Separated Graphs

- Graphs representing 2D shifts of finite type X are inherently follower-separated.



Ex: Follower-Separated Graphs

- 2D (strictly) sofic shift can also have follower-separated graph.



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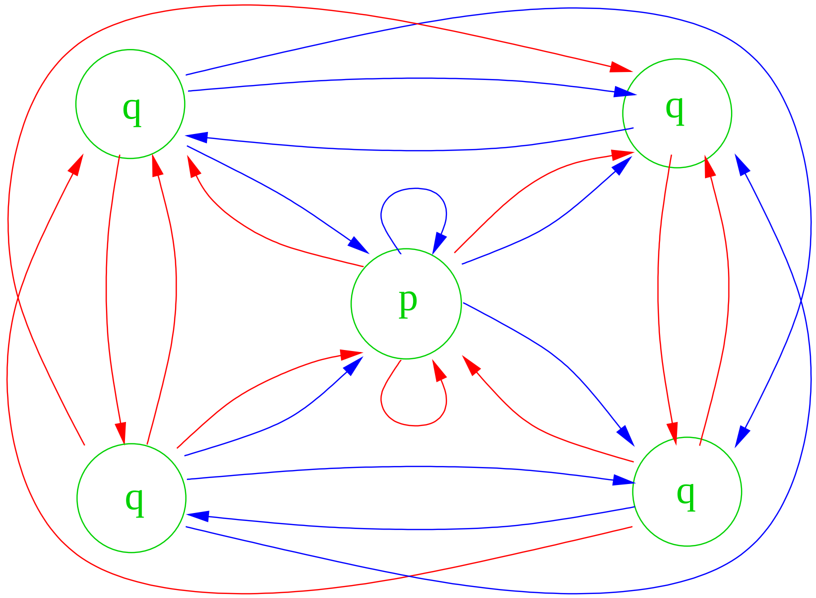


- Intersect folower sets with set $B = \{B_0, B_1, B_2\}$, where

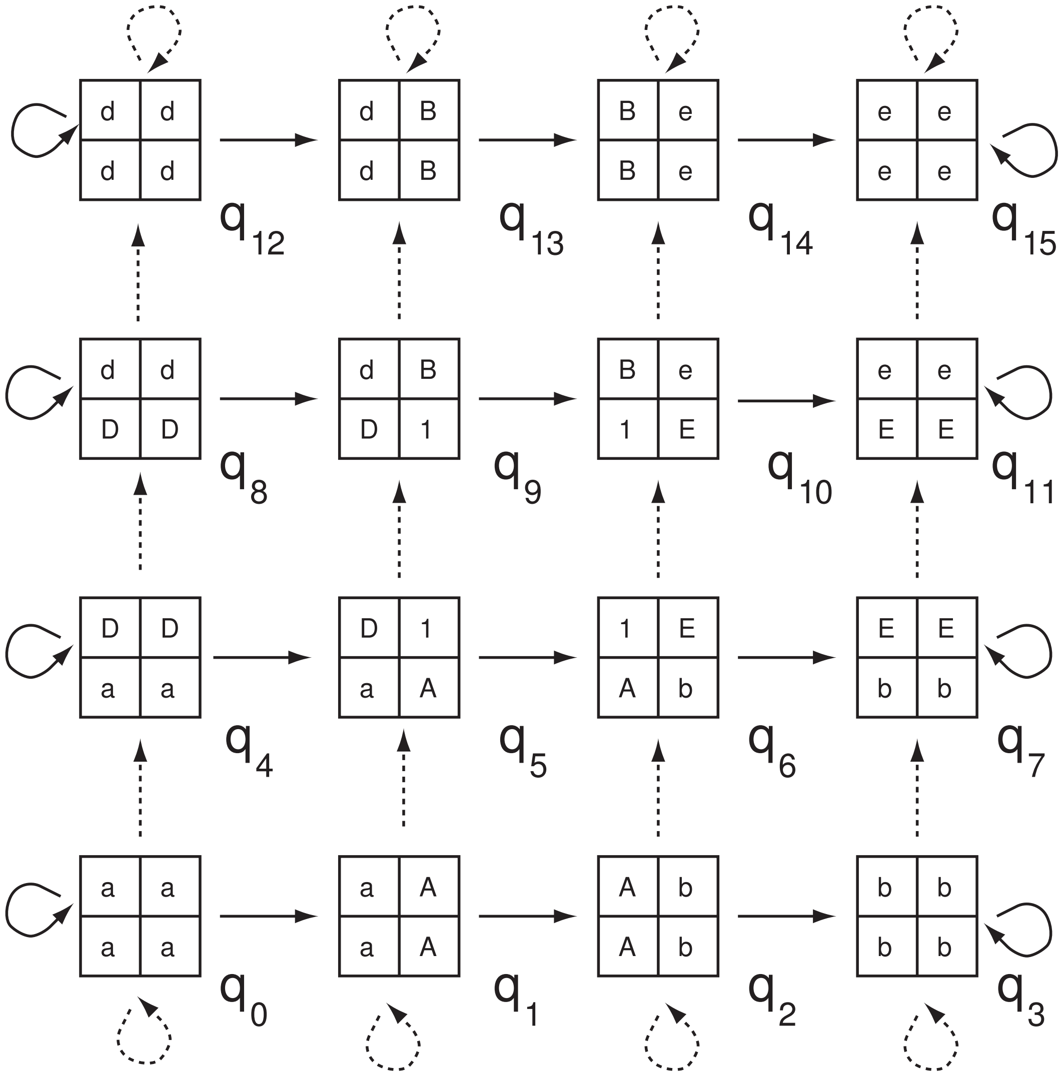
$$B_0 := \begin{matrix} p & p \\ p & p \end{matrix} \qquad B_1 := \begin{matrix} p & p \\ q & q \end{matrix} \qquad B_2 := \begin{matrix} q & p \\ q & p \end{matrix}$$

Proposition

The graph size of $\mathcal{M}^{\Phi}_{F(X)}$ can be reduced by combining states having the same follower sets without affecting the represented factor language $F(X)$.

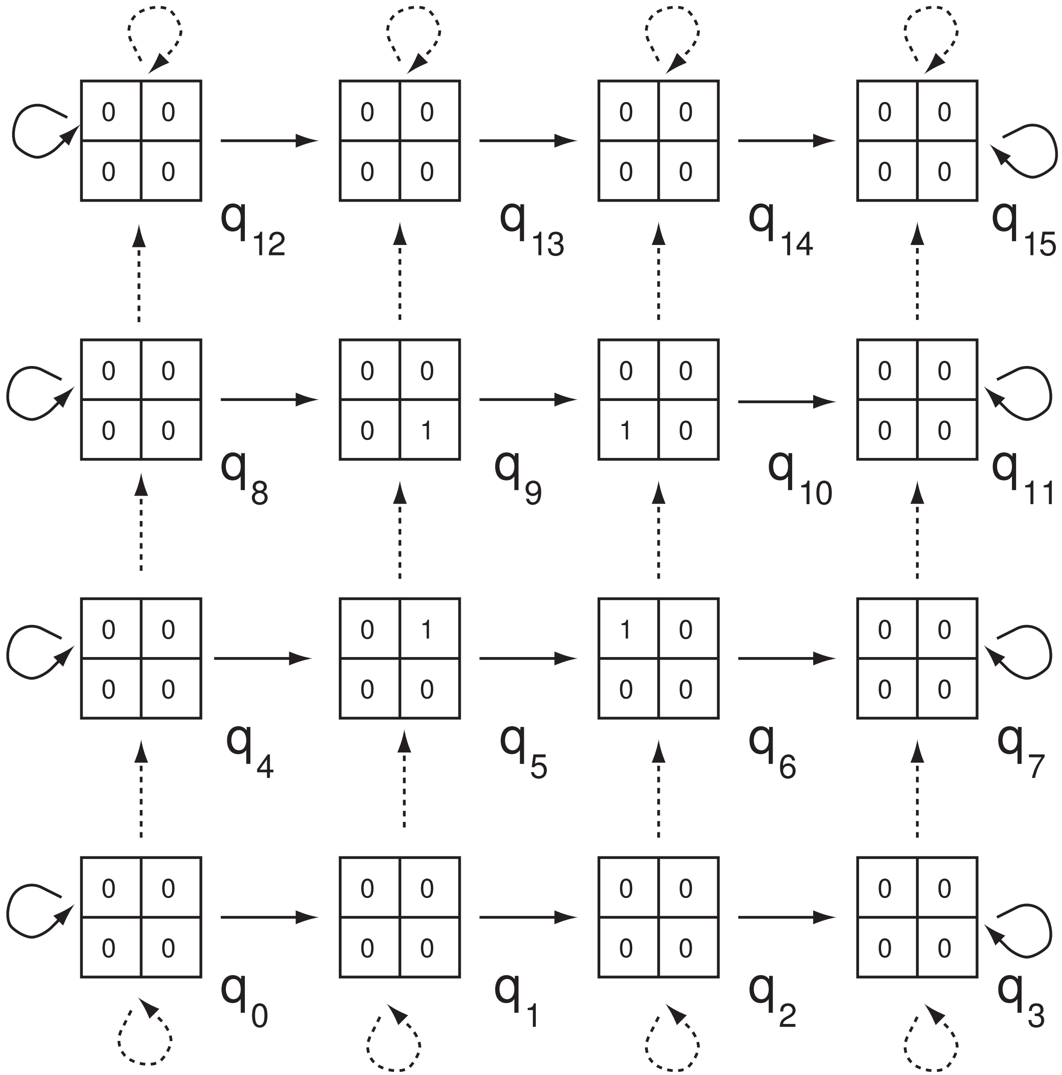
Ex: Reducing Graph Size

- Graph is follower-separated.



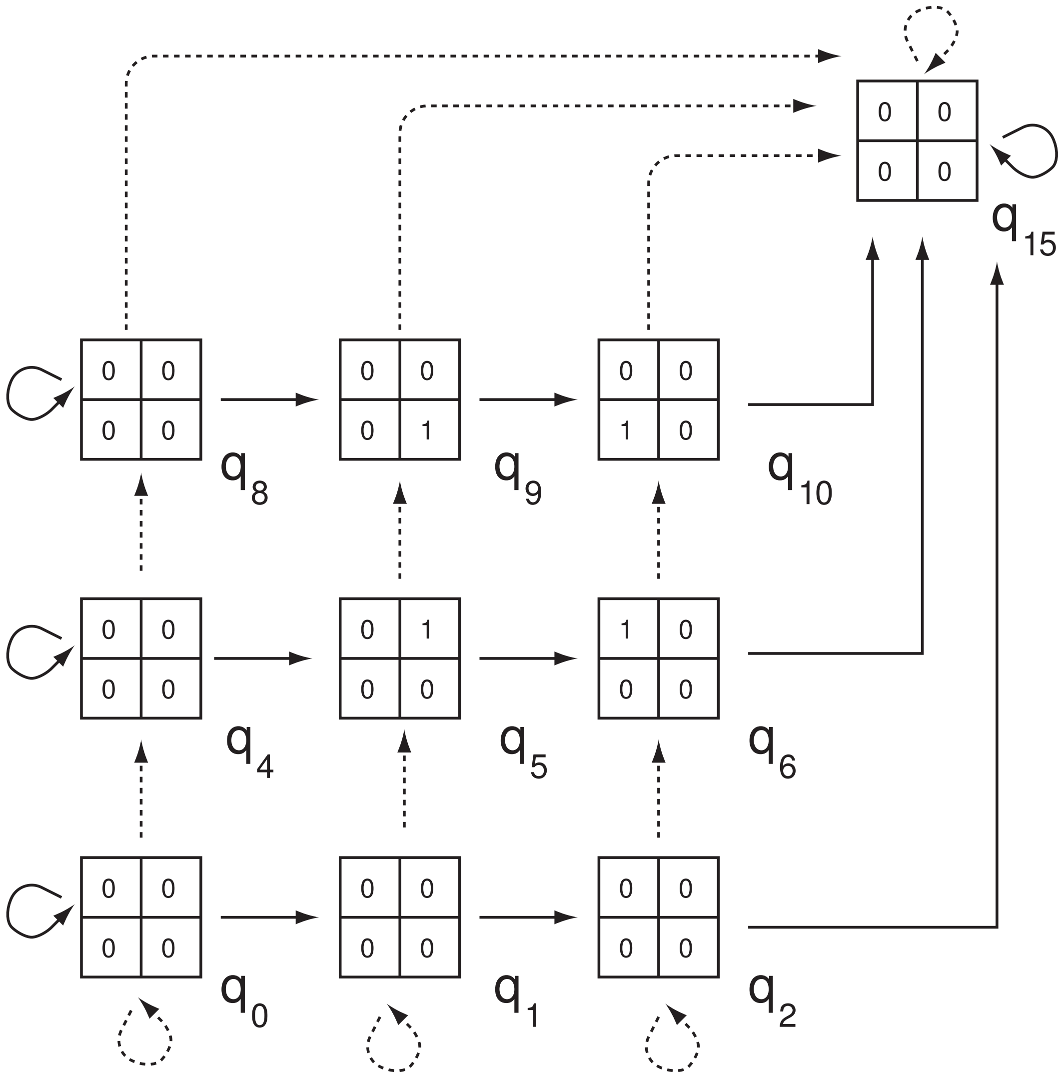
Ex: Reducing Graph Size

- Relabeled graph is not follower-separated.



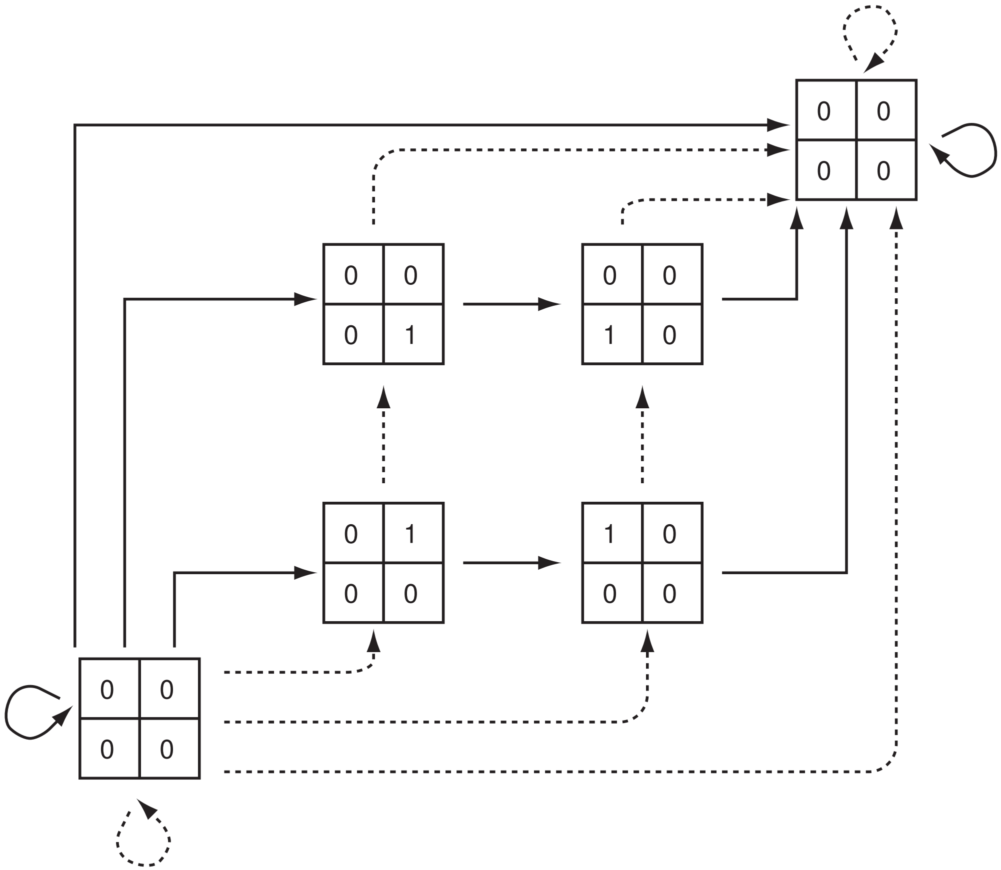
Ex: Reducing Graph Size

- Reduced graph represents same subshift.



Ex: Reducing Graph Size

- Further reduced; same subshift



Open Questions

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- Is there a notion of 2D synchronizing words for subshifts having property $F(X) = A(X)$?