FINITE SUBGROUPS OF FINITE DIMENSIONAL
DIVISION ALGEBRAS

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Let D be a finite dimensional division algebra with center K and let G be a finite odd order subgroup of the multiplicative group D^* of D. This note is concerned with the following:

CONJECTURE. If K contains no nonidentity odd order roots of unity, then G is cyclic.

We announce here some results and raise several questions about this conjecture. In [3] and [4] we proved this conjecture if K is either an algebraic number field or the completion of an algebraic number field. (The converse is also proved in [4]; if K is an algebraic number field which does contain an odd order nonidentity root of unity, then there is a finite dimensional division algebra central over K containing a noncyclic odd order subgroup.) In this note we will consider the more general case where K is an arbitrary field of characteristic zero.

By a K-division ring we mean a finite dimensional division algebra with center K. Let G be a finite subgroup of the multiplicative group of a K-division ring D and, for L a subfield of D, denote by $\mathscr{V}_L(G)$ the division subring of D generated by L and G. Let $\mathscr{Z}_L$ denote the center of $\mathscr{V}_L(G)$ and e_L the exponent of $\mathscr{V}_L(G)$. The following result is basic to our approach to the above conjecture:

THEOREM 1. *With notation as above, let ζ be a primitive e_Lth root of unity and let ϕ be an L-automorphism of $\mathscr{V}_L(G)$. Then $\phi(\zeta)=\zeta$.*

The proof of Theorem 1 involves an explicit computation using the description of $\mathscr{V}_Q(G)$ given by Amitsur in [2], where Q denotes the rational field.

Suppose G is a finite subgroup of the K-division ring D. Then $C_D(\mathscr{Z}_K)\cong \mathscr{V}_K(G) \otimes_{\mathscr{Z}_K} A$ where A is a $\mathscr{Z}_K$-division ring and $C_D(\mathscr{Z}_K)$ denotes the centralizer in D of $\mathscr{Z}_K$. Suppose $\zeta \notin K$ where ζ is a primitive e_Kth root of unity. Then there is an automorphism ϕ of $C_D(\mathscr{Z}_K)$ with $\phi(\zeta)\neq\zeta$.

In view of Theorem 1 we are led to consider whether $C_D(\mathscr{Z}_K)$ may be assumed to be $\mathscr{V}_K(G)$.

DEFINITION. Let L be a finite algebraic extension of K and let D be an L-division ring which is a subalgebra of a K-division ring B. We say that D is *maximally embedded* in B if $C_B(L) = D$.

This definition raises the following question:

Question 1. Suppose L is a finite algebraic extension of K, and D is an L-division ring which is a subalgebra of some K-division ring D_0. Does there exist a K-division ring in which D is maximally embedded?

The answer to Question 1 is affirmative if D_0 has its exponent equal to its index (which occurs, for example, if K is a local or global field). We have:

THEOREM 2. *Suppose L is a finite algebraic extension of K and let D be an L-division ring which is a subalgebra of a K-division ring D_0 having equal exponent and index. Then there exists a K-division ring B having equal exponent and index in which D is maximally embedded.*

Combining Theorems 1 and 2 we obtain:

THEOREM 3. *Let D be a K-division ring and let G be a finite subgroup of D^*. Suppose the exponent of D equals the index of D. Then K contains a primitive e_Kth root of unity. In particular, if K contains no nonidentity odd order roots of unity and D is a K-division ring having its exponent equal to its index, then all odd order subgroups of D^* are cyclic.*

Returning to the general situation for arbitrary D, we have $C_D(\mathscr{Z}_K) \cong \mathscr{V}_K(G) \otimes_{\mathscr{Z}_K} A$. If the exponents of A and $\mathscr{V}_K(G)$ are relatively prime, then the automorphism ϕ of $C_D(\mathscr{Z}_K)$ can be modified to yield an automorphism of $\mathscr{V}_K(G)$ with $\phi(\zeta) \neq \zeta$. This contradiction shows:

THEOREM 4. *The conjecture is true if there are no odd primes whose squares divide the index of D.*

Attempts to modify the automorphism ϕ above lead to the consideration of the following question:

Question 2. Let D be an E-division ring and let ψ be an automorphism of E. Can ψ be extended either to D or to a maximal subfield of D?

The answer to Question 2 is affirmative if E is a local or global field. Moreover, we have:

THEOREM 5. *An affirmative answer to Question 2 proves the conjecture.*

Let n be the index of $\mathscr{V}_Q(G)$. Then n is also the index of $\mathscr{V}_L(G)$ for any $L \subset K$ where G is a subgroup of the K-division ring D. Since $\mathscr{Z}_Q$ is an algebraic number field, $e_Q = n$. This leads to our next question:

Question 3. With notation as above, does $e_L = n$?

With regard to Question 3, it should be noted that Albert has given an example in [1] of a quadratic extension L of a field K and a K-division ring D of exponent and index 4 such that $D \otimes_K L$ is a division ring of exponent 2.

Finally, we mention that the conjecture is false if the requirement that D be finite dimensional over its center is dropped. Using results of P. M. Cohn we have:

THEOREM 6.   *Let G be a finite group which is a subgroup of some division ring. Then G is a subgroup of the multiplicative group of a division ring with center Q. However, if K is an algebraic number field containing no non-identity odd order roots of unity, then G cannot be a subgroup of the multiplicative group of a locally finite division ring with center K.*

Theorem 6 leads us to our final question:

Question 4. Let G be a noncyclic odd order subgroup of some division ring. Does there exist an algebraic division ring with center Q containing G?

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