FINITE SYMPLECTIC GEOMETRY IN DIMENSION FOUR AND CHARACTERISTIC TWO[1]

BY

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1. Introduction

The purpose of this article is to develop the point and line geometry of a nondegenerate, four-dimensional symplectic space over a finite field of characteristic two, and the additional geometry associated with the quadric of a quadratic form. This geometry furnishes a foundation for the study of the symplectic groups $PSp_4(2^n)$ and their maximal subgroups. The propositions and theorems are stated using the language of the associated projective space. The first of the two major theorems characterizes the totally isotropic reguli as being the quadrics of unique quadratic forms on the symplectic space. The second associates a unique totally isotropic line to each pair of distinct, nonpolar, hyperbolic lines.

We first introduce definitions, facts, and notation. Some basic references are [1], [2], [4], and [5].

Let n be a positive integer, F the Galois field with $q = 2^n$ elements, and F^* the nonzero elements in F. So $a = -a$ for any scalar a in F, and the square root function is an automorphism of the field F.

Let V be a four-dimensional vector space over F and f a nondegenerate, alternate, bilinear form on V. The *projective space* associated with V is the lattice of subspaces of V together with the incidence relation given by set inclusion. The one-, two-, and three-dimensional subspaces of V are called *points*, *lines*, and *planes*, respectively. If $S_1, \ldots, S_k$ are subsets of V, then $\langle S_1, \ldots, S_k \rangle$ denotes the subspace of V spanned by $S_1 \cup \cdots \cup S_k$.

Two vectors u and v in V are *orthogonal* (denoted $u \perp v$) provided $f(u, v) = 0$. The orthogonality relation on V induces a null polarity, yielding the *projective symplectic space* associated with the pair (V, f). If S is a k-dimensional subspace of V, then the *polar* of S (denoted $S^\perp$) is the $(4 - k)$-dimensional subspace $\{v \in V \mid f(v, s) = 0 \text{ for all } s \in S\}$. Two points P and Q are *orthogonal* (denoted $P \perp Q$) provided P lies in $Q^\perp$.

A line is either *totally isotropic* if every pair of its points is orthogonal or *hyperbolic* if it contains no two distinct orthogonal points. A totally isotropic line coincides with its polar, whereas a hyperbolic line h and its polar (also hyperbolic) are *skew* (that is, $h \cap h^\perp = \{0\}$). If h is a hyperbolic line, then

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$\{h, h^{\perp}\}$ is a *polar pair*, and every transversal to $\{h, h^{\perp}\}$ is totally isotropic. The totally isotropic lines which either contain a given point P or lie in $P^{\perp}$ are precisely those lines in $P^{\perp}$ which contain P.

From [6] we note that there are $q^3 + q^2 + q + 1 = (q^2 + 1)(q + 1)$ points in V, $q^2 + q + 1$ points in a plane, and $q + 1$ points on a line. Through a given point there are $q + 1$ totally isotropic lines and q^2 hyperbolic lines. All together there are $(q^2 + 1)(q + 1)$ totally isotropic lines and $(q^2 + 1)q^2$ hyperbolic lines.

2. Totally isotropic reguli

There are several point-line configurations which are significant in the geometry of a projective symplectic space: totally isotropic triangles, totally isotropic quadrilaterals, and totally isotropic reguli.

A *totally isotropic triangle* is a nondegenerate triangle whose three sides are totally isotropic lines. Since all totally isotropic lines in a plane intersect in the polar of that plane, we conclude that totally isotropic triangles *do not exist* in the projective symplectic space (V, f).

A *quadrilateral* is *totally isotropic* provided its four sides are totally isotropic lines. The vertices of a totally isotropic quadrilateral form a tetrahedron (that is, span V), and the diagonals form a polar pair.

We will use square brackets to denote an ordered set. Any ordered basis $[x_1, \ldots, x_4]$ for V such that

$$f(\sum a_i x_i, \sum b_j x_j) = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1$$

is called a *symplectic basis* for (V, f). The basis vectors of a symplectic basis $[x_1, \ldots, x_4]$ form a totally isotropic quadrilateral with diagonals $\langle x_1, x_4 \rangle$ and $\langle x_2, x_3 \rangle$. Conversely, one may select (with two degrees of freedom) a symplectic basis whose vectors span the vertices of any given totally isotropic quadrilateral.

Before defining the totally isotropic regulus, we state two simple facts about skew, totally isotropic lines and leave the proofs to the reader.

PROPOSITION 1. *If k and m are two skew, totally isotropic lines, then the totally isotropic transversals to k and m form a set of $q + 1$ mutually skew lines and determine a bijection from the points on k onto the points on m.*

PROPOSITION 2. *Any pair of totally isotropic transversals to different pairs of opposite sides of a totally isotropic quadrilateral must meet.*

If Q is a totally isotropic quadrilateral with opposite sides k, m and u, v, then each of the $q + 1$ totally isotropic transversals (*rulers*) to k and m meets each of the $q + 1$ totally isotropic transversals (*directrices*) to u and v. The set R of $(q + 1)^2$ distinct points of intersection is called a *totally isotropic regulus*. (This definition departs slightly from the classical use of the word "regulus" to refer to a certain family of lines.) Any point on the regulus is the intersection of a unique ruler and a unique directrix. Since there exist no totally isotropic triangles, any totally isotropic line meeting the regulus R in at least two distinct points must lie in R and be a ruler or a directrix. A totally isotropic regulus R is determined as the set of all points lying on the totally isotropic transversals to any given pair of skew, totally isotropic lines in R. In the next section we will show that the totally isotropic reguli are precisely the quadrics of maximal index quadratic forms on (V, f).

PROPOSITION 3. *Let R be a totally isotropic regulus. Then any totally isotropic line either contains exactly one point of R or is a ruler or a directrix of R, and any hyperbolic line contains exactly zero or two points of the regulus.*

Proof. For the first claim it suffices, by the preceeding discussion, to show that every totally isotropic line contains at least one point of the regulus R. For each of the $(q + 1)^2$ points X in R there are $q - 1$ totally isotropic lines which meet R in exactly the point X. Adding the $q + 1$ rulers and $q + 1$ directrices yields a number equal to the total number of totally isotropic lines.

Suppose k is a hyperbolic line meeting the regulus R in three distinct points X, Y, and Z. The rulers and directrices of R through X, Y, and Z will be six distinct, totally isotropic lines. A symplectic basis $[x_1, \ldots, x_4]$ for (V, f) can

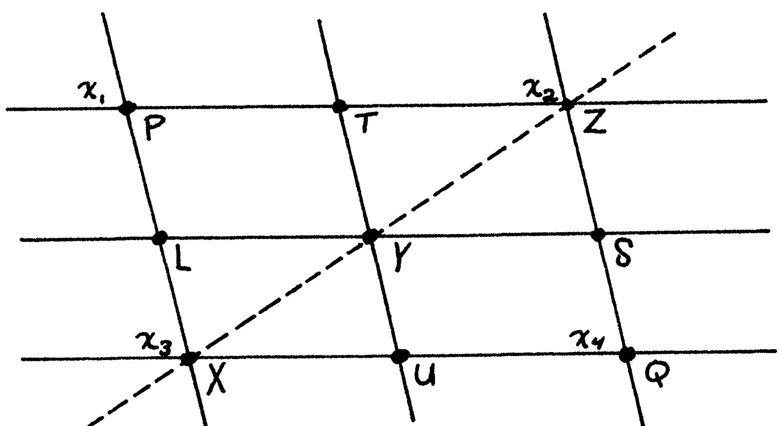

FIGURE 1

be selected such that the incidence diagram in Figure 1 applies. So $L = \langle x_1 + bx_3 \rangle$ and $T = \langle x_1 + ax_2 \rangle$ for some a and b in F^*. Hence

$$S = \langle x_2 + bx_4 \rangle, \quad U = \langle x_3 + ax_4 \rangle \quad \text{and} \quad Y = \langle x_1 + ax_2 + bx_3 + abx_4 \rangle,$$

contrary to Y being on the line $\langle x_2, x_3 \rangle$. Thus, no hyperbolic line meets a totally isotropic regulus in more than two distinct points. A count shows that all of the q^2 hyperbolic lines containing a given point on the regulus meet the regulus in exactly two points.

3. Quadratic forms

The projective symplectic geometry (V, f) can be enriched by the introduction of a quadratic form. Let Q be a *quadratic form* on the symplectic space (V, f), that is, a function $Q\colon V \to F$ such that

$$Q(au + bv) = a^2Q(u) + b^2Q(v) + abf(u, v)$$

for all $a, b \in F$ and $u, v \in V$. A *singular point* is a point spanned by a *singular vector* v (for which $Q(v) = 0$). The set of all singular points for Q is called the *quadric* for Q. A subspace is *totally singular* if all its vectors are singular. The quadratic form Q is either a *maximal index* quadratic form if each maximal totally singular subspace has dimension two, or a *nonmaximal index* quadratic form if there are no totally singular lines. See [5] and [3] for further details.

Any totally singular line is also totally isotropic. Conversely, if a totally isotropic line contains two distinct singular points, then it is also totally singular. A hyperbolic line contains exactly zero or two singular points.

We leave it to the reader to verify the following elementary facts: (1) the singular points in the polar of a singular point S form precisely the set $\{S\}$ if Q has nonmaximal index and precisely the set of points on two totally singular lines intersecting in S if Q has maximal index, and (2) the singular points in the polar of a nonsingular point N form an oval (set of $q + 1$ points in a plane, no three of which are collinear) whose tangent lines are precisely the totally isotropic lines through N.

We can clearly select a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) such that $Q(\sum a_i x_i) = a_1^2 + a_1a_4 + a_4^2Q(x_4) + a_2a_3$. Then Q has maximal index if and only if the polynomial $X^2 + X + Q(x_4)$ is reducible over F.

From Theorem 3.2 in [7], we note that a maximal index quadric contains $(q + 1)^2$ points and $2(q + 1)$ totally singular lines, whereas a nonmaximal index quadric contains $q^2 + 1$ points and no totally singular lines.

PROPOSITION 4. *Let Q_1 and Q_2 be quadratic forms on (V, f) having the same quadric of singular points. Then $Q_1 = Q_2$.*

Proof. Suppose $Q_1 \neq Q_2$. Then there is a nonsingular vector x_1 such that $Q_1(x_1) \neq Q_2(x_1)$ and a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) such that x_2 and x_3 are singular. If $y = x_1 + Q_1(x_1)x_2 + x_3$, then $Q_1(y) = 0$, while $Q_2(y) \neq 0$, contrary to hypothesis. Thus, $Q_1 = Q_2$.

THEOREM 1. *There is a one-to-one correspondence between the set of totally isotropic reguli in (V, f) and the set of maximal index quadratic forms on (V, f) given by $R \leftrightarrow Q$ if and only if R is the quadric for Q.*

Proof. Let Q be a maximal index quadratic form on (V, f). There is a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) such that $Q(\sum a_i x_i) = a_1a_4 + a_2a_3$. The skew, totally singular lines $k = \langle x_1, x_2 \rangle$ and $m = \langle x_3, x_4 \rangle$ determine a totally isotropic regulus R as the set of points on the totally isotropic transversals to k and m. Any point $\langle \sum a_i x_i \rangle$ not on k or m lies in R if and only if it lies on the line joining the points where k and m meet $\langle \sum a_i x_i \rangle^{\perp}$, that is, if and only if $\langle a_3 x_1 + a_4 x_2 \rangle$ and $\langle a_1 x_3 + a_2 x_4 \rangle$ are orthogonal. Thus, the singular points for Q are precisely the points of R, and the totally singular lines for Q are the rulers and directrices of R.

Conversely, any totally isotropic regulus R contains a totally isotropic quadrilateral from whose vertices we can select a symplectic basis $[x_1, \ldots, x_4]$ for (V, f). The function $Q\colon V \to F$ given by $Q(\sum a_i x_i) = a_1 a_4 + a_2 a_3$ is easily seen to be a quadratic form on (V, f) which has R as its quadric. Proposition 4 implies that Q is the unique quadratic form on (V, f) with quadric R.

4. Pairs of hyperbolic lines

If k and m are distinct, nonpolar hyperbolic lines, then one of the following occurs: (i) m meets k in a point X and hence $m^{\perp}$ meets $k^{\perp}$ in a point Y; (ii) m meets $k^{\perp}$ in a point Y and hence $m^{\perp}$ meets k in a point X; or (iii) the lines in $\{k, m, k^{\perp}, m^{\perp}\}$ are pairwise skew. When examining distinct, nonpolar hyperbolic lines k and m, it is more advantageous to consider the distinct polar pairs $\{k, k^{\perp}\}$ and $\{m, m^{\perp}\}$, which are either (1) *skew* if k, m, $k^{\perp}$, and $m^{\perp}$ are pairwise skew, or (2) *intersecting* if k, m, $k^{\perp}$, and $m^{\perp}$ are not pairwise skew.

THEOREM 2. (1) *If $\{k, k^{\perp}\}$ and $\{m, m^{\perp}\}$ are skew polar pairs, then the $q + 1$ totally isotropic transversals to k, m, $k^{\perp}$, and $m^{\perp}$ are pairwise skew, and there is a unique totally isotropic line which meets all these totally isotropic transversals.*

(2) *If $\{k, k^{\perp}\}$ and $\{m, m^{\perp}\}$ are intersecting polar pairs, then among the lines k, m, $k^{\perp}$, and $m^{\perp}$ there are exactly two pairwise intersections in orthogonal points X and Y, and the totally isotropic transversals to k, m, $k^{\perp}$, and $m^{\perp}$ are precisely the $2q + 1$ totally isotropic lines containing either X or Y. The totally isotropic line $\langle X, Y \rangle$ is uniquely determined by the lines k and m on the basis of incidence.*

(3) *In both cases above the totally isotropic transversals to any pair of nonpolar lines from the set $\{k, m, k^{\perp}, m^{\perp}\}$ are precisely the totally isotropic transversals to the entire set $\{k, m, k^{\perp}, m^{\perp}\}$.*

Proof. Part (3) will be proved in the course of demonstrating parts (1) and (2).

For part (1), let $\{k, k^{\perp}\}$ and $\{m, m^{\perp}\}$ be skew polar pairs. Without loss of generality we can use k and m as an arbitrary pair of nonpolar lines from the set $\{k, m, k^{\perp}, m^{\perp}\}$. For each of the $q + 1$ points X on m, the hyperbolic line k meets $X^{\perp}$ in a unique point X', and $\langle X, X' \rangle$ is the unique totally isotropic transversal to k and m through X. Clearly, the totally isotropic transversals to k and m are pairwise skew. Application of this argument using k and $m^{\perp}$ instead of k and m, together with the nonexistence of totally isotropic triangles, guarantees that any totally isotropic transversal to k and m must also intersect $k^{\perp}$ and $m^{\perp}$.

The second claim in part (1) is more difficult to prove. Let u, v, and w be three of the pairwise skew, totally isotropic transversals to k and m. Let R be the totally isotropic regulus with rulers u and v. By Proposition 3, the totally isotropic line w meets R in exactly one point Z, which lies on a unique directrix t of R. Let t meet u and v in X and Y, respectively. We wish to find a suitable totally isotropic quadrilateral on which to base computations using a symplectic basis. Let the intersection points of the various lines be labeled as in the incidence diagram in Figure 2. It is easy to verify that at most one of the points

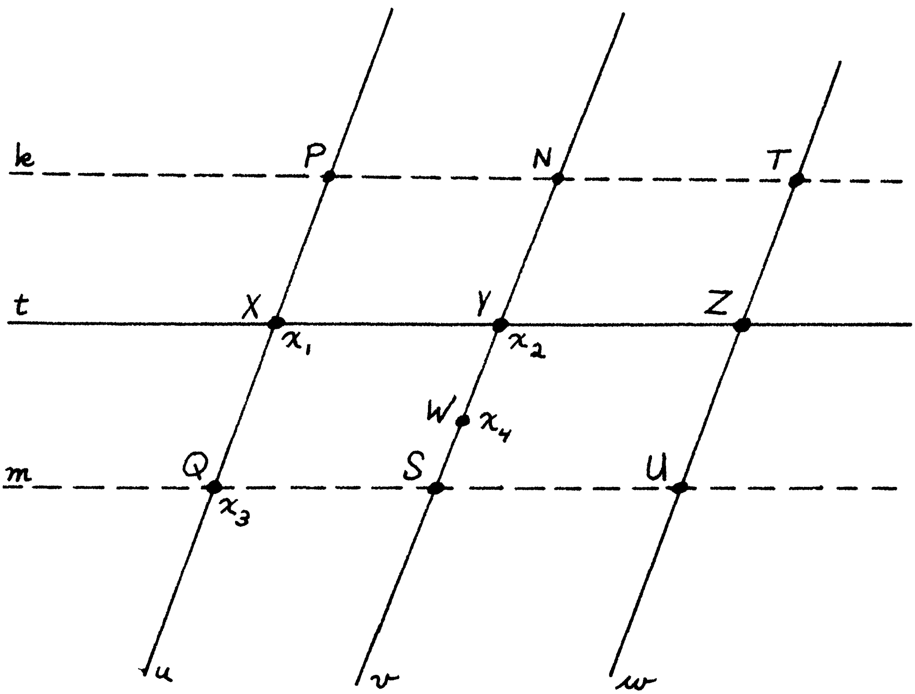

FIGURE 2

labeled "X", "Y", or "Z" might lie on k or m and coincide with another labeled point. After possible relabeling of u, v, w, and k, m, we can assume that all labeled points are distinct, except Y and N might possibly coincide. Using the portion already proved, the fact that t, u, v, and w are totally isotropic lines and k and m hyperbolic lines, and the nonexistence of totally isotropic triangles, it is easy to verify each of the following: $P \not\perp S$, $Q \not\perp N$, $Q \not\perp S$, $Q \not\perp Y$, $X \perp Y$, and $X \perp Q$. Since v does not lie in $Q^{\perp}$, it must meet $Q^{\perp}$ in a unique point W different from N, Y, and S. Hence $W \perp Q$ and $W \perp Y$, while $W \not\perp X$. Thus, $[X, Y, W, Q]$ is a totally isotropic quadrilateral from which we can select a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) which satisfies the additional condition that $x_1 + x_3$ spans P. So $N = \langle x_2 + ax_4 \rangle$ and $S = \langle x_2 + bx_4 \rangle$ for some a

in F and b in F^*. Further, $b \neq 1$ since $P \not\perp S$. Since $U = \langle x_2 + dx_3 + bx_4 \rangle$ for some d in F^*, the point T (being $U^\perp \cap k$) must be spanned by

$$\left(\frac{d}{b+1}\right) x_1 + x_2 + \left(\frac{d}{b+1}\right) x_3 + ax_4,$$

and the point Z (being $U^\perp \cap t$) must be spanned by $x_1 + (b/d)x_2$. Since $Z \perp T$, we conclude that $a = b/(b + 1)$, and so $N = \langle (b + 1)x_2 + bx_4 \rangle$.

The totally isotropic transversals to k and m different from u and v are the $q - 1$ lines

$$\langle \varepsilon x_1 + (b + 1)x_2 + \varepsilon x_3, x_2 + \varepsilon x_3 + bx_4 \rangle$$

for each ε in F^*. Each of these lines meets t in the point $\langle \varepsilon x_1 + bx_2 \rangle$. We have thus shown the existence of a totally isotropic line t meeting all the totally isotropic transversals to k and m.

If t' is a totally isotropic line which is different from t and meets all the totally isotropic transversals to k and m, then t and t' are skew and form the directrices of a totally isotropic regulus whose rulers are the totally isotropic transversals to k and m, yielding a contradiction to Proposition 3. Hence t is the unique totally isotropic line which meets all the totally isotropic transversals to k and m.

For the proof of part (2), let $\{k, k^\perp\}$ and $\{m, m^\perp\}$ be intersecting polar pairs. The first claim of part (2) is trivial. Without loss of generality, we may suppose that k and m meet in the point X, while $k^\perp$ and $m^\perp$ meet in the point Y. The plane $Y^\perp$ is spanned by k and m. Any totally isotropic line meeting k and m in distinct points lies in $Y^\perp$ and so contains Y. The totally isotropic lines which meet k and m in their intersection point are precisely the totally isotropic lines through X. Similarly the totally isotropic transversals to $k^\perp$ and $m^\perp$ are precisely the totally isotropic lines containing either X or Y. For each point W on k and different from X, the line $m^\perp$ intersects $W^\perp$ in a unique point Y, since m does not contain W. Thus, there is a unique totally isotropic transversal to k and $m^\perp$ through W. Similar arguments show that the totally isotropic transversals to k and $m^\perp$ (or to $k^\perp$ and m) are precisely the totally isotropic lines containing X or Y.

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