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Finite temperature effective action, AdS_5 black holes, and $1/N$ expansion

Luis Alvarez-Gaume[b], Cesar Gomez[♮], Hong Liu[*], and Spenta R. Wadia[b,⋆]

[b] *Physics Dept.-Theory Division, CERN, CH-1211,Geneva 23, Switzerland*

[♮] *Instituto de Fsica Terica, Universidad Autnoma de Madrid, Cantoblanco 28049 Madrid, Spain*

[*] *Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139*

[⋆] *Dept. of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai, 400 005, India[1]*

We propose a phenomenological matrix model to study string theory in $AdS_5 \times S_5$ in the canonical ensemble. The model reproduces all the known qualitative features of the theory. In particular, it gives a simple effective potential description of Euclidean black hole nucleation and the tunnelling between thermal AdS and the big black hole. It also has some interesting predictions. We find that there exists a critical temperature at which the Euclidean small black hole undergoes a Gross-Witten phase transition. We identify the phase transition with the Horowitz-Polchinski point where the black hole horizon size becomes comparable to the string scale. The appearance of the Hagedorn divergence of thermal AdS is due to the merger of saddle points corresponding to the Euclidean small black hole and thermal AdS. The merger can be described in terms of a cusp (A_3) catastrophe and divergences at the perturbative string level are smoothed out at finite string coupling using standard techniques of catastrophe theory.

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[1] Permanent address

1. Introduction

The AdS/CFT correspondence has enabled us to begin understanding various aspects of quantum gravity in a more quantitative way [1]. In particular reliable computations of black hole thermodynamics have been possible using the AdS_3/CFT correspondence [1,2]. In this paper we would like to address some aspects of AdS_5 black hole physics in the context of type IIB string theory in $AdS_5 \times S^5$, using the dual gauge theory at finite temperature.

The thermodynamic aspects of quantum gravity in AdS spacetime were discussed long ago in an important paper by Hawking and Page [3], who realized that it is possible to define a canonical ensemble for quantum gravity and in particular for Schwarzschild black holes. They found that an asymptotic AdS spacetime allows two Schwarzschild black hole solutions, which are since called small black hole (SBH) and big black hole (BBH). As the names suggest SBH can have a horizon radius that can be very small compared to the size of AdS and BBH has a horizon radius which is comparable to (or much larger than) the radius of AdS_5. Further SBH has negative specific heat and is unstable, while BBH has positive specific heat and is stable (meta-stable.) Hawking and Page also found that the system undergoes a first order phase transition at a temperature T_1 comparable to the inverse curvature radius of the spacetime. Below T_1, the system is described by a thermal gas in AdS, while above T_1 it is described by a BBH. With the discovery of the AdS/CFT correspondence [4,5,6], Witten [6,7] realized that a BBH in AdS_5 is naturally described by the deconfinement phase of $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory on $S^3 \times S^1$. He argued that the Hawking-Page transition corresponds to a large N deconfinement transition in the gauge theory at strong coupling.

Several authors [8,9,10] have discussed the partition function of the free $\mathcal{N} = 4$ SYM theory[2] and found that the large N deconfinement transition persists at zero coupling. In particular it was found that the deconfinement transition happens exactly at the Hagedorn temperature of the low temperature thermal AdS phase[3]. Near the Hagedorn temperature, the free energies of both high and low temperature phases become divergent and string

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[2] See [11,12,13,14,15,16,17,18] for other recent discussions of phase transitions in weakly coupled Yang-Mills theory.

[3] At strong coupling, the Hagedorn temperature for the thermal AdS is much higher than the Hawking-Page temperature.

perturbation theory[4] breaks down. In [19] the smoothening of the Hagedorn transition at finite string coupling was discussed. This requires a careful understanding of nonperturbative effects in $1/N$. It was found that the divergences in perturbation theory are removed by two distinct mechanisms. The divergent terms in the high temperature phase can be resummed, leading to a noncritical string description. This happens at a scale $T - T_H \sim N^{-\frac{4}{3}}$. The Hagedorn divergence of the low temperature phase is removed by summing over the contributions from the thermal AdS and the noncritical string background, happening at $T - T_H \sim N^{-2}$.

In this paper we extend the analysis of [19] to finite 't Hooft coupling and study various non-perturbative aspects of black hole physics in AdS_5 using the boundary gauge theory. The latter is precisely formulated but technically difficult to deal with in the strong coupling region, where one can make contact with gravity. In such circumstances one is, naturally led to an effective action approach, relying on the choice of an order parameter and its symmetries. The difficulty is then transferred to the coupling and temperature dependence of coefficients of the effective action coding the microscopic theory. In spite of this difficulty one can hope to make progress. Above all, one is encouraged by the success of a similar programme in QCD.

The strategy has two parts. First one may try to extract certain *universal* features of string theory in AdS_5 from the effective action. The hope is that universal features do not depend on the exact details of the effective action and can be extracted by exact analysis of a tractable model. Secondly one can approximately determine the coefficients of the effective action by explicitly matching, in our case, with data in the dual supergravity description.

String theory backgrounds like thermal AdS, BBH and SBH appear as saddle points in the Euclidean path integral of Yang-Mills theory. Perturbative string expansion around each of them is given by the large N expansion around the corresponding saddle point in the boundary theory[5]. As one varies the temperature, such expansions break down at various places where their coefficients develop nonanalytic behavior. One example is the Hagedorn temperature of thermal AdS. The other is the temperature (called T_0 by [3]) at

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[4] Since the $1/N$ expansion in the boundary theory corresponds to the perturbative string expansion in the bulk, in this paper we will use the word "large N expansion" and "perturbative string expansion" interchangeably.

[5] For the perturbative $1/N$ expansion around the saddle point, it is not important whether the saddle of interest is stable, metastable or unstable.

which SBH and BBH saddles appear. While such nonanalytic behavior is rather puzzling and hard to understand from the perturbative string point of view, in the dual Yang-Mills theory they arise due to the fact that in the large N limit the number of degrees of freedom goes to infinity. The non-analyticity occurs for the same reason as in the thermodynamic limit of classical statistical physics. At these temperatures, as in the case analyzed in [19], a non-perturbative treatment is required *no matter how large N is.* Thus these non-analyticities are excellent probes of the non-perturbative structure of the theory. Moreover, since their appearance is intrinsically tied to the large N limit, it is expected thnat they possess a certain degree of universality, just as in critical behavior in condensed matter systems. One may hope that the *qualitative* behavior at these critical temperatures sould be insensitive to the precise details of the theory and could be captured by studying much simpler systems. This gives us hope that we can study critical behavior, and hence non-perturbative aspects of large N Yang-Mills theory at finite or strong coupling, and in turn yielding insights into string theory in AdS_5.

It was discussed in [10] that the partition function of the SYM theory can be written as a matrix integral over the effective action of the Polyakov loop. For Yang-Mills theory at finite coupling, we have no way of computing this effective action explicitly. Nevertheless, with universality in mind, here we propose a class of effective actions as "phenomenological models" to approximate the full theory. We show that models in the class have a large N phase structure resembling that of a weakly coupled string theory in $AdS_5 \times S_5$. This gives strong indication that strongly coupled $\mathcal{N} = 4$ SYM theory belongs to the same universality class. This also gives us reason to believe that critical behaviors of the bulk string theory at places where string perturbation theory breaks down can indeed be captured by much simpler models.

The simplest model within the class, which we will refer to as (a, b) model, can be considered as a truncation of the full effective action of the theory to the lowest nonlinear terms[6]. Being exactly solvable to all order in N, this model provides an ideal representative to study the critical behaviors of the universality class. We proceed to perform a detailed study of various non-perturbative aspects of this model. The results, when translated into the language of bulk string theory, can be summarized as follows:

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[6]   The model contains two parameters (a, b), both are functions of the 't Hooft coupling λ and temperature T. We will assume some qualitative dependence of (a, b) on T as part of the phenomenological input data. This model has been briefly discussed earlier in the mean field approximation as a toy model for weakly coupled gauge theories in [10].

1. We give an effective potential description of the tunnelling between thermal AdS and the BBH. The Euclidean SBH plays the role of the bounce (also called a thermalon). We compute the tunnelling rate in our effective theory.

2. We find that the Euclidean SBH undergoes a third order Gross-Witten [20,21,22] phase transition in the large N limit at a temperature T_c below the Hagedorn temperature. We identify the phase transition with the Horowitz-Polchinski correspondence point [23] where the event horizon size of the SBH becomes comparable to string scale.

3. The breakdown of the perturbative string expansion of thermal AdS at the Hagedorn temperature is due to the merger of the saddle points corresponding to the SBH and thermal AdS. The merger can be described in terms of a cusp (A_3) catastrophe. The simplest possibility allowed by the symmetry. Similarly the breakdown of perturbative string theory around the BBH when it merges with the SBH saddle can be understood in terms of a fold (A_2) catastrophe. The divergences at the perturbative level are smoothed out at finite N using the standard techniques of catastrophe theory.

4. A common theme in our study of the critical behavior when a CFT approaches a singular point is that there always exists a double scaling limit and it is likely that the theory in the double scaling limit is described by a noncritical string background. This also resonates with the result of [19] and the behavior of other singular CFTs discussed in [24,25,26].

While these features are studied explicitly only in the simplest (a, b) model, we believe they persist for all models in the class due to universality of the large N phase transition and the catastrophe.

The plan of the paper is as follows. In the next section we review some aspects of the thermodynamics of quantum gravity in $AdS_5 \times S_5$ which we aim to reproduce in the large N limit of our "phenomenological" models. In section 3, we review some aspects of the computation of the Yang-Mills partition function using the effective action of the Polyakov loop and present the truncated models. Section 4 is devoted to a detailed study of the phase structure of the (a, b) model at large N. We discuss in detail the thermal history of the theory in the canonical ensemble. We also show that the sharp Hawking-Page transition in supergravity is smoothed out to a finite cross region at finite N. In section 5 we elucidate the role of SBH as the bounce which mediates the tunnelling between BBH and thermal AdS (and vice versa depending on the temperature) and calculate the tunnelling rate. We also connect the bounce and the large order behavior of perturbative theory. In section 6 we study the critical behaviors of the theory at temperatures where the perturbative

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string expansion breaks down around at least one of the three backgrounds. They can be understood using catastrophe theory. We conclude in section 7 with a discussion of future directions. We have also include a few appendices which contain details of some calculations.

2. Review of Hawking-Page transition in Euclidean Quantum gravity

In this section we review the results of [3], to be reproduced using the matrix models in later sections.

The canonical ensemble for quantum gravity in AdS can be defined as a path integral over the metric and all other fields asymptotic to AdS with time direction periodically identified with a period $\beta = 1/T$. At semi-classical level, i.e. $R^2/l_p^2 \gg 1$, where R is the curvature radius of AdS, such a path integral is dominated by configurations near the saddle points, i.e. classical solutions to the Einstein equations. If we assume spherical symmetry and zero charge, there are three possible critical points, which are thermal AdS$_5$ (Euclidean AdS with time direction periodically identified), a big (Schwarzschild) black hole (BBH) and a small black hole (SBH). Among them thermal AdS and BBH are locally stable, while SBH has a negative mode and it is unstable. The thermal AdS background has topology $S^1 \times R^4$, while SBH and BBH have topology $R^2 \times S^3$, all of them with a common boundary $S^1 \times S^3$. The Euclidean time direction in black hole backgrounds are contractible and the winding numbers are not conserved. In contrast the time circle in thermal AdS is noncontractible and the winding number is conserved.

The classical action for thermal AdS is $I_1 = 0$. This is standard in string theory: with a noncontractible time circle, there is no genus zero contribution to the free energy. A Schwarzschild black hole solution exists in AdS only for a Hawking temperature greater than

$$T_0 = \frac{\sqrt{2}}{\pi R}, \qquad \beta_0 = \frac{1}{T_0} = \frac{\pi R}{\sqrt{2}} \tag{2.1}$$

For $T > T_0$, there are two possible black holes, whose horizon sizes are given by

$$\frac{r_+}{R} = \frac{1}{\sqrt{2}} \left[\frac{\beta_0}{\beta} \pm \sqrt{\frac{\beta_0^2}{\beta^2} - 1} \right] \tag{2.2}$$

The corresponding classical Euclidean action is given by

$$I = \frac{R^3}{2\kappa^2} 2\pi \Omega_3 \left(\frac{r_+}{R} \right)^3 \frac{1 - \left(\frac{r_+}{R} \right)^2}{1 + 2 \left(\frac{r_+}{R} \right)^2} \, , \tag{2.3}$$

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where $2\kappa^2$ is the five-dimensional Newton's constant[7]. We will denote I_+, I_- the classical actions for large and small black hole respectively. The specific heat of the large black hole is positive and thus it is thermodynamically stable (i.e. it can reach locally stable thermal equilibrium with thermal radiation). The small black hole has a negative specific heat. The action I_- of the small black hole is always greater than the action of thermal AdS and of the big black hole. At temperature

$$T_1 = \frac{3}{2\pi R} > T_0 \tag{2.4}$$

the action for the big black hole is $I_+ = 0 = I_1$. When $T_0 < T < T_1$, $I_+ > 0$, and the saddle corresponding to thermal AdS dominates. When $T > T_1$, $I_+ < 0$, the big black hole (BBH) dominates. There is a change of dominance at T_1. This is the Hawking-Page transition. In the classical limit $\kappa^2 \to 0$, this is a sharp first order transition. We expect that at finite κ^2 the transition should be smoothed out. This we will see explicitly in the gauge theory description.

In the Minkowski description the spectrum of fluctuations around AdS_5 and BBH (all of positive frequency in Euclidean and Minkowski descriptions) can be interpreted in terms of physical particles and they constitute the meta-stable thermal ensemble around these backgrounds. The SBH on the other hand has a negative eigenvalue in the spectrum of small fluctuations in the Euclidean description. Hence the SBH fits the description of an instanton relevant for the tunnelling between thermal AdS_5 and the BBH. For example, one expects the rate for a BBH to tunnel into a thermal AdS is expected to be

$$\Gamma_1 = A_1 e^{-(I_- - I_+)} \tag{2.5}$$

That is, through thermal fluctuation, a big black hole can turn into a small black hole. The small black hole (since it has a negative specific heat) then can either shrink to thermal AdS by emitting thermal radiation or grow back into a big black hole by absorbing radiation. Similarly, the thermal AdS background has also a nonzero probability to nucleate a small black hole with probability

$$\Gamma_2 = A_2 e^{-I_-} \tag{2.6}$$

Afterwards the small black hole can shrink back to the thermal AdS or grow into a big black hole. The prefactors A_1, A_2 in (2.5) and (2.6) are given by the determinants of small

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[7] Note that $\frac{R^3}{2\kappa^2} \propto N^2$, where N is the rank of the gauge group in the boundary Yang-Mills theory.

fluctuations around the relevant background. In thermal equilibrium, the probability to go from a typical state in thermal AdS to that of a big black hole or back should be the same.

When $T_0 < T < T_1$, the big black hole phase is metastable, since it has a higher free energy than that of that of thermal AdS and $\Gamma_1 > \Gamma_2$. But string perturbation theory around it is well defined until T_0 is reached where we expect the perturbation theory to break down. Similarly, when $T > T_1$, thermal AdS becomes metastable and $\Gamma_2 > \Gamma_1$. For a large AdS with $R \gg l_s$ (l_s is the string length) the perturbation theory around thermal AdS breaks down at a much higher Hagedorn temperature $T_H \sim \frac{1}{l_s}$. In the Hawking-Page discussion, there also exists a temperature T_2 beyond which the thermal graviton gas in AdS will collapse into a big black hole. For a weakly coupled string theory in $AdS_5 \times S_5$, T_2 is of order $\frac{1}{(Rl_p^4)^{\frac{1}{5}}}$ and is much higher than the Hagedorn temperature $T_H \sim \frac{1}{l_s}$ for thermal AdS.

3. Effective action at finite temperature

In this section we will introduce a phenomenological matrix model for understanding string theory in $AdS_5 \times S_5$ at finite temperature.

We first give some general discussion of the partition function of $\mathcal{N} = 4$ SYM theory on S^3. We consider the theory in the canonical ensemble, i.e. the Euclidean time direction is periodically identified with a period of $\beta = \frac{1}{T}$. It was pointed out in [10] (see also [27]) that the Yang-Mills theory partition function on S^3 at a temperature T can be reduced to an integral over a unitary $U(N)$ matrix U, which is the zero mode of Polyakov loop on S^3,

$$\mathcal{Z}(\lambda, T) = \int dU \, e^{S_{eff}(U)} \tag{3.1}$$

with

$$U = P\exp\left(i \int_0^\beta A d\tau\right) \tag{3.2}$$

where $A(\tau)$ is the zero mode of the time component of the gauge field in S^3. This follows from the fact that apart from A all modes of $\mathcal{N} = 4$ SYM on S^3 are massive. Hence one can integrate them out to obtain an effective action for A. Gauge invariance requires that the effective action must be expressed in terms of products of $\mathrm{tr}U^n$ with n an integer, since

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these are the only gauge invariant quantity that can be constructed from A alone. $S_{eff}(U)$ has a Z_N symmetry

$$U \to e^{\frac{2\pi i}{N}} U$$

due to global gauge transformations which are periodic in the Euclidean time direction up to Z_N factors. A generic term in $S_{eff}(U)$ will have the form

$$\mathrm{tr}U^{n_1}\mathrm{tr}U^{n_2}\cdots\mathrm{tr}U^{n_k}, \qquad n_1 + \cdots n_k = 0 \,(\mathrm{mod}\,N), \qquad k > 1$$

We can expand S_{eff} in terms of a complete set of such operators, with the first few terms

$$S_{eff}(U) = a_1\mathrm{tr}U\mathrm{tr}U^{-1} + b_1\left(\mathrm{tr}U\mathrm{tr}U^\dagger\right)^2 + a_2\mathrm{tr}U^2\mathrm{tr}U^{-2} + c_1\mathrm{tr}U^2\mathrm{tr}U^\dagger\mathrm{tr}U^\dagger + \cdots \quad (3.3)$$

The coefficients in the expansion are functions of 't Hooft's coupling λ, and T. While these coefficients are in principle calculable at weak coupling, the explicit computations are in general very complicated (see e.g. [28]). At finite or large 't Hooft coupling, there is no available tool at the moment to attempt such a computation. Even one were able to find the expansion (3.3) explicitly, to perform a finite N computation of the matrix integral (3.1) is still a daunting task, if not impossible.

In order to make progress, in this paper, we will consider the truncation of (3.3) to terms containing only powers of $\mathrm{tr}U\mathrm{tr}U^{-1}$, i.e. we consider an effective action of the form

$$\begin{aligned} S_{eff}(U) = S(x), \qquad x &= \frac{1}{N^2}\mathrm{tr}U\mathrm{tr}U^\dagger \\ &= a\mathrm{tr}U\mathrm{tr}U^{-1} + \frac{b}{N^2}\left(\mathrm{tr}U\mathrm{tr}U^\dagger\right)^2 + \frac{c}{N^4}\left(\mathrm{tr}U\mathrm{tr}U^\dagger\right)^3 + \cdots \end{aligned} \quad (3.4)$$

Our consideration is phenomenological, motivated by the AdS/CFT correspondence to search for effective actions which lie within the same universality class as that of the SYM theory at finite coupling. At a heuristic level, one may also consider (3.4) as arising from (3.3) by "integrating out" all the higher moments $\mathrm{tr}U^n$, $\mathrm{tr}U^{-n}$ ($n > 1$). In [27] it was also argued based on group theory considerations that terms of the form (3.4) dominate over other types terms in (3.3) in the large N limit.

We will consider a class of matrix models of the form (3.4) satisfying the conditions that $S(x)$ is convex and $S'(x)$ is concave. We show in the next section and in Appendix A that in this class the large N phase structure appears to be universal. In particular,

the phase structure precisely reproduces[8] the phase structure of a weakly coupled string theory in AdS. We believe this is a strong indication that strongly coupled $\mathcal{N} = 4$ SYM theory also lies in this class. The simplest model in this universality class is given by the first two terms in (3.4)

$$\mathcal{Z}(a,b) = \int dU \exp\left[a\mathrm{tr}U\mathrm{tr}U^\dagger + \frac{b}{N^2}\left(\mathrm{tr}U\mathrm{tr}U^\dagger\right)^2\right] \tag{3.5}$$

with $b > 0$.

As discussed in the introduction, we are interested in the critical behaviors in regions of parameter space where large N expansions around various saddle points break down. Being exactly solvable at finite N, (3.5) provides a simple, nice representative to study the critical behaviors of the universality class.

To conclude this section, we note that (3.5) was discussed in [10] in the Hartree-Fock approximation. It was noted that for $b > 0$ a first-order phase transition resembling the Hawking-Page transition occurs. This observation was a motivation for the investigations in this paper.

4. Large N phase structure of the universality class

In this section we study the large N phase structure of (3.4) and (3.5) ($b > 0$). For our later purpose of studying the critical behaviors of (3.5), we give a detailed discussion of the phase structure of (3.5) using a method suitable for finite N analysis. We point out general matrix model (3.4)

$$\mathcal{Z} = \int dU\, e^{N^2 S(x)}, \qquad x = \frac{1}{N^2}\mathrm{tr}U\mathrm{tr}U^\dagger \tag{4.1}$$

has the same large N phase structure as (3.5) provided that $S(x)$ is convex and $S'(x)$ is concave. For completeness we have included in Appendix A an alternative discussion of the phase structure of (4.1) using the Hartree-Fock method.

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[8] We will assume some qualitative dependence of $S(x)$ on T as part of the phenomenological input data.

4.1. Effective potential

For $b > 0$, equation (3.5) can be rewritten using a Lagrange multiplier μ:

$$\mathcal{Z}(a,b) = \frac{N}{2\sqrt{\pi b}} \int_{-\infty}^{\infty} d\mu \, e^{-\frac{N^2}{4b}(\mu-a)^2} \int dU \exp\left[\mu \mathrm{tr} U \mathrm{tr} U^\dagger\right] . \tag{4.2}$$

The matrix integral in (4.2) can be further simplified by introducing another Lagrange multiplier g. For example for $\mu 0$, one finds

$$
\begin{aligned}
e^{N^2 \mathcal{F}(\mu)} &= \int dU \exp\left[\mu \mathrm{tr} U \mathrm{tr} U^\dagger\right] \\
&= \frac{N^2}{2\mu} \int_0^\infty dg \, g \, e^{-\frac{N^2 g^2}{4\mu} + N^2 F(g)}
\end{aligned}
\tag{4.3}
$$

with

$$e^{N^2 F(g)} = \int dU \exp\left[\frac{Ng}{2}\left(\mathrm{tr} U + \mathrm{tr} U^\dagger\right)\right] \tag{4.4}$$

The formula for $\mu < 0$ is obtained by taking $g \to ig$.

The large N expansion of (4.4) and the corresponding third order phase transition is well known [20,21,29]

$$F(g) = \begin{cases} \frac{g^2}{4} + \text{nonperturbative} & g \leq 1 \quad \text{or} \quad g \text{ imaginary} \\ g - \frac{1}{2}\log g - \frac{3}{4} + O(1/N^2) & g > 1 \end{cases} \tag{4.5}$$

The order parameter of (4.4) can be taken to be

$$
\begin{aligned}
\rho_1(g) &= \frac{1}{N}\left\langle \mathrm{Tr} U\right\rangle_g = \frac{1}{N}\left\langle \mathrm{Tr} U^\dagger\right\rangle_g = \frac{\partial F}{\partial g} \\
&= \begin{cases} \frac{g}{2} + \cdots & g < 1 \quad \text{or} \quad g \text{ imaginary} \\ 1 - \frac{1}{2g} + \cdots & g > 1 \end{cases}
\end{aligned}
\tag{4.6}
$$

characterizing the eigenvalue distribution of U. When $0 \leq \rho_1 < \frac{1}{2}$ ($g < 1$), the system is in a phase whose eigenvalue distribution does not have a gap on the unit circle. In particular, for $g = 0$, the distribution is uniform. When for $1 > \rho_1 > \frac{1}{2}$ ($g > 1$), the distribution develops a gap. (4.5) and (4.6) do not apply to $g \approx 1$, where the system undergoes a third order phase transition [20] in the large N limit. At finite N the third order discontinuity in (4.5) is smoothened out by non-perturbative effects. They will be discussed in later sections when needed.

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One can now rewrite (3.5) as a two dimensional integral

$$\mathcal{Z} = \frac{N^3}{4\sqrt{\pi b}} \int_{-\infty}^{\infty} \frac{d\mu}{\mu} \int_0^{\infty} g dg\, e^{-N^2 V(\mu,g)} \tag{4.7}$$

with

$$V(\mu, g) = \begin{cases} \frac{1}{4b}(\mu - a)^2 - \frac{g^2}{4}\frac{1-\mu}{\mu} & \mu < 0 \\[2mm] \frac{1}{4b}(\mu - a)^2 + \frac{g^2}{4}\frac{1-\mu}{\mu} & \mu > 0,\ 0 \leq g < 1 \\[2mm] \frac{1}{4b}(\mu - a)^2 + \frac{g^2}{4\mu} - g + \frac{1}{2}\log g + \frac{3}{4} + O(1/N^2) & \mu > 0,\ g > 1 \end{cases} \tag{4.8}$$

It is often convenient to integrate out g to reduce (4.7) to a one dimensional integral

$$\mathcal{Z}(a, b) = \frac{N}{2\sqrt{\pi b}} \int_{-\infty}^{\infty} d\mu\, e^{-N^2 Q(\mu)} \tag{4.9}$$

with

$$Q(\mu) = \frac{1}{4b}(\mu - a)^2 - \mathcal{F}(\mu) \tag{4.10}$$

and $\mathcal{F}(\mu)$ was defined in (4.3). The large N expansion for $\mathcal{F}(\mu)$ was found in [19], e.g. the leading order terms are

$$\mathcal{F}(\mu) = \begin{cases} 0 - \frac{1}{N^2}\log(1 - \mu) + \cdots & \mu < 1 \\[2mm] \frac{1}{2}\frac{w}{1-w} + \frac{1}{2}\log(1 - w) + O\left(\frac{1}{N^2}\right) & \mu > 1 \end{cases} \tag{4.11}$$

where for $\mu > 1$ we have introduced

$$w = \sqrt{1 - \frac{1}{\mu}}\,. \tag{4.12}$$

Inherited from (4.5), (4.8) has a third order discontinuity at $g = 1$ which needs to be supplemented with a non-perturbative treatment. (4.10) with $\mathcal{F}$ given by (4.11) has divergences[9] and first order discontinuity at $\mu = 1$. Again a non-perturbative treatment is necessary, as discussed in detail in [19]. Part of the subtlety at $\mu = 1$ in (4.10) has to do with that at $\mu = 1$, g becomes massless and the effective potential $V(\mu, g)$ is flat in the range $0 < g < 1$. Thus near $\mu = 1$ it is more convenient to use the two dimensional effective potential (4.8). Also in (4.11) we have suppressed a subdominant term which should be taken into account in the analysis of the phase structure. This is automatically

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[9] The second and higher derivatives of $\mathcal{F}(\mu)$ are also divergent for $\mu \to 1_+$.

taken care of in the two-dimensional integral (4.7). In this and following sections we will use both forms of the effective potential (4.8) and (4.10) depending on convenience. The one-dimensional effective potential $Q(\mu)$ is easier to visualize than the two-dimensional potential $V(\mu, g)$. But as we mentioned, equation (4.8) is more convenient around $\mu = 1$.

We note that the trick used in (4.2) to reduce (3.5) to an integral transform of (4.3) can be generalized to find the large N phase structure of general matrix models (4.1) with $S(x)$ convex. Since $S(x)$ is convex, it admits a Legendre transform[10]:

$$\mathcal{S}(\mu) = Max_x(\mu x - S(x))$$

The Legendre transform is involutive. If we do it twice we get back the same function. Hence $\mathcal{S}(\mu)$ is also convex. We can then write (4.1) as

$$\mathcal{Z} = \int dU \, e^{N^2 S(x)} = \int dU \int d\mu \, e^{N^2(\mu x - \mathcal{S}(\mu))} \tag{4.13}$$

and the second integral over μ is carried out using saddle points. For large N this will give an excellent approximation. Convexity in fact guarantees that there is a unique saddle point contributing. If we now exchange the order of integration in (4.13) and use (4.3), we find that

$$\mathcal{Z} = \int d\mu \, e^{-N^2 \mathcal{Q}(\mu)} \tag{4.14}$$

where

$$\mathcal{Q}(\mu) = \mathcal{S}(\mu) - \mathcal{F}(\mu) \tag{4.15}$$

and $\mathcal{F}(\mu)$ is given by (4.11) in the large N limit. We will show in next subsection that $\mathcal{Q}(\mu)$ leads to the same large N phase structure as (4.10) provided $\mathcal{S}'(\mu)$ is also convex[11].

4.2. Phase structure

In the large N limit the critical points of V in (4.7) describe different phases of the theory which in turn correspond to different bulk string theory geometries. The minima correspond to (meta)stable phases, while saddle points[12] (or maxima) to unstable phases. Note that in the large N limit, the eigenvalue distribution of the Polyakov loop U at a critical point follows from (4.4) with g given by its value at the critical point. Since there

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[10] For general properties of Legendre transformations see for instance the book [30].

[11] That $\mathcal{S}'(\mu)$ is convex means that $S'(x)$ is concave.

[12] By saddle here we refer to saddle points of $V(\mu, g)$ on the real $\mu - g$ plane.

is a one-to-one map (4.6) between ρ_1 and g, g can be considered as the order parameter of the theory. After integrating out g one can also interpret μ in (4.9) as the order parameter (at least in the range $\mu 1$).

Before discussing the critical points of the theory in detail (which is somewhat involved), we note that qualitative features of the critical point structure of (3.5) can be conveniently visualized by plotting the one dimensional effective potential $Q(\mu)$ (4.10) in the large N limit. Depending on the values of (a, b), Q can have one or three critical points (see fig. 2). The critical point structure in the (a, b) plane is plotted in fig. 1 (see below). Below curve I in fig. 1, Q has one minimum. Between curve I and curve H, it has three critical points $\mu_1 < \mu_2 < \mu_3$, with two minima (μ_1 and μ_3) and one maximum (μ_2). The two minima change dominance on curve II. On curve I, μ_2 and μ_3 merge. At curve H, μ_1 and μ_2 merge. To the right of curve H, in addition to μ_3, (4.8) also has a tachyonic saddle which is not visible in the leading order Q-plot here.
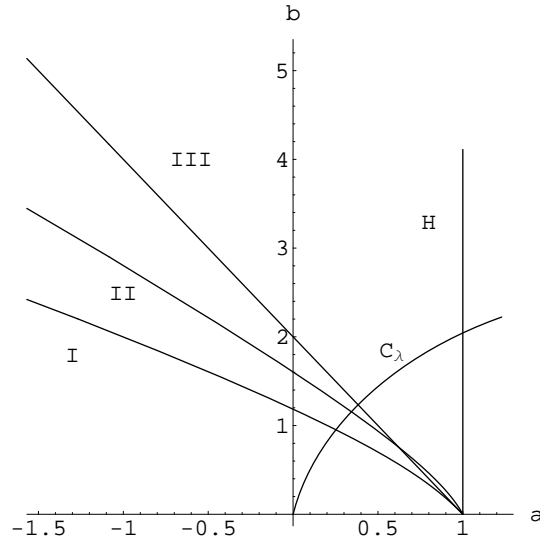

Fig. 1: This figure plots the critical point structure of the theory in the $a - b$ plane. Below line I, there is one critical point. There are three critical points between line I and line H, two minima, one maximum. At line II, two minima exchange dominance.

We now describe the critical points of (4.8) in detail:

1. From the first two lines of (4.8) one finds the following critical point
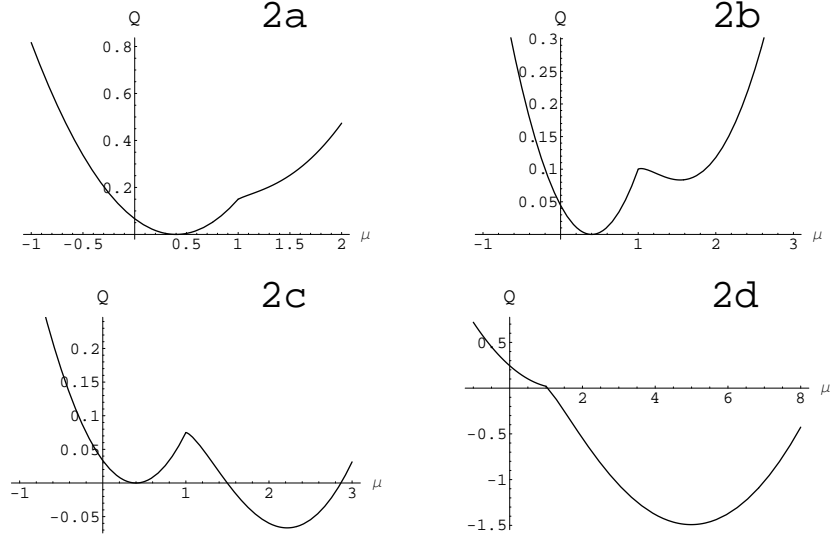
$$\mu_1 = a, \qquad g_1 = 0 \tag{4.16}$$

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Fig. 2: Plot 2a corresponds to (a,b) below curve I of fig. 1, 2b to (a,b) between curve I and curve II, 2c to (a,b) between curve II and curve H, and 2d to (a,b) lying to the right of curve H. Note the discontinuity in the first derivative at $\mu = 1$ is due to the large N approximation. It is smoothened out by non-perturbative effects.

with

$$V(\mu_1, g_1) = 0 \tag{4.17}$$

and

$$V'' = \begin{pmatrix} V_{\mu\mu} & V_{\mu g} \\ V_{\mu g} & V_{gg} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1-a}{a} \end{pmatrix} \tag{4.18}$$

where V_{gg} denotes $\frac{\partial^2 V}{\partial g^2}$ and so on. For $a < 1$, it is a local minimum. V'' becomes singular for $a = 1$ and tachyonic for $a > 1$. Since $g_1 = 0$, it describes a uniform eigenvalue distribution in the unit circle.

2. For $a < 1$ and $c = \frac{2(1-a)}{b} < 1$, there is an additional saddle point from the second line of equation (4.8) at

$$\mu_2 = 1, \qquad g_2 = \sqrt{c}, \qquad c = \frac{2(1-a)}{b} < 1 \tag{4.19}$$

with

$$V(\mu_1, g_1) = \frac{(1-a)^2}{4b} \tag{4.20}$$

and

$$V'' = \frac{1}{2} \begin{pmatrix} \frac{1}{b} + c & -\sqrt{c} \\ -\sqrt{c} & 0 \end{pmatrix} \tag{4.21}$$

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Note that V'' has a negative eigenvalue. Since $\rho_1(\mu_2, g_2) = \frac{1}{2}\sqrt{c} < \frac{1}{2}$, μ_2 describes a gapless phase in the eigenvalue distribution of U. As $a \to 1$, (4.16) and (4.19) merge, after which (4.19) disappears while (4.16) becomes tachyonic.

3. From the third line of (4.8) the equations for the critical points are given by

$$\frac{1}{2b}(\mu - a) - \frac{g^2}{4\mu^2} = 0, \qquad \frac{g}{2\mu} - 1 + \frac{1}{2g} = 0 \tag{4.22}$$

Note that the second equation in (4.22) only has real solutions for $\mu > 1$, in which case one finds that[13]

$$g = \frac{1}{1 - w}, \tag{4.23}$$

where w was introduced in (4.12). The eigenvalue distribution for such a g is given by

$$\rho_1 = 1 - \frac{1}{2g} = \frac{1 + w}{2} > \frac{1}{2} . \tag{4.24}$$

Substituting (4.23) into the first equation of (4.22) one finds an equation for μ,

$$\frac{\mu - a}{2b} = \frac{(w + 1)^2}{4} \tag{4.25}$$

Of course (4.25) can also be obtained directly by extremizing the leading order term of the effective potential (4.10) for $\mu > 1$, namely

$$Q(\mu) = V(\mu, g(\mu)) = \frac{(\mu - a)^2}{4b} - \frac{1}{2}\frac{w}{1 - w} - \frac{1}{2}\log(1 - w) + O\left(\frac{1}{N^2}\right) \tag{4.26}$$

Note that

$$Q''(\mu) = -\frac{1}{4}\left(-\frac{2}{b} + \frac{1}{\mu^2} + \frac{1}{\sqrt{\mu - 1}\mu^{3/2}}\right) \tag{4.27}$$

Deriving μ from (4.25), g and ρ_1 can then be found from (4.23) and (4.24). Since ρ_1 is a monotonic function of μ, one can treat $Q(\mu)$ as an effective potential for ρ_1.

4. Equation (4.25) can be easily solved by consider the intersections of two functions $f_1(\mu) = \frac{\mu - a}{2b}$ and $f_2(\mu) = \frac{(1+w)^2}{4}$. Note that f_2 is concave in the range $\mu \in [1, \infty)$, while f_1 is a straight line. Thus (4.25) can have at most two real solutions in the allowed range. The result is as follows. Below curve I in fig. 1 on the $a - b$ plane, which is determined by

$$Q'(\mu) = 0, \quad \text{and} \quad Q''(\mu) = 0 , \tag{4.28}$$

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[13] The other real solution has $g < 1$ and so is discarded.

(4.25) has no solutions in the desired range. Between curve I and the straight line (curve III in fig. 1)

$$c = \frac{2(1-a)}{b} = 1 \; , \tag{4.29}$$

there are two solutions[14] $1 < \mu_2 < \mu_3$. It can be checked from (4.27) that μ_2 is a maximum of Q with $Q''(\mu_2) < 0$ while μ_3 is a local minimum with $Q''(\mu_3) > 0$. At curve I, μ_2 and μ_3 merge and move into the complex plane below curve I. As one approaches (4.29) from below ($c \to 1_+$), $\mu_2 \to 1_+$. Above line $c = 1$, μ_2 moves outside the $\mu > 1$ region to becomes (4.19) and (4.25) has only one solution μ_3.

5. Below curve III, $\rho_1(\mu_2) > \frac{1}{2}$. Above curve III, μ_2 becomes (4.19) with $\rho_1(\mu_2) < \frac{1}{2}$. Thus μ_2 undergoes a Gross-Witten type phase transition in the large N limit. We will show in section 5.2 that the transition is precisely the third order Gross-Witten transition.

6. There is an additional curve (curve II in fig. 1), in the $a - b$ plane, determined by equation

$$Q'(\mu_3) = 0, \qquad \text{and} \qquad Q(\mu_3) = 0, \tag{4.30}$$

where the two minima μ_1 and μ_3 become of equal height

$$0 = V(\mu_1, g_1) = V(\mu_3, g_3) < V(\mu_2, g_2) \; .$$

Below curve II, one has

$$0 = V(\mu_1, g_1) < V(\mu_3, g_3) < V(\mu_2, g_2) \; ,$$

and above curve II

$$V(\mu_3, g_3) < 0 = V(\mu_1, g_1) < V(\mu_2, g_2) \; .$$

To summarize, the structure of critical points for (4.7) in the $a - b$ plane is plotted in fig. 1. Below curve I (4.28), V has a unique minimum (4.16). Between curve I and line $a = 1$ (curve H in fig. 1), there are three critical points $\mu_1 < \mu_2 < \mu_3$ with μ_1, μ_3 minima, while μ_2 a saddle point in the $\mu - g$ plane with one negative eigenvalue. At curve I, μ_2 and μ_3 merge together. At curve II, μ_1 and μ_3 exchange dominance and the system has a first

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[14] We do not give their explicit expressions in terms of (a, b), since they are complicated and not illuminating. We will specify their qualitative behavior below.

order phase transition. At curve III, the saddle point μ_2 undergoes a Gross-Witten phase transition. At the vertical line H, μ_1 and μ_2 merge together. To the right of line H, μ_1 becomes tachyonic, μ_2 disappears, and $\mu_3 > 1$ remains a minimum. Note that curve II and curve III always lie between curve I and line $a = 1$, but they can be above or below each other. Close to $a = 1, b = 0$, line III lies below curve II and then intersects with and rises above it.

4.3. Critical points for general models

We now consider the critical point structure of the general effective action (4.1) and (4.14). As is clear from the derivation above, the overall phase structure of (4.10) presented in fig. 2 to a large extent only depends on the convexity of the function $\frac{(\mu-a)^2}{4b}$ and its derivative. Our discussion above for (4.10) goes through for a convex $\mathcal{S}(\mu)$ in (4.15) provided that $\mathcal{S}'(\mu)$ is also convex. For example, for $\mu < 1$, with $\mathcal{F}(\mu)$ given by (4.11), $\mathcal{Q}(\mu)$ has just one critical point given by the minimum of $\mathcal{S}(\mu)$. For $\mu > 1$, the critical points of (4.15) satisfy the equation (which is a generalization of (4.25))

$$\mathcal{S}'(\mu) = \frac{(1+w)^2}{4} \tag{4.31}$$

Since the right hand side of (4.31) is concave and $\mathcal{S}'(\mu)$ is convex, $\mathcal{Q}$ can have at most two critical points in the range $\mu \in (1, \infty)$. The pattern and the evolution of the critical points with the parameters of the theory also precisely resemble that of (4.10) including a Gross-Witten phase transition for μ_2 at $\mu = 1$. Note that convexity of $\mathcal{S}'(\mu)$ implies that $S'(x)$ is concave. We thus conclude that the structure of critical points of (4.1) is *universal* if $S(x)$ is convex and $S'(x)$ is concave. In appendix A, we give a more detailed description of the phase structure of (4.1) using the Hartree-Fock approximation.

4.4. Thermal History

As discussed in section 3, we would like to use the matrix model (3.5) as a phenomenological model to study weakly coupled string theory in $AdS_5 \times S_5$ at finite temperature. The parameters a, b are functions of the 't Hooft coupling λ and the temperature T. In the last subsection, we analyzed its large N critical point structure. In this subsection we will show that with some weak assumptions about the λ and T dependence of a, b, the model captures all the essential features of the bulk story, which we reviewed in sec. 2.

We first identify the critical points of V in the large N limit with the saddle points of the Euclidean gravity. $\mu_1 = a$ has $\rho_1 = 0$ in large N limit, i.e. the winding in the Euclidean time direction is a good quantum number. We thus identify it with the thermal AdS background. μ_3, which is a minimum, can then be identified with the Euclidean big black hole phase (BBH). μ_2, which has a unique negative eigenvalue can be identified with the Euclidean small black hole phase (SBH) in AdS. That a small black hole in AdS has a unique negative eigenvalue was pointed out in [31]. Moreover, one can show from the effective potential (4.26) and (4.20) that μ_2 has a negative specific heat while μ_3 has a positive specific heat, without knowing their explicit dependence of (a, b) on T. The derivation is given in Appendix B. This matches well with the thermodynamic properties of the small and big black holes.

For fixed 't Hooft coupling λ, as one varies the temperature T, $(a(\lambda, T), b(\lambda, T))$ trace a curve in the $a-b$ plane, which we will denote by $\mathcal{C}_\lambda$. Any curve $\mathcal{C}_\lambda$ in the $a-b$ plane which starts below curve I in fig. 1 at low enough temperature and ends up to the right of the vertical line $a = 1$ at sufficiently high temperature (assuming it intersects curves I,II,III and H only once) reproduces qualitatively the Hawking-Page picture[15]. The thermal history following such a $\mathcal{C}_\lambda$ can be described as follows. At sufficiently low temperature, we start with some point below curve I, where the theory has a unique critical point μ_1, corresponding to thermal AdS. As T increases to a temperature T_0, $\mathcal{C}_\lambda$ will intersect the curve I (4.28), where new critical points μ_2 (SBH) and μ_3 (BBH) come into existence. At $T_1 > T_0$, it intersects with curve II, at which μ_1 and μ_3 change dominance. This is the Hawking-Page transition. Above T_1, μ_3 (BBH phase) dominates and thermal AdS becomes only metastable. As T increases further $\mathcal{C}_\lambda$ will eventually hit $a = 1$ (line H) from the left, where the large N expansion (perturbative string expansion) around μ_1 (thermal AdS) breaks down. This temperature should be identified with the Hagedorn temperature T_H in AdS string theory.

There are a few other important features of our matrix model not visible in the bulk supergravity analysis:

1. We found that there exists a line III where the SBH phase undergoes a Gross-Witten transition from a gapped phase to a gapless one in the eigenvalue distribution of U. We will denote T_c the temperature where $\mathcal{C}_\lambda$ intersects with line III. Since line III lies between line I and line $a = 1$, we should have $T_0 < T_c < T_H$. T_c can

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[15] We have drawn such a hypothetic curve in fig. 1.

be lower or higher than the Hawking-Page temperature T_1 depending on where $\mathcal{C}_\lambda$ hits line III. This phase transition for SBH, while not visible in supergravity, has a natural interpretation in string theory. We would like to identify it with the so-called Horowitz-Polchinski correspondence point [23] for SBH, i.e. the point at which the horizon size of SBH is comparable to the string scale[16]. In [23] it was argued that as one adiabatically decreases the string coupling, a black hole makes a transition to a state of highly excited strings with the same quantum numbers (such as mass, charge, angular momentum etc). Here we fix the string coupling (i.e. N), but raise the temperature adiabatically. The horizon radius of a small black hole decreases and eventually reaches the string scale. Following [23] we would like to argue that beyond T_c, it is more appropriate to view the critical point (4.19) as describing a set of highly excited string states. That the phase transition is third order suggests that the energy, entropy and specific heat of a SBH vary continuously across the correspondence point as it becomes a highly excited string states, but derivatives of the specific heat jump in the large N limit. In the limit of large λ, i.e. $R \gg l_s$, we expect that T_c should be close to T_H and much greater than the Hawking-Page temperature T_1. That T_c is below T_H appears to be consistent with the physical picture of the microcanonical ensemble (see e.g. [32,10]). For notational simplicity, below we will continue to refer to (4.19) as the SBH phase, keeping in mind that it should really correspond to a highly excited string state.

2. Our matrix model indicates that at the Hagedorn temperature T_H, the critical points associated with thermal AdS and the SBH merge together. This appears natural since very close to the Hagedorn temperature, thermal AdS will be dominated by a few long string states. One expects that the distinction between thermal AdS and the highly excited string phase which the SBH becomes above T_c disappears at the Hagedorn temperature. This is again consistent with the physical picture of the microcanonical ensemble of [32,10].

3. Above T_H, i.e. when the $\mathcal{C}_\lambda$ curve goes to the right of line $a = 1$, thermal AdS becomes tachyonic and the critical point corresponding to SBH disappears. BBH remains the only stable phase. The physical interpretation of a tachyonic thermal AdS is not completely clear to us. In the past, it has been argued that string theory above the Hagedorn temperature can be interpreted again as some kind of long string phase

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[16] Note that the SBH should be considered as a ten-dimensional black hole.

which can be analyzed by analytic continuation[17] (see e.g. [33]). Our result does not contradict this point of view. But we should note that this tachyonic thermal AdS phase cannot be reached from the microcanonical ensemble. Whether it plays any role in the canonical ensemble is not clear to us.

We now make some comments on the possible dependence of curve $\mathcal{C}_\lambda$ (i.e. curve $(a(\lambda, T), b(\lambda, T))$ with λ fixed) on the 't Hooft coupling λ. At $\lambda = 0$, i.e. free theory, $b(T) = 0$ [8,10]. $\mathcal{C}_0$ moves along the a-axis from $a = 0$ at $T = 0$ to $a \to \infty$ at $T \to \infty$. It crosses all lines in fig. 1 at a single point $(a = 1, b = 0)$. This was the case analyzed in [19]. At weak coupling it has been shown in [28] that for pure gauge theory at weak coupling, $b(\lambda, T) = O(\lambda^2) > 0$ and $a(\lambda, T) = a(0, T) + O(\lambda^2)$. If the result also holds for $\mathcal{N} = 4$ SYM theory, then, $\mathcal{C}_{\lambda \ll 1}$ corresponds to a curve slightly rising above the horizontal $a-$axis. In this case $T_c < T_1$. In the supergravity limit $\lambda \gg 1$, $T_0, T_1 \sim \frac{1}{R}$ while $T_c, T_H \sim \frac{1}{l_s}$, i.e. there is a big hierarchy between these temperature scales. One can in principle determine part of $\mathcal{C}_\lambda$ for $\lambda \to \infty$ by equating the free energy in the gauge theory with the corresponding free energy in supergravity. To leading order in large N we equate the Euclidean actions of the SBH and BBH to the corresponding actions of μ_2 and μ_3 in the gauge theory. More explicitly,

$$Q(\mu_2) = \frac{I_-}{N^2}, \qquad Q(\mu_3) = \frac{I_+}{N^2} \tag{4.32}$$

where I_- and I_+ are the Euclidean actions (given in (2.3)) of the SBH and BBH respectively and Q is given by (4.26). Since I_- and I_+ are functions of the single variable $t = T/T_0$, we can use (4.32) to determine $a(t)$ and $b(t)$ in the $\lambda \to \infty$ limit. Note that this comparison is only valid between curve I and curve III, since above curve III, μ_2 undergoes a large N phase transition which is not visible in supergravity.

To summarize, our phenomenological (a, b) model reproduces all the important features of string theory in $AdS_5 \times S_5$ at finite temperature. In fact we got more. We found a description of the Horowitz-Polchinski correspondence point in terms of a Gross-Witten transition and a non-perturbative picture at and beyond the Hagedorn temperature for thermal AdS.

After finding the critical points one can then use (4.2) to find the large N expansion of $\mathcal{Z}(a, b)$ around them. These expansions should correspond to perturbative string expansions around the corresponding bulk geometries. In the rest of the paper we will examine

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[17] This argument was made in flat space. When the radius of AdS is much larger than the string scale, we expect the behavior of thermal AdS to be similar to that of flat space.

in detail various regions of the parameter space where these expansions break down and study the physics there. Such regions include:

1. At T_0, where the BBH and SBH saddles merge together. Perturbative string expansions around BBH and SBH are not valid.

2. At the Hawking-Page temperature T_1, where there is a first order phase transition between thermal AdS and BBH. Although the large N expansion around each critical point does not break down, the large N expansion of the full partition function requires a special treatment.

3. At T_c, where the Gross-Witten transition for SBH takes place.

4. At T_H, the Hagedorn temperature of thermal AdS.

Note that in 1 and 4 above, the breakdown in the large N expansion happens for the metastable phases. In the terminology of the first order phase transitions, T_0 and T_H are the spinodal temperatures for the BBH and thermal AdS respectively, beyond which the metastable phase become unstable.

4.5. Full partition function and smoothening of Hawking-Page transition

An immediate consequence of considering the theory at finite N is that the sharp Hawking-Page transition is smoothed out to a region of width of order N^{-2}.

In the infinite N limit, the partition function of the system is

$$
\log \mathcal{Z} = \begin{cases} \log K_1 + O(1/N^2) & T < T_1 \\ -N^2 Q(\mu_3) + \log K_3 + O(1/N^2) & T > T_1 \end{cases} \tag{4.33}
$$

where K_1 and K_3 are Gaussian factors computed from the integral (4.7). Recall that $Q(\mu_1) = 0$. $Q(\mu_3)$ equals to zero at T_1 and become negative (positive) above (below) T_1. The transition is first order with a nonzero latent heat given by

$$
E = N^2 \frac{\partial Q(\mu_3)}{\partial \beta}\bigg|_{T_1} + O(1) \tag{4.34}
$$

The expectation value of the Polyakov loop also jumps at T_1

$$
\rho_1^2(T) = \begin{cases} O(1/N^2) & T < T_1 \\ \frac{1}{4}\left(1 + \sqrt{1 - \frac{1}{\mu_3}}\right)^2 & T > T_1 \end{cases} \tag{4.35}
$$

At finite N we need to include contributions from both geometries. The full partition function of the system between temperature T_0 and T_H can then be written in terms of the following asymptotic expansion

$$\mathcal{Z} \approx e^{-N^2 Q(\mu_1)} A_1 + e^{-N^2 Q(\mu_3)} A_3 \tag{4.36}$$

where A_1, A_3 are asymptotic series around μ_1 and μ_3 respectively

$$
\begin{aligned}
A_1 &= K_1 \left(1 + \sum_{n=1}^{\infty} N^{-2n} c_n(a, b) \right) \\
A_3 &= K_3 \left(1 + \sum_{n=1}^{\infty} N^{-2n} d_n(a, b) \right)
\end{aligned} \tag{4.37}
$$

Note that there is no contribution from μ_2, which is a maximum of $Q(\mu)$. Below the Hawking-Page temperature T_1, the μ_1 saddle dominates and μ_3 is only metastable. The contribution of the second term in (4.36) is exponentially small compared with the first. Their roles reverse above T_1.

The sharp transition at T_1 in smoothened out at finite N into a finite region $T - T_1 \sim O(N^{-2})$. We now examine this cross over region in some detail. At $\frac{T - T_1}{T_1} = \epsilon \ll 1$, we can expand $Q(\mu_3)$ as

$$Q(\mu_3(a(T), b(T)), a(T), b(T)) = -\nu \epsilon + O(\epsilon^2), \tag{4.38}$$

where

$$\nu = \rho^2 p_1 + \rho^4 q_1 > 0, \qquad p_1 = T_1 \frac{\partial a}{\partial T}\bigg|_{T_1}, \qquad q_1 = T_1 \frac{\partial b}{\partial T}\bigg|_{T_1} \tag{4.39}$$

and $\rho^2 = \rho_1^2(\mu_3(T_1))$. Note the latent heat (4.34) is related to ν by $E = N^2 \nu T_1$.

To focus on the transition region we consider the following limit

$$N \to \infty, \qquad \epsilon \to 0, \qquad t = \epsilon N^2 = \text{finite}$$

In this limit we find that

$$\log \mathcal{Z} = \log \left(K_1(T_1) + K_3(T_1) e^{\nu t} \right) + O(N^{-2}) \tag{4.40}$$

(4.40) smoothly interpolates between the first and second line of (4.33) as t varies from $-\infty$ to $+\infty$. The expectation value of the Polyakov loop is given by

$$\rho_1^2(t) = \frac{1}{N^2} \frac{\partial \log Z}{\partial a} = \rho^2 \frac{K_3(T_1) e^{\nu t}}{K_1(T_1) + K_3(T_1) e^{\nu t}} + O(1/N^2) \tag{4.41}$$

ρ_1^2 smoothly interpolates between the first and the second line of (4.35).

5. SBH and tunnelling

As pointed out in [3] when $T > T_0$[18] thermal AdS and BBH can tunnel into each other with the Euclidean SBH as the instanton bounce [34]. Our effective potential $Q(\mu)$ (see fig. 2) gives a concrete realization of the physical picture. Through thermal fluctuations, BBH or thermal AdS can jump to the top of the barrier to become a SBH. Since it has negative specific heat, the small black hole then can either become thermal AdS by emitting thermal Hawking radiation or become a big black hole by absorbing radiation. In the Euclidean description, this corresponds to rolling down the hill from the two sides of the effective potential. In thermal equilibrium, of course the total probability to go from thermal AdS to BBH or vice versa should be the same.

5.1. Tunnelling between thermal AdS and BBH

In this subsection we will calculate the tunnelling rates between thermal AdS and BBH using our effective potential description. For definiteness, we will restrict to the temperature range $T_c > T T_0$ (i.e for (a, b) lying between line I and line III). In this range the details of smoothening of singular behavior of the effective potential (4.10) at $\mu = 1$ by non-perturbative effects will not be relevant and we will use (4.10) to perform the analysis. The discussion for the range $T_H > T > T_c$ is similar, and will not be repeated. The only difference is that since the SBH is precisely located at $\mu = 1$ for $T_H > T > T_c$, it is therefore more convenient to use the two-dimensional effective potential (4.8). The perturbative expansion around SBH breaks down near T_c, and will be discused in next subsection.

The tunnelling rates between thermal AdS and BBH can be readily computed using the effective potential following the standard procedure [35,36,37]. To be definite, let us first consider the tunnelling rate for thermal AdS over the barrier. In computing (4.2), instead of using the μ contour going from $-\infty$ to $+\infty$ along the real axis, we consider a contour C_1 along the real axis from $-\infty$ to μ_2 and then deform the contour at μ_2 along a steepest descent contour to the complex μ plane (see Fig. 3a). The partition function obtained using contour C_1 is given by

$$\mathcal{Z}_1 \approx e^{-N^2 Q(\mu_1)} K_1 \left(1 + O(N^{-2})\right) + \frac{i}{2} e^{-N^2 Q(\mu_2)} K_2 \left(1 + O(N^{-2})\right) \qquad (5.1)$$

---

[18] In contrast, the flat space has a non-perturbative instability at any finite temperature [34].

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where K_1 and K_2 arise from the Gaussian factor in the saddle point approximation. The resulting free energy (let us call it F_1) has an imaginary part given by

$$\mathrm{Im}F_1 \approx \frac{1}{2\beta}e^{-N^2(Q(\mu_2)-Q(\mu_1))}\frac{K_2}{K_1}\left(1+O(N^{-2})\right) \tag{5.2}$$

The tunnelling rate is then obtained from (5.2) by [37]

$$\Gamma_1 \approx \frac{\omega_0\beta}{\pi}\mathrm{Im}F_1 = \frac{\omega_0}{2\pi}e^{-N^2(Q(\mu_2)-Q(\mu_1))}\frac{K_2}{K_1}\left(1+O(N^{-2})\right) \tag{5.3}$$

where ω_0 is the frequency for the unstable mode around the SBH background. Similarly the tunnelling rate from BBH to thermal AdS can be obtained by computing (4.2) along a contour C_2 (see Fig. 3b) to be

$$\Gamma_2 \approx \frac{\omega_0}{2\pi}e^{-N^2(Q(\mu_2)-Q(\mu_3))}\frac{K_2}{K_3}\left(1+O(N^{-2})\right) \tag{5.4}$$
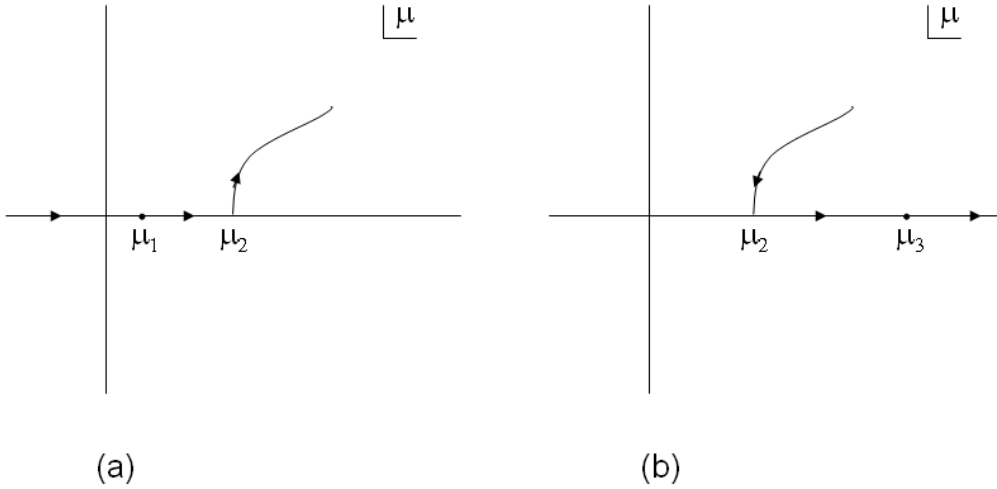

(a)   (b)

Fig. 3: (a) plots the contour C_1 which is used to compute (5.2) and (b) plots the contour C_2 used to compute (5.4).

Note that the imaginary part of the partition functions with contours C_1, C_2 can also be obtained by a Borel resummation over the perturbative expansion around the BBH and thermal AdS respectively. For example, the asymptotic expansion around the BBH μ_3 has the form

$$e^{-N^2Q(\mu_3)}K_3\left(1+\sum_{n=1}^{\infty}N^{-2n}d_n(a,b)\right) \tag{5.5}$$

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Due to the presence of a maximum at μ_2, the coefficients d_n at large n are given by

$$d_n \approx \frac{K_2}{2\pi K_3} \frac{\Gamma(n)}{(Q(\mu_2) - Q(\mu_3))^n} \qquad (5.6)$$

The asymptotic expansion (5.5) is clearly divergent and not Borel summable. In fact the Borel transform $\sum_n \frac{d_n z^n}{n!}$ of $\sum_n \frac{d_n}{N^{2n}}$ is singular at $z = Q(\mu_2) - Q(\mu_3)$, thus preventing Borel summability if the difference between the action of SBH and BBH $Q(\mu_2) - Q(\mu_3)$ is positive. This is a familiar situation in instanton physics, where we can interpret the singularity in the Borel transform in the positive axis as real instantons. One can integrate the Borel transform just above the singularity on the real positive axis, producing an imaginary part. Using this procedure, we find that (5.5) becomes[19]

$$e^{-N^2 Q(\mu_3)} K_3 \left(1 + O(N^{-2})\right) - \frac{i}{2} e^{-N^2 Q(\mu_2)} K_2 \left(1 + O(N^{-2})\right) \qquad (5.7)$$

which is precisely what we get by computing the partition function following contour C_2 in Fig. 3b and leads to the tunnelling rate (5.4). Similar discussions apply to the perturbative expansion around thermal AdS. Note that the instanton effect we have obtained associated with the SBH is of order $e^{-\frac{1}{g_s^2}}$ for $g_s = \frac{1}{N}$ and not of order $e^{-\frac{1}{g_s}}$ typical of D-instantons. In some sense, the effect we are describing could be interpreted as a collective state of N D-instantons.

Of course the total partition function is real and therefore cannot contain any imaginary part. Physically this is equivalent to a balance rule between the probability to nucleate a SBH from thermal AdS and the probability of decay of the metastable BBH. The two imaginary parts that we obtain in contours C_1 and C_2 should cancel in the complete partition function that is real. This is in fact the case once we realize that the sum of contours C_1 and C_2 gives the real contour which defined the total partition function $\mathcal{Z}$.

To conclude this subsection we will briefly comment on the asymptotic expansion for $T < T_0$. Below line I the two saddles μ_2 and μ_3 move into the complex plane, in other words the real instanton that we have associated with the SBH above line I becomes complex. This in particular means that the corresponding singularity in the Borel transform is not any more on the real positive axis and Borel summability is potentially restored[20]. From the physical point of view this means that the thermal AdS saddle is stable below line I.

---

[19] One keeps only finite number of terms in the first term below. Also there is a sign ambiguity here. We take the sign to be the same as that of contour C_2.

[20] This phenomena is well known in quantum field theory. For instance for the $\lambda\phi^4$ theory we have complex instanton solutions that produce a priori harmless singularities in the Borel transform on the negative real axis.

5.2. A Gross-Witten transition for small black hole

In this subsection we examine in detail the behavior of SBH as it crosses line III ($c = 1$) in the $a - b$ plane. It undergoes a third order phase transition in the large N limit and the large N expansion around SBH breaks down there. We will define a double scaling limit to smooth out the transition.

When $c = 1 + \epsilon$ with $0 < \epsilon \ll 1$, we find from (4.23)–(4.25) that the critical point corresponding to the SBH is given by

$$\mu_2 = 1 + \frac{\epsilon^2}{4} + \cdots, \qquad g_2 = 1 + \frac{\epsilon}{2} + \cdots, \qquad \rho_1(g_2) = \frac{1}{2} + \frac{\epsilon}{4} + \cdots \tag{5.8}$$

When $c = 1 - \epsilon$, one finds from (4.19) that

$$\mu_2 = 1, \qquad g_2 = 1 - \frac{\epsilon}{2} + \cdots, \qquad \rho_1(g_2) = \frac{1}{2} - \frac{\epsilon}{4} + \cdots \tag{5.9}$$

Thus as c crosses 1, g_2 crosses 1 and the eigenvalue distribution of μ_2 crosses $\rho_1 = \frac{1}{2}$. One can also check that the second derivatives around the SBH varies smoothly across $c = 1$.

As we commented after equation (4.12), (4.8) has a third order discontinuity at $g = 1$ which is smoothened out at finite N. More precisely, let

$$g = 1 - N^{-\frac{2}{3}} y \tag{5.10}$$

Then V should be replaced by[21]

$$V = \frac{1}{4b}(\mu - a)^2 + \frac{g^2}{4}\frac{1 - \mu}{\mu} - \sum_{n=0}^{\infty} N^{-\frac{2}{3}n} F_n(y) \tag{5.11}$$

where the $F_n(y)$'s are smooth functions of y. In particular, F_0 describes the doubling scaling limit of (4.4) with the following asymptotic expansion

$$F_0(y) = \begin{cases} \frac{y^3}{6} - \frac{1}{8}\log(-y) + \cdots & -y \gg 1 \\ \frac{1}{2\pi} e^{-\frac{4\sqrt{2}}{3} y^{\frac{3}{2}}} \left(-\frac{1}{8\sqrt{2} y^{\frac{3}{2}}} + \cdots \right) & y \gg 1 \end{cases} \tag{5.12}$$

Note that (5.11) smoothly interpolates between the second and the third line of (4.8).

---

[21] See e.g. sec. 3.1 of [19]. F_n below correspond to $F_n^{(2)}$ there.

Equations (5.8)–(5.11) suggest that to study the physics of the SBH near $c = 1$, we can consider the following scaling

$$a(T) = a_0 + a_1\epsilon q, \qquad b(T) = b_0 + b_1 q\epsilon, \qquad \frac{2(1 - a_0)}{b_0} = 1 \qquad (5.13)$$

$$\mu = 1 + \epsilon^2 x, \qquad g = 1 - \epsilon y, \qquad \epsilon = N^{-\frac{2}{3}} \qquad (5.14)$$

where $a_0 = a(T_c), a_1 = T_c a'(T_c)$ (similar for b) and $\epsilon q = \frac{T - T_c}{T_c}$. Thus we have

$$c = \frac{2(1 - a)}{b} = 1 - c_1\epsilon q, \qquad c_1 = \frac{a_1}{1 - a_0} + \frac{b_1}{b_0} = \frac{1}{b_0}(2a_1 + b_1)$$

Note $c_1 > 0$ according to our assumptions on the thermal history of the model and $q < 0$ as we approach line III form below.

Plugging (5.13) and (5.14) into (5.11), we find that

$$N^2 V = \frac{N^2(1 - a)^2}{4b} + \frac{1}{2}x\left(y - \frac{c_1 q}{2}\right) - F_0(y) + O(\epsilon) \qquad (5.15)$$

Note that in the last expression, x simply plays the role of a Lagrange multiplier. The integral over x, y in (4.7) is not well defined since we are integrating around the neighborhood of a saddle point in the $x - y$ plane. If we rotate the integration contour of x to be along the imaginary axis, the x integral will result in a delta function for y and we find that the partition function around the SBH is given by

$$\mathcal{Z}_{SBH} = iN\sqrt{\frac{\pi}{b}}e^{-\frac{N^2(1-a)^2}{4b} + F_0\left(\frac{c_1 q}{2}\right)}(1 + O(\epsilon)) \qquad (5.16)$$

The prefator i can be understood as due to the tachyonic mode of the SBH. We find that around T_c, one can define a double scaling limit where the SBH is described by F_0. It was argued in [38] that $F_0(t)$ describes the full partition function of the type 0B theory in $d = 0$ dimension, i.e. pure 2-d supergravity. The parameter t is proportional to the cosmological constant μ in the super-Liouville interaction. We are then led to the conclusion that in a double scaling limit around T_c (as we argued earlier in the Horowitz-Polchinski correspondence point) the SBH appears to be described by type 0B theory in zero dimension.

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6. Catastrophes and the break down of perturbative string expansions

In this section we examine the critical behavior of the BBH at T_0 where its saddle merges with that of the SBH and the critical behavior of the thermal AdS at the Hagedorn temperature T_H. We find in both places that the breakdown of the large N expansion can be understood in terms of the simplest type of catastrophes allowed by the symmetry. The divergences at the perturbative level can be smoothened out at finite N using the standard techniques of catastrophe theory.

6.1. Nucleation of black holes

The large N expansion around the big black hole saddle breaks down near line I, where it coalesces with the unstable small black hole saddle. We will show that the critical behavior there is given by the fold catastrophe. One can define a double scaling limit in which the partition function for this sector is given by an Airy function.

From (4.28), curve I can be parameterized by

$$
\begin{aligned}
a(w) &= \frac{1 - 2w}{(1 - w)^2(1 + w)}, \\
b(w) &= \frac{2w}{(1 - w)^2(1 + w)^3}, \qquad w \in [0, 1] \, .
\end{aligned}
\tag{6.1}
$$

Suppose that at temperature T_0, the curve $(a(T), b(T))$ intersects the curve I (6.1) at a point labelled by w_0, i.e. $a(T_0) = a(w_0)$, $b(T_0) = b(w_0)$. At the intersection point, (4.25) has a double root given by w_0, which is an inflection point of $Q(w)$. We will consider the behavior of $Q(w)$ (4.26) near $w = w_0$ and $T = T_0$. Let

$$
a(T) = a(T_0) + a_1\epsilon, \qquad b(T) = b(T_0) + b_1\epsilon, \qquad w = w_0 + y
\tag{6.2}
$$

with $\epsilon = \frac{T - T_0}{T_0}$ and $a_1 = T_0 a'(T_0)$, $b_1 = T_0 b'(T_0)$. We will consider the regime $|\epsilon| \ll 1$ and $|y| \sim \sqrt{|\epsilon|} \ll 1$. Plugging (6.2) into (4.26) and expanding in ϵ and y we find that[22]

$$
Q = C_0(\epsilon, w_0) - f\left(-\frac{1}{3}y^3 + q\epsilon y\right) + O(\epsilon^2, y^4, y^2\epsilon)
\tag{6.3}
$$

---

[22] The expansion below breaks down at $w_0 = 0$, where it can be checked that C_0 and various higher order terms become singular. At $w_0 = 0$, we have $a_0 = 1$, $b_0 = 0$ and the physics goes over to that of [19].

with

$$f = -\frac{1}{2}a'(w_0), \qquad q = \frac{(1+w_0)a_1}{b'(w_0)}\left(\tan\theta_0 - \frac{b_1}{a_1}\right) \tag{6.4}$$

where $\tan\theta_0$ is the slope of line I at w_0. Note that

$$a'(w_0) < 0, \qquad b'(w_0) > 0, \qquad \tan\theta_0 = \frac{b'(w_0)}{a'(w_0)} = -\frac{2}{(1+w_0)^2}, \qquad \theta_0 \in (\pi/2, \pi)$$

From our assumption of the thermal history of the theory, q is positive. Note that $C_0(\epsilon, w_0)$ is analytic in ϵ.

For $\epsilon > 0$, from (6.3) Q has one maximum and one minimum at $y = \pm\sqrt{\epsilon q}$. The two extrema merge at $q = 0$ and move to complex values for $\epsilon < 0$. The values of Q at the minimum and the maximum are given by ($\epsilon > 0$)

$$Q_0 = C_0(\epsilon) \mp \frac{2fq^{\frac{3}{2}}}{3}\epsilon^{\frac{3}{2}} + O(\epsilon^2) \tag{6.5}$$

The second term gives the leading nonanalytic term in ϵ and the specific heat has a critical exponent $\gamma = \frac{1}{2}$.

It is instructive to compare (6.5) with the result (2.3) from supergravity. Expanding (2.3) in ϵ around β_0 for big and small black holes we find

$$I = \widetilde{C}_0(\epsilon) \mp 2\epsilon^{\frac{3}{2}} + \cdots \tag{6.6}$$

with $\widetilde{C}_0$ an analytic function of ϵ. We see exact agreement between (6.5) and (6.6) in the critical exponent. The critical exponent $\frac{1}{2}$ is universal, depending only on the fact that a maximum and a minimum merge together (fold catastrophe).

As $T \to T_0$, the large N expansion around the big and small black holes break down. The physics around them can be captured by a double scaling limit. Since we are interested only in the BBH and SBH, we will again consider (4.2) along contour C_2 in Fig. 3b. From (6.3) we introduce a new variable

$$z = q\epsilon(N^2 f)^{\frac{2}{3}}$$

and consider the scaling limit

$$\epsilon \to 0, \qquad N \to \infty, \qquad z = \text{finite}$$

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In this limit, the partition function becomes

$$\mathcal{Z}_2 = \mathcal{C}_0 \int_{C_2} ds \, e^{-\frac{1}{3}s^3 + zs} \tag{6.7}$$

which is given by an Airy function. $\mathcal{C}_0$ is a non-universal factor

$$\mathcal{C}_0 = (N^2 f)^{-\frac{1}{3}} \frac{2w_0}{(1 - w_0^2)^2} e^{\mathcal{F}_1(w_0)} e^{-N^2 C_0}$$

where $\mathcal{F}_1$ is the $O(1/N^2)$ term in $\mathcal{F}(\mu)$ (4.11) (not given explicitly there). It is easy to check that our choice of contour gives the Airy function

$$\mathcal{Z}_2 = 2\pi i \, \mathcal{C}_0 e^{-\frac{2\pi i}{3}} Ai(ze^{-\frac{2\pi i}{3}}) \tag{6.8}$$

(6.8) smoothes out the divergences in perturbative expansions. Also note that the argument of the Airy functions precisely sits on the Stokes line. This is a consequence of the fact that for μ_2, μ_3 real, $\mathrm{Im}Q(\mu_2) = \mathrm{Im}Q(\mu_3) = 0$. Although (6.8) is complex, the full partition function should be real when including the contribution from the contour C_1 of fig. 3a.[23]

6.2. Hagedorn behavior for thermal AdS

We will now examine the merger of thermal AdS and the SBH (or more precisely long string phase). For this purpose, let us first look at the free energy near the thermal AdS background for $a < 1$, which can be found by expanding the partition function around the saddle (4.16). Around (4.16), the partition function can be approximated as

$$\begin{aligned}
\mathcal{Z}_1 &= \frac{N}{2\sqrt{\pi b}} \int_{-\infty}^{1} d\mu \, e^{-\frac{N^2}{4b}(\mu - a)^2} \frac{1}{(1 - a) - (\mu - a)} \\
&\approx \frac{N}{2\sqrt{\pi b}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dx \, e^{-\frac{N^2}{4b}x^2} \frac{x^n}{(1 - a)^{n+1}} \\
&= \frac{1}{1 - a} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}} \left(\frac{2\sqrt{b}}{N(1 - a)} \right)^{2n}
\end{aligned} \tag{6.9}$$

There are also nonperturbative corrections which are omitted here. Thus the free energy around the thermal AdS background can be written as

$$\log \mathcal{Z}_1 = -\log(1 - a) + \frac{2b}{(1 - a)^2 N^2} + + \frac{10b^2}{(1 - a)^4 N^4} + \frac{296b^3}{3(1 - a)^6 N^6} + \cdots \tag{6.10}$$

---

[23] We would like to thank G. Festuccia and A. Scardicchio for extensive discussions regarding this point.

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Note that in contrast to the free theory case analyzed in [19], the free energy now receives perturbative contributions to all orders. The free energy diverges as $a \to 1$. The leading order divergence, arising from genus one contribution, is

$$\log \mathcal{Z}_1 = -\log(T - T_H) + \text{const} \tag{6.11}$$

since as $a \to 1$, $1 - a \propto T - T_H$. (6.11) is precisely the Hagedorn divergence for string theory in a spacetime with all directions compactified (recall that AdS behaves like a box). By a Laplace transform of (6.11) one finds that the density of states is given by

$$\Omega(E) \approx \text{const } e^{\beta_H E}\left(1 + O(1/E^2)\right) \tag{6.12}$$

Also note that as $T \to T_H$, the free energy around SBH is given by

$$\log \mathcal{Z}_1 \propto -N^2(T - T_H)^2 + \cdots \tag{6.13}$$

The perturbative expansion (6.10) breaks down at $1 - a \sim N^{-1}$. To explore the physics near this point, we let[24]

$$a = 1 - N^{-1}q, \qquad \mu = 1 - N^{-1}x, \qquad g = 2N^{-\frac{1}{2}}y \tag{6.14}$$

and consider a double scaling limit with $N \to \infty$ with q, x, y finite. We find that

$$Q = N^{-2}P + O(N^{-3})$$

with

$$P = \frac{(x-q)^2}{4b} + xy^2 \ . \tag{6.15}$$

Note that P has two critical points for $q > 0$

$$(1) \qquad x_1 = q, \qquad y_1 = 0, \qquad P'' = \begin{pmatrix} \frac{1}{2b} & 0 \\ 0 & 2q \end{pmatrix}$$

$$(2) \qquad x_2 = 0, \qquad y_2 = \sqrt{\frac{q}{2b}}, \qquad P'' = \begin{pmatrix} \frac{1}{2b} & \sqrt{\frac{2q}{b}} \\ \sqrt{\frac{2q}{b}} & 0 \end{pmatrix}$$

---

[24] Below for notational simplicity we will assume b does not change as $a = 1$ is crossed. To incorporate the change in b is straightforward.

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corresponding to (μ_1, g_1) and (μ_2, g_2). They merge at $q = 0$ and for $q < 0$ only the first solution remains. The relevant part of the partition function (4.7) then becomes

$$
\begin{aligned}
\mathcal{Z}_1 &= \frac{N}{\sqrt{\pi b}} \int_{-\infty}^{\infty} dx \int_0^{\infty} y dy\, e^{-P(x,y)} \\
&= 2N \int y dy\, e^{-qy^2 + by^4}
\end{aligned}
\tag{6.16}
$$

Introducing a (0-dimensional) complex scalar ϕ with $y = |\phi|$, the second line of (6.16) can also be written as

$$
\mathcal{Z}_1 = \int d\phi d\phi^* \exp\left[-m^2(\beta)\phi^*\phi + b(\phi\phi^*)^2\right]
\tag{6.17}
$$

with $m^2(\beta) = q \propto T - T_H$. It is tempting to identify ϕ with the so-called thermal scalar (a winding tachyon) in string theory[25]. Indeed (6.17) coincides with the effective action one expects for a thermal scalar near Hagedorn temperature [39]. It is clear from the second line of (6.16) or (6.17) that the merger of thermal AdS and SBH is described by a cusp catastrophe with $q = 0$ (i.e. $a = 1$) corresponding to the cusp point. Note that the appearance of effective action (6.17) is forced on us by the $U(1)$ symmetry of the complex scalar field ϕ, which corresponds to the Z_N symmetry of the boundary effective action in the large N limit. The cusp catastrophe is the simplest possibility consistent with this symmetry. The only nontrivial dynamical input in (6.17) is that $b > 0$, following from the existence of a first order Hawking-Page transition.

The integral in (6.16) is not bounded as is expected since we are focusing in the neighborhood of the effective potential containing only thermal AdS and SBH. To define the integral in (6.16) we will choose an integration contour analogous to C_1 of Fig. 3a in section 5. We take the contour in the y-plane to go from $y = 0$ to the maximum $y_2 = \sqrt{\frac{q}{2b}}$ along the real axis and then go straight up to the complex value at y_2. The integration along the real axis will give us an error function, which smoothes out the divergences of the perturbative expansion at $q = 0$.

---

[25] Note that we see a zero dimensional scalar since AdS may be considered as a box.

7. Conclusions and discussions

In this paper we introduced a class of phenomenological models to understand string theory in $AdS_5 \times S_5$ in the canonical ensemble. Our models reproduces all the known qualitative features of the theory. They also have some interesting predictions including the existence of a third order phase transition for SBH, which we identify with the Horowitz-Polchinski point. We studied the simplest model (3.5) in great detail. We found the Hagedorn behavior of thermal AdS at T_H and the critical behavior of nucleation of Euclidean SBH and BBH at T_0 are governed respectively by cusp (A_3) and fold (A_2) catastrophe. It is clear these features persist for all models in the class due to universality of the catastrophe. We believe this gives strong indication that they capture qualitative behaviors of a weakly coupled string theory in a large AdS spacetime. Since for a large radius AdS, the SBH resembles a ten dimensional Schwarzschild black hole in flat spacetime, and the Hagedorn behavior for strings in AdS resembles that of flat space, we expect that the behaviors we observe here may yield clues to answers for similar questions in flat spacetime.

There are many other questions which can be explored along the lines of our investigation. For example, it would be nice to have a better understanding of our proposal that the Horowitz-Polchinski point for a small black hole should correspond to a Gross-Witten transition in the boundary theory. It would be interesting to understand whether the process involves changing the spacetime topology. Also, since the saddles corresponding to thermal AdS and SBH merge at T_H, we expect that in the worldsheet sigma model of thermal AdS, turning on the marginal operator corresponding to the thermal scalar at T_H, the theory can be deformed into a SBH background. It would also be interesting to understand from the worldsheet point of view what happens when the tachyon in thermal AdS above the Hagedorn temperature condenses[26]. The discussion of [40] might be useful for this purpose.

We believe that the phenomenological approach developed here can have many other applications. For example, it would be interesting to develop an effective potential approach for the tunnelling discussed in [41]. It would also be interesting to see whether one can use our methods to address the problem of black hole information loss.

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[26] It appears clear from the effective potential picture that the theory will flow to the BBH background.

Appendix A. Large N phase structure and thermal history for general matrix model (4.1)

For completeness, in this section we discuss the phase structure and thermal history of matrix model (4.1) using the Hartree-Fock approximation. The Hartree-Fock treatment of double trace operators was earlier discussed in [42]. The Hartree-Fock approximation gives equations of motion which can be solved to find the critical points of the theory and the value of the action evaluated at the critical points. However from this method one cannot find the off-shell effective potential (essential for our purposes).

In the infinite N limit, it is convenient to introduce the density of eigenvalues

$$\rho(\theta) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i), \qquad -\pi \leq \theta < \pi \tag{A.1}$$

with

$$\frac{1}{N} \mathrm{Tr} U^n = \rho_n = \int_{-\pi}^{\pi} d\theta \, \rho(\theta) \, e^{in\theta}$$

(4.1) can be written as

$$\mathcal{Z} = \int [D\rho] \, e^{-N^2 V[\rho]} \tag{A.2}$$

where $V[\rho]$ has the form

$$V[\rho] = -\frac{1}{2} \int d\theta d\phi \, \rho(\theta)\rho(\phi) \, P \log \left(2 \sin \frac{\theta - \phi}{2} \right)^2 - S(|\rho_1|^2) \tag{A.3}$$

Since the potential is symmetric, we can take ρ_1 to be real. The equations of motion following from (A.3) can be written as

$$\int d\phi\, \rho(\phi) \cot \frac{\theta - \phi}{2} = \kappa \sin \theta \tag{A.4}$$

with

$$\kappa = 2S'(x)\rho_1 , \quad x = \rho_1^2 .$$

The solutions to the above equation are well known [20,21,22], leading to the self-consistent equations for ρ_1 (using (4.6) with g replaced by κ above)

$$
\begin{aligned}
\rho_1 &= S'(x)\rho_1, & 0 \le \rho_1 &\le \frac{1}{2} \\
S'(x) &= \frac{1}{4\rho_1(1 - \rho_1)}, & \frac{1}{2} \le \rho_1 &\le 1
\end{aligned}
\tag{A.5}
$$

Note that the first line of (A.5) implies that

$$\rho_1 = 0 \tag{A.6}$$

or

$$S'(x) = 1, \quad x = \rho_1^2 \in [0, 1/4] \tag{A.7}$$

We can slightly rewrite the second equation of (A.5) as

$$S'(x) = f(x), \quad f(x) = \frac{1}{4\sqrt{x}(1 - \sqrt{x})}, \quad x \in [1/4, 1] \tag{A.8}$$

Note $S(x)$ are also functions of 't Hooft coupling λ and temperature T. We will show that given the following assumptions about $S(x; \lambda, T)$, (A.2) has exactly the same large N phase structure as that of the (a, b) model analyzed in the main text and thus that of the AdS supergravity:

1. $S(x)$ is convex, i.e. $S'(x)$ is monotonically increasing;
2. $S'(x)$ is concave;
3. For sufficiently low temperature, $S'(x)$ lies below $f(x)$ defined in (A.8) in $x \in [1/4, 1]$. For sufficiently high temperature $S'(0) > 1$;
4. $S'(1/4)$ is a monotonically increasing function of T.

We note that for (a, b) model (3.5), conditions 1 and 2 are automatically satisfied. Condition 3 corresponds to our assumption that the $\mathcal{C}_\lambda$ curve starts below line I of fig. 1 at sufficiently low temperature and ends to the right of line H at high enough temperature.

Condition 4 makes sure that $\mathcal{C}_\lambda$ intersects all lines in fig. 1 only once as T is varied. Note that conditions 4 may be further relaxed.

We first note that $f(x)$ is a monotonically increasing and convex function which takes values

$$f(1/4) = 1, \qquad f(1) = +\infty \tag{A.9}$$

At a sufficiently low T, condition 3 implies (A.8) has no solution. Since $S'(1/4) < 1$ and S' is monotonically increasing, equation (A.7) also does not have a solution. The only phase of the system at this temperature is thus given by (A.6), i.e.

$$x_1 = 0$$

This is the thermal AdS. It is a minimum of V for $S'(0) < 1$. As we increase the temperature, the curve $S'(x)$ will start intersecting[27] with $f(x)$. The temperature at which they become tangent is T_0, where the critical points corresponding to SBH and BBH start appearing. Immediately above T_0, (A.8) will have two solutions $\frac{1}{4} < x_2 < x_3 < 1$ and from condition 2 it can only have two solutions. Since V is bounded from below, x_3 should be minimum (BBH), while x_2 a maximum (SBH). At this temperature (A.7) again does not have any solution since $S'(1/4) < 1$. At a temperature $T_c > T_0$, when $S'(1/4) = 1$, $x_2 = \frac{1}{4}$ is both a solution of (A.7) and (A.8). When $T > T_c$, $S'(1/4) > 1$, (A.8) only has one solution x_3. x_2 moves to the region $x < 1/4$ and becomes a solution to (A.7). Convexity of $S(x)$ implies that the solution to (A.7) is unique. At a temperature T_H, when $S'(0) = 1$, x_2 and x_1 coincide. Above T_H, x_2 no longer exists and x_1 becomes tachyonic due to that $S'(0) > 1$. Since at T_H, x_1 and x_2 coincide, we have $V(x_2) = V(x_1) = 0$ in the large N. Since $V(x_2) > V(x_3)$, $V(x_3)$ must be smaller than zero at T_H. We thus conclude that there must be a first order Hawking-Page transition at some temperature $T_0 < T_1 < T_H$.

To summarize, the general model satisfying the four assumptions above have exactly the same phase structure as that of the (a, b) model including the Gross-Witten phase transition for SBH and the merger of SBH and thermal AdS at T_H.

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[27] This is guaranteed following our assumptions.

Appendix B. Specific heat of small and big black holes

In this appendix, we show that the phases corresponding to μ_2 and μ_3 have negative and positive specific heat respectively.

We first show that (4.19) has a negative specific heat. From (4.20),

$$
\begin{aligned}
c_v(\mu_2) &= -N^2\beta^2 \frac{\partial^2 V}{\partial\beta^2} \\
&= -\frac{N^2\beta^2}{2b}\left(\frac{\partial a}{\partial\beta} + \frac{1-a}{b}\frac{\partial b}{\partial\beta}\right)^2 < 0
\end{aligned}
\tag{B.1}
$$

We now look at the specific heat of a solution to equation (4.25). Evaluated at a critical point $\mu_c(\beta)$, the action is given by $Q(\beta, \mu_c(\beta))$. We first note an identity

$$
-\frac{d^2}{d\beta^2}Q(\beta, \mu_c(\beta)) = \frac{\left(\frac{\partial Q^2(\beta,\mu)}{\partial\mu\partial\beta}\right)^2}{\frac{\partial^2 Q}{\partial\mu^2}}\Bigg|_{\mu_c} - \frac{\partial^2 Q}{\partial\beta^2}\Bigg|_{\mu_c}
\tag{B.2}
$$

In deriving the above equation we have used the equation of motion $\frac{\partial Q}{\partial\mu}\big|_{\mu_c} = 0$ and

$$
\frac{\partial\mu_c}{\partial\beta} = -\frac{\frac{\partial^2 Q}{\partial\mu\partial\beta}}{\frac{\partial^2 Q}{\partial\mu^2}}\Bigg|_{\mu=\mu_c}
\tag{B.3}
$$

We note from (4.26) that

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial\beta^2}\Bigg|_{\mu_c} &= \frac{1}{2b}\left(\frac{\partial a}{\partial\beta} + \frac{\mu_c - a}{b}\frac{\partial b}{\partial\beta}\right)^2 \\
\frac{\partial^2 Q}{\partial\beta\partial\mu}\Bigg|_{\mu_c} &= -\frac{1}{2b}\left(\frac{\partial a}{\partial\beta} + \frac{\mu_c - a}{b}\frac{\partial b}{\partial\beta}\right)
\end{aligned}
\tag{B.4}
$$

Thus the specific heat for μ_c can be written as

$$
\begin{aligned}
c_v(\mu_c(\beta)) &= -N^2\beta^2 \frac{d^2}{d\beta^2}Q(\beta, \mu_c(\beta)) \\
&= \frac{N^2\beta^2}{4b^2}\left(\frac{\partial a}{\partial\beta} + \frac{\mu_c - a}{b}\frac{\partial b}{\partial\beta}\right)^2\left(\frac{1}{\frac{\partial^2 Q}{\partial\mu^2}}\Bigg|_{\mu=\mu_c} - 2b\right)
\end{aligned}
\tag{B.5}
$$

For $\mu_c = \mu_2$, $\frac{\partial^2 Q}{\partial\mu^2}\big|_{\mu_2} < 0$, and we find

$$
c_v(\mu_2) < 0 \ .
$$

For $\mu_c = \mu_3$, it follows from (4.27) that

$$
0 < Q''(\mu_3) = \frac{1}{2b} - \frac{1}{4\mu^2} - \frac{1}{4\sqrt{\mu-1}\mu^{3/2}} < \frac{1}{2b}
\tag{B.6}
$$

Plugging (B.6) into (B.5) we find:

$$
c_v(\mu_3) > 0 \ .
$$

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