TENSOR CATEGORIES

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These are lecture notes for the course 18.769 "Tensor categories", taught by P. Etingof at MIT in the spring of 2009.

In these notes we will assume that the reader is familiar with the basic theory of categories and functors; a detailed discussion of this theory can be found in the book [ML]. We will also assume the basics of the theory of abelian categories (for a more detailed treatment see the book [F]).

If $\mathcal{C}$ is a category, the notation $X \in \mathcal{C}$ will mean that X is an object of $\mathcal{C}$, and the set of morphisms between $X, Y \in \mathcal{C}$ will be denoted by $\mathsf{Hom}(X, Y)$.

Throughout the notes, for simplicity we will assume that the ground field k is algebraically closed unless otherwise specified, even though in many cases this assumption will not be needed.

1. Monoidal categories

1.1. **The definition of a monoidal category.** A good way of thinking about category theory (which will be especially useful throughout these notes) is that category theory is a refinement (or "categorification") of ordinary algebra. In other words, there exists a dictionary between these two subjects, such that usual algebraic structures are recovered from the corresponding categorical structures by passing to the set of isomorphism classes of objects.

For example, the notion of a (small) category is a categorification of the notion of a set. Similarly, abelian categories are a categorification of abelian groups [1] (which justifies the terminology).

This dictionary goes surprisingly far, and many important constructions below will come from an attempt to enter into it a categorical "translation" of an algebraic notion.

In particular, the notion of a monoidal category is the categorification of the notion of a monoid.

Recall that a monoid may be defined as a set C with an associative multiplication operation $(x, y) \to x \cdot y$ (i.e., a semigroup), with an element 1 such that $1^2 = 1$ and the maps $1\cdot, \cdot 1 : C \to C$ are bijections. It is easy to show that in a semigroup, the last condition is equivalent to the usual unit axiom $1 \cdot x = x \cdot 1 = x$.

[1] To be more precise, the set of isomorphism classes of objects in a (small) abelian category $\mathcal{C}$ is a commutative monoid, but one usually extends it to a group by considering "virtual objects" of the form $X - Y$, $X, Y \in \mathcal{C}$.

As usual in category theory, to categorify the definition of a monoid, we should replace the equalities in the definition of a monoid (namely, the associativity equation $(xy)z = x(yz)$ and the equation $1^2 = 1$) by isomorphisms satisfying some consistency properties, and the word "bijection" by the word "equivalence" (of categories). This leads to the following definition.

Definition 1.1.1. A monoidal category is a quintuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called the *tensor product* bifunctor, $a : \bullet \otimes (\bullet \otimes \bullet) \xrightarrow{\sim} \bullet \otimes (\bullet \otimes \bullet)$ is a functorial isomorphism:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathcal{C} \tag{1.1.1}$$

called the *associativity constraint* (or *associativity isomorphism*), $\mathbf{1} \in \mathcal{C}$ is an object of $\mathcal{C}$, and $\iota : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ is an isomorphism, subject to the following two axioms.

1. The pentagon axiom. The diagram

(1.1.2)

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{a_{W \otimes X, Y, Z}} & & \searrow^{a_{W,X,Y} \otimes \mathrm{Id}_Z} & \\ (W \otimes X) \otimes (Y \otimes Z) & & & & (W \otimes (X \otimes Y)) \otimes Z \\ \downarrow{\scriptstyle a_{W,X,Y \otimes Z}} & & & & \downarrow{\scriptstyle a_{W,X \otimes Y,Z}} \\ W \otimes (X \otimes (Y \otimes Z)) & & \xleftarrow{\mathrm{Id}_W \otimes a_{X,Y,Z}} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

is commutative for all objects W, X, Y, Z in $\mathcal{C}$.

2. The unit axiom. The functors $L_{\mathbf{1}}$ and $R_{\mathbf{1}}$ of left and right multiplication by $\mathbf{1}$ are equivalences $\mathcal{C} \to \mathcal{C}$.

The pair $(\mathbf{1}, \iota)$ is called the *unit object* of $\mathcal{C}$.[2]

We see that the set of isomorphism classes of objects in a small monoidal category indeed has a natural structure of a monoid, with multiplication $\otimes$ and unit $\mathbf{1}$. Thus, in the categorical-algebraic dictionary, monoidal categories indeed correspond to monoids (which explains their name).

Definition 1.1.2. A monoidal subcategory of a monoidal category $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ is a quintuple $(\mathcal{D}, \otimes, a, \mathbf{1}, \iota)$, where $\mathcal{D} \subset \mathcal{C}$ is a subcategory closed under the tensor product of objects and morphisms and containing $\mathbf{1}$ and ι.

[2]We note that there is no condition on the isomorphism ι, so it can be chosen arbitrarily.

1.2. **Basic properties of unit objects in monoidal categories.** Let $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ be a monoidal category. Define the isomorphism $l_X : \mathbf{1} \otimes X \to X$ by the formula

$$l_X = L_{\mathbf{1}}^{-1}((\iota \otimes \mathrm{Id}) \circ a_{\mathbf{1},\mathbf{1},X}^{-1}),$$

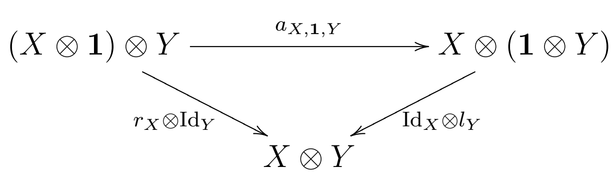
and the isomorphism $r_X : X \otimes \mathbf{1} \to X$ by the formula

$$r_X = R_{\mathbf{1}}^{-1}((\mathrm{Id} \otimes \iota) \circ a_{X,\mathbf{1},\mathbf{1}}).$$

This gives rise to functorial isomorphisms $l : L_{\mathbf{1}} \to \mathrm{Id}_{\mathcal{C}}$ and $r : R_{\mathbf{1}} \to \mathrm{Id}_{\mathcal{C}}$. These isomorphisms are called the *unit constraints* or *unit isomorphisms*. They provide the categorical counterpart of the unit axiom $1X = X1 = X$ of a monoid in the same sense as the associativity isomorphism provides the categorical counterpart of the associativity equation.

Proposition 1.2.1. *The "triangle" diagram*

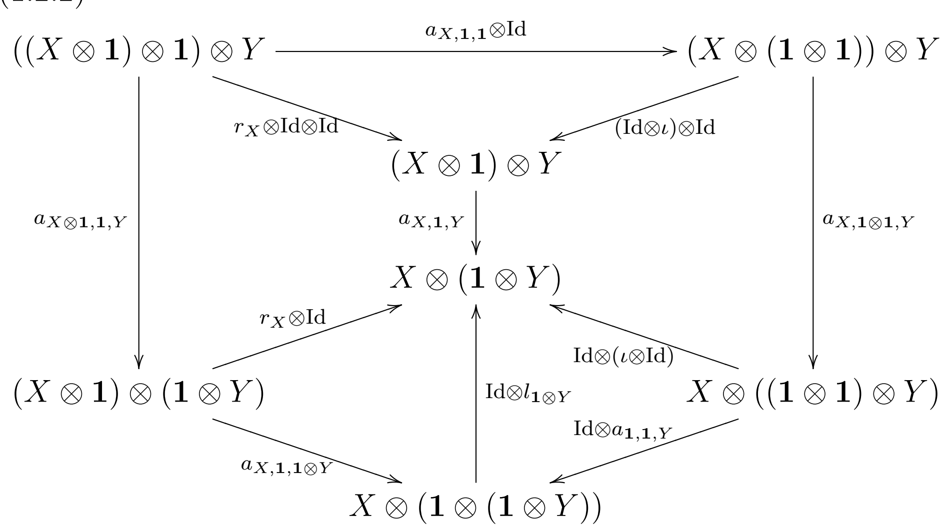
(1.2.1)



is commutative for all $X, Y \in \mathcal{C}$. In particular, one has $r_{\mathbf{1}} = l_{\mathbf{1}} = \iota$.

Proof. This follows by applying the pentagon axiom for the quadruple of objects $X, \mathbf{1}, \mathbf{1}, Y$. More specifically, we have the following diagram:

(1.2.2)

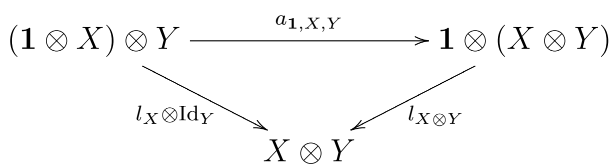


To prove the proposition, it suffices to establish the commutativity of the bottom left triangle (as any object of $\mathcal{C}$ is isomorphic to one of the form $\mathbf{1} \otimes Y$). Since the outside pentagon is commutative (by the pentagon axiom), it suffices to establish the commutativity of the other parts of the pentagon. Now, the two quadrangles are commutative due to the functoriality of the associativity isomorphisms, the commutativity of the upper triangle is the definition of r, and the commutativity of the lower right triangle is the definition of l.

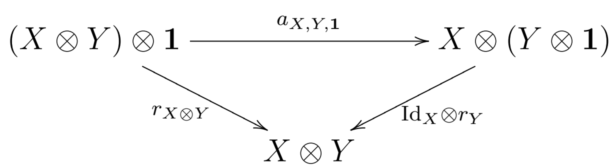
The last statement is obtained by setting $X = Y = \mathbf{1}$ in (1.2.1). $\square$

Proposition 1.2.2. *The following diagrams commute for all objects $X, Y \in \mathcal{C}$:*

(1.2.3)

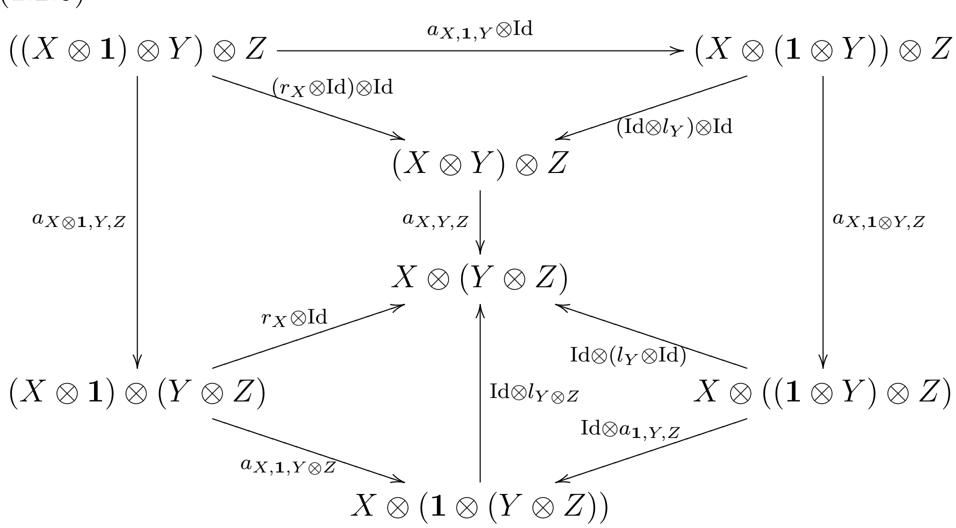


(1.2.4)



Proof. Consider the diagram

(1.2.5)



where X, Y, Z are objects in $\mathcal{C}$. The outside pentagon commutes by the pentagon axiom (1.1.2). The functoriality of a implies the commutativity of the two middle quadrangles. The triangle axiom (1.2.1) implies the commutativity of the upper triangle and the lower left triangle. Consequently, the lower right triangle commutes as well. Setting

$X = \mathbf{1}$ and applying the functor $L_{\mathbf{1}}^{-1}$ to the lower right triangle, we obtain commutativity of the triangle (1.2.3). The commutativity of the triangle (1.2.4) is proved similarly. □

Proposition 1.2.3. *For any object X in $\mathcal{C}$ one has the equalities $l_{\mathbf{1}\otimes X} = \mathrm{Id} \otimes l_X$ and $r_{X\otimes\mathbf{1}} = r_X \otimes \mathrm{Id}$.*

Proof. It follows from the functoriality of l that the following diagram commutes

(1.2.6)
$$\begin{array}{ccc} \mathbf{1}\otimes(\mathbf{1}\otimes X) & \xrightarrow{\mathrm{Id}\otimes l_X} & \mathbf{1}\otimes X \\ {\scriptstyle l_{\mathbf{1}\otimes X}}\downarrow & & \downarrow{\scriptstyle l_X} \\ \mathbf{1}\otimes X & \xrightarrow[l_X]{} & X \end{array}$$

Since l_X is an isomorphism, the first identity follows. The second identity follows similarly from the functoriality of r. □

Proposition 1.2.4. *The unit object in a monoidal category is unique up to a unique isomorphism.*

Proof. Let $(\mathbf{1}, \iota), (\mathbf{1}', \iota')$ be two unit objects. Let (r, l), (r', l') be the corresponding unit constraints. Then we have the isomorphism $\eta := l_{\mathbf{1}'} \circ (r'_{\mathbf{1}})^{-1} : \mathbf{1} \to \mathbf{1}'$.

It is easy to show using commutativity of the above triangle diagrams that η maps ι to ι'. It remains to show that η is the only isomorphism with this property. To do so, it suffices to show that if $b : \mathbf{1} \to \mathbf{1}$ is an isomorphism such that the diagram

(1.2.7)
$$\begin{array}{ccc} \mathbf{1}\otimes\mathbf{1} & \xrightarrow{b\otimes b} & \mathbf{1}\otimes\mathbf{1} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathbf{1} & \xrightarrow[b]{} & \mathbf{1} \end{array}$$

is commutative, then $b = \mathrm{Id}$. To see this, it suffices to note that for any morphism $c : \mathbf{1} \to \mathbf{1}$ the diagram

(1.2.8)
$$\begin{array}{ccc} \mathbf{1}\otimes\mathbf{1} & \xrightarrow{c\otimes \mathrm{Id}} & \mathbf{1}\otimes\mathbf{1} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathbf{1} & \xrightarrow[c]{} & \mathbf{1} \end{array}$$

is commutative (as $\iota = r_{\mathbf{1}}$), so $b \otimes b = b \otimes \mathrm{Id}$ and hence $b = \mathrm{Id}$. □

Exercise 1.2.5. Verify the assertion in the proof of Proposition 1.2.4 that η maps ι to ι'.

Hint. Use Propositions 1.2.1 and 1.2.2.

The results of this subsection show that a monoidal category can be alternatively defined as follows:

Definition 1.2.6. A monoidal category is a sextuple $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ satisfying the pentagon axiom (1.1.2) and the triangle axiom (1.2.1).

This definition is perhaps more traditional than Definition 1.1.1, but Definition 1.1.1 is simpler. Besides, Proposition 1.2.4 implies that for a triple $(\mathcal{C}, \otimes, a)$ satisfying a pentagon axiom (which should perhaps be called a "semigroup category", as it categorifies the notion of a semigroup), being a monoidal category is a property and not a structure (similarly to how it is for semigroups and monoids).

Furthermore, one can show that the commutativity of the triangles implies that in a monoidal category one can safely identify $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ with X using the unit isomorphisms, and assume that the unit isomorphism are the identities (which we will usually do from now on).[3]

In a sense, all this means that in constructions with monoidal categories, unit objects and isomorphisms always "go for the ride", and one need not worry about them especially seriously. For this reason, below we will typically take less care dealing with them than we have done in this subsection.

Proposition 1.2.7. *(*[SR, 1.3.3.1]*) The monoid* $\mathsf{End}(\mathbf{1})$ *of endomorphisms of the unit object of a monoidal category is commutative.*

Proof. The unit isomorphism $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ induces the isomorphism $\psi : \mathsf{End}(\mathbf{1} \otimes \mathbf{1}) \xrightarrow{\sim} \mathsf{End}(\mathbf{1})$. It is easy to see that $\psi(a \otimes 1) = \psi(1 \otimes a) = a$ for any $a \in \mathsf{End}(\mathbf{1})$. Therefore,

$$ab = \psi((a \otimes 1)(1 \otimes b)) = \psi((1 \otimes b)(a \otimes 1)) = ba, \tag{1.2.9}$$

for any $a, b \in \mathsf{End}(\mathbf{1})$. □

1.3. **First examples of monoidal categories.** Monoidal categories are ubiquitous. You will see one whichever way you look. Here are some examples.

Example 1.3.1. The category **Sets** of sets is a monoidal category, where the tensor product is the Cartesian product and the unit object is a one element set; the structure morphisms a, ι, l, r are obvious. The same holds for the subcategory of finite sets, which will be denoted by Sets [4]. This example can be widely generalized: one can take the

[3] We will return to this issue later when we discuss MacLane's coherence theorem.

[4] Here and below, the absence of a finiteness condition condition is indicated by the boldface font, while its presence is indicated by the Roman font.

category of sets with some structure, such as groups, topological spaces, etc.

Example 1.3.2. Any additive category is monoidal, with $\otimes$ being the direct sum functor $\oplus$, and $\mathbf{1}$ being the zero object.

The remaining examples will be especially important below.

Example 1.3.3. Let k be any field. The category $k-\mathbf{Vec}$ of all $k-$vector spaces is a monoidal category, where $\otimes = \otimes_k$, $\mathbf{1} = k$, and the morphisms a, ι, l, r are the obvious ones. The same is true about the category of finite dimensional vector spaces over k, denoted by $k-\mathrm{Vec}$. We will often drop k from the notation when no confusion is possible.

More generally, if R is a commutative unital ring, then replacing k by R we can define monoidal categories $R-\mathbf{mod}$ of R-modules and $R-\mathrm{mod}$ of R-modules of finite type.

Example 1.3.4. Let G be a group. The category $\mathbf{Rep}_k(G)$ of all representations of G over k is a monoidal category, with $\otimes$ being the tensor product of representations: if for a representation V one denotes by ρ_V the corresponding map $G \to GL(V)$, then

$$\rho_{V\otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

The unit object in this category is the trivial representation $\mathbf{1} = k$. A similar statement holds for the category $\mathrm{Rep}_k(G)$ of finite dimensional representations of G. Again, we will drop the subscript k when no confusion is possible.

Example 1.3.5. Let G be an affine (pro)algebraic group over k. The categories $\mathrm{Rep}(G)$ of all algebraic representations of G over k is a monoidal category (similarly to Example 1.3.4).

Similarly, if $\mathfrak{g}$ is a Lie algebra over k, then the category of its representations $\mathbf{Rep}(\mathfrak{g})$ and the category of its finite dimensional representations $\mathrm{Rep}(\mathfrak{g})$ are monoidal categories: the tensor product is defined by

$$\rho_{V\otimes W}(a) = \rho_V(a) \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \rho_W(a)$$

(where $\rho_Y : \mathfrak{g} \to \mathfrak{gl}(Y)$ is the homomorphism associated to a representation Y of $\mathfrak{g}$), and $\mathbf{1}$ is the 1-dimensional representation with the zero action of $\mathfrak{g}$.

Example 1.3.6. Let G be a monoid (which we will usually take to be a group), and let A be an abelian group (with operation written multiplicatively). Let $\mathcal{C}_G = \mathcal{C}_G(A)$ be the category whose objects δ_g are labeled by elements of G (so there is only one object in each isomorphism class), $\mathsf{Hom}(\delta_{g_1}, \delta_{g_2}) = \emptyset$ if $g_1 \neq g_2$, and $\mathsf{Hom}(\delta_g, \delta_g) = A$, with the functor $\otimes$ defined by $\delta_g \otimes \delta_h = \delta_{gh}$, and the tensor tensor product of morphisms defined by $a \otimes b = ab$. Then $\mathcal{C}_G$ is a monoidal category with the associativity isomorphism being the identity, and $\mathbf{1}$ being the unit element of G. This shows that in a monoidal category, $X \otimes Y$ need not be isomorphic to $Y \otimes X$ (indeed, it suffices to take a non-commutative monoid G).

This example has a "linear" version. Namely, let k be a field, and $k - \mathbf{Vec}_G$ denote the category of G-graded vector spaces over k, i.e. vector spaces V with a decomposition $V = \oplus_{g \in G} V_g$. Morphisms in this category are linear operators which preserve the grading. Define the tensor product on this category by the formula

$$(V \otimes W)_g = \oplus_{x,y \in G: xy=g} V_x \otimes W_y,$$

and the unit object $\mathbf{1}$ by $\mathbf{1}_1 = k$ and $\mathbf{1}_g = 0$ for $g \neq 1$. Then, defining a, ι in an obvious way, we equip $k - \mathbf{Vec}_G$ with the structure of a monoidal category. Similarly one defines the monoidal category $k - \mathrm{Vec}_G$ of finite dimensional G-graded k-vector spaces.

In the category $k - \mathrm{Vec}_G$, we have pairwise non-isomorphic objects δ_g, $g \in G$, defined by the formula $(\delta_g)_x = k$ if $x = g$ and $(\delta_g)_x = 0$ otherwise. For these objects, we have $\delta_g \otimes \delta_h \cong \delta_{gh}$. Thus the category $\mathcal{C}_G(k^\times)$ is a (non-full) monoidal subcategory of $k - \mathrm{Vec}_G$. This subcategory can be viewed as a "basis" of Vec_G (and Vec_G as "the linear span" of $\mathcal{C}_G$), as any object of Vec_G is isomorphic to a direct sum of objects δ_g with nonnegative integer multiplicities.

When no confusion is possible, we will denote the categories $k - \mathrm{Vec}_G$, $k - \mathbf{Vec}_G$ simply by Vec_G, $\mathbf{Vec}_G$.

Example 1.3.7. This is really a generalization of Example 1.3.6, which shows that the associativity isomorphism is not always "the obvious one".

Let G be a group, A an abelian group, and ω be a 3-cocycle of G with values in A. This means that $\omega : G \times G \times G \to A$ is a function satisfying the equation

(1.3.1)
$$\omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2 g_3, g_4)\omega(g_2, g_3, g_4),$$

for all $g_1, g_2, g_3, g_4 \in G$.

Let us define the monoidal category $\mathcal{C}_G^\omega = \mathcal{C}_G^\omega(A)$ as follows. As a category, it is the same as the category $\mathcal{C}_G$ defined above. The bifunctor $\otimes$ and the unit object $(\mathbf{1}, \iota)$ in this category is also the same as those in $\mathcal{C}_G$. The only difference is in the new associativity isomorphism a^ω, which is not "the obvious one" (i.e., the identity) like in $\mathcal{C}_G$, but rather is defined by the formula

$$a^{\omega}_{\delta_g,\delta_h,\delta_m} = \omega(g,h,m) : (\delta_g \otimes \delta_h) \otimes \delta_m \to \delta_g \otimes (\delta_h \otimes \delta_m), \tag{1.3.2}$$

where $g, h, m \in G$.

The fact that $\mathcal{C}_G^{\omega}$ with these structures is indeed a monoidal category follows from the properties of ω. Namely, the pentagon axiom (1.1.2) follows from equation (1.3.1), and the unit axiom is obvious.

Similarly, for a field k, one can define the category $(k-)\mathbf{Vec}_G^{\omega}$, which differs from $\mathbf{Vec}_G$ just by the associativity isomorphism. This is done by extending the associativity isomorphism of $\mathcal{C}_G^{\omega}$ by additivity to arbitrary direct sums of objects δ_g. This category contains a monoidal subcategory Vec_G^{ω} of finite dimensional G-graded vector spaces with associativity defined by ω.

Remark 1.3.8. It is straightforward to verify that the unit morphisms l, r in $\mathbf{Vec}_G^{\omega}$ are given on 1-dimensional spaces by the formulas

$$l_{\delta_g} = \omega(1,1,g)^{-1}\mathrm{Id}_g, \; r_{\delta_g} = \omega(g,1,1)\mathrm{Id}_g,$$

and the triangle axiom says that $\omega(g,1,h) = \omega(g,1,1)\omega(1,1,h)$. Thus, we have $l_X = r_X = \mathrm{Id}$ if and only if

$$\omega(g,1,1) = \omega(1,1,g), \tag{1.3.3}$$

for any $g \in G$ or, equivalently,

$$\omega(g,1,h) = 1, \; g,h \in G. \tag{1.3.4}$$

A cocycle satisfying this condition is said to be *normalized*.

Example 1.3.9. Let $\mathcal{C}$ be a category. Then the category $\mathsf{End}(\mathcal{C})$ of all functors from $\mathcal{C}$ to itself is a monoidal category, where $\otimes$ is given by composition of functors. The associativity isomorphism in this category is the identity. The unit object is the identity functor, and the structure morphisms are obvious. If $\mathcal{C}$ is an abelian category, the same is true about the categories of additive, left exact, right exact, and exact endofunctors of $\mathcal{C}$.

Example 1.3.10. Let A be an associative ring with unit. Then the category $A-\mathbf{bimod}$ of bimodules over A is a monoidal category, with tensor product $\otimes = \otimes_A$, over A. The unit object in this category is the ring A itself (regarded as an A-bimodule).

If A is commutative, this category has a full monoidal subcategory $A-\mathbf{mod}$, consisting of A-modules, regarded as bimodules in which the left and right actions of A coincide. More generally, if X is a scheme, one can define the monoidal category $\mathbf{QCoh}(X)$ of quasicoherent sheaves on X; if X is affine and $A = \mathcal{O}_X$, then $\mathbf{QCoh}(X) = A-\mathbf{mod}$.

Similarly, if A is a finite dimensional algebra, we can define the monoidal category $A-$bimod of finite dimensional A-bimodules. Other similar examples which often arise in geometry are the category $\mathrm{Coh}(X)$ of coherent sheaves on a scheme X, its subcategory $\mathrm{VB}(X)$ of vector bundles (i.e., locally free coherent sheaves) on X, and the category $\mathrm{Loc}(X)$ of locally constant sheaves of finite dimensional k-vector spaces (also called local systems) on any topological space X. All of these are monoidal categories in a natural way.

Example 1.3.11. The category of tangles.

Let $S_{m,n}$ be the disjoint union of m circles $\mathbb{R}/\mathbb{Z}$ and n intervals $[0,1]$. A *tangle* is a piecewise smooth embedding $f : S_{m,n} \to \mathbb{R}^2 \times [0,1]$ such that the boundary maps to the boundary and the interior to the interior. We will abuse the terminology by also using the term "tangle" for the image of f.

Let x, y, z be the Cartesian coordinates on $\mathbb{R}^2 \times [0,1]$. Any tangle has inputs (points of the image of f with $z = 0$) and outputs (points of the image of f with $z = 1$). For any integers $p, q \geq 0$, let $\widetilde{T}_{p,q}$ be the set of all tangles which have p inputs and q outputs, all having a vanishing y-coordinate. Let $T_{p,q}$ be the set of isotopy classes of elements of $\widetilde{T}_{p,q}$; thus, during an isotopy, the inputs and outputs are allowed to move (preserving the condition $y = 0$), but cannot meet each other. We can define a canonical composition map $T_{p,q} \times T_{q,r} \to T_{p,r}$, induced by the concatenation of tangles. Namely, if $s \in T_{p,q}$ and $t \in T_{q,r}$, we pick representatives $\widetilde{s} \in \widetilde{T}_{p,q}, \widetilde{t} \in \widetilde{T}_{q,r}$ such that the inputs of $\widetilde{t}$ coincide with the outputs of $\widetilde{s}$, concatenate them, perform an appropriate reparametrization, and rescale $z \to z/2$. The obtained tangle represents the desired composition ts.

We will now define a monoidal category $\mathcal{T}$ called the category of tangles (see [K, T, BaKi] for more details). The objects of this category are nonnegative integers, and the morphisms are defined by $\mathsf{Hom}_{\mathcal{T}}(p,q) = T_{p,q}$, with composition as above. The identity morphisms are the elements $\mathrm{id}_p \in T_{p,p}$ represented by p vertical intervals and no circles (in particular, if $p = 0$, the identity morphism id_p is the empty tangle).

Now let us define the monoidal structure on the category $\mathcal{T}$. The tensor product of objects is defined by $m \otimes n = m + n$. However, we also need to define the tensor product of morphisms. This tensor product is induced by union of tangles. Namely, if $t_1 \in T_{p_1,q_1}$ and $t_2 \in T_{p_2,q_2}$, we pick representatives $\widetilde{t_1} \in \widetilde{T}_{p_1,q_1}, \widetilde{t_2} \in \widetilde{T}_{p_2,q_2}$ in such a way that any point of $\widetilde{t_1}$ is to the left of any point of $\widetilde{t_2}$ (i.e., has a smaller x-coordinate). Then $t_1 \otimes t_2$ is represented by the tangle $\widetilde{t_1} \cup \widetilde{t_2}$.

We leave it to the reader to check the following:

1. The product $t_1 \otimes t_2$ is well defined, and its definition makes $\otimes$ a bifunctor.

2. There is an obvious associativity isomorphism for $\otimes$, which turns $\mathcal{T}$ into a monoidal category (with unit object being the empty tangle).

1.4. **Monoidal functors, equivalence of monoidal categories.** As we have explained, monoidal categories are a categorification of monoids. Now we pass to categorification of morphisms between monoids, namely *monoidal functors.*

Definition 1.4.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', \iota')$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and $J = \{J_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y) | X, Y \in \mathcal{C}\}$ is a natural isomorphism, such that $F(\mathbf{1})$ is isomorphic to $\mathbf{1}'$. and the diagram

(1.4.1)

$$\begin{array}{ccc} (F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\ {\scriptstyle J_{X,Y} \otimes' \mathrm{Id}_{F(Z)}} \downarrow & & {\scriptstyle \mathrm{Id}_{F(X)} \otimes' J_{Y,Z}} \downarrow \\ F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\ {\scriptstyle J_{X \otimes Y, Z}} \downarrow & & {\scriptstyle J_{X, Y \otimes Z}} \downarrow \\ F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z)) \end{array}$$

is commutative for all $X, Y, Z \in \mathcal{C}$ ("the monoidal structure axiom").

A monoidal functor F is said to be an *equivalence of monoidal categories* if it is an equivalence of ordinary categories.

Remark 1.4.2. It is important to stress that, as seen from this definition, a monoidal functor is not just a functor between monoidal categories, but a functor with an additional structure (the isomorphism J) satisfying a certain equation (the monoidal structure axiom). As we will see later, this equation may have more than one solution, so the same functor can be equipped with different monoidal structures.

It turns out that if F is a monoidal functor, then there is a canonical isomorphism $\varphi : \mathbf{1}' \to F(\mathbf{1})$. This isomorphism is defined by the commutative diagram

$$
\begin{array}{ccc}
\mathbf{1}' \otimes' F(\mathbf{1}) & \xrightarrow{l'_{F(\mathbf{1})}} & F(\mathbf{1}) \\
{\scriptstyle \varphi \otimes' \mathrm{Id}_{F(X)}} \Big\downarrow & & \Big\uparrow {\scriptstyle F(l_{\mathbf{1}})} \\
F(\mathbf{1}) \otimes' F(\mathbf{1}) & \xrightarrow{J_{\mathbf{1},\mathbf{1}}} & F(\mathbf{1} \otimes \mathbf{1})
\end{array} \qquad (1.4.2)
$$

where l, r, l', r' are the unit isomorphisms for $\mathcal{C}, \mathcal{C}'$ defined in Subsection 1.2.

Proposition 1.4.3. *For any monoidal functor $(F, J) : \mathcal{C} \to \mathcal{C}'$, the diagrams*

$$
\begin{array}{ccc}
\mathbf{1}' \otimes' F(X) & \xrightarrow{l'_{F(X)}} & F(X) \\
{\scriptstyle \varphi \otimes' \mathrm{Id}_{F(X)}} \Big\downarrow & & \Big\uparrow {\scriptstyle F(l_X)} \\
F(\mathbf{1}) \otimes' F(X) & \xrightarrow{J_{\mathbf{1},X}} & F(\mathbf{1} \otimes X)
\end{array} \qquad (1.4.3)
$$

and

$$
\begin{array}{ccc}
F(X) \otimes' \mathbf{1}' & \xrightarrow{r'_{F(X)}} & F(X) \\
{\scriptstyle \mathrm{Id}_{F(X)} \otimes' \varphi} \Big\downarrow & & \Big\uparrow {\scriptstyle F(r_X)} \\
F(X) \otimes' F(\mathbf{1}) & \xrightarrow{J_{X,\mathbf{1}}} & F(X \otimes \mathbf{1})
\end{array} \qquad (1.4.4)
$$

are commutative for all $X \in \mathcal{C}$.

Exercise 1.4.4. Prove Proposition 1.4.3.

Proposition 1.4.3 implies that a monoidal functor can be equivalently defined as follows.

Definition 1.4.5. A monoidal functor $\mathcal{C} \to \mathcal{C}'$ is a triple (F, J, φ) which satisfies the monoidal structure axiom and Proposition 1.4.3.

This is a more traditional definition of a monoidal functor.

Remark 1.4.6. It can be seen from the above that for any monoidal functor (F, J) one can safely identify $\mathbf{1}'$ with $F(\mathbf{1})$ using the isomorphism φ, and assume that $F(\mathbf{1}) = \mathbf{1}'$ and $\varphi = \mathrm{Id}$ (similarly to how we have identified $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ with X and assumed that $l_X = r_X = \mathrm{Id}_X$). We will usually do so from now on. Proposition 1.4.3 implies that with these conventions, one has

$$J_{\mathbf{1},X} = J_{X,\mathbf{1}} = \mathrm{Id}_X. \qquad (1.4.5)$$

Remark 1.4.7. It is clear that the composition of monoidal functors is a monoidal functor. Also, the identity functor has a natural structure of a monoidal functor.

1.5. **Morphisms of monoidal functors.** Monoidal functors between two monoidal categories themselves form a category. Namely, one has the following notion of a morphism (or natural transformation) between two monoidal functors.

Definition 1.5.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', \iota')$ be two monoidal categories, and (F^1, J^1), (F^2, J^2) two monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$. A *morphism* (or a *natural transformation*) of monoidal functors $\eta : (F^1, J^1) \to (F^2, J^2)$ is a natural transformation $\eta : F^1 \to F^2$ such that $\eta_{\mathbf{1}}$ is an isomorphism, and the diagram

$$\begin{array}{ccc} F^1(X) \otimes' F^1(Y) & \xrightarrow{J^1_{X,Y}} & F^1(X \otimes Y) \\ {\scriptstyle \eta_X \otimes' \eta_Y}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{X \otimes Y}} \\ F^2(X) \otimes' F^2(Y) & \xrightarrow{J^2_{X,Y}} & F^2(X \otimes Y) \end{array} \tag{1.5.1}$$

is commutative for all $X, Y \in \mathcal{C}$.

Remark 1.5.2. It is easy to show that $\eta_{\mathbf{1}} \circ \varphi^1 = \varphi^2$, so if one makes the convention that $\varphi^i = \mathrm{Id}$, one has $\eta_{\mathbf{1}} = \mathrm{Id}$.

Remark 1.5.3. It is easy to show that if $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of monoidal categories, then there exists a monoidal equivalence $F^{-1} : \mathcal{C}' \to \mathcal{C}$ such that the functors $F \circ F^{-1}$ and $F^{-1} \circ F$ are isomorphic to the identity functor as monoidal functors. Thus, for any monoidal category $\mathcal{C}$, the monoidal auto-equivalences of $\mathcal{C}$ up to isomorphism form a group with respect to composition.

1.6. **Examples of monoidal functors.** Let us now give some examples of monoidal functors and natural transformations.

Example 1.6.1. An important class of examples of monoidal functors is *forgetful functors* (e.g. functors of "forgetting the structure", from the categories of groups, topological spaces, etc., to the category of sets). Such functors have an obvious monoidal structure. An example important in these notes is the forgetful functor $\mathbf{Rep}_G \to \mathbf{Vec}$ from the representation category of a group to the category of vector spaces. More generally, if $H \subset G$ is a subgroup, then we have a forgetful (or restriction) functor $\mathbf{Rep}_G \to \mathbf{Rep}_H$. Still more generally, if $f : H \to G$ is a group homomorphism, then we have the pullback functor $f^* : \mathbf{Rep}_G \to \mathbf{Rep}_H$. All these functors are monoidal.

Example 1.6.2. Let $f : H \to G$ be a homomorphism of groups. Then any H-graded vector space is naturally G-graded (by pushforward of grading). Thus we have a natural monoidal functor $f_* : \mathbf{Vec}_H \to \mathbf{Vec}_G$. If G is the trivial group, then f_* is just the forgetful functor $\mathbf{Vec}_H \to \mathbf{Vec}$.

Example 1.6.3. Let A be a k-algebra with unit, and $\mathcal{C} = A - \mathbf{mod}$ be the category of left A-modules. Then we have a functor $F : A - \mathbf{bimod} \to \mathrm{End}(\mathcal{C})$ given by $F(M) = M \otimes_A$. This functor is naturally monoidal. A similar functor $F : A - \mathrm{bimod} \to \mathsf{End}(\mathcal{C})$. can be defined if A is a finite dimensional k-algebra, and $\mathcal{C} = A - \mathrm{mod}$ is the category of finite dimensional left A-modules.

Proposition 1.6.4. *The functor* $F : A - \mathrm{bimod} \to \mathrm{End}(\mathcal{C})$ *takes values in the full monoidal subcategory* $\mathrm{End}_{re}(\mathcal{C})$ *of right exact endofunctors of* $\mathcal{C}$*, and defines an equivalence between monoidal categories* $A - \mathrm{bimod}$ *and* $\mathrm{End}_{re}(\mathcal{C})$

Proof. The first statement is clear, since the tensor product functor is right exact. To prove the second statement, let us construct the quasi-inverse functor F^{-1}. Let $G \in \mathrm{End}_{re}(\mathcal{C})$. Define $F^{-1}(G)$ by the formula $F^{-1}(G) = G(A)$; this is clearly an A-bimodule, since it is a left A-module with a commuting action $\mathrm{End}_A(A) = A^{op}$ (the opposite algebra). We leave it to the reader to check that the functor F^{-1} is indeed quasi-inverse to F. □

Remark 1.6.5. A similar statement is valid without the finite dimensionality assumption, if one adds the condition that the right exact functors must commute with inductive limits.

Example 1.6.6. Let S be a monoid, and $\mathcal{C} = \mathrm{Vec}_S$, and $\mathsf{Id}_{\mathcal{C}}$ the identity functor of $\mathcal{C}$. It is easy to see that morphisms $\eta : \mathsf{Id}_{\mathcal{C}} \to \mathsf{Id}_{\mathcal{C}}$ correspond to homomorphisms of monoids: $\eta : S \to k$ (where k is equipped with the multiplication operation). In particular, $\eta(s)$ may be 0 for some s, so η does not have to be an isomorphism.

1.7. **Monoidal functors between categories $\mathcal{C}_G^\omega$.** Let G_1, G_2 be groups, A an abelian group, and $\omega_i \in Z^3(G_i, A)$, $i = 1, 2$ be 3-cocycles. Let $\mathcal{C}_i = \mathcal{C}_{G_i}^{\omega_i}$, $i = 1, 2$ (see Example 1.3.7).

Any monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defines, by restriction to simple objects, a group homomorphism $f : G_1 \to G_2$. Using the axiom (1.4.1) of a monoidal functor we see that a monoidal structure on F is given by

$$J_{g,h} = \mu(g,h)\mathrm{Id}_{\delta_{f(gh)}} : F(\delta_g) \otimes F(\delta_h) \xrightarrow{\sim} F(\delta_{gh}),\ g, h \in G_1, \tag{1.7.1}$$

where $\mu : G_1 \times G_1 \to A$ is a function such that

$$\omega_1(g,h,l)\mu(gh,l)\mu(g,h) = \mu(g,hl)\mu(h,l)\omega_2(f(g),f(h),f(l)),$$

for all $g,h,l \in G_1$. That is,

$$f^*\omega_2 = \omega_1\partial_2(\mu), \tag{1.7.2}$$

i.e., ω_1 and $f^*\omega_2$ are cohomologous in $Z^3(G_1,A)$.

Conversely, given a group homomorphism $f : G_1 \to G_2$, a function $\mu : G_1 \times G_1 \to A$ satisfying (1.7.2) gives rise to a monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defined by $F(\delta_g) = \delta_{f(g)}$ with the monoidal structure given by formula (1.7.1). This functor is an equivalence if and only if f is an isomorphism.

To summarize, monoidal functors $\mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$ correspond to pairs (f,μ), where $f : G_1 \to G_2$ is a group homomorphism such that ω_1 and $f^*\omega_2$ are cohomologous, and μ is a function satisfying (1.7.2) (such functions are in a (non-canonical) bijection with A-valued 2-cocycles on G_1). Let $F_{f,\mu}$ denote the corresponding functor.

Let us determine natural monoidal transformations between $F_{f,\mu}$ and $F_{f',\mu'}$. Clearly, such a transformation exists if and only if $f = f'$, is always an isomorphism, and is determined by a collection of morphisms $\eta_g : \delta_{f(g)} \to \delta_{f(g)}$ (i.e., $\eta_g \in A$), satisfying the equation

$$\mu'(g,h)(\eta_g \otimes \eta_h) = \eta_{gh}\mu(g,h) \tag{1.7.3}$$

for all $g,h \in G_1$, i.e.,

$$\mu' = \mu\partial_1(\eta). \tag{1.7.4}$$

Conversely, every function $\eta : G_1 \to A$ satisfying (1.7.4) gives rise to a morphism of monoidal functors $\eta : F_{f,\mu} \to F_{f,\mu'}$ defined as above. Thus, functors $F_{f,\mu}$ and $F_{f',\mu'}$ are isomorphic as monoidal functors if and only if $f = f'$ and μ is cohomologous to μ'.

Thus, we have obtained the following proposition.

Proposition 1.7.1. *(i) The monoidal isomorphisms $F_{f,\mu} \to F_{f,\mu'}$ of monoidal functors $F_{f,\mu_i} : \mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$ form a torsor over the group $H^1(G_1,k^\times) = \mathrm{Hom}(G_1,k^\times)$ of characters of G_1;*

(ii) Given f, the set of μ parametrizing isomorphism classes of $F_{f,\mu}$ is a torsor over $H^2(G_1,k^\times)$;

(iii) The structures of a monoidal category on $(\mathcal{C}_G,\otimes)$ are parametrized by $H^3(G,k^\times)/\mathrm{Out}(G)$, where $\mathrm{Out}(G)$ is the group of outer automorphisms of G. [5]

[5] Recall that the group $\mathrm{Inn}(G)$ of inner automorphisms of a group G acts trivially on $H^*(G,A)$ (for any coefficient group A), and thus the action of the group $\mathrm{Aut}(G)$ on $H^*(G,A)$ factors through $\mathrm{Out}(G)$.

Remark 1.7.2. The same results, including Proposition 1.7.1, are valid if we replace the categories $\mathcal{C}_G^\omega$ by their "linear spans" Vec_G^ω, and require that the monoidal functors we consider are additive. To see this, it is enough to note that by definition, for any morphism η of monoidal functors, $\eta_1 \neq 0$, so equation (1.7.3) (with $h = g^{-1}$) implies that all η_g must be nonzero. Thus, if a morphism $\eta : F_{f,\mu} \to F_{f',\mu'}$ exists, then it is an isomorphism, and we must have $f = f'$.

Remark 1.7.3. The above discussion implies that in the definition of the categories $\mathcal{C}_G^\omega$ and Vec_G^ω, it may be assumed without loss of generality that the cocycle ω is normalized, i.e., $\omega(g, 1, h) = 1$, and thus $l_{\delta_g} = r_{\delta_g} = \mathrm{Id}$ (which is convenient in computations). Indeed, we claim that any 3-cocycle ω is cohomologous to a normalized one. To see this, it is enough to alter ω by dividing it by $\partial_2\mu$, where μ is any 2-cochain such that $\mu(g, 1) = \omega(g, 1, 1)$, and $\mu(1, h) = \omega(1, 1, h)^{-1}$.

Example 1.7.4. Let $G = \mathbb{Z}/n\mathbb{Z}$ where $n > 1$ is an integer, and $k = \mathbb{C}$. Consider the cohomology of $\mathbb{Z}/n\mathbb{Z}$.

Since $H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}) = 0$ for all $i > 0$, writing the long exact sequence of cohomology for the short exact sequence of coefficient groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \longrightarrow 0,$$

we obtain a natural isomorphism $H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \cong H^{i+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$.

It is well known [Br] that the graded ring $H^*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ is $(\mathbb{Z}/n\mathbb{Z})[x]$ where x is a generator in degree 2. Moreover, as a module over $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^\times$, we have $H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = (\mathbb{Z}/n\mathbb{Z})^\vee$. Therefore, using the graded ring structure, we find that $H^{2m}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong H^{2m-1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = ((\mathbb{Z}/n\mathbb{Z})^\vee)^{\otimes m}$ as an $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$-module. In particular, $H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = ((\mathbb{Z}/n\mathbb{Z})^\vee)^{\otimes 2}$.

This consideration shows that if $n = 2$ then the categorification problem has 2 solutions (the cases of trivial and non-trivial cocycle), while if n is a prime greater than 2 then there are 3 solutions: the trivial cocycle, and two non-trivial cocycles corresponding (non-canonically) to quadratic residues and non-residues $\mathsf{mod}\, n$.

Let us give an explicit formula for the 3-cocycles on $\mathbb{Z}/n\mathbb{Z}$. Modulo coboundaries, these cocycles are given by

$$\phi(i, j, k) = \varepsilon^{\frac{si(j+k-(j+k)')}{n}}, \tag{1.7.5}$$

where ε is a primitive nth root of unity, $s \in \mathbb{Z}/n\mathbb{Z}$, and for an integer m we denote by m' the remainder of division of m by n.

Exercise 1.7.5. Show that when s runs over $\mathbb{Z}/n\mathbb{Z}$ this formula defines cocycles representing all cohomology classes in $H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$.

1.8. **MacLane's strictness theorem.** As we have seen above, it is much simpler to work with monoidal categories in which the associativity and unit constrains are the identity maps.

Definition 1.8.1. A monoidal category $\mathcal{C}$ is *strict* if for all objects X, Y, Z in $\mathcal{C}$ one has equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$, and the associativity and unit constraints are the identity maps.

Example 1.8.2. The category $\mathrm{End}(\mathcal{C})$ endofunctors of a category $\mathcal{C}$ is strict.

Example 1.8.3. Let $\overline{\mathrm{Sets}}$ be the category whose objects are nonnegative integers, and $\mathsf{Hom}(m, n)$ is the set of maps from $\{0, ..., m-1\}$ to $\{0, ..., n-1\}$. Define the tensor product functor on objects by $m \otimes n = mn$, and for $f_1 : m_1 \to n_1, f_2 : m_2 \to n_2$, define $f_1 \otimes f_2 : m_1 m_2 \to n_1 n_2$ by

$$(f_1 \otimes f_2)(m_2 x + y) = n_2 f_1(x) + f_2(y), 0 \le x \le m_1 - 1,\ 0 \le y \le m_2 - 1.$$

Then $\overline{\mathrm{Sets}}$ is a strict monoidal category. Moreover, we have a natural inclusion $\overline{\mathrm{Sets}} \hookrightarrow \mathrm{Sets}$, which is obviously a monoidal equivalence.

Example 1.8.4. This is really a linear version of the previous example. Let $k-\overline{\mathrm{Vec}}$ be the category whose objects are nonnegative integers, and $\mathsf{Hom}(m, n)$ is the set of matrices with m columns and n rows over some field k (and the composition of morphisms is the product of matrices). Define the tensor product functor on objects by $m \otimes n = mn$, and for $f_1 : m_1 \to n_1, f_2 : m_2 \to n_2$, define $f_1 \otimes f_2 : m_1 m_2 \to n_1 n_2$ to be the Kronecker product of f_1 and f_2. Then $k - \overline{\mathrm{Vec}}$ is a strict monoidal category. Moreover, we have a natural inclusion $k - \overline{\mathrm{Vec}} \hookrightarrow k - \mathrm{Vec}$, which is obviously a monoidal equivalence.

Similarly, for any group G one can define a strict monoidal category $k - \overline{\mathrm{Vec}}_G$, whose objects are $\mathbb{Z}_+$-valued functions on G with finitely many nonzero values, and which is monoidally equivalent to $k - \mathrm{Vec}_G$. We leave this definition to the reader.

On the other hand, some of the most important monoidal categories, such as **Sets**, **Vec**, $\mathbf{Vec}_G$, Sets, Vec, Vec_G, should be regarded as nonstrict (at least if one defines them in the usual way). It is even more indisputable that the categories $\mathbf{Vec}_G^\omega$, Vec_G^ω for cohomologically nontrivial ω are not strict.

However, the following remarkable theorem of MacLane implies that in practice, one may always assume that a monoidal category is strict.

Theorem 1.8.5. *Any monoidal category is monoidally equivalent to a strict monoidal category.*

Proof. The proof presented below was given in [JS]. We will establish an equivalence between $\mathcal{C}$ and the monoidal category of right $\mathcal{C}$-module endofunctors of $\mathcal{C}$, which we will discuss in more detail later. The non-categorical algebraic counterpart of this result is of course the fact that every monoid M is isomorphic to the monoid consisting of maps from M to itself commuting with the right multiplication.

For a monoidal category $\mathcal{C}$, let $\mathcal{C}'$ be the monoidal category defined as follows. The objects of $\mathcal{C}$ are pairs (F, c) where $F : \mathcal{C} \to \mathcal{C}$ is a functor and

$$c_{X,Y} : F(X) \otimes Y \xrightarrow{\sim} F(X \otimes Y)$$

is a functorial isomorphism, such that the following diagram is commutative for all objects X, Y, Z in $\mathcal{C}$:

(1.8.1)

$$\begin{array}{ccc} & (F(X)\otimes Y)\otimes Z & \\ {\scriptstyle c_{X,Y}\otimes \mathrm{Id}_Z}\swarrow & & \searrow{\scriptstyle a_{F(X),Y,Z}} \\ F(X\otimes Y)\otimes Z & & F(X)\otimes(Y\otimes Z) \\ {\scriptstyle c_{X\otimes Y,Z}}\downarrow & & \downarrow{\scriptstyle c_{X,Y\otimes Z}} \\ F((X\otimes Y)\otimes Z) & \xrightarrow[F(a_{X,Y,Z})]{} & F(X\otimes(Y\otimes Z)). \end{array}$$

A morphism $\theta : (F^1, c^1) \to (F^2, c^2)$ in $\mathcal{C}'$ is a natural transformation $\theta : F^1 \to F^2$ such that the following square commutes for all objects X, Y in $\mathcal{C}$:

(1.8.2)
$$\begin{array}{ccc} F^1(X)\otimes Y & \xrightarrow{c^1_{X,Y}} & F^1(X\otimes Y) \\ {\scriptstyle \theta_X\otimes \mathrm{Id}_Y}\downarrow & & \downarrow{\scriptstyle \theta_{X\otimes Y}} \\ F_2(X)\otimes Y & \xrightarrow[c^2_{X,Y}]{} & F^2(X\otimes Y) \end{array}$$

Composition of morphisms is the vertical composition of natural transformations. The tensor product of objects is given by $(F^1, c^1) \otimes (F^2, c^2) = (F^1F^2, c)$ where c is given by a composition

(1.8.3) $$F^1F^2(X)\otimes Y \xrightarrow{c^1_{F_2(X),Y}} F^1(F^2(X)\otimes Y) \xrightarrow{F_1(c^2_{X,Y})} F^1F^2(X\otimes Y)$$

for all $X, Y \in \mathcal{C}$, and the tensor product of morphisms is the horizontal composition of natural transformations. Thus $\mathcal{C}'$ is a strict monoidal category (the unit object is the identity functor).

Consider now the functor of left multiplication $L : \mathcal{C} \to \mathcal{C}'$ given by

$$L(X) = (X \otimes \bullet,\ a_{X,\bullet,\bullet}),\ L(f) = f \otimes \bullet.$$

Note that the diagram (1.8.1) for $L(X)$ is nothing but the pentagon diagram (1.1.2).

We claim that this functor L is a monoidal equivalence.

First of all, L essentially surjective: it is easy to check that for any $(F, c) \in \mathcal{C}'$, (F, c) is isomorphic to $L(F(\mathbf{1}))$.

Let us now show that L is fully faithful. Let $\theta : L(X) \to L(Y)$ be a morphism in $\mathcal{C}$. Define $f : X \to Y$ to be the composite

$$X \xrightarrow{r_X^{-1}} X \otimes \mathbf{1} \xrightarrow{\theta_{\mathbf{1}}} Y \otimes \mathbf{1} \xrightarrow{r_Y} Y. \tag{1.8.4}$$

We claim that for all Z in $\mathcal{C}$ one has $\theta_Z = f \otimes \mathrm{Id}_Z$ (so that $\theta = L(f)$ and L is full). Indeed, this follows from the commutativity of the diagram (1.8.5)

$$\begin{array}{ccccccc}
X \otimes Z & \xrightarrow{r_X^{-1} \otimes \mathrm{Id}_Z} & (X \otimes \mathbf{1}) \otimes Z & \xrightarrow{a_{X,\mathbf{1},Z}} & X \otimes (\mathbf{1} \otimes Z) & \xrightarrow{\mathrm{Id}_X \otimes l_Z} & X \otimes Z \\
{\scriptstyle f\otimes \mathrm{Id}_Z}\downarrow & & {\scriptstyle \theta_{\mathbf{1}}\otimes Z}\downarrow & & {\scriptstyle \theta_{\mathbf{1}\otimes \mathrm{Id}_Z}}\downarrow & & {\scriptstyle \theta_Z}\downarrow \\
Y \otimes Z & \xrightarrow[r_Y^{-1} \otimes \mathrm{Id}_Z]{} & (Y \otimes \mathbf{1}) \otimes Z & \xrightarrow[a_{Y,\mathbf{1},Z}]{} & Y \otimes (\mathbf{1} \otimes Z) & \xrightarrow[\mathrm{Id}_Y \otimes l_Z]{} & Y \otimes Z,
\end{array}$$

where the rows are the identity morphisms by the triangle axiom (1.2.1), the left square commutes by the definition of f, the right square commutes by naturality of θ, and the central square commutes since θ is a morphism in $\mathcal{C}'$.

Next, if $L(f) = L(g)$ for some morphisms f, g in $\mathcal{C}$ then, in particular $f \otimes \mathrm{Id}_{\mathbf{1}} = g \otimes \mathrm{Id}_{\mathbf{1}}$ so that $f = g$. Thus L is faithful.

Finally, we define a monoidal functor structure $J_{X,Y} : L(X) \circ L(Y) \xrightarrow{\sim} L(X \otimes Y)$ on L by

$$\begin{aligned} J_{X,Y} = a_{X,Y,\bullet}^{-1} &: X \otimes (Y \otimes \bullet), ((\mathrm{Id}_X \otimes a_{Y,\bullet,\bullet}) \circ a_{X,Y\otimes\bullet,\bullet}) \\ &\xrightarrow{\sim} ((X \otimes Y) \otimes \bullet,\ a_{X\otimes Y,\bullet,\bullet}). \end{aligned}$$

The diagram (1.8.2) for the latter natural isomorphism is just the pentagon diagram in $\mathcal{C}$. For the functor L the hexagon diagram (1.4.1) in the definition of a monoidal functor also reduces to the pentagon diagram in $\mathcal{C}$. The theorem is proved. □

Remark 1.8.6. The nontrivial nature of MacLane's strictness theorem is demonstrated by the following instructive example, which shows that even though a monoidal category is always **equivalent** to a strict category, it need **not** be **isomorphic** to one. (By definition, an isomorphism of monoidal categories is a monoidal equivalence which is an isomorphism of categories).

Namely, let $\mathcal{C}$ be the category $\mathcal{C}_G^\omega$. If ω is cohomologically nontrivial, this category is clearly not isomorphic to a strict one. However, by Maclane's strictness theorem, it is equivalent to a strict category $\mathcal{C}'$.

In fact, in this example a strict category $\mathcal{C}'$ monoidally equivalent to $\mathcal{C}$ can be constructed quite explicitly, as follows. Let $\widetilde{G}$ be another group with a surjective homomorphism $f : \widetilde{G} \to G$ such that the 3-cocycle $f^*\omega$ is cohomologically trivial. Such $\widetilde{G}$ always exists, e.g., a free group (recall that the cohomology of a free group in degrees higher than 1 is trivial, see [Br]). Let $\mathcal{C}'$ be the category whose objects δ_g are labeled by elements of $\widetilde{G}$, $\mathsf{Hom}(\delta_g, \delta_h) = A$ if g, h have the same image in G, and $\mathsf{Hom}(\delta_g, \delta_h) = \emptyset$ otherwise. This category has an obvious tensor product, and a monoidal structure defined by the 3-cocycle $f^*\omega$. We have an obvious monoidal functor $F : \mathcal{C}' \to \mathcal{C}$ defined by the homomorphism $\widetilde{G} \to G$, and it is an equivalence, even though not an isomorphism. However, since the cocycle $f^*\omega$ is cohomologically trivial, the category $\mathcal{C}'$ is isomorphic to the same category with the trivial associativity isomorphism, which is strict.

Remark 1.8.7. [6] A category is called *skeletal* if it has only one object in each isomorphism class. The axiom of choice implies that any category is equivalent to a skeletal one. Also, by MacLane's theorem, any monoidal category is monoidally equivalent to a strict one. However, Remark 1.8.6 shows that a monoidal category need not be monoidally equivalent to a category which is skeletal and strict at the same time. Indeed, as we have seen, to make a monoidal category strict, it may be necessary to add new objects to it (which are isomorphic, but not equal to already existing ones). In fact, the desire to avoid adding such objects is the reason why we sometimes use nontrivial associativity isomorphisms, even though MacLane's strictness theorem tells us we don't have to. This also makes precise the sense in which the categories Sets, Vec, Vec_G, are "more strict" than the category Vec_G^ω for cohomologically nontrivial ω. Namely, the first three categories are monoidally equivalent to strict skeletal categories $\overline{\mathrm{Sets}}$, $\overline{\mathrm{Vec}}$, $\overline{\mathrm{Vec}_G}$, while the category Vec_G^ω is not monoidally equivalent to a strict skeletal category.

Exercise 1.8.8. Show that any monoidal category $\mathcal{C}$ is monoidally equivalent to a skeletal monoidal category $\overline{\mathcal{C}}$. Moreover, $\overline{\mathcal{C}}$ can be chosen in such a way that $l_X, r_X = \mathrm{Id}_X$ for all objects $X \in \overline{\mathcal{C}}$.

Hint. Without loss of generality one can assume that $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X$ and $l_X, r_X = \mathrm{Id}_X$ for all objects $X \in \mathcal{C}$. Now in every

[6] This remark is borrowed from the paper [Kup2].

isomorphism class i of objects of $\mathcal{C}$ fix a representative X_i, so that $X_1 = \mathbf{1}$, and for any two classes i, j fix an isomorphism $\mu_{ij} : X_i \otimes X_j \to X_{i \cdot j}$, so that $\mu_{i1} = \mu_{1i} = \mathrm{Id}_{X_i}$. Let $\overline{\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ consisting of the objects X_i, with tensor product defined by $X_i \overline{\otimes} X_j = X_{i \cdot j}$, and with all the structure transported using the isomorphisms μ_{ij}. Then $\overline{\mathcal{C}}$ is the required skeletal category, monoidally equivalent to $\mathcal{C}$.

1.9. **The MacLane coherence theorem.** In a monoidal category, one can form n-fold tensor products of any ordered sequence of objects $X_1, ..., X_n$. Namely, such a product can be attached to any parenthesizing of the expression $X_1 \otimes ... \otimes X_n$, and such products are, in general, distinct objects of $\mathcal{C}$.

However, for $n = 3$, the associativity isomorphism gives a canonical identification of the two possible parenthesizings, $(X_1 \otimes X_2) \otimes X_3$ and $X_1 \otimes (X_2 \otimes X_3)$. An easy combinatorial argument then shows that one can identify any two parenthesized products of $X_1, ..., X_n$, $n \geq 3$, using a chain of associativity isomorphisms.

We would like to say that for this reason we can completely ignore parentheses in computations in any monoidal category, identifying all possible parenthesized products with each other. But this runs into the following problem: for $n \geq 4$ there may be two or more different chains of associativity isomorphisms connecting two different parenthesizings, and a priori it is not clear that they provide the same identification.

Luckily, for $n = 4$, this is settled by the pentagon axiom, which states exactly that the two possible identifications are the same. But what about $n > 4$?

This problem is solved by the following theorem of MacLane, which is the first important result in the theory of monoidal categories.

Theorem 1.9.1. *(MacLane's Coherence Theorem)* [ML] *Let* $X_1, \dots, X_n \in \mathcal{C}$. *Let* P_1, P_2 *be any two parenthesized products of* $X_1, ..., X_n$ *(in this order) with arbitrary insertions of unit objects* $\mathbf{1}$. *Let* $f, g : P_1 \to P_2$ *be two isomorphisms, obtained by composing associativity and unit isomorphisms and their inverses possibly tensored with identity morphisms. Then* $f = g$.

Proof. We derive this theorem as a corollary of the MacLane's strictness Theorem 1.8.5. Let $L : \mathcal{C} \to \mathcal{C}'$ be a monoidal equivalence between $\mathcal{C}$ and a strict monoidal category $\mathcal{C}'$. Consider a diagram in $\mathcal{C}$ representing f and g and apply L to it. Over each arrow of the resulting diagram representing an associativity isomorphism, let us build a rectangle as in (1.4.1), and do similarly for the unit morphisms. This way we obtain a prism one of whose faces consists of identity maps (associativity and unit isomorphisms in $\mathcal{C}'$) and whose sides are commutative. Hence, the other face is commutative as well, i.e., $f = g$. $\square$

As we mentioned, this implies that any two parenthesized products of $X_1, ..., X_n$ with insertions of unit objects are indeed canonically isomorphic, and thus one can safely identify all of them with each other and ignore bracketings in calculations in a monoidal category. We will do so from now on, unless confusion is possible.

1.10. **Rigid monoidal categories.** Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ be a monoidal category, and let X be an object of $\mathcal{C}$. In what follows, we suppress the unit morphisms l, r.

Definition 1.10.1. A *right dual* of an object X in $\mathcal{C}$ is an object X^* in $\mathcal{C}$ equipped with morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$, called the *evaluation* and *coevaluation* morphisms, such that the compositions

(1.10.1)
$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{Id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{Id}_X \otimes \mathrm{ev}_X} X,$$
(1.10.2)
$$X^* \xrightarrow{\mathrm{Id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{Id}_{X^*}} X^*$$

are the identity morphisms.

Definition 1.10.2. A *left dual* of an object X in $\mathcal{C}$ is an object ${}^*X$ in $\mathcal{C}$ equipped with morphisms $\mathrm{ev}'_X : X \otimes {}^*X \to \mathbf{1}$ and $\mathrm{coev}'_X : \mathbf{1} \to {}^*X \otimes X$ such that the compositions

(1.10.3)
$$X \xrightarrow{\mathrm{Id}_X \otimes \mathrm{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{Id}_X} X,$$
(1.10.4)
$${}^*X \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{Id}_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathrm{Id}_{{}^*X} \otimes \mathrm{ev}'_X} X^*$$

are the identity morphisms.

Remark 1.10.3. It is obvious that if X^* is a right dual of an object X then X is a left dual of X^* with $\mathrm{ev}'_{X^*} = \mathrm{ev}_X$ and $\mathrm{coev}'_{X^*} = \mathrm{coev}_X$, and vice versa. Also, in any monoidal category, $\mathbf{1}^* = {}^*\mathbf{1} = \mathbf{1}$ with the structure morphisms ι and ι^{-1}. Also note that changing the order of tensor product switches right duals and left duals, so to any statement about right duals there corresponds a symmetric statement about left duals.

Proposition 1.10.4. *If $X \in \mathcal{C}$ has a right (respectively, left) dual object, then it is unique up to a unique isomorphism.*

Proof. Let X_1^*, X_2^* be two right duals to X. Denote by e_1, c_1, e_2, c_2 the corresponding evaluation and coevaluation morphisms. Then we have a morphism $\alpha : X_1^* \to X_2^*$ defined as the composition

$$X_1^* \xrightarrow{\mathrm{Id}_{X_1^*} \otimes c_2} X_1^* \otimes (X \otimes X_2^*) \xrightarrow{a^{-1}_{X_1^*, X, X_2^*}} (X_1^* \otimes X) \otimes X_2^* \xrightarrow{e_1 \otimes \mathrm{Id}_{X_2^*}} X_2^*.$$

Similarly one defines a morphism $\beta : X_2^* \to X_1^*$. We claim that $\beta \circ \alpha$ and $\alpha \circ \beta$ are the identity morphisms, so α is an isomorphism. Indeed consider the following diagram:

$$\begin{array}{ccccc}
X_1^* & \xrightarrow{\mathsf{Id} \otimes c_1} & X_1^* \otimes X \otimes X_1^* & \overset{\mathsf{Id}}{\searrow} & \\
\downarrow{\scriptstyle \mathsf{Id} \otimes c_2} & & \downarrow{\scriptstyle \mathsf{Id} \otimes c_2 \otimes \mathsf{Id}} & & \\
X_1^* \otimes X \otimes X_2^* & \xrightarrow[\mathsf{Id} \otimes c_1]{} & X_1^* \otimes X \otimes X_2^* \otimes X \otimes X_1^* & \xrightarrow[\mathsf{Id} \otimes e_2 \otimes \mathsf{Id}]{} & X_1^* \otimes X \otimes X_1^* \\
\downarrow{\scriptstyle e_1 \otimes \mathsf{Id}} & & \downarrow{\scriptstyle e_1 \otimes \mathsf{Id}} & & \downarrow{\scriptstyle e_1 \otimes \mathsf{Id}} \\
X_2^* & \xrightarrow[\mathsf{Id} \otimes c_1]{} & X_2^* \otimes X \otimes X_1^* & \xrightarrow[e_2 \otimes \mathsf{Id}]{} & X_1^*.
\end{array}$$

Here we suppress the associativity constraints. It is clear that the three small squares commute. The triangle in the upper right corner commutes by axiom (1.10.1) applied to X_2^*. Hence, the perimeter of the diagram commutes. The composition through the top row is the identity by (1.10.2) applied to X_1^*. The composition through the bottom row is $\beta \circ \alpha$ and so $\beta \circ \alpha = \mathsf{Id}$. The proof of $\alpha \circ \beta = \mathsf{Id}$ is completely similar.

Moreover, it is easy to check that $\alpha : X_1^* \to X_2^*$ is the only isomorphism which preserves the evaluation and coevaluation morphisms. This proves the proposition for right duals. The proof for left duals is similar. □

Exercise 1.10.5. Fill in the details in the proof of Proposition 1.10.4.

If X, Y are objects in $\mathcal{C}$ which have right duals X^*, Y^* and $f : X \to Y$ is a morphism, one defines the *right dual* $f^* : Y^* \to X^*$ of f as the composition

$$\begin{aligned}
(1.10.5) \qquad Y^* &\xrightarrow{\mathrm{Id}_{Y^*} \otimes \mathrm{coev}_X} Y^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{Y^*, X, X^*}} (Y^* \otimes X) \otimes X^* \\
&\xrightarrow{(\mathrm{Id}_{Y^*} \otimes f) \otimes \mathrm{Id}_{X^*}} (Y^* \otimes Y) \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{Id}_{X^*}} X^*.
\end{aligned}$$

Similarly, if X, Y are objects in $\mathcal{C}$ which have left duals ${}^*X, {}^*Y$ and $f : X \to Y$ is a morphism, one defines the *left dual* ${}^*f : {}^*Y \to {}^*X$ of f

as the composition

$$
(1.10.6)\quad {}^*Y \xrightarrow{\mathrm{coev}'_X\otimes \mathrm{Id}_{{}^*Y}} ({}^*X\otimes X)\otimes {}^*Y \xrightarrow{a_{{}^*X,X,{}^*Y}} {}^*X\otimes (X\otimes {}^*Y) \xrightarrow{\mathrm{Id}_{{}^*X}\otimes (f\otimes \mathrm{Id}_{{}^*Y})} {}^*X\otimes (Y\otimes {}^*Y) \xrightarrow{\mathrm{Id}_{{}^*X}\otimes \mathrm{ev}'_Y} {}^*X.
$$

Exercise 1.10.6. Let $\mathcal{C},\mathcal{D}$ be monoidal categories. Suppose

$$
(F,J):\mathcal{C}\to\mathcal{D}
$$

is a monoidal functor with the corresponding isomorphism $\varphi:\mathbf{1}\to F(\mathbf{1})$. Let X be an object in $\mathcal{C}$ with a right dual X^*. Prove that $F(X^*)$ is a right dual of $F(X)$ with the evaluation and coevaluation given by

$$
\begin{aligned}
\mathrm{ev}_{F(X)} &: F(X^*)\otimes F(X) \xrightarrow{J_{X,X^*}} F(X^*\otimes X) \xrightarrow{F(\mathrm{ev}_X)} F(\mathbf{1})=\mathbf{1},\\
\mathrm{coev}_{F(X)} &: \mathbf{1}=F(\mathbf{1}) \xrightarrow{F(\mathrm{coev}_X)} F(X\otimes X^*) \xrightarrow{J^{-1}_{X,X^*}} F(X)\otimes F(X^*).
\end{aligned}
$$

State and prove a similar result for left duals.

Proposition 1.10.7. *Let $\mathcal{C}$ be a monoidal category.*

(i) Let U,V,W be objects in $\mathcal{C}$ admitting right (respectively, left) duals, and let $f:V\to W$, $g:U\to V$ be morphisms in $\mathcal{C}$. Then $(f\circ g)^*=g^*\circ f^*$ (respectively, ${}^*(f\circ g)={}^*g\circ {}^*f$).

(ii) If U,V have right (respectively, left) duals then the object $V^*\otimes U^*$ (respectively, ${}^*V\otimes {}^*U$) has a natural structure of a right (respectively, left) dual to $U\otimes V$.

Exercise 1.10.8. Prove Proposition 1.10.7.

Proposition 1.10.9. *(i) If an object V has a right dual V^* then there are natural adjunction isomorphisms*

$$
(1.10.7)\qquad \mathsf{Hom}(U\otimes V,W) \xrightarrow{\sim} \mathsf{Hom}(U,W\otimes V^*),
$$

$$
(1.10.8)\qquad \mathsf{Hom}(V^*\otimes U,W) \xrightarrow{\sim} \mathsf{Hom}(U,V\otimes W).
$$

Thus, the functor $\bullet\otimes V^*$ is right adjoint to $\bullet\otimes V$ and $V\otimes\bullet$ is right adjoint to $V^*\otimes\bullet$.

(ii) If an object V has a left dual ${}^*V$ then there are natural adjunction isomorphisms

$$
(1.10.9)\qquad \mathsf{Hom}(U\otimes {}^*V,W) \xrightarrow{\sim} \mathsf{Hom}(U,W\otimes V),
$$

$$
(1.10.10)\qquad \mathsf{Hom}(V\otimes U,W) \xrightarrow{\sim} \mathsf{Hom}(U,{}^*V\otimes W).
$$

Thus, the functor $\bullet\otimes V$ is right adjoint to $\bullet\otimes {}^*V$ and ${}^*V\otimes\bullet$ is right adjoint to $V\otimes\bullet$).

Proof. The isomorphism in (1.10.7) is given by

$$f \mapsto (f \otimes \mathrm{Id}_{V^*}) \circ (\mathrm{Id}_U \otimes \mathrm{coev}_V)$$

and has the inverse

$$g \mapsto (\mathrm{Id}_W \otimes \mathrm{ev}_V) \circ (g \otimes \mathrm{Id}_V).$$

The other isomorphisms are similar, and are left to the reader as an exercise. [7] □

Remark 1.10.10. Proposition 1.10.9 provides another proof of Proposition 1.10.4. Namely, setting $U = \mathbf{1}$ and $V = X$ in (1.10.8), we obtain a natural isomorphism $\mathsf{Hom}(X^*, W) \cong \mathsf{Hom}(\mathbf{1}, X \otimes W)$ for any right dual X^* of X. Hence, if Y_1, Y_2 are two such duals then there is a natural isomorphism $\mathsf{Hom}(Y_1, W) \cong \mathsf{Hom}(Y_2, W)$, whence there is a canonical isomorphism $Y_1 \cong Y_2$ by Yoneda's Lemma. The proof for left duals is similar.

Definition 1.10.11. A monoidal category $\mathcal{C}$ is called *rigid* if every object $X \in \mathcal{C}$ has a right dual object and a left dual object.

Example 1.10.12. The category Vec of finite dimensional k-vector spaces is rigid: the right and left dual to a finite dimensional vector space V are its dual space V^*, with the evaluation map $\mathrm{ev}_V : V^* \otimes V \to k$ being the contraction, and the coevaluation map $\mathrm{coev}_V : k \to V \otimes V^*$ being the usual embedding. On the other hand, the category **Vec** of all k-vector spaces is not rigid, since for infinite dimensional spaces there is no coevaluation maps (indeed, suppose that $c : k \to V \otimes Y$ is a coevaluation map, and consider the subspace V' of V spanned by the first component of $c(1)$; this subspace finite dimensional, and yet the composition $V \to V \otimes Y \otimes V \to V$, which is supposed to be the identity map, lands in V' - a contradiction).

Example 1.10.13. The category $\mathrm{Rep}(G)$ of finite dimensional k-representations of a group G is rigid: for a finite dimensional representation V, the (left or right) dual representation V^* is the usual dual space (with the evaluation and coevaluation maps as in Example 1.10.12), and with the G-action given by $\rho_{V^*}(g) = (\rho_V(g)^{-1})^*$. Similarly, the category $\mathrm{Rep}(\mathfrak{g})$ of finite dimensional representations of a Lie algebra $\mathfrak{g}$ is rigid, with $\rho_{V^*}(a) = -\rho_V(a)^*$.

Example 1.10.14. The category Vec_G is rigid if and only if the monoid G is a group; namely, $\delta_g^* = {}^*\delta_g = \delta_{g^{-1}}$ (with the obvious structure maps). More generally, for any group G and 3-cocycle $\omega \in Z^3(G, k^\times)$,

[7] A convenient way to do computations in this and previous Propositions is using the graphical calculus (see [K, Chapter XIV]).

the category Vec_G^ω is rigid. Namely, assume for simplicity that the cocycle ω is normalized (as we know, we can do so without loss of generality). Then we can define duality as above, and normalize the coevaluation morphisms of δ_g to be the identities. The evaluation morphisms will then be defined by the formula $\mathrm{ev}_{\delta_g} = \omega(g, g^{-1}, g)$.

It follows from Proposition 1.10.4 that in a monoidal category $\mathcal{C}$ with right (respectively, left) duals, one can define the (contravariant) *right (*respectively*, left) duality functor* $\mathcal{C} \to \mathcal{C}$ by $X \mapsto X^*$, $f \mapsto f^*$ (respectively, $X \mapsto {}^*X$, $f \mapsto {}^*f$) for every object X and morphism f in $\mathcal{C}$. By Proposition 1.10.7(ii), these functors are anti-monoidal, in the sense that they map the monoidal structure of $\mathcal{C}$ to its opposite; hence the functors $X \to X^{**}$, $X \to {}^{**}X$ are monoidal. Also, it follows from Proposition 1.10.9 that the functors of right and left duality, when they are defined, are fully faithful (it suffices to use (i) for $U = X^*, V = Y, W = \mathbf{1}$).

Moreover, it follows from Remark 1.10.3 that in a rigid monoidal category, the functors of right and left duality are mutually quasi-inverse anti-equivalences of categories (i.e. they are equivalences from $\mathcal{C}$ to its opposite category). This implies that the functors $X \to X^{**}$, $X \to {}^{**}X$ are mutually quasi-inverse monoidal autoequivalences. We will see later in Example 1.27.2 that these autoequivalences may be nontrivial; in particular, it is possible that objects V^* and ${}^*V$ are not isomorphic.

Exercise 1.10.15. Show that if $\mathcal{C}, \mathcal{D}$ are rigid monoidal categories, $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are monoidal functors, and $\eta : F_1 \to F_2$ is a morphism of monoidal functors, then η is an isomorphism.[8]

Exercise 1.10.16. Let A be an algebra. Show that $M \in A - \mathbf{bimod}$ has a left (respectively, right) dual if and only if it is finitely generated projective when considered as a left (respectively, right) A-module. Sinilarly, if A is commutative, $M \in A - \mathbf{mod}$ has a left and right dual if and only if it is finitely generated projective.

1.11. **Invertible objects.** Let $\mathcal{C}$ be a rigid monoidal category.

Definition 1.11.1. An object X in $\mathcal{C}$ is *invertible* if $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ are isomorphisms.

Clearly, this notion categorifies the notion of an invertible element in a monoid.

Example 1.11.2. The objects δ_g in Vec_G^ω are invertible.

[8]As we have seen in Remark 1.6.6, this is false for non-rigid categories.

Proposition 1.11.3. *Let X be an invertible object in $\mathcal{C}$. Then*
(i) ${}^*X \cong X^*$ and X^* is invertible;
(ii) if Y is another invertible object then $X \otimes Y$ is invertible.

Proof. Dualizing coev_X and ev_X we get isomorphisms $X \otimes {}^*X \cong \mathbf{1}$ and ${}^*X \otimes X \cong \mathbf{1}$. Hence ${}^*X \cong {}^*X \otimes X \otimes X^* \cong X^*$. In any rigid category the evaluation and coevaluation morphisms for ${}^*X$ can be defined by $\mathrm{ev}_{{}^*X} := {}^*\mathrm{coev}_X$ and $\mathrm{coev}_{{}^*X} := {}^*\mathrm{ev}_X$, so ${}^*X$ is invertible. The second statement follows from the fact that $\mathrm{ev}_{X\otimes Y}$ can be defined as a composition of ev_X and ev_Y and similarly $\mathrm{coev}_{X\otimes Y}$ can be defined as a composition of coev_Y and coev_X. □

Proposition 1.11.3 implies that invertible objects of $\mathcal{C}$ form a monoidal subcategory $\mathrm{Inv}(\mathcal{C})$ of $\mathcal{C}$.

Example 1.11.4. Gr-categories. Let us classify rigid monoidal categories $\mathcal{C}$ where all objects are invertible and all morphisms are isomorphisms. We may assume that $\mathcal{C}$ is skeletal, i.e. there is only one object in each isomorphism class, and objects form a group G. Also, by Proposition 1.2.7, $\mathsf{End}(\mathbf{1})$ is an abelian group; let us denote it by A. Then for any $g \in G$ we can identify $\mathsf{End}(g)$ with A, by sending $f \in \mathsf{End}(g)$ to $f \otimes \mathrm{Id}_{g^{-1}} \in \mathsf{End}(\mathbf{1}) = A$. Then we have an action of G on A by

$$a \in \mathsf{End}(\mathbf{1}) \mapsto g(a) := \mathrm{Id}_g \otimes a \in \mathsf{End}(g).$$

Let us now consider the associativity isomorphism. It is defined by a function $\omega : G \times G \times G \to A$. The pentagon relation gives

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2g_3, g_4)g_1(\omega(g_2, g_3, g_4)), \tag{1.11.1}$$

for all $g_1, g_2, g_3, g_4 \in G$, which means that ω is a 3-cocycle of G with coefficients in the (generally, nontrivial) G-module A. We see that any such 3-cocycle defines a rigid monoidal category, which we will call $\mathcal{C}_G^\omega(A)$. The analysis of monoidal equivalences between such categories is similar to the case when A is a trivial G-module, and yields that for a given group G and G-module A, equivalence classes of $\mathcal{C}_G^\omega$ are parametrized by $H^3(G, A)/\mathrm{Out}(G)$.

Categories of the form $\mathcal{C}_G^\omega(A)$ are called Gr-categories, and were studied in [Si].

1.12. **Tensor and multitensor categories.** Now we will start considering monoidal structures on abelian categories. For the sake of brevity, we will not recall the basic theory of abelian categories; let us just recall the Freyd-Mitchell theorem stating that abelian categories can be characterized as full subcategories of categories of left modules over rings, which are closed under taking direct sums, as well as kernels, cokernels, and images of morphisms. This allows one to visualize the main concepts of the theory of abelian categories in terms of the classical theory of modules over rings.

Recall that an abelian category $\mathcal{C}$ is said to be k-linear (or defined over k) if for any X, Y in $\mathcal{C}$, $\mathsf{Hom}(X, Y)$ is a k-vector space, and composition of morphisms is bilinear.

Definition 1.12.1. A k-linear abelian category is said to be *locally finite* if it is essentially small[9], and the following two conditions are satisfied:

(i) for any two objects X, Y in $\mathcal{C}$, the space $\mathsf{Hom}(X, Y)$ is finite dimensional;

(ii) every object in $\mathcal{C}$ has finite length.

Almost all abelain categories we will consider will be locally finite.

Proposition 1.12.2. *In a locally finite abelian category* $\mathcal{C}$*,* $\mathsf{Hom}(X, Y) = 0$ *if* X, Y *are simple and non-isomorphic, and* $\mathsf{Hom}(X, X) = k$ *for any simple object* X*.*

Proof. Recall Schur's lemma: if X, Y are simple objects of an abelian category, and $f \in \mathsf{Hom}(X, Y)$, then $f = 0$ or f is an isomorphism. This implies that $\mathsf{Hom}(X, Y) = 0$ if X, Y are simple and non-isomorphic, and $\mathsf{Hom}(X, X)$ is a division algebra; since k is algebraically closed, condition (i) implies that $\mathsf{Hom}(X, X) = k$ for any simple object $X \in \mathcal{C}$. □

Also, the Jordan-Hölder and Krull-Schmidt theorems hold in any locally finite abelian category $\mathcal{C}$.

Definition 1.12.3. Let $\mathcal{C}$ be a locally finite k-linear abelian rigid monoidal category. We will call $\mathcal{C}$ a *multitensor category* over k if the bifunctor $\otimes$ is bilinear on morphisms. If in addition $\mathsf{End}(\mathbf{1}) \cong k$ then we will call $\mathcal{C}$ a *tensor category*.

A *multifusion category* is a semisimple multitensor category with finitely many isomorphism simple objects. A *fusion category* is a semisimple tensor category with finitely many isomorphism simple objects.

Example 1.12.4. The categories Vec of finite dimensional k-vector spaces, $\mathrm{Rep}(G)$ of finite dimensional k-representations of a group G (or algebraic representations of an affine algebraic group G), $\mathrm{Rep}(\mathfrak{g})$ of finite dimensional representations of a Lie algebra $\mathfrak{g}$, and Vec_G^ω of

[9]Recall that a category is called essentially small if its isomorphism classes of objects form a set.

G-graded finite dimensional k-vector spaces with associativity defined by a 3-cocycle ω are tensor categories. If G is a finite group, Rep(G) is a fusion category. In particular, Vec is a fusion category.

Example 1.12.5. Let A be a finite dimensional semisimple algebra over k. Let $A-$bimod be the category of finite dimensional A-bimodules with bimodule tensor product over A, i.e.,

$$(M, N) \mapsto M \otimes_A N.$$

Then $\mathcal{C}$ is a multitensor category with the unit object $\mathbf{1} = A$, the left dual defined by $M \mapsto \mathsf{Hom}({}_AM, {}_AA)$, and the right dual defined by $M \mapsto \mathsf{Hom}(M_A, A_A)$.[10] The category $\mathcal{C}$ is tensor if and only if A is simple, in which case it is equivalent to k − Vec. More generally, if A has n matrix blocks, the category $\mathcal{C}$ can be alternatively described as the category whose objects are n-by-n matrices of vector spaces, $V = (V_{ij})$, and the tensor product is matrix multiplication:

$$(V \otimes W)_{il} = \oplus_{j=1}^{n} V_{ij} \otimes W_{jl}.$$

This category will be denoted by $M_n(\text{Vec})$. It is a multifusion category.

In a similar way, one can define the multitensor category $M_n(\mathcal{C})$ of n-by-n matrices of objects of a given multitensor category $\mathcal{C}$. If $\mathcal{C}$ is a multifusion category, so is $M_n(\mathcal{C})$.

1.13. **Exactness of the tensor product.**

Proposition 1.13.1. *(see* [BaKi, 2.1.8]*) Let $\mathcal{C}$ be a multitensor category. Then the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in both factors (i.e.,* biexact*).*

Proof. The proposition follows from the fact that by Proposition 1.10.9, the functors $V\otimes$ and $\otimes V$ have left and right adjoint functors (the functors of tensoring with the corresponding duals), and any functor between abelian categories which has a left and a right adjoint functor is exact. □

Remark 1.13.2. The proof of Proposition 1.13.1 shows that the biadditivity of the functor $\otimes$ holds automatically in any rigid monoidal abelian category. However, this is not the case for bilinearity of $\otimes$, and thus condition of bilinearity of tensor product in the definition of a multitensor category is not redundant.

This may be illustrated by the following example. Let $\mathcal{C}$ be the category of finite dimensional $\mathbb{C}$-bimodules in which the left and right

[10]Note that if A is a finite dimensional non-semisimple algebra then the category of finite dimensional A-bimodules is not rigid, since the duality functors defined as above do not satisfy rigidity axioms (cf. Exercise 1.10.16).

actions of $\mathbb{R}$ coincide. This category is $\mathbb{C}$-linear abelian; namely, it is semisimple with two simple objects $\mathbb{C}_+ = \mathbf{1}$ and $\mathbb{C}_-$, both equal to $\mathbb{C}$ as a real vector space, with bimodule structures $(a,b)z = azb$ and $(a,b)z = az\bar{b}$, respectively. It is also also rigid monoidal, with $\otimes$ being the tensor product of bimodules. But the tensor product functor is not $\mathbb{C}$-bilinear on morphisms (it is only $\mathbb{R}$-bilinear).

Definition 1.13.3. A *multiring category* over k is a locally finite k-linear abelian monoidal category $\mathcal{C}$ with biexact tensor product. If in addition $\mathsf{End}(\mathbf{1}) = k$, we will call $\mathcal{C}$ a *ring category.*

Thus, the difference between this definition and the definition of a (multi)tensor category is that we don't require the existence of duals, but instead require the biexactness of the tensor product. Note that Proposition 1.13.1 implies that any multitensor category is a multiring category, and any tensor category is a ring category.

Corollary 1.13.4. *For any pair of morphisms f_1, f_2 in a multiring category $\mathcal{C}$ one has* $\mathrm{Im}(f_1 \otimes f_2) = \mathrm{Im}(f_1) \otimes \mathrm{Im}(f_2)$.

Proof. Let I_1, I_2 be the images of f_1, f_2. Then the morphisms $f_i : X_i \to Y_i$, $i = 1, 2$, have decompositions $X_i \to I_i \to Y_i$, where the sequences

$$X_i \to I_i \to 0,\ 0 \to I_i \to Y_i$$

are exact. Tensoring the sequence $X_1 \to I_1 \to 0$ with I_2, by Proposition 1.13.1, we get the exact sequence

$$X_1 \otimes I_2 \to I_1 \otimes I_2 \to 0$$

Tenosring X_1 with the sequence $X_2 \to I_2 \to 0$, we get the exact sequence

$$X_1 \otimes X_2 \to X_1 \otimes I_2 \to 0.$$

Combining these, we get an exact sequence

$$X_1 \otimes X_2 \to I_1 \otimes I_2 \to 0.$$

Arguing similarly, we show that the sequence

$$0 \to I_1 \otimes I_2 \to Y_1 \otimes Y_2$$

is exact. This implies the statement. □

Proposition 1.13.5. *If $\mathcal{C}$ is a multiring category with right duals, then the right dualization functor is exact. The same applies to left duals.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence. We need to show that the sequence $0 \to Z^* \to Y^* \to X^* \to 0$ is exact. Let T be any object of $\mathcal{C}$, and consider the sequence

$$0 \to \mathsf{Hom}(T, Z^*) \to \mathsf{Hom}(T, Y^*) \to \mathsf{Hom}(T, X^*).$$

By Proposition 1.10.9, it can be written as

$$0 \to \mathsf{Hom}(T \otimes Z, \mathbf{1}) \to \mathsf{Hom}(T \otimes Y, \mathbf{1}) \to \mathsf{Hom}(T \otimes X, \mathbf{1}),$$

which is exact, since the sequence

$$T \otimes X \to T \otimes Y \to T \otimes Z \to 0$$

is exact, by the exactness of the functor $T\otimes$. This implies that the sequence $0 \to Z^* \to Y^* \to X^*$ is exact.

Similarly, consider the sequence

$$0 \to \mathsf{Hom}(X^*, T) \to \mathsf{Hom}(Y^*, T) \to \mathsf{Hom}(Z^*, T).$$

By Proposition 1.10.9, it can be written as

$$0 \to \mathsf{Hom}(\mathbf{1}, X \otimes T) \to \mathsf{Hom}(\mathbf{1}, Y \otimes T) \to \mathsf{Hom}(\mathbf{1}, Z \otimes T),$$

which is exact since the sequence

$$0 \to X \otimes T \to Y \otimes T \to Z \otimes T$$

is exact, by the exactness of the functor $\otimes T$. This implies that the sequence $Z^* \to Y^* \to X^* \to 0$ is exact. □

Proposition 1.13.6. *Let P be a projective object in a multiring category $\mathcal{C}$. If $X \in \mathcal{C}$ has a right dual, then the object $P \otimes X$ is projective. Similarly, if $X \in \mathcal{C}$ has a left dual, then the object $X \otimes P$ is projective.*

Proof. In the first case by Proposition 1.10.9 we have $\mathsf{Hom}(P \otimes X, Y) = \mathsf{Hom}(P, Y \otimes X^*)$, which is an exact functor of Y, since the functors $\otimes X^*$ and $\mathsf{Hom}(P, \bullet)$ are exact. So $P \otimes X$ is projective. The second case is similar. □

Corollary 1.13.7. *If $\mathcal{C}$ multiring category with right duals, then $\mathbf{1} \in \mathcal{C}$ is a projective object if and only if $\mathcal{C}$ is semisimple.*

Proof. If $\mathbf{1}$ is projective then for any $X \in \mathcal{C}$, $X \cong \mathbf{1} \otimes X$ is projective. This implies that $\mathcal{C}$ is semisimple. The converse is obvious. □

1.14. **Quasi-tensor and tensor functors.**

Definition 1.14.1. Let $\mathcal{C}$, $\mathcal{D}$ be multiring categories over k, and $F : \mathcal{C} \to \mathcal{D}$ be an exact and faithful functor.

(i) F is said to be a quasi-tensor functor if it is equipped with a functorial isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$, and $F(\mathbf{1}) = \mathbf{1}$.

(ii) A quasi-tensor functor (F, J) is said to be a tensor functor if J is a monoidal structure (i.e., satisfies the monoidal structure axiom).

Example 1.14.2. The functors of Examples 1.6.1,1.6.2 and Subsection 1.7 (for the categories Vec_G^ω) are tensor functors. The identity functor $\mathrm{Vec}_G^{\omega_1} \to Vec_G^{\omega_2}$ for non-cohomologous 3-cocycles ω_1, ω_2 is not a tensor functor, but it can be made quasi-tensor by any choice of J.

1.15. **Semisimplicity of the unit object.**

Theorem 1.15.1. *In any multiring category,* $\mathsf{End}(\mathbf{1})$ *is a semisimple algebra, so it is isomorphic to a direct sum of finitely many copies of* k*.*

Proof. By Proposition 1.2.7, $\mathsf{End}(\mathbf{1})$ is a commutative algebra, so it is sufficient to show that for any $a \in \mathsf{End}(\mathbf{1})$ such that $a^2 = 0$ we have $a = 0$. Let $J = \mathrm{Im}(a)$. Then by Corollary 1.13.4 $J \otimes J = \mathrm{Im}(a \otimes a) = \mathrm{Im}(a^2 \otimes 1) = 0$.

Now let $K = \mathrm{Ker}(a)$. Then by Corollary 1.13.4, $K \otimes J$ is the image of $1 \otimes a$ on $K \otimes \mathbf{1}$. But since $K \otimes \mathbf{1}$ is a subobject of $\mathbf{1} \otimes \mathbf{1}$, this is the same as the image of $a \otimes 1$ on $K \otimes \mathbf{1}$, which is zero. So $K \otimes J = 0$.

Now tensoring the exact sequence $0 \to K \to \mathbf{1} \to J \to 0$ with J, and applying Proposition 1.13.1, we get that $J = 0$, so $a = 0$. □

Let $\{p_i\}_{i \in I}$ be the primitive idempotents of the algebra $\mathsf{End}(\mathbf{1})$. Let $\mathbf{1}_i$ be the image of p_i. Then we have $\mathbf{1} = \oplus_{i \in I} \mathbf{1}_i$.

Corollary 1.15.2. *In any multiring category* $\mathcal{C}$ *the unit object* $\mathbf{1}$ *is isomorphic to a direct sum of pairwise non-isomorphic indecomposable objects:* $\mathbf{1} \cong \oplus_i \mathbf{1}_i$.

Exercise 1.15.3. One has $\mathbf{1}_i \otimes \mathbf{1}_j = 0$ for $i \neq j$. There are canonical isomorphisms $\mathbf{1}_i \otimes \mathbf{1}_i \cong \mathbf{1}_i$, and $\mathbf{1}_i \cong \mathbf{1}_i{}^*$.

Let $\mathcal{C}_{ij} := \mathbf{1}_i \otimes \mathcal{C} \otimes \mathbf{1}_j$.

Definition 1.15.4. The subcategories $\mathcal{C}_{ij}$ will be called the *component* subcategories of $\mathcal{C}$.

Proposition 1.15.5. *Let* $\mathcal{C}$ *be a multiring category.*

(1) $\mathcal{C} = \oplus_{i,j \in I} \mathcal{C}_{ij}$. *Thus every indecomposable object of* $\mathcal{C}$ *belongs to some* $\mathcal{C}_{ij}$.
(2) *The tensor product maps* $\mathcal{C}_{ij} \times \mathcal{C}_{kl}$ *to* $\mathcal{C}_{il}$*, and it is zero unless* $j = k$.
(3) *The categories* $\mathcal{C}_{ii}$ *are ring categories with unit objects* $\mathbf{1}_i$ *(which are tensor categories if* $\mathcal{C}$ *is rigid).*
(3) *The functors of left and right duals, if they are defined, map* $\mathcal{C}_{ij}$ *to* $\mathcal{C}_{ji}$.

Exercise 1.15.6. Prove Proposition 1.15.5.

Proposition 1.15.5 motivates the terms “multiring category” and “multitensor category”, as such a category gives us multiple ring categories, respectively tensor categories $\mathcal{C}_{ii}$.

Remark 1.15.7. Thus, a multiring category may be considered as a 2-category with objects being elements of I, 1-morphisms from j to i forming the category $\mathcal{C}_{ij}$, and 2-morphisms being 1-morphisms in $\mathcal{C}$.

Theorem 1.15.8. *(i) In a ring category with right duals, the unit object* $\mathbf{1}$ *is simple.*

(ii) In a multiring category with right duals, the unit object $\mathbf{1}$ *is semisimple, and is a direct sum of pairwise non-isomorphic simple objects* $\mathbf{1}_i$.

Proof. Clearly, (i) implies (ii) (by applying (i) to the component categories $\mathcal{C}_{ii}$). So it is enough to prove (i).

Let X be a simple subobject of $\mathbf{1}$ (it exists, since $\mathbf{1}$ has finite length). Let

$$0 \longrightarrow X \longrightarrow \mathbf{1} \longrightarrow Y \longrightarrow 0 \tag{1.15.1}$$

be the corresponding exact sequence. By Proposition 1.13.5, the right dualization functor is exact, so we get an exact sequence

$$0 \longrightarrow Y^* \longrightarrow \mathbf{1} \longrightarrow X^* \longrightarrow 0. \tag{1.15.2}$$

Tensoring this sequence with X on the left, we obtain

$$0 \longrightarrow X \otimes Y^* \longrightarrow X \longrightarrow X \otimes X^* \longrightarrow 0, \tag{1.15.3}$$

Since X is simple and $X \otimes X^* \neq 0$ (because the coevaluation morphism is nonzero) we obtain that $X \otimes X^* \cong X$. So we have a surjective composition morphism $\mathbf{1} \to X \otimes X^* \to X$. From this and (1.15.1) we have a nonzero composition morphism $\mathbf{1} \twoheadrightarrow X \hookrightarrow \mathbf{1}$. Since $\mathsf{End}(\mathbf{1}) = k$, this morphism is a nonzero scalar, whence $X = \mathbf{1}$. $\square$

Corollary 1.15.9. *In a ring category with right duals, the evaluation morphisms are surjective and the coevaluation morphisms are injective.*

Exercise 1.15.10. Let $\mathcal{C}$ be a multiring category with right duals. and $X \in \mathcal{C}_{ij}$ and $Y \in \mathcal{C}_{jk}$ be nonzero.

(a) Show that $X \otimes Y \neq 0$.
(b) Deduce that $\text{length}(X \otimes Y) \geq \text{length}(X)\text{length}(Y)$.
(c) Show that if $\mathcal{C}$ is a ring category with right duals then an invertible object in $\mathcal{C}$ is simple.
(d) Let X be an object in a multiring category with right duals such that $X \otimes X^* \cong \mathbf{1}$. Show that X is invertible.

Example 1.15.11. An example of a ring category where the unit object is not simple is the category $\mathcal{C}$ of finite dimensional representations of the quiver of type A_2. Such representations are triples (V, W, A), where V, W are finite dimensional vector spaces, and $A : V \to W$ is a linear operator. The tensor product on such triples is defined by the formula

$$(V, W, A) \otimes (V', W', A') = (V \otimes V', W \otimes W', A \otimes A'),$$

with obvious associativity isomorphisms, and the unit object (k, k, Id). Of course, this category has neither right nor left duals.

1.16. **Grothendieck rings.** Let $\mathcal{C}$ be a locally finite abelian category over k. If X and Y are objects in $\mathcal{C}$ such that Y is simple then we denote by $[X : Y]$ the multiplicity of Y in the Jordan-Hölder composition series of X.

Recall that the Grothendieck group $\mathsf{Gr}(\mathcal{C})$ is the free abelian group generated by isomorphism classes X_i, $i \in I$ of simple objects in $\mathcal{C}$, and that to every object X in $\mathcal{C}$ we can canonically associate its class $[X] \in \mathsf{Gr}(\mathcal{C})$ given by the formula $[X] = \sum_i [X : X_i] X_i$. It is obvious that if

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

then $[Y] = [X] + [Z]$. When no confusion is possible, we will write X instead of $[X]$.

Now let $\mathcal{C}$ be a multiring category. The tensor product on $\mathcal{C}$ induces a natural multiplication on $\mathsf{Gr}(\mathcal{C})$ defined by the formula

$$X_i X_j := [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k] X_k.$$

Lemma 1.16.1. *The above multiplication on* $\mathsf{Gr}(\mathcal{C})$ *is associative.*

Proof. Since the tensor product functor is exact,

$$[(X_i \otimes X_j) \otimes X_p : X_l] = \sum_k [X_i \otimes X_j : X_k][X_k \otimes X_p : X_l].$$

On the other hand,

$$[X_i \otimes (X_j \otimes X_p) : X_l] = \sum_k [X_j \otimes X_p : X_k][X_i \otimes X_k : X_l].$$

Thus the associativity of the multiplication follows from the isomorphism $(X_i \otimes X_j) \otimes X_p \cong X_i \otimes (X_j \otimes X_p)$. □

Thus $\mathsf{Gr}(\mathcal{C})$ is an associative ring with the unit $\mathbf{1}$. It is called the *Grothendieck ring* of $\mathcal{C}$.

The following proposition is obvious.

Proposition 1.16.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be multiring categories and $F : \mathcal{C} \to \mathcal{D}$ be a quasi-tensor functor. Then F defines a homomorphism of unital rings* $[F] : \mathsf{Gr}(\mathcal{C}) \to \mathsf{Gr}(\mathcal{D})$.

Thus, we see that (multi)ring categories categorify rings (which justifies the terminology), while quasi-tensor (in particular, tensor) functors between them categorify unital ring homomorphisms. Note that Proposition 1.15.5 may be regarded as a categorical analog of the Peirce decomposition in classical algebra.

1.17. **Groupoids.** The most basic examples of multitensor categories arise from finite groupoids. Recall that a *groupoid* is a small category where all morphisms are isomorphisms. Thus a groupoid $\mathcal{G}$ entails a set X of objects of $\mathcal{G}$ and a set G of morphisms of $\mathcal{G}$, the source and target maps $s, t : G \to X$, the composition map $\mu : G \times_X G \to G$ (where the fibered product is defined using s in the first factor and using t in the second factor), the unit morphism map $u : X \to G$, and the inversion map $i : G \to G$ satisfying certain natural axioms, see e.g. [Ren] for more details.

Here are some examples of groupoids.

(1) Any group G is a groupoid $\mathcal{G}$ with a single object whose set of morphisms to itself is G.
(2) Let X be a set and let $G = X \times X$. Then the *product groupoid* $\mathcal{G}(X) := (X, G)$ is a groupoid in which s is the first projection, t is the second projection, u is the diagonal map, and i is the permutation of factors. In this groupoid for any $x, y \in X$ there is a unique morphism from x to y.
(3) A more interesting example is the *transformation groupoid* $T(G, X)$ arising from the action of a group G on a set X. The set of objects of $T(G, X)$ is X, and arrows correspond to triples (g, x, y) where $y = gx$ with an obvious composition law. In other words, the set of morphisms is $G \times X$ and $s(g, x) = x$, $t(g, x) = gx$, $u(x) = (1, x)$, $i(g, x) = (g^{-1}, gx)$.

Let $\mathcal{G} = (X, G, \mu, s, t, u, i)$ be a finite groupoid (i.e., G is finite) and let $\mathcal{C}(\mathcal{G})$ be the category of finite dimensional vector spaces graded by the set G of morphisms of $\mathcal{G}$, i.e., vector spaces of the form $V = \oplus_{g\in G} V_g$. Introduce a tensor product on $\mathcal{C}(\mathcal{G})$ by the formula

$$(V \otimes W)_g = \bigoplus_{(g_1,g_2):g_1g_2=g} V_{g_1} \otimes W_{g_2}. \tag{1.17.1}$$

Then $\mathcal{C}(\mathcal{G})$ is a multitensor category. The unit object is $\mathbf{1} = \oplus_{x\in X} \mathbf{1}_x$, where $\mathbf{1}_x$ is a 1-dimensional vector space which sits in degree id_x in G. The left and right duals are defined by $(V^*)_g = ({}^*V)_g = V_{g^{-1}}$.

We invite the reader to check that the component subcategories $\mathcal{C}(\mathcal{G})_{xy}$ are the categories of vector spaces graded by $\mathsf{Mor}(y, x)$.

We see that $\mathcal{C}(\mathcal{G})$ is a tensor category if and only if $\mathcal{G}$ is a group, which is the case of Vec_G already considered in Example 1.3.6. Note also that if $X = \{1, ..., n\}$ then $\mathcal{C}(\mathcal{G}(X))$ is naturally equivalent to $M_n(\mathrm{Vec})$.

Exercise 1.17.1. Let C_i be isomorphism classes of objects in a finite groupoid $\mathcal{G}$, $n_i = |C_i|$, $x_i \in C_i$ be representatives of C_i, and $G_i = \mathrm{Aut}(x_i)$ be the corresponding automorphism groups. Show that $\mathcal{C}(\mathcal{G})$ is (non-canonically) monoidally equivalent to $\oplus_i M_{n_i}(\mathrm{Vec}_{G_i})$.

Remark 1.17.2. The finite length condition in Definition 1.12.3 is not superfluous: there exists a rigid monoidal k-linear abelian category with bilinear tensor product which contains objects of infinite length. An example of such a category is the category $\mathcal{C}$ of Jacobi matrices of finite dimensional vector spaces. Namely, the objects of $\mathcal{C}$ are semi-infinite matrices $V = \{V_{ij}\}_{ij \in \mathbb{Z}_+}$ of finite dimensional vector spaces V_{ij} with finitely many non-zero diagonals, and morphisms are matrices of linear maps. The tensor product in this category is defined by the formula

$$(V \otimes W)_{il} = \sum_j V_{ij} \otimes W_{jl}, \tag{1.17.2}$$

and the unit object $\mathbf{1}$ is defined by the condition $\mathbf{1}_{ij} = k^{\delta_{ij}}$. The left and right duality functors coincide and are given by the formula

$$(V^*)_{ij} = (V_{ji})^*. \tag{1.17.3}$$

The evaluation map is the direct sum of the canonical maps $V_{ij}^* \otimes V_{ij} \to \mathbf{1}_{jj}$, and the coevaluation map is a direct sum of the canonical maps $\mathbf{1}_{ii} \to V_{ij} \otimes V_{ij}^*$.

Note that the category $\mathcal{C}$ is a subcategory of the category $\mathcal{C}'$ of $\mathcal{G}(\mathbb{Z}_+)$-graded vector spaces with finite dimensional homogeneous components. Note also that the category $\mathcal{C}'$ is not closed under the tensor product defined by (1.17.2) but the category $\mathcal{C}$ is.

Exercise 1.17.3.
(1) Show that if X is a finite set then the group of invertible objects of the category $\mathcal{C}(\mathcal{G}(X))$ is isomorphic to $\mathsf{Aut}(X)$.
(2) Let $\mathcal{C}$ be the category of Jacobi matrices of vector spaces from Example 1.17.2. Show that the statement Exercise 1.15.10(d) fails for $\mathcal{C}$. Thus the finite length condition is important in Exercise 1.15.10.

1.18. **Finite abelian categories and exact faithful functors.**

Definition 1.18.1. A k-linear abelian category $\mathcal{C}$ is said to be *finite* if it is equivalent to the category $A-$ mod of finite dimensional modules over a finite dimensional k-algebra A.

Of course, the algebra A is not canonically attached to the category $\mathcal{C}$; rather, $\mathcal{C}$ determines the Morita equivalence class of A. For this reason, it is often better to use the following "intrinsic" definition, which is well known to be equivalent to Definition 1.18.1:

Definition 1.18.2. A k-linear abelian category $\mathcal{C}$ is *finite* if
(i) $\mathcal{C}$ has finite dimensional spaces of morphisms;
(ii) every object of $\mathcal{C}$ has finite length;
(iii) every simple object of $\mathcal{C}$ has a projective cover; and
(iv) there are finitely many isomorphism classes of simple objects.

Note that the first two conditions are the requirement that $\mathcal{C}$ be locally finite.

Indeed, it is clear that if A is a finite dimensional algebra then $A-$ mod clearly satisfies (i)-(iv), and conversely, if $\mathcal{C}$ satisfies (i)-(iv), then one can take $A = \mathsf{End}(P)^{op}$, where P is a projective generator of $\mathcal{C}$ (e.g., $P = \oplus_{i=1}^{n} P_i$, where P_i are projective covers of all the simple objects X_i).

A projective generator P of $\mathcal{C}$ represents a functor $F = F_P : \mathcal{C} \to \mathrm{Vec}$ from $\mathcal{C}$ to the category of finite dimensional k-vector spaces, given by the formula $F(X) = \mathsf{Hom}(P, X)$. The condition that P is projective translates into the exactness property of F, and the condition that P is a generator (i.e., covers any simple object) translates into the property that F is faithful (does not kill nonzero objects or morphisms). Moreover, the algebra $A = \mathsf{End}(P)^{op}$ can be alternatively defined as $\mathsf{End}(F)$, the algebra of functorial endomorphisms of F. Conversely, it is well known (and easy to show) that any exact faithful functor $F : \mathcal{C} \to \mathrm{Vec}$ is represented by a unique (up to a unique isomorphism) projective generator P.

Now let $\mathcal{C}$ be a finite k-linear abelian category, and F_1, $F_2 : \mathcal{C} \to \mathrm{Vec}$ be two exact faithful functors. Define the functor $F_1 \otimes F_2 : \mathcal{C} \times \mathcal{C} \to \mathrm{Vec}$ by $(F_1 \otimes F_2)(X, Y) := F_1(X) \otimes F_2(Y)$.

Proposition 1.18.3. *There is a canonical algebra isomorphism* $\alpha_{F_1,F_2} : \mathsf{End}(F_1) \otimes \mathsf{End}(F_2) \cong \mathsf{End}(F_1 \otimes F_2)$ *given by*

$$\alpha_{F_1,F_2}(\eta_1 \otimes \eta_2)|_{F_1(X)\otimes F_2(Y)} := \eta_1|_{F_1(X)} \otimes \eta_2|_{F_2(Y)},$$

where $\eta_i \in \mathsf{End}(F_i)$, $i = 1, 2$.

Exercise 1.18.4. Prove Proposition 1.18.3.

1.19. **Fiber functors.** Let $\mathcal{C}$ be a k-linear abelian monoidal category.

Definition 1.19.1. A *quasi-fiber functor* on $\mathcal{C}$ is an exact faithful functor $F : \mathcal{C} \to \mathrm{Vec}$ from $\mathcal{C}$ to the category of finite dimensional k-vector spaces, such that $F(\mathbf{1}) = k$, equipped with an isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$. If in addition J is a monoidal structure (i.e. satisfies the monoidal structure axiom), one says that F is a *fiber functor*.

Example 1.19.2. The forgetful functors $\mathrm{Vec}_G \to \mathrm{Vec}$, $\mathrm{Rep}(G) \to \mathrm{Vec}$ are naturally fiber functors, while the forgetful functor $\mathrm{Vec}_G^\omega \to \mathrm{Vec}$ is quasi-fiber, for any choice of the isomorphism J (we have seen that if ω is cohomologically nontrivial, then Vec_G^ω does not admit a fiber functor). Also, the functor $\mathrm{Loc}(X) \to \mathrm{Vec}$ on the category of local systems on a connected topological space X which attaches to a local system E its fiber E_x at a point $x \in X$ is a fiber functor, which justifies the terminology. (Note that if X is Hausdorff, then this functor can be identified with the abovementioned forgetful functor $\mathrm{Rep}(\pi_1(X, x)) \to \mathrm{Vec}$).

Exercise 1.19.3. Show that if an abelian monoidal category $\mathcal{C}$ admits a quasi-fiber functor, then it is a ring category, in which the object $\mathbf{1}$ is simple. So if in addition $\mathcal{C}$ is rigid, then it is a tensor category.

1.20. **Coalgebras.**

Definition 1.20.1. A *coalgebra* (with counit) over a field k is a k-vector space C together with a comultiplicaton (or coproduct) $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$ such that

(i) Δ is coassociative, i.e.,

$$(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$$

as maps $C \to C^{\otimes 3}$;

(ii) one has

$$(\varepsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \varepsilon) \circ \Delta = \mathrm{Id}$$

as maps $C \to C$ (the "counit axiom").

Definition 1.20.2. A left comodule over a coalgebra C is a vector space M together with a linear map $\pi : M \to C \otimes M$ (called the coaction map), such that for any $m \in M$, one has

$$(\Delta \otimes \mathrm{Id})(\pi(m)) = (\mathrm{Id} \otimes \pi)(\pi(m)), \ (\varepsilon \otimes \mathrm{Id})(\pi(m)) = m.$$

Similarly, a right comodule over C is a vector space M together with a linear map $\pi : M \to M \otimes C$, such that for any $m \in M$, one has

$$(\pi \otimes \mathrm{Id})(\pi(m)) = (\mathrm{Id} \otimes \Delta)(\pi(m)), \ (\mathrm{Id} \otimes \varepsilon)(\pi(m)) = m.$$

For example, C is a left and right comodule with $\pi = \Delta$, and so is k, with $\pi = \varepsilon$.

Exercise 1.20.3. (i) Show that if C is a coalgebra then C^* is an algebra, and if A is a finite dimensional algebra then A^* is a coalgebra.

(ii) Show that for any coalgebra C, any (left or right) C-comodule M is a (respectively, right or left) C^*-module, and the converse is true if C is finite dimensional.

Exercise 1.20.4. (i) Show that any coalgebra C is a sum of finite dimensional subcoalgebras.

Hint. Let $c \in C$, and let

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(c) = (\mathrm{Id} \otimes \Delta) \circ \Delta(c) = \sum_i c_i^1 \otimes c_i^2 \otimes c_i^3.$$

Show that $\mathrm{span}(c_i^2)$ is a subcoalgebra of C containing c.

(ii) Show that any C-comodule is a sum of finite dimensional subcomodules.

1.21. **Bialgebras.** Let $\mathcal{C}$ be a finite monoidal category, and $(F, J) : \mathcal{C} \to \mathrm{Vec}$ be a fiber functor. Consider the algebra $H := \mathsf{End}(F)$. This algebra has two additional structures: the comultiplication $\Delta : H \to H \otimes H$ and the counit $\varepsilon : H \to k$. Namely, the comultiplication is defined by the formula

$$\Delta(a) = \alpha_{F,F}^{-1}(\widetilde{\Delta}(a)),$$

where $\widetilde{\Delta}(a) \in \mathsf{End}(F \otimes F)$ is given by

$$\widetilde{\Delta}(a)_{X,Y} = J_{X,Y}^{-1} a_{X \otimes Y} J_{X,Y},$$

and the counit is defined by the formula

$$\varepsilon(a) = a_{\mathbf{1}} \in k.$$

Theorem 1.21.1. *(i) The algebra H is a coalgebra with comultiplication Δ and counit ε.*

(ii) The maps Δ and ε are unital algebra homomorphisms.

Proof. The coassociativity of Δ follows form axiom (1.4.1) of a monoidal functor. The counit axiom follows from (1.4.3) and (1.4.4). Finally, observe that for all $\eta, \nu \in \mathsf{End}(F)$ the images under $\alpha_{F,F}$ of both $\Delta(\eta)\Delta(\nu)$ and $\Delta(\eta\nu)$ have components $J_{X,Y}^{-1}(\eta\nu)_{X \otimes Y} J_{X,Y}$; hence, Δ is an algebra homomorphism (which is obviously unital). The fact that ε is a unital algebra homomorphism is clear. $\square$

Definition 1.21.2. An algebra H equipped with a comultiplication Δ and a counit ε satisfying properties (i),(ii) of Theorem 1.21.1 is called a *bialgebra.*

Thus, Theorem 1.21.1 claims that the algebra $H = \mathsf{End}(F)$ has a natural structure of a bialgebra.

Now let H be any bialgebra (not necessarily finite dimensional). Then the category $\mathbf{Rep}(H)$ of representations (i.e., left modules) of H and its subcategory $\mathrm{Rep}(H)$ of finite dimensional representations of H are naturally monoidal categories (and the same applies to right modules). Indeed, one can define the tensor product of two H-modules X, Y to be the usual tensor product of vector spaces $X \otimes Y$, with the action of H defined by the formula

$$\rho_{X\otimes Y}(a) = (\rho_X \otimes \rho_Y)(\Delta(a)),\ a \in H$$

(where $\rho_X : H \to \mathsf{End}(X), \rho_Y : H \to \mathsf{End}(Y)$), the associativity isomorphism to be the obvious one, and the unit object to be the 1-dimensional space k with the action of H given by the counit, $a \to \varepsilon(a)$. Moreover, the forgetful functor $\mathrm{Forget} : \mathrm{Rep}(H) \to \mathrm{Vec}$ is a fiber functor.

Thus we see that one has the following theorem.

Theorem 1.21.3. *The assignments* $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$*,* $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ *are mutually inverse bijections between*

1) finite abelian k-linear monoidal categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors;

2) finite dimensional bialgebras H over k up to isomorphism.

Proof. Straightforward from the above. □

Theorem 1.21.3 is called *the reconstruction theorem for finite dimensional bialgebras* (as it reconstructs the bialgebra H from the category of its modules using a fiber functor).

Exercise 1.21.4. Show that the axioms of a bialgebra are self-dual in the following sense: if H is a finite dimensional bialgebra with multiplication $\mu : H \otimes H \to H$, unit $i : k \to H$, comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to k$, then H^* is also a bialgebra, with the multiplication Δ^*, unit ε^*, comultiplication μ^*, and counit i^*.

Exercise 1.21.5. (i) Let G be a finite monoid, and $\mathcal{C} = \mathrm{Vec}_G$. Let $F : \mathcal{C} \to \mathrm{Vec}$ be the forgetful functor. Show that $H = \mathsf{End}(F)$ is the bialgebra $\mathrm{Fun}(G, k)$ of k-valued functions on G, with comultiplication $\Delta(f)(x, y) = f(xy)$ (where we identify $H \otimes H$ with $\mathrm{Fun}(G \times G, k)$), and counit $\varepsilon(f) = f(1)$.

(ii) Show that $\mathrm{Fun}(G,k)^* = k[G]$, the monoid algebra of G (with basis $x \in G$ and product $x \cdot y = xy$), with coproduct $\Delta(x) = x \otimes x$, and counit $\varepsilon(x) = 1$, $x \in G$. Note that the bialgebra $k[G]$ may be defined for any G (not necessarily finite).

Exercise 1.21.6. Let H be a k-algebra, $\mathcal{C} = H-\mathbf{mod}$ be the category of H-modules, and $F : \mathcal{C} \to \mathbf{Vec}$ be the forgetful functor (we don't assume finite dimensionality). Assume that $\mathcal{C}$ is monoidal, and F is given a monoidal structure J. Show that this endows H with the structure of a bialgebra, such that (F, J) defines a monoidal equivalence $\mathcal{C} \to \mathbf{Rep}(H)$.

Note that not only modules, but also comodules over a bialgebra H form a monoidal category. Indeed, for a finite dimensional bialgebra, this is clear, as right (respectively, left) modules over H is the same thing as left (respectively, right) comodules over H^*. In general, if X, Y are, say, right H-comodules, then the right comodule $X \otimes Y$ is the usual tensor product of X, Y with the coaction map defined as follows: if $x \in X, y \in Y$, $\pi(x) = \sum x_i \otimes a_i$, $\pi(y) = \sum y_j \otimes b_j$, then

$$\pi_{X \otimes Y}(x \otimes y) = \sum x_i \otimes y_j \otimes a_i b_j.$$

For a bialgebra H, the monoidal category of right H-comodules will be denoted by $H - \mathbf{comod}$, and the subcategory of finite dimensional comodules by $H - \mathrm{comod}$.

1.22. **Hopf algebras.** Let us now consider the additional structure on the bialgebra $H = \mathsf{End}(F)$ from the previous subsection in the case when the category $\mathcal{C}$ has right duals. In this case, one can define a linear map $S : H \to H$ by the formula

$$S(a)_X = a^*_{X^*},$$

where we use the natural identification of $F(X)^*$ with $F(X^*)$.

Proposition 1.22.1. *("the antipode axiom") Let $\mu : H \otimes H \to H$ and $i : k \to H$ be the multiplication and the unit maps of H. Then*

$$\mu \circ (\mathrm{Id} \otimes S) \circ \Delta = i \circ \varepsilon = \mu \circ (S \otimes \mathrm{Id}) \circ \Delta$$

as maps $H \to H$.

Proof. For any $b \in \mathsf{End}(F \otimes F)$ the linear map $\mu \circ (\mathrm{Id} \otimes S)(\alpha^{-1}_{F,F}(b))_X$, $X \in \mathcal{C}$ is given by
(1.22.1)

$$F(X) \xrightarrow{\mathrm{coev}_{F(X)}} F(X) \otimes F(X)^* \otimes F(X) \xrightarrow{b_{X,X^*}} F(X) \otimes F(X)^* \otimes F(X) \xrightarrow{\mathrm{ev}_{F(X)}} F(X),$$

where we suppress the identity isomorphisms, the associativity constraint, and the isomorphism $F(X)^* \cong F(X^*)$. Indeed, it suffices to check (1.22.1) for $b = \eta \otimes \nu$, where $\eta, \nu \in H$, which is straightforward.

Now the first equality of the proposition follows from the commutativity of the diagram

(1.22.2)
$$\begin{array}{ccc}
F(X) & \xrightarrow{\mathrm{coev}_{F(X)}} & F(X) \otimes F(X)^* \otimes F(X) \\
\Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle J_{X,X^*}} \\
F(X) & \xrightarrow{F(\mathrm{coev}_X)} & F(X \otimes X^*) \otimes F(X) \\
\Big\downarrow{\scriptstyle \eta_{\mathbf{1}}} & & \Big\downarrow{\scriptstyle \eta_{X \otimes X^*}} \\
F(X) & \xrightarrow{F(\mathrm{coev}_X)} & F(X \otimes X^*) \otimes F(X) \\
\Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle J^{-1}_{X,X^*}} \\
F(X) & \xleftarrow{\mathrm{ev}_{F(X)}} & F(X) \otimes F(X)^* \otimes F(X),
\end{array}$$

for any $\eta \in \mathsf{End}(F)$.

Namely, the commutativity of the upper and the lower square follows from the fact that upon identification of $F(X)^*$ with $F(X^*)$, the morphisms $\mathrm{ev}_{F(X)}$ and $\mathrm{coev}_{F(X)}$ are given by the diagrams of Exercise 1.10.6. The middle square commutes by the naturality of η. The composition of left vertical arrows gives $\varepsilon(\eta)\mathrm{Id}_{F(X)}$, while the composition of the top, right, and bottom arrows gives $\mu \circ (\mathrm{Id} \otimes S) \circ \Delta(\eta)$.

The second equality is proved similarly. $\square$

Definition 1.22.2. An *antipode* on a bialgebra H is a linear map $S : H \to H$ which satisfies the equalities of Proposition 1.22.1.

Exercise 1.22.3. Show that the antipode axiom is self-dual in the following sense: if H is a finite dimensional bialgebra with antipode S_H, then the bialgebra H^* also admits an antipode $S_{H^*} = S_H^*$.

The following is a "linear algebra" analog of the fact that the right dual, when it exists, is unique up to a unique isomorphism.

Proposition 1.22.4. *An antipode on a bialgebra H is unique if exists.*

Proof. The proof essentially repeats the proof of uniqueness of right dual. Let S, S' be two antipodes for H. Then using the antipode properties of S, S', associativity of μ, and coassociativity of Δ, we get

$$S = \mu \circ (S \otimes [\mu \circ (\mathrm{Id} \otimes S') \circ \Delta]) \circ \Delta =$$
$$\mu \circ (\mathrm{Id} \otimes \mu) \circ (S \otimes \mathrm{Id} \otimes S') \circ (\mathrm{Id} \otimes \Delta) \circ \Delta =$$
$$\mu \circ (\mu \otimes \mathrm{Id}) \circ (S \otimes \mathrm{Id} \otimes S') \circ (\Delta \otimes \mathrm{Id}) \circ \Delta =$$

$$\mu \circ ([\mu \circ (S \otimes \mathrm{Id}) \circ \Delta] \otimes S') \circ \Delta = S'.$$

□

Proposition 1.22.5. *If S is an antipode on a bialgebra H then S is an antihomomorphism of algebras with unit and of coalgebras with counit.*

Proof. Let

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(a) = (\mathrm{Id} \otimes \Delta) \circ \Delta(a) = \sum_i a_i^1 \otimes a_i^2 \otimes a_i^3,$$

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(b) = (\mathrm{Id} \otimes \Delta) \circ \Delta(b) = \sum_j b_j^1 \otimes b_j^2 \otimes b_j^3.$$

Then using the definition of the antipode, we have

$$S(ab) = \sum_i S(a_i^1 b) a_i^2 S(a_i^3) = \sum_{i,j} S(a_i^1 b_j^1) a_i^2 b_j^2 S(b_j^3) S(a_i^3) = S(b)S(a).$$

Thus S is an antihomomorphism of algebras (which is obviously unital). The fact that it is an antihomomorphism of coalgebras then follows using the self-duality of the axioms (see Exercises 1.21.4,1.22.3), or can be shown independently by a similar argument. □

Corollary 1.22.6. *(i) If H is a bialgebra with an antipode S, then the abelian monoidal category $\mathcal{C} = \mathrm{Rep}(H)$ has right duals. Namely, for any object X, the right dual X^* is the usual dual space of X, with action of H given by*

$$\rho_{X^*}(a) = \rho_X(S(a))^*,$$

and the usual evaluation and coevaluation morphisms of the category Vec.

(ii) If in addition S is invertible, then $\mathcal{C}$ also admits left duals, i.e. is rigid (in other words, $\mathcal{C}$ is tensor category). Namely, for any object X, the left dual ${}^*X$ is the usual dual space of X, with action of H given by

$$\rho_{^*X}(a) = \rho_X(S^{-1}(a))^*,$$

and the usual evaluation and coevaluation morphisms of the category Vec.

Proof. Part (i) follows from the antipode axiom and Proposition 1.22.5. Part (ii) follows from part (i) and the fact that the operation of taking the left dual is inverse to the operation of taking the right dual. □

Remark 1.22.7. A similar statement holds for finite dimensional comodules. Namely, if X is a finite dimensional right comodule over a bialgebra H with an antipode, then the right dual is the usual dual X^* with

$$(\pi_{X^*}(f), x \otimes \phi) := ((\mathrm{Id} \otimes S)(\pi_X(x)), f \otimes \phi),$$

$x \in X, f \in X^*, \phi \in H^*$. If S is invertible, then the left dual ${}^*X$ is defined by the same formula with S replaced by S^{-1}.

Remark 1.22.8. The fact that S is an antihomomorphism of coalgebras is the "linear algebra" version of the categorical fact that dualization changes the order of tensor product (Proposition 1.10.7(ii)).

Definition 1.22.9. A bialgebra equipped with an invertible antipode S is called a *Hopf algebra*.

Remark 1.22.10. We note that many authors use the term "Hopf algebra" for any bialgebra with an antipode.

Thus, Corollary 1.22.6 states that if H is a Hopf algebra then $\mathrm{Rep}(H)$ is a tensor category. So, we get the following *reconstruction theorem for finite dimensional Hopf algebras.*

Theorem 1.22.11. *The assignments* $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$, $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ *are mutually inverse bijections between*

1) finite tensor categories $\mathcal{C}$ *with a fiber functor* F*, up to monoidal equivalence and isomorphism of monoidal functors;*

2) finite dimensional Hopf algebras over k *up to isomorphism.*

Proof. Straightforward from the above. □

Exercise 1.22.12. The algebra of functions $\mathrm{Fun}(G, k)$ on a finite monoid G is a Hopf algebra if and only if G is a group. In this case, the antipode is given by the formula $S(f)(x) = f(x^{-1})$, $x \in G$.

More generally, if G is an affine algebraic group over k, then the algebra $\mathcal{O}(G)$ of regular functions on G is a Hopf algebra, with the comultiplication, counit, and antipode defined as in the finite case.

Similarly, $k[G]$ is a Hopf algebra if and only if G is a group, with $S(x) = x^{-1}$, $x \in G$.

Exercises 1.21.5 and 1.22.12 motivate the following definition:

Definition 1.22.13. In any coalgebra C, a nonzero element $g \in C$ such that $\Delta(g) = g \otimes g$ is called a *grouplike element.*

Exercise 1.22.14. Show that if g is a grouplike of a Hopf algebra H, then g is invertible, with $g^{-1} = S(g)$. Also, show that the product of two grouplike elements is grouplike. In particular, grouplike elements of any Hopf algebra H form a group, denoted $\mathbf{G}(H)$. Show that this group can also be defined as the group of isomorphism classes of 1-dimensional H-comodules under tensor multiplication.

Proposition 1.22.15. *If H is a finite dimensional bialgebra with an antipode S, then S is invertible, so H is a Hopf algebra.*

Proof. Let H_n be the image of S^n. Since S is an antihomomorphism of algebras and coalgebras, H_n is a Hopf subalgebra of H. Let m be the smallest n such that $H_n = H_{n+1}$ (it exists because H is finite dimensional). We need to show that $m = 0$. If not, we can assume that $m = 1$ by replacing H with H_{m-1}.

We have a map $S' : H_1 \to H_1$ inverse to S. For $a \in H$, let the triple coproduct of a be

$$\sum_i a_i^1 \otimes a_i^2 \otimes a_i^3.$$

Consider the element

$$b = \sum_i S'(S(a_i^1))S(a_i^2)a_i^3.$$

On the one hand, collapsing the last two factors using the antipode axiom, we have $b = S'(S(a))$. On the other hand, writing b as

$$b = \sum_i S'(S(a_i^1))S(S'(S(a_i^2)))a_i^3$$

and collapsing the first two factors using the antipode axiom, we get $b = a$. Thus $a = S'(S(a))$ and thus $a \in H_1$, so $H = H_1$, a contradiction. □

Exercise 1.22.16. Let μ^{op} and Δ^{op} be obtained from μ, Δ by permutation of components.

(i) Show that if $(H, \mu, i, \Delta, \varepsilon, S)$ is a Hopf algebra, then $H_{op} := (H, \mu^{op}, i, \Delta, \varepsilon, S^{-1})$, $H^{cop} := (H, \mu, i, \Delta^{op}, \varepsilon, S^{-1})$, $H_{op}^{cop} := (H, \mu^{op}, i, \Delta^{op}, \varepsilon, S)$ are Hopf algebras. Show that H is isomorphic to H_{op}^{cop}, and H_{op} to H^{cop}.

(ii) Suppose that a bialgebra H is a commutative ($\mu = \mu^{op}$) or cocommutative ($\Delta = \Delta^{op}$). Let S be an antipode on H. Show that $S^2 = 1$.

(iii) Assume that bialgebras H and H^{cop} have antipodes S and S'. Show that $S' = S^{-1}$, so H is a Hopf algebra.

Exercise 1.22.17. Show that if A, B are bialgebras, bialgebras with antipode, or Hopf algebras, then so is the tensor product $A \otimes B$.

Exercise 1.22.18. A finite dimensional module or comodule over a Hopf algebra is invertible if and only if it is 1-dimensional.

1.23. **Reconstruction theory in the infinite setting.** In this subsection we would like to generalize the reconstruction theory to the situation when the category $\mathcal{C}$ is not assumed to be finite.

Let $\mathcal{C}$ be any essentially small k-linear abelian category, and $F : \mathcal{C} \to \mathrm{Vec}$ an exact, faithful functor. In this case one can define the space $\mathrm{Coend}(F)$ as follows:

$$\mathrm{Coend}(F) := (\oplus_{X \in \mathcal{C}} F(X)^* \otimes F(X))/E$$

where E is spanned by elements of the form $y_* \otimes F(f)x - F(f)^* y_* \otimes x$, $x \in F(X)$, $y_* \in F(Y)^*$, $f \in \mathsf{Hom}(X, Y)$; in other words, $\mathrm{Coend}(F) = \varinjlim \mathsf{End}(F(X))^*$. Thus we have $\mathsf{End}(F) = \varprojlim \mathsf{End}(F(X)) = \mathrm{Coend}(F)^*$, which yields a coalgebra structure on $\mathrm{Coend}(F)$. So the algebra $\mathsf{End}(F)$ (which may be infinite dimensional) carries the inverse limit topology, in which a basis of neighborhoods of zero is formed by the kernels K_X of the maps $\mathsf{End}(F) \to \mathsf{End}(F(X))$, $X \in \mathcal{C}$, and $\mathrm{Coend}(F) = \mathsf{End}(F)^\vee$, the space of continuous linear functionals on $\mathsf{End}(F)$.

The following theorem is standard (see [Ta2]).

Theorem 1.23.1. *Let $\mathcal{C}$ be a k-linear abelian category with an exact faithful functor $F : \mathcal{C} \to \mathrm{Vec}$. Then F defines an equivalence between $\mathcal{C}$ and the category of finite dimensional right comodules over* $C := \mathrm{Coend}(F)$ *(or, equivalently, with the category of continuous finite dimensional left* $\mathsf{End}(F)$*-modules).*

Proof. (sketch) Consider the ind-object $Q := \oplus_{X \in \mathcal{C}} F(X)^* \otimes X$. For $X, Y \in \mathcal{C}$ and $f \in \mathsf{Hom}(X, Y)$, let

$$j_f : F(Y)^* \otimes X \to F(X)^* \otimes X \oplus F(Y)^* \otimes Y \subset Q$$

be the morphism defined by the formula

$$j_f = \mathrm{Id} \otimes f - F(f)^* \otimes \mathrm{Id}.$$

Let I be the quotient of Q by the image of the direct sum of all j_f. In other words, $I = \varinjlim (F(X)^* \otimes X)$.

The following statements are easy to verify:

(i) I represents the functor $F(\bullet)^*$, i.e. $\mathsf{Hom}(X, I)$ is naturally isomorphic to $F(X)^*$; in particular, I is injective.

(ii) $F(I) = C$, and I is naturally a left C-comodule (the comodule structure is induced by the coevaluation morphism $F(X)^* \otimes X \to F(X)^* \otimes F(X) \otimes F(X)^* \otimes X$).

(iii) Let us regard F as a functor $\mathcal{C} \to C - \mathrm{comod}$. For $M \in C - \mathrm{comod}$, let $\theta_M : M \otimes I \to M \otimes C \otimes I$ be the morphism $\pi_M \otimes \mathrm{Id} - \mathrm{Id} \otimes \pi_I$, and let K_M be the kernel of θ_M. Then the functor $G : C - \mathrm{comod} \to \mathcal{C}$ given by the formula $G(M) = \mathrm{Ker} \theta_M$, is a quasi-inverse to F.

This completes the proof. $\square$

Now assume that the abelian category $\mathcal{C}$ is also monoidal. Then the coalgebra $\mathrm{Coend}(F)$ also carries a multiplication and unit, dual to the comultiplication and counit of $\mathsf{End}(F)$. More precisely, since $\mathsf{End}(F)$ may now be infinite dimensional, the algebra $\mathsf{End}(F \otimes F)$ is in general isomorphic not to the usual tensor product $\mathsf{End}(F) \otimes \mathsf{End}(F)$, but rather to its completion $\mathsf{End}(F) \widehat{\otimes} \mathsf{End}(F)$ with respect to the inverse limit topology. Thus the comultiplication of $\mathsf{End}(F)$ is a continuous linear map $\Delta : \mathsf{End}(F) \to \mathsf{End}(F) \widehat{\otimes} \mathsf{End}(F)$. The dual Δ^* of this map defines a multiplication on $\mathrm{Coend}(F)$.

If $\mathcal{C}$ has right duals, the bialgebra $\mathrm{Coend}(F)$ acquires an antipode, defined in the same way as in the finite dimensional case. This antipode is invertible if there are also left duals (i.e. if $\mathcal{C}$ is rigid). Thus Theorem 1.23.1 implies the following "infinite" extensions of the reconstruction theorems.

Theorem 1.23.2. *The assignments* $(\mathcal{C}, F) \mapsto H = \mathrm{Coend}(F)$*,* $H \mapsto (H - \mathrm{Comod}, \mathrm{Forget})$ *are mutually inverse bijections between*

1) k-linear abelian monoidal categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors, and bialgebras over k, up to isomorphism;

2) k-linear abelian monoidal categories $\mathcal{C}$ with right duals with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors, and bialgebras over k with an antipode, up to isomorphism;

3) tensor categories $\mathcal{C}$ over k with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors, and Hopf algebras over k, up to isomorphism.

Remark 1.23.3. This theorem allows one to give a categorical proof of Proposition 1.22.4, deducing it from the fact that the right dual, when it exists, is unique up to a unique isomorphism.

Remark 1.23.4. Corollary 1.22.15 is not true, in general, in the infinite dimensional case: there exist bialgebras H with a non-invertible antipode S, see [Ta1]. Therefore, there exist ring categories with simple object $\mathbf{1}$ and right duals that do not have left duals, i.e., are not tensor categories (namely, $H - \mathrm{comod}$).

In the next few subsections, we will review some of the most important basic results about Hopf algebras. For a much more detailed treatment, see the book [Mo].

1.24. **More examples of Hopf algebras.** Let us give a few more examples of Hopf algebras. As we have seen, to define a Hopf algebra, it suffices to give an associative unital algebra H, and define a coproduct on generators of H (this determines a Hopf algebra structure on H uniquely if it exists). This is what we'll do in the examples below.

Example 1.24.1. (Enveloping algebras) Let $\mathfrak{g}$ be a Lie algebra, and let $H = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Define the coproduct on H by setting $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$. It is easy to show that this extends to the whole H, and that H equipped with this Δ is a Hopf algebra. Moreover, it is easy to see that the tensor category $\mathrm{Rep}(H)$ is equivalent to the tensor category $\mathrm{Rep}(\mathfrak{g})$.

This example motivates the following definition.

Definition 1.24.2. An element x of a bialgebra H is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. The space of primitive elements of H is denoted $\mathrm{Prim}(H)$.

Exercise 1.24.3. (i) Show that $\mathrm{Prim}(H)$ is a Lie algebra under the commutator.

(ii) Show that if x is a primitive element then $\varepsilon(x) = 0$, and in presence of an antipode $S(x) = -x$.

Exercise 1.24.4. (i) Let V be a vector space, and SV be the symmetric algebra V. Then SV is a Hopf algebra (namely, it is the universal enveloping algebra of the abelian Lie algebra V). Show that if k has characteristic zero, then $\mathrm{Prim}(SV) = V$.

(ii) What happens in characteristic p?

Hint. One can restrict to a situation when V is finite dimensional. In this case, regarding elements $f \in SV$ as polynomials on V^*, one can show that f is primitive if and only if it is additive, i.e., $f(x + y) = f(x) + f(y)$.

(iii) Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero. Show that $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$.

Hint. Identify $U(\mathfrak{g})$ with $S\mathfrak{g}$ as coalgebras by using the symmetrization map.

Example 1.24.5. (Taft algebras) Let q be a primitive n-th root of unity. Let H be the algebra (of dimension n^2) generated over k by g and x satisfying the following relations: $g^n = 1$, $x^n = 0$ and $gxg^{-1} = qx$. Define the coproduct on H by $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes g + 1 \otimes x$. It is easy to show that this extends to a Hopf algebra structure on H. This Hopf algebra H is called the *Taft algebra*. For $n = 2$, one obtains the Sweedler Hopf algebra of dimension 4. Note that H is not commutative or cocommutative, and $S^2 \neq 1$ on H (as $S^2(x) = qx$).

This example motivates the following generalization of Definition 1.24.2.

Definition 1.24.6. Let g, h be grouplike elements of a coalgebra H. A skew-primitive element of type (h, g) is an element $x \in H$ such that $\Delta(x) = h \otimes x + x \otimes g$.

Remark 1.24.7. A multiple of $h-g$ is always a skew-primitive element of type (h, g). Such a skew-primitive element is called *trivial*. Note that the element x in Example 1.24.5 is nontrivial.

Exercise 1.24.8. Let x be a skew-primitive element of type h, g in a Hopf algebra H.

(i) Show that $\varepsilon(x) = 0$, $S(x) = -h^{-1}xg^{-1}$.

(ii) Show that if $a, b \in H$ are grouplike elements, then axb is a skew-primitive element of type (ahb, agb).

Example 1.24.9. (Nichols Hopf algebras) Let $H = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \ltimes \wedge(x_1, ..., x_n)$, where the generator g of $\mathbb{Z}/2\mathbb{Z}$ acts on x_i by $gx_ig^{-1} = -x_i$. Define the coproduct on H by making g grouplike, and setting $\Delta(x_i) := x_i \otimes g + 1 \otimes x_i$ (so x_i are skew-primitive elements). Then H is a Hopf algebra of dimension 2^{n+1}. For $n = 1$, H is the Sweedler Hopf algebra from the previous example.

Exercise 1.24.10. Show that the Hopf algebras of Examples 1.24.1,1.24.5,1.24.9 are well defined.

Exercise 1.24.11. (Semidirect product Hopf algebras) Let H be a Hopf algebra, and G a group of automorphisms of H. Let A be the semidirect product $k[G] \ltimes H$. Show that A admits a unique structure of a Hopf algebra in which $k[G]$ and H are Hopf subalgebras.

1.25. **The Quantum Group $U_{\mathsf{q}}(\mathfrak{sl}_2)$.** Let us consider the Lie algebra $\mathfrak{sl}_2$. Recall that there is a basis $\mathsf{h}, \mathsf{e}, \mathsf{f} \in \mathfrak{sl}_2$ such that $[\mathsf{h}, \mathsf{e}] = 2\mathsf{e}$, $[\mathsf{h}, \mathsf{f}] = -2\mathsf{f}$, $[\mathsf{e}, \mathsf{f}] = \mathsf{h}$. This motivates the following definition.

Definition 1.25.1. Let $\mathsf{q} \in k, \mathsf{q} \neq \pm 1$. The *quantum group* $U_{\mathsf{q}}(\mathfrak{sl}_2)$ is generated by elements E, F and an invertible element K with defining relations

$$K\mathsf{E}K^{-1} = \mathsf{q}^2\mathsf{E}, K\mathsf{F}K^{-1} = \mathsf{q}^{-2}\mathsf{F}, [\mathsf{E}, \mathsf{F}] = \frac{K - K^{-1}}{\mathsf{q} - \mathsf{q}^{-1}}.$$

Theorem 1.25.2. *There exists a unique Hopf algebra structure on $U_{\mathsf{q}}(\mathfrak{sl}_2)$, given by*

- $\Delta(K) = K \otimes K$ *(thus K is a grouplike element);*
- $\Delta(\mathsf{E}) = \mathsf{E} \otimes K + 1 \otimes \mathsf{E}$*;*
- $\Delta(\mathsf{F}) = \mathsf{F} \otimes 1 + K^{-1} \otimes \mathsf{F}$ *(thus E, F are skew-primitive elements).*

Exercise 1.25.3. Prove Theorem 1.25.2.

Remark 1.25.4. Heuristically, $K = \mathsf{q}^{\mathsf{h}}$, and thus

$$\lim_{\mathsf{q}\to 1} \frac{K - K^{-1}}{\mathsf{q} - \mathsf{q}^{-1}} = \mathsf{h}.$$

So in the limit $\mathsf{q} \to 1$, the relations of $U_{\mathsf{q}}(\mathfrak{sl}_2)$ degenerate into the relations of $U(\mathfrak{sl}_2)$, and thus $U_{\mathsf{q}}(\mathfrak{sl}_2)$ should be viewed as a Hopf algebra deformation of the enveloping algebra $U(\mathfrak{sl}_2)$. In fact, one can make this heuristic idea into a precise statement, see e.g. [K].

If q is a root of unity, one can also define a finite dimensional version of $U_{\mathsf{q}}(\mathfrak{sl}_2)$. Namely, assume that the order of q is an odd number ℓ. Let $u_{\mathsf{q}}(\mathfrak{sl}_2)$ be the quotient of $U_{\mathsf{q}}(\mathfrak{sl}_2)$ by the additional relations

$$\mathsf{E}^\ell = \mathsf{F}^\ell = K^\ell - 1 = 0.$$

Then it is easy to show that $u_{\mathsf{q}}(\mathfrak{sl}_2)$ is a Hopf algebra (with the coproduct inherited from $U_{\mathsf{q}}(\mathfrak{sl}_2)$). This Hopf algebra is called *the small quantum group* attached to $\mathfrak{sl}_2$.

1.26. **The quantum group $U_{\mathsf{q}}(\mathfrak{g})$.** The example of the previous subsection can be generalized to the case of any simple Lie algebra. Namely, let $\mathfrak{g}$ be a simple Lie algebra of rank r, and let $A = (a_{ij})$ be its Cartan matrix. Recall that there exist unique relatively prime positive integers $d_i, i = 1, \dots r$ such that $d_i a_{ij} = d_j a_{ji}$. Let $\mathsf{q} \in k, \mathsf{q} \neq \pm 1$.

Definition 1.26.1.
- The q-*analog* of n is

$$[n]_{\mathsf{q}} = \frac{\mathsf{q}^n - \mathsf{q}^{-n}}{\mathsf{q} - \mathsf{q}^{-1}}.$$

- The q-*analog of the factorial* is

$$[n]_{\mathsf{q}}! = \prod_{l=1}^{n} [l]_{\mathsf{q}} = \frac{(\mathsf{q} - \mathsf{q}^{-1}) \cdots (\mathsf{q}^n - \mathsf{q}^{-n})}{(\mathsf{q} - \mathsf{q}^{-1})^n}.$$

Definition 1.26.2. The *quantum group* $U_{\mathsf{q}}(\mathfrak{g})$ is generated by elements $\mathsf{E}_i, \mathsf{F}_i$ and invertible elements K_i, with defining relations

$$K_i K_j = K_j K_i,\ K_i \mathsf{E}_j K_i^{-1} = \mathsf{q}^{a_{ij}} \mathsf{E}_j, K_i \mathsf{F}_j K_i^{-1} = \mathsf{q}^{-a_{ij}} \mathsf{F}_j,$$

$$[\mathsf{E}_i, \mathsf{F}_j] = \delta_{ij} \frac{K_i^{d_i} - K_i^{-d_i}}{\mathsf{q}^{d_i} - \mathsf{q}^{-d_i}}, \quad \text{and the } \mathsf{q}\text{-Serre relations:}$$

$$\sum_{l=0}^{1-a_{ij}} \frac{(-1)^l}{[l]_{\mathsf{q}_i}! [1 - a_{ij} - l]_{\mathsf{q}_i}!} \mathsf{E}_i^{1-a_{ij}-l} \mathsf{E}_j \mathsf{E}_i^l = 0, i \neq j \tag{1.26.1}$$

and

$$\sum_{l=0}^{1-a_{ij}} \frac{(-1)^l}{[l]_{\mathsf{q}_i}![1-a_{ij}-l]_{\mathsf{q}_i}!} \mathsf{F}_i^{1-a_{ij}-l}\mathsf{F}_j\mathsf{F}_i^l = 0, i \neq j. \tag{1.26.2}$$

More generally, the same definition can be made for any symmetrizable Kac-Moody algebra $\mathfrak{g}$.

Theorem 1.26.3. *(see e.g.* [CP]*) There exists a unique Hopf algebra structure on $U_{\mathsf{q}}(\mathfrak{g})$, given by*

- $\Delta(K_i) = K_i \otimes K_i$;
- $\Delta(\mathsf{E}_i) = \mathsf{E}_i \otimes K_i + 1 \otimes \mathsf{E}_i$;
- $\Delta(\mathsf{F}_i) = \mathsf{F}_i \otimes 1 + K_i^{-1} \otimes \mathsf{F}_i$.

Remark 1.26.4. Similarly to the case of $\mathfrak{sl}_2$, in the limit $\mathsf{q} \to 1$, these relations degenerate into the relations for $U(\mathfrak{g})$, so $U_{\mathsf{q}}(\mathfrak{g})$ should be viewed as a Hopf algebra deformation of the enveloping algebra $U(\mathfrak{g})$.

1.27. **Categorical meaning of skew-primitive elements.** We have seen that many interesting Hopf algebras contain nontrivial skew-primitive elements. In fact, the notion of a skew-primitive element has a categorical meaning. Namely, we have the following proposition.

Proposition 1.27.1. *Let g, h be grouplike elements of a coalgebra C, and* $\mathrm{Prim}_{h,g}(C)$ *be the space of skew-primitive elements of type h, g. Then the space* $\mathrm{Prim}_{h,g}(H)/k(h-g)$ *is naturally isomorphic to* $\mathrm{Ext}^1(g,h)$*, where g, h are regarded as 1-dimensional right C-comodules.*

Proof. Let V be a 2-dimensional H-comodule, such that we have an exact sequence

$$0 \to h \to V \to g \to 0.$$

Then V has a basis v_0, v_1 such that

$$\pi(v_0) = v_0 \otimes h,\ \pi(v_1) = v_1 \otimes x + v_0 \otimes g.$$

The condition that this is a comodule yields that x is a skew-primitive element of type (h, g). So any extension defines a skew-primitive element and vice versa. Also, we can change the basis by $v_0 \to v_0$, $v_1 \to v_1 + \lambda v_0$, which modifies x by adding a trivial skew-primitive element. This implies the result. □

Example 1.27.2. The category $\mathcal{C}$ of finite dimensional comodules over $u_{\mathsf{q}}(\mathfrak{sl}_2)$ is an example of a finite tensor category in which there are objects V such that V^{**} is not isomorphic to V. Namely, in this category, the functor $V \mapsto V^{**}$ is defined by the squared antipode S^2, which is conjugation by K: $S^2(x) = KxK^{-1}$. Now, we have

$\mathrm{Ext}^1(K, 1) = Y = \langle E, FK \rangle$, a 2-dimensional space. The set of isomorphism classes of nontrivial extensions of K by 1 is therefore the projective line $\mathbb{P}Y$. The operator of conjugation by K acts on Y with eigenvalues q^2, q^{-2}, hence nontrivially on $\mathbb{P}Y$. Thus for a generic extension V, the object V^{**} is not isomorphic to V.

However, note that *some power* of the functor $**$ on $\mathcal{C}$ is isomorphic (in fact, monoidally) to the identity functor (namely, this power is the order of q). We will later show that this property holds in any finite tensor category.

Note also that in the category $\mathcal{C}$, $V^{**} \cong V$ if V is simple. This clearly has to be the case in any tensor category where all simple objects are invertible. We will also show (see Proposition 1.41.1 below) that this is the case in any semisimple tensor category. An example of a tensor category in which V^{**} is not always isomorphic to V even for simple V is the category of finite dimensional representations of the the Yangian $H = Y(\mathfrak{g})$ of a simple complex Lie algebra $\mathfrak{g}$, see [CP, 12.1]. Namely, for any finite dimensional representation V of H and any complex number z one can define the shifted representation $V(z)$ (such that $V(0) = V$). Then $V^{*}{*} \cong V(2h^\vee)$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, see [CP, p.384]. If V is a non-trivial irreducible finite dimensional representation then $V(z) \ncong V$ for $z \neq 0$. Thus, $V^{**} \ncong V$. Moreover, we see that the functor $**$ has infinite order even when restricted to simple objects of $\mathcal{C}$.

However, the representation category of the Yangian is infinite, and the answer to the following question is unknown to us.

Question 1.27.3. Does there exist a *finite* tensor category, in which there is a simple object V such that V^{**} is not isomorphic to V? (The answer is unknown to the authors).

Theorem 1.27.4. *Assume that k has characteristic 0. Let $\mathcal{C}$ be a finite ring category over k with simple object* $\mathbf{1}$. *Then* $\mathrm{Ext}^1(\mathbf{1}, \mathbf{1}) = 0$.

Proof. Assume the contrary, and suppose that V is a nontrivial extension of $\mathbf{1}$ by itself. Let P be the projective cover of $\mathbf{1}$. Then $\mathsf{Hom}(P, V)$ is a 2-dimensional space, with a filtration induced by the filtration on V, and both quotients naturally isomorphic to $E := \mathsf{Hom}(P, \mathbf{1})$. Let v_0, v_1 be a basis of $\mathsf{Hom}(P, V)$ compatible to the filtration, i.e. v_0 spans the 1-dimensional subspace defined by the filtration. Let $A = \mathsf{End}(P)$ (this is a finite dimensional algebra). Let $\varepsilon : A \to k$ be the character defined by the (right) action of A on E. Then the matrix of $a \in A$ in the basis v_0, v_1 has the form

$$[a]_1 = \begin{pmatrix} \varepsilon(a) & \chi_1(a) \\ 0 & \varepsilon(a) \end{pmatrix} \tag{1.27.1}$$

where $\chi_1 \in A^*$ is nonzero. Since $a \to [a]_1$ is a homomorphism, χ_1 is a derivation: $\chi_1(xy) = \chi_1(x)\varepsilon(y) + \varepsilon(x)\chi_1(y)$.

Now consider the representation $V \otimes V$. Using the exactness of tensor products, we see that the space $\mathsf{Hom}(P, V \otimes V)$ is 4-dimensional, and has a 3-step filtration, with successive quotients $E, E \oplus E, E$, and basis $v_{00}; v_{01}, v_{10}; v_{11}$, consistent with this filtration. The matrix of $a \in \mathsf{End}(P)$ in this basis is

$$[a]_2 = \begin{pmatrix} \varepsilon(a) & \chi_1(a) & \chi_1(a) & \chi_2(a) \\ 0 & \varepsilon(a) & 0 & \chi_1(a) \\ 0 & 0 & \varepsilon(a) & \chi_1(a) \\ 0 & 0 & 0 & \varepsilon(a) \end{pmatrix} \tag{1.27.2}$$

Since $a \to [a]_2$ is a homomorphism, we find

$$\chi_2(ab) = \varepsilon(a)\chi_2(b) + \chi_2(a)\varepsilon(b) + 2\chi_1(a)\chi_1(b).$$

We can now proceed further (i.e. consider $V \otimes V \otimes V$ etc.) and define for every positive n, a linear function $\chi_n \in A^*$ which satisfies the equation

$$\chi_n(ab) = \sum_{j=0}^{n} \binom{n}{j} \chi_j(a)\chi_{n-j}(b),$$

where $\chi_0 = \varepsilon$. Thus for any $s \in k$, we can define $\phi_s : A \to k((t))$ by $\phi_s(a) = \sum_{m \geq 0} \chi_m(a) s^m t^m / m!$, and we find that ϕ_s is a family of pairwise distinct homomorphisms. This is a contradiction, as A is a finite dimensional algebra. We are done. $\square$

Corollary 1.27.5. *If a finite ring category $\mathcal{C}$ over a field of characteristic zero has a unique simple object* $\mathbf{1}$*, then $\mathcal{C}$ is equivalent to the category Vec.*

Corollary 1.27.6. *A finite dimensional bialgebra H over a field of characteristic zero cannot contain nonzero primitive elements.*

Proof. Apply Theorem 1.27.4 to the category $H-$ comod and use Proposition 1.27.1. $\square$

Remark 1.27.7. Here is a "linear algebra" proof of this corollary. Let x be a nonzero primitive element of H. Then we have a family of grouplike elements $e^{stx} \in H((t))$, $s \in k$, i.e., an infinite collection of characters of $H^*((t))$, which is impossible, as H is finite dimensional.

Corollary 1.27.8. *If H is a finite dimensional* commutative *Hopf algebra over an algebraically closed field k of characteristic 0, then $H = \mathrm{Fun}(G, k)$ for a unique finite group G.*

Proof. Let $G = \mathrm{Spec}(H)$ (a finite group scheme), and $x \in T_1G = (\mathfrak{m}/\mathfrak{m}^2)^*$ where $\mathfrak{m}$ is the kernel of the counit. Then x is a linear function on $\mathfrak{m}$. Extend it to H by setting $x(1) = 0$. Then x s a derivation:

$$x(fg) = x(f)g(1) + f(1)x(g).$$

This implies that x is a primitive element in H^*. So by Corollary 1.27.6, $x = 0$. this implies that G is reduced at the point 1. By using translations, we see that G is reduced at all other points. So G is a finite group, and we are done. □

Remark 1.27.9. Theorem 1.27.4 and all its corollaries fail in characteristic $p > 0$. A counterexample is provided by the Hopf algebra $k[x]/(x^p)$, where x is a primitive element.

1.28. **Pointed tensor categories and pointed Hopf algebras.**

Definition 1.28.1. A coalgebra C is *pointed* if its category of right comodules is pointed, i.e., if any simple right C-comodule is 1-dimensional.

Remark 1.28.2. A finite dimensional coalgebra C is pointed if and only if the algebra C^* is basic, i.e., the quotient $C^*/\mathrm{Rad}(C^*)$ of C^* by its radical is commutative. In this case, simple C-comodules are points of $\mathrm{Specm}(H^*/\mathrm{Rad}(H^*))$, which justifies the term "pointed".

Definition 1.28.3. A tensor category $\mathcal{C}$ is pointed if every simple object of $\mathcal{C}$ is invertible.

Thus, the category of right comodules over a Hopf algebra H is pointed if and only if H is pointed.

Example 1.28.4. The category Vec_G^ω is a pointed category. If G is a p-group and k has characteristic p, then $\mathrm{Rep}_k(G)$ is pointed. Any cocommutative Hopf algebra, the Taft and Nichols Hopf algebras, as well as the quantum groups $U_\mathfrak{q}(\mathfrak{g})$ are pointed Hopf algebras.

1.29. **The coradical filtration.** Let $\mathcal{C}$ be a locally finite abelian category.

Any object $X \in \mathcal{C}$ has a canonical filtration

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X \qquad (1.29.1)$$

such that X_{i+1}/X_i is the socle (i.e., the maximal semisimple subobject) of X/X_i (in other words, X_{i+1}/X_i is the sum of all simple subobjects of X/X_i). This filtration is called the *socle filtration*, or the *coradical* *filtration* of X. It is easy to show by induction that the coradical filtration is a filtration of X of the smallest possible length, such that the successive quotients are semisimple. The length of the coradical filtration of X is called the *Loewy length* of X, and denoted $\mathrm{Lw}(X)$. Then we have a filtration of the category $\mathcal{C}$ by Loewy length of objects: $\mathcal{C}_0 \subset \mathcal{C}_1 \subset ...$, where $\mathcal{C}_i$ denotes the full subcategory of objects of $\mathcal{C}$ of Loewy length $\le i+1$. Clearly, the Loewy length of any subquotient of an object X does not exceed the Loewy length of X, so the categories $\mathcal{C}_i$ are closed under taking subquotients.

Definition 1.29.1. The filtration of $\mathcal{C}$ by $\mathcal{C}_i$ is called the *coradical filtration* of $\mathcal{C}$.

If $\mathcal{C}$ is endowed with an exact faithful functor $F : \mathcal{C} \to \mathrm{Vec}$ then we can define the coalgebra $C = \mathrm{Coend}(F)$ and its subcoalgebras $C_i = \mathrm{Coend}(F|_{\mathcal{C}_i})$, and we have $C_i \subset C_{i+1}$ and $C = \cup_i C_i$ (alternatively, we can say that C_i is spanned by matrix elements of C-comodules $F(X)$, $X \in \mathcal{C}_i$). Thus we have defined an increasing filtration by subcoalgebras of any coalgebra C. This filtration is called the *coradical filtration*, and the term C_0 is called the *coradical* of C.

The "linear algebra" definition of the coradical filtration is as follows. One says that a coalgebra is *simple* if it does not have nontrivial subcoalgebras, i.e. if it is finite dimensional, and its dual is a simple (i.e., matrix) algebra. Then C_0 is the sum of all simple subcoalgebras of C. The coalgebras C_{n+1} for $n \ge 1$ are then defined inductively to be the spaces of those $x \in C$ for which

$$\Delta(x) \in C_n \otimes C + C \otimes C_0.$$

Exercise 1.29.2. (i) Suppose that C is a finite dimensional coalgebra, and I is the Jacobson radical of C^*. Show that $C_n^\perp = I^{n+1}$. This justifies the term "coradical filtration".

(ii) Show that the coproduct respects the coradical filtration, i.e. $\Delta(C_n) \subset \sum_{i=0}^n C_i \otimes C_{n-i}$.

(iii) Show that C_0 is the **direct** sum of simple subcoalgebras of C. In particular, grouplike elements of any coalgebra C are linearly independent.

Hint. Simple subcoalgebras of C correspond to finite dimensional irrreducible representations of C^*.

Denote by $\mathrm{gr}(C)$ the associated graded coalgebra of a coalgebra C with respect to the coradical filtration. Then $\mathrm{gr}(C)$ is a $\mathbb{Z}_+$-graded coalgebra. It is easy to see from Exercise 1.29.2(i) that the coradical filtration of $\mathrm{gr}(C)$ is induced by its grading. A graded coalgebra $\bar{C}$

with this property is said to be *coradically graded*, and a coalgebra C such that $\mathrm{gr}(C) = \bar{C}$ is called a *lifting* of C.

Definition 1.29.3. A coalgebra C is said to be *cosemisimple* if C is a direct sum of simple subcoalgebras.

Clearly, C is cosemisimple if and only if $C - \mathrm{comod}$ is a semisimple category.

Proposition 1.29.4. *(i) A category $\mathcal{C}$ is semisimple if and only if $\mathcal{C}_0 = \mathcal{C}_1$.*

(ii) A coalgebra C is cosemisimple if and only if $C_0 = C_1$.

Proof. (ii) is a special case of (i), and (i) is clear, since the condition means that $\mathrm{Ext}^1(X,Y) = 0$ for any simple X, Y, which implies (by the long exact sequence of cohomology) that $\mathrm{Ext}^1(X,Y) = 0$ for all $X, Y \in \mathcal{C}$. □

Corollary 1.29.5. *(The Taft-Wilson theorem) If C is a pointed coalgebra, then C_0 is spanned by (linearly independent) grouplike elements g, and $C_1/C_0 = \oplus_{h,g}\mathrm{Prim}_{h,g}(C)/k(h-g)$. In particular, any non-cosemisimple pointed coalgebra contains nontrivial skew-primitive elements.*

Proof. The first statement is clear (the linear independence follows from Exercise 1.29.2(iii). Also, it is clear that any skew-primitive element is contained in C_1. Now, if $x \in C_1$, then x is a matrix element of a C-comodule of Loewy length ≤ 2, so it is a sum of matrix elements 2-dimensional comodules, i.e. of grouplike and skew-primitive elements.

It remains to show that the sum $\sum_{h,g} \mathrm{Prim}_{h,g}(C)/k(h-g) \subset C/C_0$ is direct. For this, it suffices to consider the case when C is finite dimensional. Passing to the dual algebra $A = C^*$, we see that the statement is equivalent to the claim that I/I^2 (where I is the radical of A) is isomorphic (in a natural way) to $\oplus_{g,h}\mathrm{Ext}^1(g,h)^*$.

Let p_g be a complete system of orthogonal idempotents in A/I^2, such that $h(p_g) = \delta_{hg}$. Define a pairing $I/I^2 \times \mathrm{Ext}^1(g,h) \to k$ which sends $a \otimes \alpha$ to the upper right entry of the 2-by-2 matrix by which a acts in the extension of g by h defined by α. It is easy to see that this pairing descends to a pairing $B : p_h(I/I^2)p_g \times \mathrm{Ext}^1(g,h) \to k$. If the extension α is nontrivial, the upper right entry cannot be zero, so B is right-nondegenerate. Similarly, if a belongs to the left kernel of B, then a acts by zero in any A-module of Loewy length 2, so $a = 0$. Thus, B is left-nondegenerate. This implies the required isomorphism. □

Proposition 1.29.6. *If C, D are coalgebras, and $f : C \to D$ is a coalgebra homomorphism such that $f|_{C_1}$ is injective, then f is injective.*

Proof. One may assume that C and D are finite dimensional. Then the statement can be translated into the following statement about finite dimensional algebras: if A, B are finite dimensional algebras and $f : A \to B$ is an algebra homomorphism which descends to a surjective homomorphism $A \to B/\mathrm{Rad}(B)^2$, then f is surjective.

To prove this statement, let $b \in B$. Let $I = \mathrm{Rad}(B)$. We prove by induction in n that there exists $a \in A$ such that $b - f(a) \in I^n$. The base of induction is clear, so we only need to do the step of induction. So assume $b \in I^n$. We may assume that $b = b_1...b_n$, $b_i \in I$, and let $a_i \in A$ be such that $f(a_i) = b_i$ modulo I^2. Let $a = a_1...a_n$. Then $b - f(a) \in I^{n+1}$, as desired. □

Corollary 1.29.7. *If H is a Hopf algebra over a field of characteristic zero, then the natural map $\xi : U(\mathrm{Prim}(H)) \to H$ is injective.*

Proof. By Proposition 1.29.6, it suffices to check the injectivity of ξ in degree 1 of the coradical filtration. Thus, it is enough to check that ξ is injective on primitive elements of $U(\mathrm{Prim}(H))$. But by Exercise 1.24.4, all of them lie in $\mathrm{Prim}(H)$, as desired. □

1.30. **Chevalley's theorem.**

Theorem 1.30.1. *(Chevalley) Let k be a field of characteristic zero. Then the tensor product of two simple finite dimensional representations of any group or Lie algebra over k is semisimple.*

Proof. Let V be a finite dimensional vector space over a field k (of any characteristic), and $G \subset GL(V)$ be a Zariski closed subgroup.

Lemma 1.30.2. *If V is a completely reducible representation of G, then G is reductive.*

Proof. Let V be a nonzero rational representation of an affine algebraic group G. Let U be the unipotent radical of G. Let $V^U \subset V$ be the space of invariants. Since U is unipotent, $V^U \neq 0$. So if V is irreducible, then $V^U = V$, i.e., U acts trivially. Thus, U acts trivially on any completely reducible representation of G. So if V is completely reducible and $G \subset GL(V)$, then G is reductive. □

Now let G be any group, and V, W be two finite dimensional irreducible representations of G. Let G_V, G_W be the Zariski closures of the images of G in $GL(V)$ and $GL(W)$, respectively. Then by Lemma 1.30.2 G_V, G_W are reductive. Let G_{VW} be the Zariski closure of the image of G in $G_V \times G_W$. Let U be the unipotent radical of G_{VW}. Let $p_V : G_{VW} \to G_V$, $p_W : G_{VW} \to G_W$ be the projections. Since p_V is surjective, $p_V(U)$ is a normal unipotent subgroup of G_V, so $p_V(U) = 1$. Similarly, $p_W(U) = 1$. So $U = 1$, and G_{VW} is reductive.

Let G'_{VW} be the closure of the image of G in $GL(V \otimes W)$. Then G'_{VW} is a quotient of G_{VW}, so it is also reductive. Since $\mathrm{char} k = 0$, this implies that the representation $V \otimes W$ is completely reducible as a representation of G'_{VW}, hence of G.

This proves Chevalley's theorem for groups. The proof for Lie algebras is similar. □

1.31. **Chevalley property.**

Definition 1.31.1. A tensor category $\mathcal{C}$ is said to have *Chevalley property* if the category $\mathcal{C}_0$ of semisimple objects of $\mathcal{C}$ is a tensor subcategory.

Thus, Chevalley theorem says that the category of finite dimensional representations of any group or Lie algebra over a field of characteristic zero has Chevalley property.

Proposition 1.31.2. *A pointed tensor category has Chevalley property.*

Proof. Obvious. □

Proposition 1.31.3. *In a tensor category with Chevalley property,*

$$\mathrm{Lw}(X \otimes Y) \leq \mathrm{Lw}(X) + \mathrm{Lw}(Y) - 1. \tag{1.31.1}$$

Thus $\mathcal{C}_i \otimes \mathcal{C}_j \subset \mathcal{C}_{i+j}$.

Proof. Let $X(i)$, $0 \leq i \leq m$, $Y(j)$, $0 \leq j \leq n$, be the successive quotients of the coradical filtrations of X, Y. Then $Z := X \otimes Y$ has a filtration with successive quotients $Z(r) = \oplus_{i+j=r} X(i) \otimes Y(j)$, $0 \leq r \leq m+n$. Because of the Chevalley property, these quotients are semisimple. This implies the statement. □

Remark 1.31.4. It is clear that the converse to Proposition 1.31.3 holds as well: equation (1.31.3) (for simple X and Y) implies the Chevalley property.

Corollary 1.31.5. *In a pointed Hopf algebra H, the coradical filtration is a Hopf algebra filtration, i.e. $H_i H_j \subset H_{i+j}$ and $S(H_i) = H_i$, so $\mathrm{gr}(H)$ is a Hopf algebra.*

In this situation, the Hopf algebra H is said to be a *lifting* of the coradically graded Hopf algebra $\mathrm{gr}(H)$.

Example 1.31.6. The Taft algebra and the Nichols algebras are coradically graded. The associated graded Hopf algebra of $U_{\mathfrak{q}}(\mathfrak{g})$ is the Hopf algebra defined by the same relations as $U_{\mathsf{q}}(\mathfrak{g})$, except that the commutation relation between E_i and F_j is replaced with the condition that E_i and F_j commute (for all i, j). The same applies to the small quantum group $u_{\mathsf{q}}(\mathfrak{sl}_2)$.

Exercise 1.31.7. Let k be a field of characteristic p, and G a finite group. Show that the category $\mathrm{Rep}_k(G)$ has Chevalley property if and only if G has a normal p-Sylow subgroup.

1.32. **The Andruskiewitsch-Schneider conjecture.** It is easy to see that any Hopf algebra generated by grouplike and skew-primitive elements is automatically pointed.

On the other hand, there exist pointed Hopf algebras which are not generated by grouplike and skew-primitive elements. Perhaps the simplest example of such a Hopf algebra is the algebra of regular functions on the Heisenberg group (i.e. the group of upper triangular 3 by 3 matrices with ones on the diagonal). It is easy to see that the commutative Hopf algebra H is the polynomial algebra in generators x, y, z (entries of the matrix), so that x, y are primitive, and

$$\Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y.$$

Since the only grouplike element in H is 1, and the only skew-primitive elements are x, y, H is not generated by grouplike and skew-primitive elements.

However, one has the following conjecture, due to Andruskiewitsch and Schneider.

Conjecture 1.32.1. Any finite dimensional pointed Hopf algebra over a field of characteristic zero is generated in degree 1 of its coradical filtration, i.e., by grouplike and skew-primitive elements.

It is easy to see that it is enough to prove this conjecture for coradically graded Hopf algebras; this has been done in many special cases (see [AS]).

The reason we discuss this conjecture here is that it is essentially a categorical statement. Let us make the following definition.

Definition 1.32.2. We say that a tensor category $\mathcal{C}$ is *tensor-generated* by a collection of objects X_α if every object of $\mathcal{C}$ is a subquotient of a finite direct sum of tensor products of X_α.

Proposition 1.32.3. *A pointed Hopf algebra H is generated by grouplike and skew-primitive elements if and only if the tensor category* $H-\mathrm{comod}$ *is tensor-generated by objects of length* 2.

Proof. This follows from the fact that matrix elements of the tensor product of comodules V, W for H are products of matrix elements of V, W. □

Thus, one may generalize Conjecture 1.32.1 to the following conjecture about tensor categories.

Conjecture 1.32.4. Any finite pointed tensor category over a field of characteristic zero is tensor generated by objects of length 2.

As we have seen, this property fails for infinite categories, e.g., for the category of rational representations of the Heisenberg group. In fact, this is very easy to see categorically: the center of the Heisenberg group acts trivially on 2-dimensional representations, but it is not true for a general rational representation.

1.33. **The Cartier-Kostant theorem.**

Theorem 1.33.1. *Any cocommutative Hopf algebra H over an algebraically closed field of characteristic zero is of the form $k[G] \ltimes U(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra, and G is a group acting on $\mathfrak{g}$.*

Proof. Let G be the group of grouplike elements of H. Since H is cocommutative, it is pointed, and $\mathrm{Ext}^1(g, h) = 0$ if $g, h \in G$, $g \neq h$. Hence the category $\mathcal{C} = H-$comod splits into a direct sum of blocks $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$, where $\mathcal{C}_g$ is the category of objects of $\mathcal{C}$ which have a filtration with successive quotients isomorphic to g. So $H = \oplus_{g \in G} H_g$, where $\mathcal{C}_g = H_g-$comod, and $H_g = gH_1$. Moreover, $A = H_1$ is a Hopf algebra, and we have an action of G on A by Hopf algebra automorphisms.

Now let $\mathfrak{g} = \mathrm{Prim}(A) = \mathrm{Prim}(H)$. This is a Lie algebra, and the group G acts on it (by conjugation) by Lie algebra automorphisms. So we need just to show that the natural homomorphism $\psi : U(\mathfrak{g}) \to A$ is actually an isomorphism.

It is clear that any morphism of coalgebras preserves the coradical filtration, so we can pass to the associated graded morphism $\psi_0 : S\mathfrak{g} \to A_0$, where $A_0 = \mathrm{gr}(A)$. It is enough to check that ψ_0 is an isomorphism.

The morphism ψ_0 is an isomorphism in degrees 0 and 1, and by Corollary 1.29.7, it is injective. So we only need to show surjectivity.

We prove the surjectivity in each degree n by induction. To simplify notation, let us identify $S\mathfrak{g}$ with its image under ψ_0. Suppose that the surjectivity is known in all degrees below n. Let z be a homogeneous element in A_0 of degree n. Then it is easy to see from the counit axiom that

$$\Delta(z) - z \otimes 1 - 1 \otimes z = u \qquad (1.33.1)$$

where $u \in S\mathfrak{g} \otimes S\mathfrak{g}$ is a symmetric element (as Δ is cocommutative).

Equation 1.33.1 implies that the element u satisfies the equation

$$(\Delta \otimes \mathrm{Id})(u) + u \otimes 1 = (\mathrm{Id} \otimes \Delta)(u) + 1 \otimes u. \tag{1.33.2}$$

Lemma 1.33.2. *Let V be a vector space over a field k of characteristic zero. Let $u \in SV \otimes SV$ be a symmetric element satisfying equation (1.33.2). Then $u = \Delta(w) - w \otimes 1 - 1 \otimes w$ for some $w \in SV$.*

Proof. Clearly, we may assume that V is finite dimensional. Regard u as a polynomial function on $V^* \times V^*$; our job is to show that

$$u(x, y) = w(x + y) - w(x) - w(y)$$

for some polynomial w.

If we regard u as a polynomial, equation (1.33.2) takes the form of the 2-cocycle condition

$$u(x + y, t) + u(x, y) = u(x, y + t) + u(y, t).$$

Thus u defines a group law on $U := V^* \oplus k$, given by

$$(x, a) + (y, b) = (x + y, a + b + u(x, y)).$$

Clearly, we may assume that u is homogeneous, of some degree $d \neq 1$. Since u is symmetric, the group U is abelian. So in U we have

$$((x, 0) + (x, 0)) + ((y, 0) + (y, 0)) = ((x, 0) + (y, 0)) + ((x, 0) + (y, 0))$$

Computing the second component of both sides, we get

$$u(x, x) + u(y, y) + 2^d u(x, y) = 2u(x, y) + u(x + y, x + y).$$

So one can take $w(x) = (2^d - 2)^{-1} u(x, x)$, as desired. □

Now, applying Lemma 1.33.2, we get that there exists $w \in A_0$ such that $z - w$ is a primitive element, which implies that $z - w \in A_0$, so $z \in A_0$. □

Remark 1.33.3. The Cartier-Kostant theorem implies that any cocommutative Hopf algebra over an algebraically closed field of characteristic zero in which the only grouplike element is 1 is of the form $U(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra (a version of the Milnor-Moore theorem), in particular is generated by primitive elements. The latter statement is false in positive charactersitic. Namely, consider the commutative Hopf algebra $\mathbb{Q}[x, z]$ where x, z are primitive, and set $y = z + x^p/p$, where p is a prime. Then

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i \otimes x^{p-i}. \tag{1.33.3}$$

Since the numbers $\frac{1}{p}\binom{p}{i}$ are integers, this formula (together with $\Delta(x) = x\otimes 1+1\otimes x$, $S(x) = -x$, $S(y) = -y$) defines a Hopf algebra structure on $H = k[x, y]$ for any field k, in particular, one of characteristic p. But if k has characteristic p, then it is easy to see that H is not generated by primitive elements (namely, the element y is not in the subalgebra generated by them).

The Cartier-Kostant theorem implies that any affine pro-algebraic group scheme over a field of characteristic zero is in fact a pro-algebraic group. Namely, we have

Corollary 1.33.4. *Let H be a commutative Hopf algebra over a field k of characteristic zero. Then H has no nonzero nilpotent elements.*

Proof. It is clear that H is a union of finitely generated Hopf subalgebras (generated by finite dimensional subcoalgebras of H), so we may assume that H is finitely generated. Let $\mathfrak{m}$ be the kernel of the counit of H, and $B = \cup_{n=1}^{\infty}(H/\mathfrak{m}^n)^*$ (i.e., B is the continuous dual of the formal completion of H near the ideal $\mathfrak{m}$). It is easy to see that B is a cocommutative Hopf algebra, and its only grouplike element is 1. So by the Cartier-Kostant theorem $B = U(\mathfrak{g})$, where $\mathfrak{g} = (\mathfrak{m}/\mathfrak{m}^2)^*$. This implies that $G = \mathrm{Spec}(H)$ is smooth at $1 \in G$, i.e. it is an algebraic group, as desired. □

Remark 1.33.5. Note that Corollary 1.33.4 is a generalization of Corollary 1.27.6.

1.34. **Quasi-bialgebras.** Let us now discuss reconstruction theory for quasi-fiber functors. This leads to the notion of quasi-bialgebras and quasi-Hopf algebras, which were introduced by Drinfeld in [Dr1] as linear algebraic counterparts of abelian monoidal categories with quasi-fiber functors.

Definition 1.34.1. Let $\mathcal{C}$ be an abelian monoidal category over k, and $(F, J) : \mathcal{C} \to \mathrm{Vec}$ be a quasi-fiber functor. (F, J) is said to be *normalized* if $J_{\mathbf{1}X} = J_{X\mathbf{1}} = \mathrm{Id}_{F(X)}$ for all $X \in \mathcal{C}$.

Definition 1.34.2. Two quasi-fiber functors (F, J_1) and (F, J_2) are said to be *twist equivalent* (by the twist $J_1^{-1}J_2$).

Since for a quasi-fiber functor (unlike a fiber functor), the isomorphism J is not required to satisfy any equations, it typically does not carry any valuable structural information, and thus it is more reasonable to classify quasi-fiber functors not up to isomorphism, but rather up to twist equivalence combined with isomorphism.

Remark 1.34.3. It is easy to show that any quasi-fiber functor is equivalent to a normalized one.

Now let $\mathcal{C}$ be a finite abelian monoidal category over k, and let (F, J) be a normalized quasi-fiber functor. Let $H = \mathsf{End}\, F$ be the corresponding finite dimensional algebra. Then H has a coproduct Δ and a counit ε defined exactly as in the case of a fiber functor, which are algebra homomorphisms. The only difference is that, in general, Δ is not coassociative, since J does not satisfy the monoidal structure axiom. Rather, there is an invertible element $\Phi \in H^{\otimes 3}$, defined by the commutative diagram

(1.34.1)

$$\begin{array}{ccc}
(F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{\Phi_{F(X),F(Y),F(Z)}} & F(X)\otimes (F(Y)\otimes F(Z)) \\
\Big\downarrow{\scriptstyle J_{X,Y}\otimes \mathrm{Id}_{F(Z)}} & & \Big\downarrow{\scriptstyle \mathrm{Id}_{F(X)}\otimes J_{Y,Z}} \\
F(X\otimes Y)\otimes F(Z) & & F(X)\otimes F(Y\otimes Z) \\
\Big\downarrow{\scriptstyle J_{X\otimes Y,Z}} & & \Big\downarrow{\scriptstyle J_{X,Y\otimes Z}} \\
F((X\otimes Y)\otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X\otimes (Y\otimes Z))
\end{array}$$

for all $X, Y, Z \in \mathcal{C}$, and we have the following proposition.

Proposition 1.34.4. *The following identities hold:*

$$(\mathrm{Id}\otimes\Delta)(\Delta(h)) = \Phi(\Delta\otimes\mathrm{Id})(\Delta(h))\Phi^{-1}, \quad h \in H, \tag{1.34.2}$$

(1.34.3)

$$(\mathrm{Id}\otimes\mathrm{Id}\otimes\Delta)(\Phi)(\Delta\otimes\mathrm{Id}\otimes\mathrm{Id})(\Phi) = (1\otimes\Phi)(\mathrm{Id}\otimes\Delta\otimes\mathrm{Id})(\Phi)(\Phi\otimes 1),$$

$$(\varepsilon\otimes\mathrm{Id})(\Delta(h)) = h = (\mathrm{Id}\otimes\varepsilon)(\Delta(h)), \tag{1.34.4}$$

$$(\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id})(\Phi) = 1\otimes 1. \tag{1.34.5}$$

Proof. The first identity follows from the definition of Φ, the second one from the pentagon axiom for $\mathcal{C}$, the third one from the condition that (F, J) is normalized, and the fourth one from the triangle axiom and the condition that (F, J) is normalized. □

Definition 1.34.5. An associative unital k-algebra H equipped with unital algebra homomorphisms $\Delta : H \to H \otimes H$ (the coproduct) and $\varepsilon : H \to k$ (the counit) and an invertible element $\Phi \in H^{\otimes 3}$ satisfying the identities of Proposition 1.34.4 is called a *quasi-bialgebra*. The element Φ is called the *associator* of H.

Thus, the notion of a quasi-bialgebra is a generalization of the notion of a bialgebra; namely, a bialgebra is a quasi-bialgebra with $\Phi = 1$.[11]

For a quasi-bialgebra H, the tensor product of (left) H-modules V and W is an H-module via Δ, i.e., in the same way as for bialgebras. Also, it follows from (1.34.2) that for any H-modules U, V, W the mapping

$$(1.34.6)\quad a_{U,V,W} : (U \otimes V) \otimes W \cong U \otimes (V \otimes W) : u \otimes v \otimes w \mapsto \Phi(u \otimes v \otimes w)$$

is an H-module isomorphism. The axiom (1.34.4) implies that the natural maps $l_V = \mathrm{Id} : \mathbf{1} \otimes V \xrightarrow{\sim} V$ and $r_V = \mathrm{Id} : V \otimes \mathbf{1} \xrightarrow{\sim} V$ are also H-module isomorphisms. Finally, equations (1.34.3) and (1.34.5) say, respectively, that the pentagon axiom (1.1.2) and the triangle axiom (1.2.1) are satisfied for $\mathbf{Rep}(H)$. In other words, $\mathbf{Rep}(H)$ is a monoidal category.

Definition 1.34.6. A *twist* for a quasi-bialgebra H is an invertible element $J \in H \otimes H$ such that $(\varepsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \varepsilon)(J) = 1$. Given a twist, we can define a new quasi-bialgebra H^J which is H as an algebra, with the same counit, the coproduct given by

$$\Delta^J(x) = J^{-1}\Delta(x)J,$$

and the associator given by

$$\Phi^J = (\mathrm{Id} \otimes J)^{-1}(\mathrm{Id} \otimes \Delta)(J)^{-1}\Phi(\Delta \otimes \mathrm{Id})(J)(J \otimes \mathrm{Id})$$

The algebra H^J is called *twist equivalent* to H, by the twist J.

It is easy to see that twist equivalent quasi-fiber functors produce twist-equivalent quasi-bialgebras, and vice versa. Also, we have the following proposition.

Proposition 1.34.7. *If a finite k-linear abelian monoidal category $\mathcal{C}$ admits a quasi-fiber functor, then this functor is unique up to twisting.*

Proof. Let X_i, $i = 1, ..., n$ be the simple objects of $\mathcal{C}$. The functor F is exact, so it is determined up to isomorphism by the numbers $d_i = \dim F(X_i)$. So our job is to show that these numbers are uniquely determined by $\mathcal{C}$.

Let $N_i = (N_{ij}^k)$ be the matrix of left multiplication by X_i in the Grothendieck ring of $\mathcal{C}$ in the basis $\{X_j\}$, i.e.

$$X_iX_j = \sum N_{ij}^k X_k$$

(so, k labels the rows and j labels the columns of N_i).

[11]However, note that Δ can be coassociative even if $\Phi \neq 1$.

We claim that d_i is the spectral radius of N_i. Indeed, on the one hand, we have

$$\sum N_{ij}^m d_m = d_i d_j,$$

so d_i is an eigenvalue of N_i^T, hence of N_i. On the other hand, if e_j is the standard basis of $\mathbb{Z}^n$ then for any $r \geq 0$ the sum of the coordinates of the vector $N_i^r e_j$ is the length of the object $X_i^{\otimes r} \otimes X_j$, so it is dominated by $d_i^r d_j$. This implies that the spectral radius of N_i is at most d_i. This means that the spectral radius is exactly d_i, as desired. □

Therefore, we have the following reconstruction theorem.

Theorem 1.34.8. *The assignments* $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$*,* $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ *are mutually inverse bijections between*

1) finite k*-linear abelian monoidal categories* $\mathcal{C}$ *admitting a quasi-fiber functor, up to monoidal equivalence of categories.*

2) finite dimensional quasi-bialgebras H *over* k *up to twist equivalence and isomorphism.*

Proof. Straightforward from the above. □

Exercise 1.34.9. Suppose that in the situation of Exercise 1.21.6, the functor F is equipped with a quasi-monoidal structure J, i.e. an isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$, such that $J_{\mathbf{1}X} = J_{X\mathbf{1}} = \mathrm{Id}_{F(X)}$. Show that this endows H with the structure of a quasi-bialgebra, such that (F, J) defines a monoidal equivalence $\mathcal{C} \to \mathbf{Rep}(H)$.

Remark 1.34.10. Proposition 1.34.7 is false for infinite categories. For example, it is known that if $\mathcal{C} = \mathrm{Rep}(SL_2(\mathbb{C}))$, and $V \in \mathcal{C}$ is a 2-dimensional repesentation, then there exists a for any positive integer $n \geq 2$ there exists a fiber functor on $\mathcal{C}$ with $\dim F(V) = n$ (see [Bi]).

1.35. **Quasi-bialgebras with an antipode and quasi-Hopf algebras.** Now consider the situation of the previous subsection, and assume that the category $\mathcal{C}$ has right duals. In this case, by Proposition 1.13.5, the right dualization functor is exact; it is also faithful by Proposition 1.10.9. Therefore, the functor $F(V^*)^*$ is another quasi-fiber functor on $\mathcal{C}$. So by Proposition 1.34.7, this functor is isomorphic to F. Let us fix such an isomorphism $\xi = (\xi_V)$, $\xi_V : F(V) \to F(V^*)^*$. Then we have natural linear maps $k \to F(V) \otimes F(V^*)$, $F(V^*) \otimes F(V) \to k$ constructed as in Exercise 1.10.6, which can be regarded as linear maps $\hat{\alpha} : F(V) \to F(V^*)^*$ and $\hat{\beta} : F(V^*)^* \to F(V)$. Thus, the quasi-bialgebra $H = \mathsf{End}(F)$ has the following additional structures.

1. The elements $\alpha, \beta \in H$ such that for any $V \in \mathcal{C}$, $\alpha_V = \xi_V^{-1} \circ \hat{\alpha}_V$, $\beta_V = \hat{\beta}_V \circ \xi_V$. Note that α and β are not necessarily invertible.

2. The antipode $S : H \to H$, which is a unital algebra antihomomorphism such that if $\Delta(a) = \sum_i a_i^1 \otimes a_i^2$, $a \in H$, then

$$\sum_i S(a_i^1)\alpha a_i^2 = \varepsilon(a)\alpha, \qquad \sum_i a_i^1 \beta S(a_i^2) = \varepsilon(a)\beta. \tag{1.35.1}$$

Namely, for $a \in H$ $S(a)$ acts on $F(V)$ by $\xi^{-1} \circ a^*_{F(V^*)} \circ \xi$.

Let us write the associator as $\Phi = \sum_i \Phi_i^1 \otimes \Phi_i^2 \otimes \Phi_i^3$ and its inverse as $\sum \bar{\Phi}_i^1 \otimes \bar{\Phi}_i^2 \otimes \bar{\Phi}_i^3$.

Proposition 1.35.1. *One has*

$$\sum \Phi_i^1 \beta S(\Phi_i^2)\alpha \Phi_i^3 = 1, \qquad \sum S(\bar{\Phi}_i^1)\alpha \bar{\Phi}_i^2 \beta S(\bar{\Phi}_i^3) = 1. \tag{1.35.2}$$

Proof. This follows directly from the duality axioms. □

Definition 1.35.2. An *antipode* on a quasi-bialgebra H is a triple (S, α, β), where $S : H \to H$ is a unital antihomomorphism and $\alpha, \beta \in H$, satisfying identities (1.35.1) and (1.35.2).

A *quasi-Hopf algebra* is a quasi-bialgebra $(H, \Delta, \varepsilon, \Phi)$ for which there exists an antipode (S, α, β) such that S is bijective.

Thus, the notion of a quasi-Hopf algebra is a generalization of the notion of a Hopf algebra; namely, a Hopf algebra is a quasi-Hopf algebra with $\Phi = 1$, $\alpha = \beta = 1$.

We see that if in the above setting $\mathcal{C}$ has right duals, then $H = \mathsf{End}(F)$ is a finite dimensional bialgebra admitting antipode, and if $\mathcal{C}$ is rigid (i.e., a tensor category), then H is a quasi-Hopf algebra.

Conversely, if H is a quasi-bialgebra with an antipode, then the category $\mathcal{C} = \mathrm{Rep}(H)$ admits right duals. Indeed, the right dual module of an H-module V is defined as in the Hopf algebra case: it is the dual vector space V^* with the action of H given by

$$\langle h\phi, v \rangle = \langle \phi, S(h)v \rangle, \qquad v \in V, \phi \in V^*, h \in H.$$

Let $\sum v_i \otimes f_i$ be the image of Id_V under the canonical isomorphism $\mathsf{End}(V) \xrightarrow{\sim} V \otimes V^*$. Then the evaluation and coevaluation maps are defined using the elements α and β:

$$\mathrm{ev}_V(f \otimes v) = f(\alpha v), \mathrm{coev}_V(1) = \sum \beta v_i \otimes f_i.$$

Axiom (1.35.1) is then equivalent to ev_V and coev_V being H-module maps. Equations (1.35.2) are equivalent, respectively, to axioms (1.10.1) and (1.10.2) of a right dual.

If S is invertible, then the right dualization functor is an equivalence of categories, so the representation category $\mathrm{Rep}(H)$ of a quasi-Hopf algebra H is rigid, i.e., a tensor category.

Exercise 1.35.3. Let $H := (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ be a quasi-bialgebra with an antipode, and $u \in H$ be an invertible element.

(i) Show that if one sets

$$\overline{S}(h) = uS(h)u^{-1}, \qquad \overline{\alpha} = u\alpha, \quad \text{and} \quad \overline{\beta} = \beta u^{-1} \tag{1.35.3}$$

then the triple $(\overline{S}, \overline{\alpha}, \overline{\beta})$ is an antipode.

(ii) Conversely, show that any $\overline{S}$, $\overline{\alpha}$, and $\overline{\beta}$ satisfying conditions (1.35.1) and (1.35.2) are given by formulas (1.35.3) for a uniquely defined u.

Hint. If H is finite dimensional, (ii) can be formally deduced from the uniqueness of the right dual in a tensor category up to a unique isomorphism. Use this approach to obtain the unique possible formula for u, and check that it does the job for any H.

Remark 1.35.4. The non-uniqueness of S, α, and β observed in Exercise 1.35.3 reflects the freedom in choosing the isomorphism ξ.

Example 1.35.5. (cf. Example 1.10.14) Let G be a finite group and let $\omega \in Z^3(G, k^\times)$ be a normalized 3-cocycle, see (1.3.1). Consider the algebra $H = \mathrm{Fun}(G, k)$ of k-valued functions on G with the usual coproduct and counit. Set

$$\Phi = \sum \omega(f, g, h) p_f \otimes p_g \otimes p_h, \quad \alpha = \sum \omega(g, g^{-1}, g) p_g, \quad \beta = 1,$$

where p_g is the primitive idempotent of H corresponding to $g \in G$. It is straightforward to check that these data define a commutative quasi-Hopf algebra, which we denote $\mathrm{Fun}(G, k)_\omega$. The tensor category $\mathrm{Rep}(\mathrm{Fun}(G, k)_\omega)$ is obviously equivalent to Vec_G^ω.

It is easy to show that a twist of a quasi-bialgebra H with an antipode is again a quasi-bialgebra with an antipode (this reflects the fact that in the finite dimensional case, the existence of an antipode for H is the property of the category of finite dimensional representations of H). Indeed, if the twist J and its inverse have the form

$$J = \sum_i a_i \otimes b_i, \;\; J^{-1} = \sum_i a_i' \otimes b_i'$$

then H^J has an antipode (S^J, α^J, β^J) with $S^J = S$ and $\alpha^J = \sum_i S(a_i)\alpha b_i$, $\beta^J = \sum_i a_i' \beta S(b_i')$. Thus, we have the following reconstruction theorem.

Theorem 1.35.6. *The assignments* $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$*,* $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ *are mutually inverse bijections between*

(i) finite abelian k-linear monoidal categories $\mathcal{C}$ with right duals admitting a quasi-fiber functor, up to monoidal equivalence of categories,

and finite dimensional quasi-bialgebras H over k with an antipode, up to twist equivalence and isomorphism;

(ii) finite tensor categories $\mathcal{C}$ admitting a quasi-fiber functor, up to monoidal equivalence of categories, and finite dimensional quasi-Hopf algebras H over k, up to twist equivalence and isomorphism.

Remark 1.35.7. One can define the dual notions of a coquasi-bialgebra and coquasi-Hopf algebra, and prove the corresponding reconstruction theorems for tensor categories which are not necessarily finite. This is straightforward, but fairly tedious, and we will not do it here.

1.36. **Twists for bialgebras and Hopf algebras.** Let H be a bialgebra. We can regard it as a quasi-bialgebra with $\Phi = 1$. Let J be a twist for H.

Definition 1.36.1. J is called a bialgebra twist if H^J is a bialgebra, i.e. $\Phi^J = 1$.

Thus, a bialgebra twist for H is an invertible element $J \in H \otimes H$ such that $(\varepsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \varepsilon)(J) = 1$, and J satisfies the *twist equation*

$$(\mathrm{Id} \otimes \Delta)(J)(\mathrm{Id} \otimes J) = (\Delta \otimes \mathrm{Id})(J)(J \otimes \mathrm{Id}). \tag{1.36.1}$$

Exercise 1.36.2. Show that if a bialgebra H has an antipode S, and J is a bialgebra twist for H, then the bialgebra H^J also has an antipode. Namely, let $J = \sum a_i \otimes b_i$, $J^{-1} = \sum a_i' \otimes b_i'$, and set $Q_J = \sum_i S(a_i)b_i$. Then Q_J is invertible with $Q_J^{-1} = \sum_i a_i' S(b_i')$, and the antipode of H^J is defined by $S^J(x) = Q_J^{-1} S(x) Q_J$. In particular, a bialgebra twist of a Hopf algebra is again a Hopf algebra.

Remark 1.36.3. Twisting does not change the category of H-modules as a monoidal category, and the existence of an antipode (for finite dimensional H) is a categorical property (existence of right duals). This yields the above formulas, and then one easily checks that they work for any H.

Any twist on a bialgebra H defines a fiber functor (Id, J) on the category $\mathrm{Rep}(H)$. However, two different twists J_1, J_2 may define isomorphic fiber functors. It is easy to see that this happens if there is an invertible element $v \in H$ such that

$$J_2 = \Delta(v) J_1 (v^{-1} \otimes v^{-1}).$$

In this case the twists J_1 and J_2 are called *gauge equivalent* by the gauge transformation v, and the bialgebras H^{J_1}, H^{J_2} are isomorphic (by conjugation by v). So, we have the following result.

Proposition 1.36.4. *Let H be a finite dimensional bialgebra. Then $J \mapsto (\mathrm{Id}, J)$ is a bijection between:*
1) gauge equivalence classes of bialgebra twists for H, and
2) fiber functors on $\mathcal{C} = \mathrm{Rep}(H)$, up to isomorphism.

Proof. By Proposition 1.34.7, any fiber functor on $\mathcal{C}$ is isomorphic to the forgetful functor F as an additive functor. So any fiber functor, up to an isomorphism, has the form (F, J), where J is a bialgebra twist. Now it remains to determine when (F, J_1) and (F, J_2) are isomorphic. Let $v : (F, J_1) \to (F, J_2)$ be an isomorphism. Then $v \in H$ is an invertible element, and it defines a gauge transformation mapping J_1 to J_2. □

Proposition 1.36.5. *Let G be a group. Then fiber functors on Vec_G up to an isomorphism bijectively correspond to $H^2(G, k^\times)$.*

Proof. A monoidal structure on the forgetful functor F is given by a function $J(g, h) : \delta_g \otimes \delta_h \to \delta_g \otimes \delta_h$, $J(g, h) \in k^\times$. It is easy to see that the monoidal structure condition is the condition that J is a 2-cocycle, and two 2-cocycles define isomorphic monoidal structures if and only if they differ by a coboundary. Thus, equivalence classes of monoidal structures on F are parametrized by $H^2(G, k^\times)$, as desired. □

Remark 1.36.6. Proposition 1.36.5 shows that there may exist non-isomorphic fiber functors on a given finite tensor category $\mathcal{C}$ defining isomorphic Hopf algebras. Indeed, all fiber functors on Vec_G yield the same Hopf algebra $\mathrm{Fun}(G, k)$. These fiber functors are, however, all equivalent to each other by monoidal autoequivalences of $\mathcal{C}$.

Remark 1.36.7. Since Vec_G^ω does not admit fiber functors for cohomologically nontrivial ω, Proposition 1.36.5 in fact classifies fiber functors on all categories Vec_G^ω.

1.37. **Quantum traces.** Let $\mathcal{C}$ be a rigid monoidal category, V be an object in $\mathcal{C}$, and $a \in \mathsf{Hom}(V, V^{**})$. Define the left quantum trace

$$\mathrm{Tr}_V^L(a) := \mathrm{ev}_{V^*} \circ (a \otimes \mathrm{Id}_{V^*}) \circ \mathrm{coev}_V \in \mathsf{End}(\mathbf{1}). \tag{1.37.1}$$

Similarly, if $a \in \mathsf{Hom}(V, {}^{**}V)$ then we can define the right quantum trace

$$\mathrm{Tr}_V^R(a) := \mathrm{ev}_{{}^{**}V} \circ (\mathrm{Id}_{{}^*V} \otimes a) \circ \mathrm{coev}_{{}^*V} \in \mathsf{End}(\mathbf{1}). \tag{1.37.2}$$

In a tensor category over k, $\mathrm{Tr}^L(a)$ and $\mathrm{Tr}^R(a)$ can be regarded as elements of k.

When no confusion is possible, we will denote Tr_V^L by Tr_V.

The following proposition shows that usual linear algebra formulas hold for the quantum trace.

Proposition 1.37.1. *If* $a \in \mathsf{Hom}(V, V^{**})$, $b \in \mathsf{Hom}(W, W^{**})$ *then*

(1) $Tr^L_V(a) = Tr^R_{V^*}(a^*)$;
(2) $Tr^L_{V \oplus W}(a \oplus b) = Tr^L_V(a) + Tr^L_W(b)$ *(in additive categories);*
(3) $Tr^L_{V \otimes W}(a \otimes b) = Tr^L_V(a)\, Tr^L_W(b)$;
(4) *If* $c \in \mathsf{Hom}(V, V)$ *then* $Tr^L_V(ac) = Tr^L_V(c^{**}a)$, $Tr^R_V(ac) = Tr^R_V({}^{**}ca)$.

Similar equalities to (2),(3) also hold for right quantum traces.

Exercise 1.37.2. Prove Proposition 1.37.1.

If $\mathcal{C}$ is a multitensor category, it is useful to generalize Proposition 1.37.1(2) as follows.

Proposition 1.37.3. *If* $a \in \mathsf{Hom}(V, V^{**})$ *and* $W \subset V$ *such that* $a(W) \subset W^{**}$ *then* $Tr^L_V(a) = Tr^L_W(a) + Tr^L_{V/W}(a)$. *That is, Tr is additive on exact sequences. The same statement holds for right quantum traces.*

Exercise 1.37.4. Prove Proposition 1.37.3.

1.38. **Pivotal categories and dimensions.**

Definition 1.38.1. Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is an isomorphism of monoidal functors $a : \mathrm{Id} \xrightarrow{\sim} ?^{**}$.

That is, a pivotal structure is a collection of morphisms $a_X : X \xrightarrow{\sim} X^{**}$ natural in X and satisfying $a_{X \otimes Y} = a_X \otimes a_Y$ for all objects X, Y in $\mathcal{C}$.

Definition 1.38.2. A rigid monoidal category $\mathcal{C}$ equipped with a pivotal structure is said to be *pivotal*.

Exercise 1.38.3. (1) If a is a pivotal structure then $a_{V^*} = (a_V)^{*-1}$. Hence, $a_{V^{**}} = a_V^{**}$.
(2) Let $\mathcal{C} = \mathrm{Rep}(H)$, where H is a finite dimensional Hopf algebra. Show that pivotal structures on $\mathcal{C}$ bijectively correspond to group-like elements of H such that $gxg^{-1} = S^2(x)$ for all $x \in H$.

Let a be a pivotal structure on a rigid monoidal category $\mathcal{C}$.

Definition 1.38.4. The *dimension* of an object X with respect to a is $\dim_a(X) := \mathrm{Tr}(a_X) \in \mathsf{End}(\mathbf{1})$.

Thus, in a tensor category over k, dimensions are elements of k. Also, it follows from Exercise 1.38.3 that $\dim_a(V) = \dim_a(V^{**})$.

Proposition 1.38.5. *If* $\mathcal{C}$ *is a tensor category, then the function* $X \mapsto \dim_a(X)$ *is a character of the Grothendieck ring* $\mathsf{Gr}(\mathcal{C})$.

Proof. Proposition 1.37.3 implies that $\dim_a$ is additive on exact sequences, which means that it gives rise to a well-defined linear map from $\mathsf{Gr}(\mathcal{C})$ to k. The fact that this map is a character follows from the obvious fact that $\dim_a(\mathbf{1}) = 1$ and Proposition 1.37.1(3). $\square$

Corollary 1.38.6. *Dimensions of objects in a pivotal finite tensor category are algebraic integers in k.* [12]

Proof. This follows from the fact that a character of any ring that is finitely generated as a $\mathbb{Z}$-module takes values in algebraic integers. $\square$

1.39. **Spherical categories.**

Definition 1.39.1. A pivotal structure a on a tensor category $\mathcal{C}$ is *spherical* if $\dim_a(V) = \dim_a(V^*)$ for any object V in $\mathcal{C}$. A tensor category is *spherical* if it is equipped with a spherical structure.

Since $\dim_a$ is additive on exact sequences, it suffices to require the property $\dim_a(V) = \dim_a(V^*)$ only for simple objects V.

Theorem 1.39.2. *Let $\mathcal{C}$ be a spherical category and V be an object of $\mathcal{C}$. Then for any $x \in \mathsf{Hom}(V, V)$ one has $Tr^L_V(a_V x) = Tr^R_V(x a_V^{-1})$.*

Proof. We first note that $\mathrm{Tr}^R_X(a_X^{-1}) = \dim_a(X^*)$ for any object X by Proposition 1.37.1(1) and Exercise 1.38.3(1). Now let us prove the proposition in the special case when V is semisimple. Thus $V = \oplus_i Y_i \otimes V_i$, where V_i are vector spaces and Y_i are simple objects. Then $x = \oplus_i x_i \otimes \mathrm{Id}_{V_i}$ with $x_i \in \mathsf{End}_k(Y_i)$ and $a = \oplus \mathrm{Id}_{Y_i} \otimes a_{V_i}$ (by the functoriality of a). Hence

$$\begin{aligned} \mathrm{Tr}^L_V(ax) &= \sum \mathrm{Tr}(x_i) \dim(V_i), \\ \mathrm{Tr}^R_V(xa^{-1}) &= \sum \mathrm{Tr}(x_i) \dim(V_i^*). \end{aligned}$$

This implies the result for a semisimple V.

Consider now the general case. Then V has the coradical filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \tag{1.39.1}$$

(such that V_{i+1}/V_i is a maximal semisimple subobject in V/V_i). This filtration is preserved by x and by a (i.e., $a : V_i \to V_i^{**}$). Since traces are additive on exact sequences by Proposition 1.37.3, this implies that the general case of the required statement follows from the semisimple case. $\square$

[12] If k has positive characteristic, by an algebraic integer in k we mean an element of a finite subfield of k.

Exercise 1.39.3. (i) Let $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})$ be the group of isomorphism classes of monoidal automorphisms of a monoidal category $\mathcal{C}$. Show that the set of isomorphism classes of pivotal structures on $\mathcal{C}$ is a torsor over $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})$, and the set of isomorphism classes of spherical structures on $\mathcal{C}$ is a torsor over the subgroup $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})_2$ in $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})$ of elements which act by ± 1 on simple objects.

1.40. **Semisimple multitensor categories.** In this section we will more closely consider semisimple multitensor categories which have some important additional properties compared to the general case.

1.41. **Isomorphism between V^{**} and V.**

Proposition 1.41.1. *Let $\mathcal{C}$ be a semisimple multitensor category and let V be an object in $\mathcal{C}$. Then ${}^*V \cong V^*$. Hence, $V \cong V^{**}$.*

Proof. We may assume that V is simple.

We claim that the unique simple object X such that $\mathsf{Hom}(\mathbf{1}, V \otimes X) \neq 0$ is V^*. Indeed, $\mathsf{Hom}(\mathbf{1}, V \otimes X) \cong \mathsf{Hom}({}^*X, V)$ which is non-zero if and only if ${}^*X \cong V$, i.e., $X \cong V^*$. Similarly, the unique simple object X such that $\mathsf{Hom}(V \otimes X, \mathbf{1}) \neq 0$ is ${}^*V$. But since $\mathcal{C}$ is semisimple, $\dim_k \mathsf{Hom}(\mathbf{1}, V \otimes X) = \dim_k \mathsf{Hom}(V \otimes X, \mathbf{1})$, which implies the result. □

Remark 1.41.2. As noted in Remark 1.27.2, the result of Proposition 1.41.1 is false for non-semisimple categories.

Remark 1.41.3. Proposition 1.41.1 gives rise to the following question.

Question 1.41.4. Does any semisimple tensor category admit a pivotal structure? A spherical structure?

This is the case for all known examples. The general answer is unknown to us at the moment of writing (even for ground fields of characteristic zero).

Proposition 1.41.5. *If $\mathcal{C}$ is a semisimple tensor category and $a : V \xrightarrow{\sim} V^{**}$ for a simple object V then $Tr(a) \neq 0$.*

Proof. $Tr(a)$ is the composition morphism of the diagram $\mathbf{1} \to V \otimes V^* \to \mathbf{1}$ where both morphisms are non-zero. If the composition morphism is zero then there is a non-zero morphism $(V \otimes V^*)/\mathbf{1} \to \mathbf{1}$ which means that the $[V \otimes V^* : 1] \geq 2$. Since $\mathcal{C}$ is semisimple, this implies that $\dim_k \mathsf{Hom}(1,\, V \otimes V^*)$ is at least 2. Hence, $\dim_k \mathsf{Hom}(V,\, V) \geq 2$ which contradicts the simplicity of V. □

Remark 1.41.6. The above result is false for non-semisimple categories. For example, let $\mathcal{C} = \mathrm{Rep}_k(GL_p(\mathbb{F}_p))$, the representation category of the group $GL_p(\mathbb{F}_p)$ over a field k of characteristic p. Let V be the p dimensional vector representation of $GL_p(\mathbb{F}_p)$ (which is clearly irreducible). Let $a : V \to V^{**}$ be the identity map. Then $\mathrm{Tr}(a) = \dim_k(V) = p = 0$ in k.

1.42. **Grothendieck rings of semisimple tensor categories.**

Definition 1.42.1. (i) A $\mathbb{Z}_+-$*basis* of an algebra free as a module over $\mathbb{Z}$ is a basis $B = \{b_i\}$ such that $b_i b_j = \sum_k c_{ij}^k b_k,\ c_{ij}^k \in \mathbb{Z}_+$.

(ii) A $\mathbb{Z}_+-$*ring* is an algebra over $\mathbb{Z}$ with unit equipped with a fixed $\mathbb{Z}_+-$basis.

Definition 1.42.2. (1) A $\mathbb{Z}_+-$ring A with basis $\{b_i\}_{i\in I}$ is called a *based ring* if the following conditions hold

[a] There exists a subset $I_0 \subset I$ such that $1 = \sum_{i\in I_0} b_i$.

[b] Let $\tau : A \to \mathbb{Z}$ be the group homomorphism defined by

$$\tau(b_i) = \begin{cases} 1 & \text{if} \quad i \in I_0 \\ 0 & \text{if} \quad i \notin I_0 \end{cases} \tag{1.42.1}$$

There exists an involution $i \mapsto i^*$ of I such that induced map $a = \sum_{i\in I} a_i b_i \mapsto a^* = \sum_{i \in I} a_i b_{i^*},\ a_i \in \mathbb{Z}$ is an anti-involution of ring A and such that

$$\tau(b_i b_j) = \begin{cases} 1 & \text{if} \quad i = j^* \\ 0 & \text{if} \quad i \neq j^*. \end{cases} \tag{1.42.2}$$

(2) A *unital* $\mathbb{Z}_+$*-ring* is a $\mathbb{Z}_+$-ring A such that 1 belongs to the basis.
(3) A *multifusion ring* is a based ring of finite rank. A *fusion ring* is a unital based ring of finite rank.

Remark 1.42.3. (1) It follows easily from definition that $i, j \in I_0,\ i \neq j$ implies that $b_i^2 = b_i,\ b_i b_j = 0,\ i^* = i$.
(2) It is easy to see that for a given $\mathbb{Z}_+-$ring A, being a (unital) based ring is a *property*, not an additional structure.
(3) Note that any $\mathbb{Z}_+$-ring is assumed to have a unit, and is not necessarily a unital $\mathbb{Z}_+$-ring.

Proposition 1.42.4. *If $\mathcal{C}$ is a semisimple multitensor category then $\mathsf{Gr}(\mathcal{C})$ is a based ring. If $\mathcal{C}$ is semisimple tensor category then $\mathsf{Gr}(\mathcal{C})$ is a unital based ring. If $\mathrm{Gr}(\mathcal{C})$ is a (multi)fusion category, then $\mathrm{Gr}_{\mathcal{C}}$ is a (multi)fusion ring.*

Proof. The $\mathbb{Z}_+$-basis in $\mathsf{Gr}(\mathcal{C})$ consists of isomorphism classes of simple objects of $\mathcal{C}$. The set I_0 consists of the classes of simple subobjects of

1. The involution $*$ is the duality map (by Proposition 1.41.1 it does not matter whether to use left or right duality). This implies the first two statements. The last statement is clear. □

Example 1.42.5. Let $\mathcal{C}$ be the category of finite dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Then the simple objects of this category are irreducible representations V_m of dimension $m+1$ for $m = 0, 1, 2, \ldots$; $V_0 = \mathbf{1}$. The Grothendieck ring of $\mathcal{C}$ is determined by the well-known Clebsch-Gordon rule

$$V_i \otimes V_j = \bigoplus_{l=|i-j|, i+j-l \in 2\mathbb{Z}}^{i+j} V_l. \tag{1.42.3}$$

The duality map on this ring is the identity. The same is true if $\mathcal{C} = \mathrm{Rep}(U_{\mathsf{q}}(\mathfrak{sl}_2))$ when q is not a root of unity, see [K].

Let $\mathcal{C}$ be a semisimple multitensor category with simple objects $\{X_i\}_{i \in I}$. Let I_0 be the subset of I such that $\mathbf{1} = \oplus_{i \in I_0} X_i$. Let $H^l_{ij} := \mathsf{Hom}(X_l, X_i \otimes X_j)$ (if $X_p \in \mathcal{C}_{ij}$ with $p \in I$ and $i, j \in I_0$, we will identify spaces H^p_{pi} and H^p_{ip} with k using the left and right unit morphisms).

We have $X_i \otimes X_j = \bigoplus_l H^l_{ij} \otimes X_l$. Hence,

$$\begin{aligned}(X_{i_1} \otimes X_{i_2}) \otimes X_{i_3} &\cong \bigoplus_{i_4} \bigoplus_{j} H^j_{i_1 i_2} \otimes H^{i_4}_{j i_3} \otimes X_{i_4} \\ X_{i_1} \otimes (X_{i_2} \otimes X_{i_3}) &\cong \bigoplus_{i_4} \bigoplus_{l} H^{i_4}_{i_1 l} \otimes H^{l}_{i_2 i_3} \otimes X_{i_4}.\end{aligned}$$

Thus the associativity constraint reduces to a collection of linear isomorphisms

$$\Phi^{i_4}_{i_1 i_2 i_3} : \bigoplus_j H^j_{i_1 i_2} \otimes H^{i_4}_{j i_3} \cong \bigoplus_l H^{i_4}_{i_1 l} \otimes H^l_{i_2 i_3}. \tag{1.42.4}$$

The matrix blocks of these isomorphisms,

$$(\Phi^{i_4}_{i_1 i_2 i_3})_{jl} : H^j_{i_1 i_2} \otimes H^{i_4}_{j i_3} \to H^{i_4}_{i_1 l} \otimes H^l_{i_2 i_3} \tag{1.42.5}$$

are called $6j$-*symbols* because they depend on six indices.

Example 1.42.6. Let $\mathcal{C}$ be the category of finite dimensional representations of the Lie algebra $sl_2(\mathbb{C})$. Then the spaces H^l_{ij} are 0- or 1-dimensional. In fact, it is obvious from the Clebsch-Gordan rule that the map $(\Phi^{i_4}_{i_1 i_2 i_3})_{jl}$ is a map between nonzero (i.e., 1-dimensional) spaces if and only if the numbers i_1, i_2, i_3, i_4, j, l are edge lengths of a tetrahedron with faces corresponding to the four H-spaces ($i_1 i_2 j$, $j i_3 i_4$,$i_1 l i_4$, $i_2 i_3 l$, such that the perimeter of every face is even (this tetrahedron is allowed to be in the Euclidean 3-space, Euclidean plane, or hyperbolic 3-space, so the only conditions are the triangle inequalities on the faces). In this case, the $6j$-symbol can be regarded as a number, provided we choose a basis vector in every non-zero H^l_{ij}. Under an appropriate normalization of basis vectors these numbers are the *Racah coefficients* or *classical* $6j$*-symbols*. More generally, if $\mathcal{C} = U_{\mathsf{q}}(\mathfrak{sl}_2)$, where q is not a root of unity, then the numbers $(\Phi^{i_4}_{i_1i_2i_3})_{jl}$ are called q-*Racah coefficients* or *quantum* $6j$*-symbols* [CFS].

Further, the evaluation and coevaluation maps define elements

$$\alpha_{ij} \in (H^j_{ii^*})^* \quad \text{and} \quad \beta_{ij} \in H^j_{ii^*},\ j \in I_0. \tag{1.42.6}$$

Now the axioms of a rigid monoidal category, i.e., the triangle and pentagon identities and the rigidity axioms translate into non-linear algebraic equations with respect to the $6j$-symbols $(\Phi^{i_4}_{i_1i_2i_3})_{jl}$ and vectors α_{ij}, β_{ij}.

Exercise 1.42.7. Write down explicitly the relation on $6j$ symbols coming from the pentagon identity. If $\mathcal{C} = \mathrm{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ this relation is called the *Elliott-Biedenharn* relation ([CFS]).

Proposition 1.42.4 gives rise to the following general problem of *categorification* of based rings which is one of the main problems in the structure theory of tensor categories.

Problem 1.42.8. Given a based ring R, describe (up to equivalence) all multitensor categories over k whose Grothendieck ring is isomorphic to R.

It is clear from the above explanations that this problem is equivalent to finding all solutions of the system of algebraic equations coming from the axioms of the rigid monoidal category modulo the group of automorphisms of the spaces H^k_{ij} ("gauge transformations"). In general, this problem is very difficult because the system of equations involved is nonlinear, contains many unknowns and is usually over-determined. In particular, it is not clear a priori whether for a given R this system has at least one solution, and if it does, whether the set of these solutions is finite. It is therefore amazing that the theory of tensor categories allows one to solve the categorification problem in a number of nontrivial cases. This will be done in later parts of these notes; now we will only mention the simplest result in this direction, which follows from the results of Subsection 1.7.

Let $\mathbb{Z}[G]$ be the group ring of a group G, with basis $\{g \in G\}$ and involution $g^* = g^{-1}$. Clearly, $\mathbb{Z}[G]$ is a unital based ring.

Proposition 1.42.9. *The categorifications of* $\mathbb{Z}[G]$ *are* Vec_G^ω*, and they are parametrized by* $H^3(G, k^\times)/\mathrm{Out}(G)$.

Remark 1.42.10. It is tempting to say that any $\mathbb{Z}_+$-ring R has a canonical categorification over any field k: one can take the skeletal semisimple category $\mathcal{C} = \mathcal{C}_R$ over k whose Grothendieck group is R, define the tensor product functor on $\mathcal{C}$ according to the multiplication in R, and then "define" the associativity isomorphism to be the identity (which appears to make sense because the category is skeletal, and therefore by the associativity of R one has $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$). However, a more careful consideration shows that this approach does not actually work. Namely, such "associativity isomorphism" fails to be functorial with respect to morphisms; in other words, if $g : Y \to Y$ is a morphism, then $(\mathrm{Id}_X \otimes g) \otimes \mathrm{Id}_Z$ is not always equal to $\mathrm{Id}_X \otimes (g \otimes \mathrm{Id}_Z)$.

To demonstrate this explicitly, denote the simple objects of the category $\mathcal{C}$ by X_i, $i = 1, ..., r$, and let $X_i \otimes X_j = \oplus_k N_{ij}^l X_l$. Take $X = X_i$, $Y = mX_j$, and $Z = X_l$; then g is an m by m matrix over k. The algebra $\mathsf{End}((X \otimes Y) \otimes Z) = \mathsf{End}(X \otimes (Y \otimes Z))$ is equal to $\oplus_s \mathrm{Mat}_{mn_s}(k)$, where

$$n_s = \sum_p N_{ij}^p N_{pl}^s = \sum_q N_{iq}^s N_{jl}^q,$$

and in this algebra we have

$$(\mathrm{Id}_X \otimes g) \otimes \mathrm{Id}_Z = \oplus_{p=1}^r \mathrm{Id}_{N_{ij}^p} \otimes g \otimes \mathrm{Id}_{N_{pl}^s},$$
$$\mathrm{Id}_X \otimes (g \otimes \mathrm{Id}_Z) = \oplus_{q=1}^r \mathrm{Id}_{N_{iq}^s} \otimes g \otimes \mathrm{Id}_{N_{jl}^q},$$

We see that these two matrices are, in general, different, which shows that the identity "associativity isomorphism" is not functorial.

1.43. **Semisimplicity of multifusion rings.**

Definition 1.43.1. A $*$-algebra is an associative algebra B over $\mathbb{C}$ with an antilinear anti-involution $* : B \to B$ and a linear functional $\tau : B \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$, and the Hermitian form $\tau(ab^*)$ is positive definite.

Obviously, any semisimple algebra $B = \oplus_{i=1}^r \mathrm{Mat}_i(\mathbb{C})$ is a $*$-algebra. Namely, if $p_i > 0$ are any positive numbers for $i = 1, ..., r$ then one can define $*$ to be the usual hermitian adjoint of matrices, and set $\tau(a_1, ..., a_r) = \sum_i p_i \mathrm{Tr}(a_i)$. Conversely, it is easy to see that any 8-algebra structure on a finite dimensional semisimple algebra has this form up to an isomorphism (and the numbers p_i are uniquely determined, as traces of central idempotents of B).

It turns out that this is the most general example of a finite dimensional $*$-algebra. Namely, we have

Proposition 1.43.2. *Any finite dimensional $*$-algebra B is semisimple.*

Proof. If $M \subset B$ is a subbimodule, and $M^\perp$ is the orthogonal complement of M under the form $\tau(ab^*)$, then $M^\perp$ is a subbimodule of B, and $M \cap M^\perp = 0$ because of the positivity of the form. So we have $B = M \oplus M^\perp$. Thus B is a semisimple B-bimodule, which implies the proposition. □

Corollary 1.43.3. *If e is a nonzero idempotent in a finite dimensional $*$-algebra B then $\tau(e) > 0$.*

The following proposition is obvious.

Proposition 1.43.4. *Let A be a based ring. Then the algebra $A \otimes_{\mathbb{Z}} \mathbb{C}$ is canonically a $*$-algebra.*

Corollary 1.43.5. *Let A be a multifusion ring. Then the algebra $A \otimes_{\mathbb{Z}} \mathbb{C}$ is semsimiple.*

Corollary 1.43.6. *Let X be a basis element of a fusion ring A. Then there exists $n > 0$ such that $\tau(X^n) > 0$.*

Proof. Since $\tau(X^n(X^*)^n) > 0$ for all $n > 0$, X is not nilpotent. Let

$$q(x) := \prod_{i=0}^{r} (x - a_i)^{m_i}$$

be the minimal polynomial of X (a_i are distinct). Assume that $a_0 \neq 0$ (we can do so since X is not nilpotent). Let

$$g(t) = \prod_{i=1}^{r} (x - a_i)^{m_i} x h(x),$$

where h is a polynomial chosen in such a way that $g(a_0) = 1$, $g^{(j)}(a_0) = 0$ for $j = 1, ..., m_0 - 1$ (this is clearly possible). Then $g(X)$ is an idempotent, so by Corollary 1.43.3, $\tau(g(X)) > 0$. Hence there exists $n > 0$ such that $\tau(X^n) \neq 0$, as desired. □

1.44. **The Frobenius-Perron theorem.** The following classical theorem from linear algebra [Ga, XIII.2] plays a crucial role in the theory of tensor categories.

Theorem 1.44.1. *Let B be a square matrix with nonnegative entries.*

(1) *B has a nonnegative real eigenvalue. The largest nonnegative real eigenvalue $\lambda(B)$ of B dominates the absolute values of all other eigenvalues μ of B: $|\mu| \leq \lambda(B)$ (in other words, the spectral radius of B is an eigenvalue). Moreover, there is an eigenvector of B with nonnegative entries and eigenvalue $\lambda(B)$.*

(2) *If B has strictly positive entries then $\lambda(B)$ is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries. Moreover, $|\mu| < \lambda(B)$ for any other eigenvalue μ of B.*

(3) *If B has an eigenvector $\mathbf{v}$ with strictly positive entries, then the corresponding eigenvalue is $\lambda(B)$.*

Proof. Let B be an n by n matrix with nonnegative entries. Let us first show that B has a nonnegative eigenvalue. If B has an eigenvector $\mathbf{v}$ with nonnegative entries and eigenvalue 0, then there is nothing to prove. Otherwise, let Σ be the set of column vectors $\mathbf{x} \in \mathbb{R}^n$ with with nonnegative entries x_i and $s(\mathbf{x}) := \sum x_i$ equal to 1 (this is a simplex). Define a continuous map $f_B : \Sigma \to \Sigma$ by $f_B(\mathbf{x}) = \frac{B\mathbf{x}}{s(B\mathbf{x})}$. By the Brouwer fixed point theorem, this map has a fixed point $\mathbf{f}$. Then $B\mathbf{f} = \lambda \mathbf{f}$, where $\lambda > 0$. Thus the eigenvalue $\lambda(B)$ is well defined, and B always has a nonnegative eigenvector $\mathbf{f}$ with eigenvalue $\lambda = \lambda(B)$.

Now assume that B has strictly positive entries. Then $\mathbf{f}$ must have strictly positive entries f_i. If $\mathbf{d}$ is another real eigenvector of B with eigenvalue λ, let z be the smallest of the numbers of d_i/f_i. Then the vector $\mathbf{v} = \mathbf{d} - z\mathbf{f}$ satisfies $B\mathbf{v} = \lambda \mathbf{v}$, has nonnegative entries and has one entry equal to zero. Hence $\mathbf{v} = 0$ and λ is a simple eigenvalue.

Now let $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{C}^n$ be a row vector. Define the norm $|\mathbf{y}| := \sum |y_j| f_j$. Then

$$|\mathbf{y}B| = \sum_j |\sum_i y_i b_{ij}| f_j \leq \sum_{i,j} |y_i| b_{ij} f_j = \lambda |\mathbf{y}|,$$

and the equality holds if and only if all the complex numbers $\sum y_i b_{ij}$ which are nonzero have the same argument. So if $\mathbf{y}B = \mu \mathbf{y}$, then $|\mu| \leq \lambda$, and if $|\mu| = \lambda$ then all y_i which are nonzero have the same argument, so we can renormalize $\mathbf{y}$ to have nonnegative entries. This implies that $\mu = \lambda$. Thus, (ii) is proved.

Now consider the general case (B has nonnegative entries). Assume that B has a row eigenevector $\mathbf{y}$ with strictly positive entries and eigenvalue μ. Then

$$\mu \mathbf{y}\mathbf{f} = \mathbf{y}B\mathbf{f} = \lambda \mathbf{y}\mathbf{f},$$

which implies $\mu = \lambda$, as $\mathbf{y}\mathbf{f} \neq 0$. This implies (iii).

It remains to finish the proof of (i) (i.e. to prove that $\lambda(B)$ dominates all other eigenvalues of B). Let Γ_B be the oriented graph whose vertices are labeled by $1, ..., n$, and there is an edge from j to i if and only if $b_{ij} > 0$. Let us say that i is accessible from j if there is a path in Γ_B leading from j to i. Let us call B irreducible if any vertex is accessible from any other. By conjugating B by a permutation matrix if necessary, we can get to a situation when $i \geq j$ implies that i is accessible from j. This means that B is a block upper triangular matrix, whose diagonal blocks are irreducible. So it suffices to prove the statement in question for irreducible B.

But if B is irreducible, then either B is the zero 1-by-1 matrix, or some power of B has strictly positive entries. So the result follows from (ii). □

1.45. **Tensor categories with finitely many simple objects. Frobenius-Perron dimensions.** Let A be a $\mathbb{Z}_+$-ring with $\mathbb{Z}_+$-basis I.

Definition 1.45.1. We will say that A is *transitive* if for any $X, Z \in I$ there exist $Y_1, Y_2 \in I$ such that XY_1 and Y_2X involve Z with a nonzero coefficient.

Proposition 1.45.2. *If $\mathcal{C}$ is a ring category with right duals then* $\mathsf{Gr}(\mathcal{C})$ *is a transitive unital $\mathbb{Z}_+$-ring.*

Proof. Recall from Theorem 1.15.8 that the unit object $\mathbf{1}$ in $\mathcal{C}$ is simple. So $\mathsf{Gr}(\mathcal{C})$ is unital. This implies that for any simple objects X, Z of $\mathcal{C}$, the object $X \otimes X^* \otimes Z$ contains Z as a composition factor (as $X \otimes X^*$ contains $\mathbf{1}$ as a composition factor), so one can find a simple object Y_1 occurring in $X^* \otimes Z$ such that Z occurs in $X \otimes Y_1$. Similarly, the object $Z \otimes X^* \otimes X$ contains Z as a composition factor, so one can find a simple object Y_2 occurring in $Z \otimes X^*$ such that Z occurs in $Y_2 \otimes X$. Thus $\mathsf{Gr}(\mathcal{C})$ is transitive. □

Let A be a transitive unital $\mathbb{Z}_+$-ring of finite rank. Define the group homomorphism $\mathsf{FPdim} : A \to \mathbb{C}$ as follows. For $X \in I$, let $\mathsf{FPdim}(X)$ be the maximal nonnegative eigenvalue of the matrix of left multiplication by X. It exists by the Frobenius-Perron theorem, since this matrix has nonnegative entries. Let us extend FPdim from the basis I to A by additivity.

Definition 1.45.3. The function FPdim is called the *Frobenius-Perron dimension.*

In particular, if $\mathcal{C}$ is a ring category with right duals and finitely many simple objects, then we can talk about Frobenius-Perron dimensions of objects of $\mathcal{C}$.

Proposition 1.45.4. *Let $X \in I$.*

(1) *The number $\alpha = \mathsf{FPdim}(X)$ is an algebraic integer, and for any algebraic conjugate α' of α we have $\alpha \geq |\alpha'|$.*
(2) $\mathsf{FPdim}(X) \geq 1$.

Proof. (1) Note that α is an eigenvalue of the integer matrix N_X of left multiplication by X, hence α is an algebraic integer. The number α' is a root of the characteristic polynomial of N_X, so it is also an eigenvalue of N_X. Thus by the Frobenius-Perron theorem $\alpha \geq |\alpha'|$.

(2) Let r be the number of algebraic conjugates of α. Then $\alpha^r \geq N(\alpha)$ where $N(\alpha)$ is the norm of α. This implies the statement since $N(\alpha) \geq 1$. □

Proposition 1.45.5. (1) *The function* $\mathsf{FPdim} : A \to \mathbb{C}$ *is a ring homomorphism.*

(2) *There exists a unique, up to scaling, element* $R \in A_{\mathbb{C}} := A \otimes_{\mathbb{Z}} \mathbb{C}$ *such that* $XR = \mathsf{FPdim}(X)R$*, for all* $X \in A$*. After an appropriate normalization this element has positive coefficients, and satisfies* $\mathsf{FPdim}(R) > 0$ *and* $RY = \mathsf{FPdim}(Y)R$*,* $Y \in A$*.*

(3) FPdim *is a unique nonzero character of* A *which takes nonnegative values on* I*.*

(4) *If* $X \in A$ *has nonnegative coefficients with respect to the basis of* A*, then* $\mathsf{FPdim}(X)$ *is the largest nonnegative eigenvalue* $\lambda(N_X)$ *of the matrix* N_X *of multiplication by* X*.*

Proof. Consider the matrix M of right multiplication by $\sum_{X \in I} X$ in A in the basis I. By transitivity, this matrix has strictly positive entries, so by the Frobenius-Perron theorem, part (2), it has a unique, up to scaling, eigenvector $R \in A_{\mathbb{C}}$ with eigenvalue $\lambda(M)$ (the maximal positive eigenvalue of M). Furthermore, this eigenvector can be normalized to have strictly positive entries.

Since R is unique, it satisfies the equation $XR = d(X)R$ for some function $d : A \to \mathbb{C}$. Indeed, XR is also an eigenvector of M with eigenvalue $\lambda(M)$, so it must be proportional to R. Furthermore, it is clear that d is a character of A. Since R has positive entries, $d(X) = \mathsf{FPdim}(X)$ for $X \in I$. This implies (1). We also see that $\mathsf{FPdim}(X) > 0$ for $X \in I$ (as R has strictly positive coefficients), and hence $\mathsf{FPdim}(R) > 0$.

Now, by transitivity, R is the unique, up to scaling, solution of the system of linear equations $XR = \mathsf{FPdim}(X)R$ (as the matrix N of left multiplication by $\sum_{X \in I} X$ also has positive entries). Hence, $RY = d'(Y)R$ for some character d'. Applying FPdim to both sides and using that $\mathsf{FPdim}(R) > 0$, we find $d' = \mathsf{FPdim}$, proving (2).

If χ is another character of A taking positive values on I, then the vector with entries $\chi(Y)$, $Y \in I$ is an eigenvector of the matrix N of the left multiplication by the element $\sum_{X \in I} X$. Because of transitivity of A the matrix N has positive entries. By the Frobenius-Perron theorem there exists a positive number λ such that $\chi(Y) = \lambda\,\mathsf{FPdim}(Y)$. Since χ is a character, $\lambda = 1$, which completes the proof.

Finally, part (4) follows from part (2) and the Frobenius-Perron theorem (part (3)). □

Example 1.45.6. Let $\mathcal{C}$ be the category of finite dimensional representations of a quasi-Hopf algebra H, and A be its Grothendieck ring. Then by Proposition 1.10.9, for any $X, Y \in \mathcal{C}$

$$\dim \mathsf{Hom}(X \otimes H, Y) = \dim \mathsf{Hom}(H, {}^*X \otimes Y) = \dim(X)\dim(Y),$$

where H is the regular representation of H. Thus $X \otimes H = \dim(X)H$, so $\mathsf{FPdim}(X) = \dim(X)$ for all X, and $R = H$ up to scaling.

This example motivates the following definition.

Definition 1.45.7. The element R will be called a *regular element* of A.

Proposition 1.45.8. *Let A be as above and $* : I \to I$ be a bijection which extends to an anti-automorphism of A. Then* FPdim *is invariant under $*$.*

Proof. Let $X \in I$. Then the matrix of right multiplication by X^* is the transpose of the matrix of left multiplication by X modified by the permutation $*$. Thus the required statement follows from Proposition 1.45.5(2). □

Corollary 1.45.9. *Let $\mathcal{C}$ be a ring category with right duals and finitely many simple objects, and let X be an object in $\mathcal{C}$. If* $\mathsf{FPdim}(X) = 1$ *then X is invertible.*

Proof. By Exercise 1.15.10(d) it is sufficient to show that $X \otimes X^* = \mathbf{1}$. This follows from the facts that $\mathbf{1}$ is contained in $X \otimes X^*$ and $\mathsf{FPdim}(X \otimes X^*) = \mathsf{FPdim}(X)\,\mathsf{FPdim}(X^*) = 1$. □

Proposition 1.45.10. *Let $f : A_1 \to A_2$ be a unital homomorphism of transitive unital $\mathbb{Z}_+$-rings of finite rank, whose matrix in their $\mathbb{Z}_+$-bases has non-negative entries. Then*

(1) *f preserves Frobenius-Perron dimensions.*
(2) *Let I_1, I_2 be the $\mathbb{Z}_+$-bases of A_1, A_2, and suppose that for any Y in I_2 there exists $X \in I_1$ such that the coefficient of Y in $f(X)$ is non-zero. If R is a regular element of A_1 then $f(R)$ is a regular element of A_2.*

Proof. (1) The function $X \mapsto \mathsf{FPdim}(f(X))$ is a nonzero character of A_1 with nonnegative values on the basis. By Proposition 1.45.5(3),

$\mathsf{FPdim}(f(X)) = \mathsf{FPdim}(X)$ for all X in I. (2) By part (1) we have

$$f(\sum_{X \in I_1} X) f(R_1) = \mathsf{FPdim}(f(\sum_{X \in I_1} X)) f(R_1). \tag{1.45.1}$$

But $f(\sum_{X \in I_1} X)$ has strictly positive coefficients in I_2, hence $f(R_1) = \beta R_2$ for some $\beta > 0$. Applying FPdim to both sides, we get the result. □

Corollary 1.45.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories with finitely many classes of simple objects. If $F : \mathcal{C} \to \mathcal{D}$ be a quasi-tensor functor, then $\mathsf{FPdim}_{\mathcal{D}}(F(X)) = \mathsf{FPdim}_{\mathcal{C}}(X)$ for any X in $\mathcal{C}$.*

Example 1.45.12. (Tambara-Yamagami fusion rings) Let G be a finite group, and TY_G be an extension of the unital based ring $\mathbb{Z}[G]$:

$$TY_G := \mathbb{Z}[G] \oplus \mathbb{Z}X,$$

where X is a new basis vector with $gX = Xg = X$, $X^2 = \sum_{g \in G} g$. This is a fusion ring, with $X^* = X$. It is easy to see that $\mathsf{FPdim}(g) = 1$, $\mathsf{FPdim}(X) = |G|^{1/2}$. We will see later that these rings are categorifiable if and only if G is abelian.

Example 1.45.13. (Verlinde rings for $\mathfrak{sl}_2$). Let k be a nonnegative integer. Define a unital $\mathbb{Z}_+$-ring $\mathrm{Ver}_k = \mathrm{Ver}_k(\mathfrak{sl}_2)$ with basis V_i, $i = 0, ..., k$ ($V_0 = 1$), with duality given by $V_i^* = V_i$ and multiplication given by the truncated Clebsch-Gordan rule:

$$V_i \otimes V_j = \bigoplus_{l = |i-j|, i+j-l \in 2\mathbb{Z}}^{\min(i+j, 2k-(i+j))} V_l. \tag{1.45.2}$$

It other words, one computes the product by the usual Clebsch-Gordan rule, and then deletes the terms that are not defined (V_i with $i > k$) and also their mirror images with respect to point $k+1$. We will show later that this ring admits categorifications coming from quantum groups at roots of unity.

Note that $\mathrm{Ver}_0 = \mathbb{Z}$, $\mathrm{Ver}_1 = \mathbb{Z}[\mathbb{Z}_2]$, $\mathrm{Ver}_2 = TY_{\mathbb{Z}_2}$. The latter is called the *Ising fusion ring*, as it arises in the Ising model of statistical mechanics.

Exercise 1.45.14. Show that $\mathsf{FPdim}(V_j) = [j+1]_q := \frac{q^{j+1} - q^{-j-1}}{q - q^{-1}}$, where $q = e^{\frac{\pi i}{k+2}}$.

Note that the Verlinde ring has a subring Ver_k^0 spanned by V_j with even j. If $k = 3$, this ring has basis $1, X = V_2$ with $X^2 = X + 1, X^* = X$. This ring is called the *Yang-Lee* fusion ring. In the Yang-Lee ring, $\mathsf{FPdim}(X)$ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

Note that one can define the generalized Yang-Lee fusion rings YL_n $n \in \mathbb{Z}_+$, with basis $1, X$, multiplication $X^2 = 1 + nX$ and duality $X^* = X$. It is, however, shown in [O2] that these rings are not categorifiable when $n > 1$.

Proposition 1.45.15. *(Kronecker) Let B be a matrix with nonnegative integer entries, such that $\lambda(BB^T) = \lambda(B)^2$. If $\lambda(B) < 2$ then $\lambda(B) = 2\cos(\pi/n)$ for some integer $n \geq 2$.*

Proof. Let $\lambda(B) = q + q^{-1}$. Then q is an algebraic integer, and $|q| = 1$. Moreover, all conjugates of $\lambda(B)^2$ are nonnegative (since they are eigenvalues of the matrix BB^T, which is symmetric and nonnegative definite), so all conjugates of $\lambda(B)$ are real. Thus, if q_* is a conjugate of q then $q_* + q_*^{-1}$ is real with absolute value < 2 (by the Frobenius-Perron theorem), so $|q_*| = 1$. By a well known result in elementary algebraic number theory, this implies that q is a root of unity: $q = e^{2\pi ik/m}$, where k and m are coprime. By the Frobenius-Perron theorem, so $k = \pm 1$, and m is even (indeed, if $m = 2p+1$ is odd then $|q^p + q^{-p}| > |q + q^{-1}|$). So $q = e^{\pi i/n}$ for some integer $n \geq 2$, and we are done. □

Corollary 1.45.16. *Let A be a fusion ring, and $X \in A$ a basis element. Then if $FPdim(X) < 2$ then $FPdim(X) = 2\cos(\pi/n)$, for some integer $n \geq 3$.*

Proof. This follows from Proposition 1.45.15, since $FPdim(XX^*) = FPdim(X)^2$. □

1.46. **Deligne's tensor product of finite abelian categories.** Let $\mathcal{C}, \mathcal{D}$ be two finite abelian categories over a field k.

Definition 1.46.1. *Deligne's tensor product* $\mathcal{C} \boxtimes \mathcal{D}$ is an abelian category which is universal for the functor assigning to every k-linear abelian category $\mathcal{A}$ the category of right exact in both variables bilinear bifunctors $\mathcal{C} \times \mathcal{D} \to \mathcal{A}$. That is, there is a bifunctor $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D} : (X, Y) \mapsto X \boxtimes Y$ which is right exact in both variables and is such that for any right exact in both variables bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ there exists a unique right exact functor $\bar{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ satisfying $\bar{F} \circ \boxtimes = F$.

Proposition 1.46.2. *(cf.* [D, Proposition 5.13]*) (i) The tensor product $\mathcal{C} \boxtimes \mathcal{D}$ exists and is a finite abelian category.*

(ii) It is unique up to a unique equivalence.

(iii) Let C, D be finite dimensional algebras and let $\mathcal{C} = C - \mathrm{mod}$ and $\mathcal{D} = D - \mathrm{mod}$. Then $\mathcal{C} \boxtimes \mathcal{D} = C \otimes D - \mathrm{mod}$.

(iv) The bifunctor $\boxtimes$ is exact in both variables and satisfies

$$\mathsf{Hom}_{\mathcal{C}}(X_1, Y_1) \otimes \mathsf{Hom}_{\mathcal{D}}(X_2, Y_2) \cong \mathsf{Hom}_{\mathcal{C} \boxtimes \mathcal{D}}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2).$$

(v) any bilinear bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ exact in each variable defines an exact functor $\bar{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$.

Proof. (sketch). (ii) follows from the universal property in the usual way.

(i) As we know, a finite abelian category is equivalent to the category of finite dimensional modules over an algebra. So there exist finite dimensional algebras C, D such that $\mathcal{C} = C - \text{mod}$, $\mathcal{D} = D - \text{mod}$. Then one can define $\mathcal{C} \boxtimes \mathcal{D} = C \otimes D - \text{mod}$, and it is easy to show that it satisfies the required conditions. This together with (ii) also implies (iii).

(iv),(v) are routine. □

A similar result is valid for locally finite categories.

Deligne's tensor product can also be applied to functors. Namely, if $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ are additive right exact functors between finite abelian categories then one can define the functor $F \boxtimes G : \mathcal{C} \boxtimes D \to \mathcal{C}' \boxtimes \mathcal{D}'$.

1.47. **Finite (multi)tensor categories.** In this subsection we will study general properties of finite multitensor and tensor categories.

Recall that in a finite abelian category, every simple object X has a projective cover $P(X)$. The object $P(X)$ is unique up to a non-unique isomorphism. For any Y in $\mathcal{C}$ one has

$$\dim \mathsf{Hom}(P(X), Y) = [Y : X]. \quad (1.47.1)$$

Let $K_0(\mathcal{C})$ denote the free abelian group generated by isomorphism classes of indecomposable projective objects of a finite abelian category $\mathcal{C}$. Elements of $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ will be called *virtual* projective objects. We have an obvious homomorphism $\gamma : K_0(\mathcal{C}) \to \mathsf{Gr}(\mathcal{C})$. Although groups $K_0(\mathcal{C})$ and $\mathsf{Gr}(\mathcal{C})$ have the same rank, in general γ is neither surjective nor injective even after tensoring with $\mathbb{C}$. The matrix C of γ in the natural basis is called the *Cartan matrix* of $\mathcal{C}$; its entries are $[P(X) : Y]$, where X, Y are simple objects of $\mathcal{C}$.

Now let $\mathcal{C}$ be a finite multitensor category, let I be the set of isomorphism classes of simple objects of $\mathcal{C}$, and let $i^*, {}^*i$ denote the right and left duals to i, respectively. Let $\mathsf{Gr}(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$, spanned by isomorphism classes of the simple objects X_i, $i \in I$. In this ring, we have $X_i X_j = \sum_k N_{ij}^k X_k$, where N_{ij}^k are nonnegative integers. Also, let P_i denote the projective covers of X_i.

Proposition 1.47.1. *Let $\mathcal{C}$ be a finite multitensor category. Then $K_0(\mathcal{C})$ is a $\mathsf{Gr}(\mathcal{C})$-bimodule.*

Proof. This follows from the fact that the tensor product of a projective object with any object is projective, Proposition 1.13.6. □

Let us describe this bimodule explicitly.

Proposition 1.47.2. *For any object Z of $\mathcal{C}$,*

$$P_i \otimes Z \cong \oplus_{j,k} N^i_{kj^*}[Z : X_j]P_k, \ \ Z \otimes P_i \cong \oplus_{j,k} N^i_{*jk}[Z : X_j]P_k.$$

Proof. $\mathsf{Hom}(P_i \otimes Z, X_k) = \mathsf{Hom}(P_i, X_k \otimes Z^*)$, and the first formula follows from Proposition 1.13.6. The second formula is analogous. □

Proposition 1.47.3. *Let P be a projective object in a multitensor category $\mathcal{C}$. Then P^* is also projective. Hence, any projective object in a multitensor category is also injective.*

Proof. We need to show that the functor $\mathsf{Hom}(P^*, \bullet)$ is exact. This functor is isomorphic to $\mathsf{Hom}(\mathbf{1}, P \otimes \bullet)$. The functor $P \otimes \bullet$ is exact and moreover, by Proposition 1.13.6, any exact sequence splits after tensoring with P, as an exact sequence consisting of projective objects. The Proposition is proved. □

Proposition 1.47.3 implies that an indecomposable projective object P has a unique simple subobject, i.e. that the socle of P is simple.

For any finite tensor category $\mathcal{C}$ define an element $R_{\mathcal{C}} \in K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ by

$$R_{\mathcal{C}} = \sum_{i \in I} \mathsf{FPdim}(X_i)P_i. \tag{1.47.2}$$

Definition 1.47.4. The virtual projective object $R_{\mathcal{C}}$ is called the *regular object* of $\mathcal{C}$.

Definition 1.47.5. Let $\mathcal{C}$ be a finite tensor category. Then the *Frobenius-Perron* dimension of $\mathcal{C}$ is defined by

$$\mathsf{FPdim}(\mathcal{C}) := \mathsf{FPdim}(R_{\mathcal{C}}) = \sum_{i \in I} \mathsf{FPdim}(X_i)\,\mathsf{FPdim}(P_i). \tag{1.47.3}$$

Example 1.47.6. Let H be a finite dimensional quasi-Hopf algebra. Then $\mathsf{FPdim}(\mathrm{Rep}(H)) = \dim(H)$.

Proposition 1.47.7.
1. *$Z \otimes R_{\mathcal{C}} = R_{\mathcal{C}} \otimes Z = \mathsf{FPdim}(Z)R_{\mathcal{C}}$ for all $Z \in \mathsf{Gr}(\mathcal{C})$.*
2. *The image of $R_{\mathcal{C}}$ in $\mathsf{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a regular element.*

Proof. We have $\sum_i \mathsf{FPdim}(X_i) \dim \mathsf{Hom}(P_i, Z) = \mathsf{FPdim}(Z)$ for any object Z of $\mathcal{C}$. Hence,

$$\begin{aligned}\sum_i \mathsf{FPdim}(X_i) \dim \mathsf{Hom}(P_i \otimes Z, Y) &= \sum_i \mathsf{FPdim}(X_i) \dim \mathsf{Hom}(P_i, Y \otimes Z^*) \\ &= \mathsf{FPdim}(Y \otimes Z^*) \\ &= \mathsf{FPdim}(Y)\,\mathsf{FPdim}(Z^*) \\ &= \mathsf{FPdim}(Y)\,\mathsf{FPdim}(Z) \\ &= \mathsf{FPdim}(Z) \sum_i \mathsf{FPdim}(X_i) \dim \mathsf{Hom}(P_i, Y).\end{aligned}$$

Now, $P(X) \otimes Z$ are projective objects by Proposition 1.13.6. Hence, the formal sums $\sum_i \mathsf{FPdim}(X_i) P_i \otimes Z = R_{\mathcal{C}} \otimes Z$ and $\mathsf{FPdim}(Z) \sum_i \mathsf{FPdim}(X_i) P_i = \mathsf{FPdim}(Z) R_{\mathcal{C}}$ are linear combinations of P_j, $j \in I$ with the same coefficients. □

Remark 1.47.8. We note the following useful inequality:

$$\mathsf{FPdim}(\mathcal{C}) \geq N\,\mathsf{FPdim}(P), \tag{1.47.4}$$

where N is the number of simple objects in $\mathcal{C}$, and P is the projective cover of the neutral object $\mathbf{1}$. Indeed, for any simple object V the projective object $P(V) \otimes {}^*V$ has a nontrivial homomorphism to $\mathbf{1}$, and hence contains P. So $\mathsf{FPdim}(P(V))\,\mathsf{FPdim}(V) \geq \mathsf{FPdim}(P)$. Adding these inequalities over all simple V, we get the result.

1.48. Integral tensor categories.

Definition 1.48.1. A transitive unital $\mathbb{Z}_+$-ring A of finite rank is said to be integral if $\mathsf{FPdim} : A \to \mathbb{Z}$ (i.e. the Frobenius-Perron dimnensions of elements of $\mathcal{C}$ are integers). A tensor category $\mathcal{C}$ is integral if $\mathrm{Gr}(\mathcal{C})$ is integral.

Proposition 1.48.2. *A finite tensor category $\mathcal{C}$ is integral if and only if $\mathcal{C}$ is equivalent to the representation category of a finite dimensional quasi-Hopf algebra.*

Proof. The "if" part is clear from Example 1.45.6. To prove the "only if" part, it is enough to construct a quasi-fiber functor on $\mathcal{C}$. Define $P = \oplus_i \mathsf{FPdim}(X_i) P_i$, where X_i are the simple objects of $\mathcal{C}$, and P_i are their projective covers. Define $F = \mathsf{Hom}(P, \bullet)$. Obviously, F is exact and faithful, $F(\mathbf{1}) \cong \mathbf{1}$, and $\dim F(X) = \mathsf{FPdim}(X)$ for all $X \in \mathcal{C}$. Using Proposition 1.46.2, we continue the functors $F(\bullet \otimes \bullet)$ and $F(\bullet) \otimes F(\bullet)$ to the functors $\mathcal{C} \boxtimes \mathcal{C} \to \mathrm{Vec}$. Both of these functors are exact and take the same values on the simple objects of $\mathcal{C} \boxtimes \mathcal{C}$. Thus these functors are isomorphic and we are done. □

Corollary 1.48.3. *The assignment $H \mapsto \mathrm{Rep}(H)$ defines a bijection between integral finite tensor categories $\mathcal{C}$ over k up to monoidal equivalence, and finite dimensional quasi-Hopf algebras H over k, up to twist equivalence and isomorphism.*

1.49. **Surjective quasi-tensor functors.** Let $\mathcal{C}$, $\mathcal{D}$ be abelian categories. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor.

Definition 1.49.1. We will say that F is *surjective* if any object of $\mathcal{D}$ is a subquotient in $F(X)$ for some $X \in \mathcal{C}$. [13]

Exercise 1.49.2. Let A, B be coalgebras, and $f : A \to B$ a homomorphism. Let $F = f^* : A - \mathrm{comod} \to B - \mathrm{comod}$ be the corresponding pushforward functor. Then F is surjective if and only if f is surjective.

Now let $\mathcal{C}$, $\mathcal{D}$ be finite tensor categories.

Theorem 1.49.3. *(*[EO]*) Let $F : \mathcal{C} \to \mathcal{D}$ be a surjective quasi-tensor functor. Then F maps projective objects to projective ones.*

Proof. Let $\mathcal{C}$ be a finite tensor category, and $X \in \mathcal{C}$. Let us write X as a direct sum of indecomposable objects (such a representation is unique). Define the *projectivity defect* $p(X)$ of X to be the sum of Frobenius-Perron dimensions of all the non-projective summands in this sum (this is well defined by the Krull-Schmidt theorem). It is clear that $p(X \oplus Y) = p(X) + p(Y)$. Also, it follows from Proposition 1.13.6 that $p(X \otimes Y) \le p(X)p(Y)$.

Let P_i be the indecomposable projective objects in $\mathcal{C}$. Let $P_i \otimes P_j \cong \oplus_k B_{ij}^k P_k$, and let B_i be the matrix with entries B_{ij}^k. Also, let $B = \sum B_i$. Obviously, B has strictly positive entries, and the Frobenius-Perron eigenvalue of B is $\sum_i \mathsf{FPdim}(P_i)$.

On the other hand, let $F : \mathcal{C} \to \mathcal{D}$ be a surjective quasi-tensor functor between finite tensor categories. Let $p_j = p(F(P_j))$, and $\mathbf{p}$ be the vector with entries p_j. Then we get $p_i p_j \ge \sum_k B_{ij}^k p_k$, so $(\sum_i p_i)\mathbf{p} \ge B\mathbf{p}$. So, either p_i are all zero, or they are all positive, and the norm of B with respect to the norm $|x| = \sum p_i |x_i|$ is at most $\sum p_i$. Since $p_i \le \mathsf{FPdim}(P_i)$, this implies $p_i = \mathsf{FPdim}(P_i)$ for all i (as the largest eigenvalue of B is $\sum_i \mathsf{FPdim}(P_i)$).

Assume the second option is the case. Then $F(P_i)$ do not contain nonzero projective objects as direct summands, and hence for any projective $P \in \mathcal{C}$, $F(P)$ cannot contain a nonzero projective object as a direct summand. However, let Q be a projective object of $\mathcal{D}$. Then,

[13] This definition does not coincide with a usual categorical definition of surjectivity of functors which requires that every object of $\mathcal{D}$ be isomorphic to some $F(X)$ for an object X in $\mathcal{C}$.

since F is surjective, there exists an object $X \in \mathcal{C}$ such that Q is a subquotient of $F(X)$. Since any X is a quotient of a projective object, and F is exact, we may assume that $X = P$ is projective. So Q occurs as a subquotient in $F(P)$. As Q is both projective and injective, it is actually a direct summand in $F(P)$. Contradiction.

Thus, $p_i = 0$ and $F(P_i)$ are projective. The theorem is proved. $\square$

1.50. **Categorical freeness.** Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories, and $F : \mathcal{C} \to \mathcal{D}$ be a quasi-tensor functor.

Theorem 1.50.1. *One has*

$$F(R_{\mathcal{C}}) = \frac{\mathsf{FPdim}(\mathcal{C})}{\mathsf{FPdim}(\mathcal{D})} R_{\mathcal{D}}. \tag{1.50.1}$$

Proof. By Theorem 1.49.3, $F(R_{\mathcal{C}})$ is a virtually projective object. Thus, $F(R_{\mathcal{C}})$ must be proportional to $R_{\mathcal{D}}$, since both (when written in the basis P_i) are eigenvectors of a matrix with strictly positive entries with its Frobenius-Perron eigenvalue. (For this matrix we may take the matrix of multiplication by $F(X)$, where X is such that $F(X)$ contains as composition factors all simple objects of $\mathcal{D}$; such exists by the surjectivity of F). The coefficient is obtained by computing the Frobenius-Perron dimensions of both sides. $\square$

Corollary 1.50.2. *In the above situation, one has* $\mathsf{FPdim}(\mathcal{C}) \geq \mathsf{FPdim}(\mathcal{D})$, *and* $\mathsf{FPdim}(\mathcal{D})$ *divides* $\mathsf{FPdim}(\mathcal{C})$ *in the ring of algebraic integers. In fact,*

$$\frac{\mathsf{FPdim}(\mathcal{C})}{\mathsf{FPdim}(\mathcal{D})} = \sum \mathsf{FPdim}(X_i) \dim \mathsf{Hom}(F(P_i), \mathbf{1}_{\mathcal{D}}), \tag{1.50.2}$$

where X_i runs over simple objects of $\mathcal{C}$.

Proof. The statement is obtained by computing the dimension of $\mathsf{Hom}(\bullet, \mathbf{1}_{\mathcal{D}})$ for both sides of (1.50.1). $\square$

Suppose now that $\mathcal{C}$ is integral, i.e., by Proposition 1.48.2, it is the representation category of a quasi-Hopf algebra H. In this case, $R_{\mathcal{C}}$ is an honest (not only virtual) projective object of $\mathcal{C}$, namely the free rank 1 module over H. Theorefore, multiples of $R_{\mathcal{C}}$ are free H-modules of finite rank, and vice versa.

Then Theorem 1.49.3 and the fact that $F(R_{\mathcal{C}})$ is proportional to $R_{\mathcal{D}}$ implies the following categorical freeness result.

Corollary 1.50.3. *If $\mathcal{C}$ is integral, and $F : \mathcal{C} \to \mathcal{D}$ is a surjective quasi-tensor functor then $\mathcal{D}$ is also integral, and the object $F(R_{\mathcal{C}})$ is free of rank* $\mathsf{FPdim}(\mathcal{C})/\mathsf{FPdim}(\mathcal{D})$ *(which is an integer).*

Proof. The Frobenius-Perron dimensions of simple objects of $\mathcal{D}$ are coordinates of the unique eigenvector of the positive integer matrix of multiplication by $F(R_{\mathcal{C}})$ with integer eigenvalue $\mathsf{FPdim}(\mathcal{C})$, normalized so that the component of $\mathbf{1}$ is 1. Thus, all coordinates of this vector are rational numbers, hence integers (because they are algebraic integers). This implies that the category $\mathcal{D}$ is integral. The second statement is clear from the above. □

Corollary 1.50.4. *(*[Scha]*; for the semisimple case see* [ENO1]*) A finite dimensional quasi-Hopf algebra is a free module over its quasi-Hopf subalgebra.*

Remark 1.50.5. In the Hopf case Corollary 1.50.3 is well known and much used; it is due to Nichols and Zoeller [NZ].

1.51. **The distinguished invertible object.** Let $\mathcal{C}$ be a finite tensor category with classes of simple objects labeled by a set I. Since duals to projective objects are projective, we can define a map $D : I \to I$ such that $P_i^* = P_{D(i)}$. It is clear that $D^2(i) = i^{**}$.

Let 0 be the label for the unit object. Let $\rho = D(0)$. (In other words, ${}^*L_\rho$ is the socle of $P_0 = P(\mathbf{1})$). We have

$$\mathsf{Hom}(P_i^*, L_j) = \mathsf{Hom}(\mathbf{1}, P_i \otimes L_j) = \mathsf{Hom}(\mathbf{1}, \oplus_k N_{kj^*}^i P_k).$$

This space has dimension $N_{\rho j^*}^i$. Thus we get

$$N_{\rho j^*}^i = \delta_{D(i),j}.$$

Let now L_ρ be the corresponding simple object. By Proposition 1.47.2, we have

$$L_\rho^* \otimes P_m \cong \oplus_k N_{\rho m}^k P_k \cong P_{D(m)^*}.$$

Lemma 1.51.1. *L_ρ is an invertible object.*

Proof. The last equation implies that the matrix of action of L_{ρ^*} on projectives is a permutation matrix. Hence, the Frobenius-Perron dimension of L_{ρ^*} is 1, and we are done. □

Lemma 1.51.2. *One has: $P_{D(i)} = P_{^*i} \otimes L_\rho$; $L_{D(i)} = L_{^*i} \otimes L_\rho$.*

Proof. It suffices to prove the first statement. Therefore, our job is to show that $\dim \mathsf{Hom}(P_i^*, L_j) = \dim \mathsf{Hom}(P_{^*i}, L_j \otimes L_{\rho^*})$. The left hand side was computed before, it is $N_{\rho j^*}^i$. On the other hand, the right hand side is $N_{j,\rho^*}^{^*i}$ (we use that $\rho^* = {}^*\rho$ for an invertible object ρ). These numbers are equal by the properies of duality, so we are done. □

Corollary 1.51.3. *One has: $P_{i^{**}} = L_\rho^* \otimes P_{^{**}i} \otimes L_\rho$; $L_{i^{**}} = L_\rho^* \otimes L_{^{**}i} \otimes L_\rho$.*

Proof. Again, it suffices to prove the first statement. We have

$$P_{i^{**}} = P_i^{**} = (P_{*i} \otimes L_\rho)^* = L_\rho^* \otimes P_{*i}^* = L_\rho^* \otimes P_{**i} \otimes L_\rho$$

□

Definition 1.51.4. L_ρ is called the *distinguished invertible object* of $\mathcal{C}$.

We see that for any i, the socle of P_i is $\hat{L}_i := L_\rho^* \otimes^{**} L_i = L_i^{**} \otimes L_\rho^*$. This implies the following result.

Corollary 1.51.5. *Any finite dimensional quasi-Hopf algebra H is a Frobenius algebra, i.e. H is isomorphic to H^* as a left H-module.*

Proof. It is easy to see that that a Frobenius algebra is a quasi-Frobenius algebra (i.e. a finite dimensionbal algebra for which projective and injective modules coincide), in which the socle of every indecomposable projective module has the same dimension as its cosocle (i.e., the simple quotient). As follows from the above, these conditions are satisfied for finite dimensional quasi-Hopf algebras (namely, the second condition forllows from the fact that L_ρ is 1-dimensional). □

1.52. **Integrals in quasi-Hopf algebras.**

Definition 1.52.1. A *left integral* in an algebra H with a counit $\varepsilon : H \to k$ is an element $I \in H$ such that $xI = \varepsilon(x)I$ for all $x \in H$. Similarly, a *right integral* in H is an element $I \in H$ such that $Ix = \varepsilon(x)I$ for all $x \in H$.

Remark 1.52.2. Let H be the convolution algebra of distributions on a compact Lie group G. This algebra has a counit ε defined by $\varepsilon(\xi) = \xi(1)$. Let dg be a left-invariant Haar measure on G. Then the distribution $I(f) = \int_G f(g)dg$ is a left integral in H (unique up to scaling). This motivates the terminology.

Note that this example makes sense for a finite group G over any field k. In this case, $H = k[G]$, and $I = \sum_{g \in G} g$ is both a left and a right integral.

Proposition 1.52.3. *Any finite dimensional quasi-Hopf algebra admits a unique nonzero left integral up to scaling and a unique nonzero right integral up to scaling.*

Proof. It suffices to prove the statement for left integrals (for right integrals the statement is obtained by applying the antipode). A left integral is the same thing as a homomorphism of left modules $k \to H$. Since H is Frobenius, this is the same as a homomorphism $k \to H^*$, i.e. a homomorphism $H \to k$. But such homomorphisms are just multiples of the counit. □

Note that the space of left integrals of an algebra H with a counit is a right H-module (indeed, if I is a left integral, then so is Iy for all $y \in H$). Thus, for finite dimensional quasi-Hopf algebras, we obtain a character $\chi : H \to k$, such that $Ix = \chi(x)I$ for all $x \in H$. This character is called *the distinguished character* of H (if H is a Hopf algebra, it is commonly called the distinguished grouplike element of H^*, see [Mo]).

Proposition 1.52.4. *Let H be a finite dimensional quasi-Hopf algebra, and $\mathcal{C} = \mathsf{Rep}(H)$. Then L_ρ coincides with the distinguished character χ.*

Proof. Let I be a nonzero left integral in H. We have $xI = \varepsilon(x)I$ and $Ix = \chi(x)I$. This means that for any $V \in \mathcal{C}$, I defines a morphism from $V \otimes \chi^{-1}$ to V.

The element I belongs to the submodule P_i of H, whose socle is the trivial H-module. Thus, $P_i^* = P(\mathbf{1})$, and hence by Lemma 1.51.2, $i = \rho$. Thus, I defines a nonzero (but rank 1) morphism $P_\rho \otimes \chi^{-1} \to P_\rho$. The image of this morphism, because of rank 1, must be $L_0 = \mathbf{1}$, so $\mathbf{1}$ is a quotient of $P_\rho \otimes \chi^{-1}$, and hence χ is a quotient of P_ρ. Thus, $\chi = L_\rho$, and we are done. □

Proposition 1.52.5. *The following conditions on a finite dimensional quasi-Hopf algebra H are equivalent:*
(i) H is semisimple;
(ii) $\varepsilon(I) \neq 0$ (where I is a left integral in H);
(iii) $I^2 \neq 0$;
(iv) I can be normalized to be an idempotent.

Proof. (ii) implies (i): If $\varepsilon(I) \neq 0$ then $k = \mathbf{1}$ is a direct summand in H as a left H-module. This implies that $\mathbf{1}$ is projective, hence Rep(H) is semisimple (Corollary 1.13.7).

(i) implies (iv): If H is semisimple, the integral is a multiple of the projector to the trivial representation, so the statement is obvious.

(iv) implies (iii): obvious.

(iii) implies (ii): clear, since $I^2 = \varepsilon(I)I$. □

Definition 1.52.6. A finite tensor category $\mathcal{C}$ is *unimodular* if $L_\rho = \mathbf{1}$. A finite dimensional quasi-Hopf algebra H is unimodular if Rep(H) is a unimodular category, i.e. if left and right integrals in H coincide.

Remark 1.52.7. This terminology is motivated by the notion of a unimodular Lie group, which is a Lie group on which a left invariant Haar measure is also right invariant, and vice versa.

Remark 1.52.8. Obviously, every semisimple category is automatically unimodular.

Exercise 1.52.9. (i) Let H be the Nichols Hopf algebra of dimension 2^{n+1} (Example 1.24.9). Find the projective covers of simple objects, the distinguished invertible object, and show that H is not unimodular. In particular, Sweedler's finite dimensional Hopf algebra is not unimodular.

(ii) Do the same if H is the Taft Hopf algebra (Example 1.24.5).

(iii) Let $H = u_{\mathfrak{q}}(\mathfrak{sl}_2)$ be the small quantum group at a root of unity q of odd order (see Subsection 1.25). Show that H is unimodular, but H^* is not. Find the distinguished character of H^* (i.e., the distinguished grouplike element of H). What happens for the corresponding graded Hopf algebra $\mathrm{gr}(H)$?

1.53. **Dimensions of projective objects and degeneracy of the Cartan matrix.** The following result in the Hopf algebra case was proved by M.Lorenz [L]; our proof in the categorical setting is analogous to his.

Let $C_{ij} = [P_i : L_j]$ be the entries of the Cartan matrix of a finite tensor category $\mathcal{C}$.

Theorem 1.53.1. *Suppose that $\mathcal{C}$ is not semisimple, and admits an isomorphism of additive functors $u : \mathrm{Id} \to **$. Then the Cartan matrix C is degenerate over the ground field k.*

Proof. Let $\dim(V) = \mathrm{Tr}|_V(u)$ be the dimension function defined by the (left) categorical trace of u. This function is additive on exact sequences, so it is a linear functional on $\mathrm{Gr}(\mathcal{C})$.

On the other hand, the dimension of every projective object P with respect to this function is zero. Indeed, the dimension of P is the composition of maps $\mathbf{1} \to P \otimes P^* \to P^{**} \otimes P^* \to \mathbf{1}$, where the maps are the coevaluation, $u \otimes \mathrm{Id}$, and the evaluation. If this map is nonzero then $\mathbf{1}$ is a direct summand in $P \otimes P^*$, which is projective. Thus $\mathbf{1}$ is projective, So $\mathcal{C}$ is semisimple by Corollary 1.13.7. Contradiction.

Since the dimension of the unit object $\mathbf{1}$ is not zero, $\mathbf{1}$ is not a linear combination of projective objects in the Grothendieck group tensored with k. We are done. □

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