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Finite Time Analyticity for the Two and Three Dimensional Kelvin-Helmholtz Instability

C. Sulem[1], P. L. Sulem[2], C. Bardos[3],* and U. Frisch[2]

1 CNRS, Dept. de Mathématiques, Université de Nice, Parc Valrose, F-06034 Nice, France
2 CNRS, Observatoire de Nice, B.P. 252, F-06007 Nice-Cedex, France
3 Dept. de Mathématiques, Université de Paris XIII, Villetaneuse, France

Abstract. The well-posed property for the finite time vortex sheet problem with analytic initial data was first conjectured by Birkhoff in two dimensions and is shown here to hold both in two and three dimensions. Incompressible, inviscid and irrotational flow with a velocity jump across an interface is assumed. In two dimensions, global existence of a weak solution to the Euler equation with such initial conditions is established. In three dimensions, a Lagrangian representation of the vortex sheet analogous to the Birkhoff equation in two dimensions is presented.

1. Introduction

A velocity discontinuity (vortex sheet) in an ideal incompressible fluid is subject to the Kelvin-Helmholtz instability [see Birkhoff (1962) and Saffman and Baker (1979) for a general introduction]. A simple illustration is provided by a flow with uniform velocity U in the x-direction above the (x, y) plane and with the same velocity in the opposite direction below this plane. Such a motion constitutes a stationary but unstable solution to the equations of fluid dynamics (see e.g. Chandrashekar, 1961). When a slight disturbance preserving the irrotationality of the flow outside the interface is considered, a linear analysis indicates that the amplitude of the k-Fourier mode of the interface corrugation increases exponentially in time at the rate $|k \cdot U|$. The linear problem therefore requires analytic initial data to be well posed and will generally be so only for a finite time. Birkhoff (1962) conjectures that the nonlinear problem with analytic initial data is well posed at least for a finite time. Richtmyer and Morton (1967) make a similar conjecture for piecewise analytic data.

The present paper is devoted to the nonlinear problem with analytic initial data. We shall not, as is mostly the case in studies of Kelvin-Helmholtz instability,

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restrict ourselves to two dimensional flow to take advantage of the vorticity conservation. A system of equations for the interface and the vorticity density is derived from the Euler equation written in the sense of distribution in Sect. 2. This system involves first order differential operators and also zero order pseudo-differential operators. Such operators arise when expressing the interface velocity in terms of the vorticity density. A proof of finite time analyticity of the interface in two and three dimensions is given in Sect. 4. It is based on an abstract Cauchy-Kowalewski theorem in scale of Banach spaces (here spaces of analytic functions) in the formulation of Nishida (1977); see also Baouendi and Goulaouic (1977) which improves a result of Ovsjannikov (1971) and Nirenberg (1972). In two dimensions, our finite time analyticity result is a special case of a result by Babenko and Petrovich (1979) which deals with the Rayleigh-Taylor problem. The Babenko and Petrovich proof (which in places is only sketched) uses also scales of Banach spaces (à la Ovsyannikov, 1971); it puts more restriction on the initial data. Note that all these results leave room for improvement because the theorems are restricted to situations where a coordinate of the interface can be resolved in terms of the other (others). In Sect. 5, we prove existence for all times of a weak solution of the two dimensional Euler equation with vorticity concentrated on an analytic line. The last section is devoted to a Lagrangian representation of the vortex sheet which is an extension of the Birkhoff equation (1954, 1955, 1962) from two to three dimensions.

2. Equation of Motion of a Vortex Sheet

We consider an ideal three-dimensional flow in a domain without boundary and we assume that the vorticity is concentrated on a surface (vortex sheet). We derive from the Euler equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \\ &\nabla \cdot u = 0 \end{aligned} \tag{2.1}$$

a system which governs the time evolution of the vortex sheet and the vorticity density on it.

The vortex sheet $\mathcal{S}(t)$ is represented by the equation

$$r = r(\lambda, \mu, t) \qquad (\lambda, \mu) \in \mathbb{R}^2, \tag{2.2}$$

or when cartesian coordinates are used

$$x_i = x_i(\lambda, \mu, t) \qquad i = 1, 2, 3. \tag{2.3}$$

The vorticity density $\Omega(\lambda, \mu, t)$ is defined by:

$$\int \varphi(r)\omega(r,t)dr = \int \varphi(r(\lambda,\mu,t))\Omega(\lambda,\mu,t)d\lambda d\mu, \tag{2.4}$$

where $\omega = \text{Curl } u$ is the vorticity distribution and φ a test function. Both $\mathcal{S}(t)$ and $\Omega(\cdot, t)$ are assumed to be smooth.

In the absence of irrotational contribution, the velocity at a point exterior to the vortex sheet is given by

$$u(r,t) = -\frac{1}{4\pi}\int \frac{r - r(\lambda,\mu,t)}{|r - r(\lambda,\mu,t)|^3} \wedge \Omega(\lambda,\mu,t)\,d\lambda\,d\mu \tag{2.5}$$

and has limits $u^+(\lambda,\mu,t)$ and $u^-(\lambda,\mu,t)$ when the point r tends in an arbitrary way from one or the other side of the vortex sheet to a point $r(\lambda,\mu,t) \in \mathscr{S}(t)$. In addition,

$$V(\lambda,\mu,t) = \frac{u^+(\lambda,\mu,t) + u^-(\lambda,\mu,t)}{2} = -\frac{1}{4\pi}⨍ \frac{r(\lambda,\mu,t) - r(\lambda',\mu',t)}{|r(\lambda,\mu,t) - r(\lambda',\mu',t)|^3} \wedge \Omega(\lambda',\mu',t)\,d\lambda'\,d\mu', \tag{2.6}$$

where ⨍ indicates that the integral is taken in the sense of Cauchy principal value.

When written in the sense of distributions, the Euler equation reads

$$\langle u, \operatorname{Curl}\varphi \rangle = \langle \omega, \varphi \rangle, \tag{2.7a}$$

$$\left\langle u_i, \frac{\partial \varphi_i}{\partial x_i} \right\rangle = 0, \tag{2.7b}$$

$$\left\langle u, \operatorname{Curl}\frac{\partial \varphi}{\partial t} \right\rangle + \left\langle u_i u, \frac{\partial}{\partial x_i}\operatorname{Curl}\varphi \right\rangle = 0, \tag{2.7c}$$

where φ is a test function in $(\mathscr{D}(\mathbb{R}^3 \times \mathbb{R}^+))^3$. In each of the two domains separated by the vortex sheet $\operatorname{Curl} u = 0$ and $\operatorname{div} u = 0$, and Eqs. (2.7a) and (2.7b) readily imply that the velocity jump across the vortex sheet

$$[u] = u^+ - u^- \tag{2.8}$$

satisfies

$$[u] \cdot N = 0, \tag{2.9}$$

$$[u] \wedge N = \Omega, \tag{2.10}$$

where the normal vector N is defined by (subscripts t, λ, μ denote partial derivatives)

$$N = r_\lambda \wedge r_\mu \tag{2.11}$$

and has components (β, γ, δ).

We introduce the three-dimensional manifold $\mathscr{M} = \{t = t, r = r(\lambda,\mu,t)\}$ and a normal vector v to this manifold with components $(\alpha = -r_t \cdot N, \beta, \gamma, \delta)$. The volume element on $\mathscr{M}$ is $dv = \|v\|\,dt\,d\lambda\,d\mu$. Using the Green formula and Eq. (2.9), Eq. (2.7c) is rewritten

$$\left.\begin{aligned} \int \Big\{ ([u] \wedge N) \cdot \frac{\partial \varphi}{\partial t} + ([u_i u] \wedge N) \cdot \frac{\partial \varphi}{\partial x_i} + N_i [\nabla u_i \wedge u] \cdot \varphi \\ + N \cdot \left(\frac{\partial r}{\partial t} - V\right)([\operatorname{Curl} u] \cdot \varphi) \Big\}\, d\lambda\, d\mu\, dt = 0, \end{aligned}\right\} \tag{2.12}$$

where the last term vanishes because $[\mathrm{Curl}\, u]=0$. In the first term of (2.12) we do the substitution

$$\frac{\partial\varphi}{\partial t}(r(\lambda,\mu,t),t)=\frac{d\varphi}{dt}(r(\lambda,\mu,t),t)-(r_t\cdot\nabla)\varphi(r(\lambda,\mu,t),t). \tag{2.13}$$

$\nabla\varphi_i$ at a point of the vortex sheet is expressed as follows:

$$\begin{aligned}\nabla\varphi_i&=\frac{(\nabla\varphi_i\cdot N)}{\|N\|^2}N+\frac{(\nabla\varphi_i\cdot r_\lambda)}{\|r_\lambda\|^2}r_\lambda+\frac{(\nabla\varphi_i,r_\lambda,N)}{\|r_\lambda\|^2\,\|N\|^2}(r_\lambda\wedge N)\\&=\frac{1}{\|N\|}\frac{\partial\varphi_i}{\partial n}N+\frac{1}{\|r_\lambda\|^2}\frac{\partial\varphi_i}{\partial\lambda}r_\lambda+\frac{(r_\lambda\cdot r_\mu)}{\|N\|^2\,\|r_\lambda\|^2}\frac{\partial\varphi_i}{\partial\lambda}(r_\lambda\wedge N)-\frac{1}{\|N\|^2}\frac{\partial\varphi_i}{\partial\mu}(r_\lambda\wedge N)\end{aligned}$$

and finally

$$\nabla\varphi_i=\frac{1}{\|N\|}\frac{\partial\varphi_i}{\partial n}+\frac{1}{\|N\|^2}\frac{\partial\varphi_i}{\partial\lambda}(r_\mu\wedge N)-\frac{1}{\|N\|^2}\frac{\partial\varphi_i}{\partial\mu}(r_\lambda\wedge N). \tag{2.14}$$

Thus

$$\begin{aligned}\int([u]\wedge N)\cdot\frac{\partial\varphi}{\partial t}d\lambda d\mu dt&=\int\left\{-\frac{\partial\Omega}{\partial t}+\frac{\partial}{\partial\lambda}\left(\frac{\Omega}{\|N\|^2}(r_t,r_\mu,N)\right)-\frac{\partial}{\partial\mu}\left(\frac{\Omega}{\|N\|^2}(r_t,r_\lambda,N)\right)\right\}\\&\quad\cdot\varphi d\lambda d\mu dt-\int(r_t\cdot N)\left(\frac{\Omega}{\|N\|}\cdot\frac{\partial\varphi}{\partial n}\right)d\lambda d\mu dt,\end{aligned} \tag{2.15}$$

where (a,b,c) denotes the triple scalar product $a\cdot(b\wedge c)$.

$$\begin{aligned}\int([u_iu]\wedge N)\cdot\frac{\partial}{\partial x_i}\varphi d\lambda d\mu dt&=\int\frac{\partial\varphi}{\partial n}\cdot\frac{1}{\|N\|}\{[u_iu]\wedge N\}N_i d\lambda d\mu dt\\&\quad+\int\varphi\cdot\left\{-\frac{\partial}{\partial\lambda}\left(\frac{1}{\|N\|^2}(r_\mu\wedge N)_i[u_iu]\wedge N\right)\right.\\&\quad\left.+\frac{\partial}{\partial\mu}\left(\frac{1}{\|N\|^2}(r_\lambda\wedge N)_i[u_iu]\wedge N\right)\right\}.\end{aligned} \tag{2.16}$$

Replacing (2.15) in Eq. (2.12) and writing that the coefficients of φ and $\frac{\partial\varphi}{\partial n}$ in the resulting equation vanish, we obtain the system

$$\left(\frac{\partial r}{\partial t}-V\right)\cdot N=0, \tag{2.17a}$$

$$\begin{aligned}&-\frac{\partial\Omega}{\partial t}+\frac{\partial}{\partial\lambda}\left\{\frac{\Omega}{\|N\|^2}(r_t,r_\mu,N)\right\}-\frac{\partial}{\partial\mu}\left\{\frac{\Omega}{\|N\|^2}(r_t,r_\lambda,N)\right\}+[\nabla u_i\wedge u]N_i\\&-\frac{\partial}{\partial\lambda}\left\{\frac{1}{\|N\|^2}(r_\mu\wedge N)_i[u_iu]\wedge N\right\}+\frac{\partial}{\partial\mu}\left\{\frac{1}{\|N\|^2}(r_\lambda\wedge N)_i[u_iu]\wedge N\right\}=0.\end{aligned} \tag{2.17b}$$

Further simplifications result from the following transformations

$$[\nabla u_i\wedge u]N_i=N_i[\nabla u_i]\wedge V+N_i\frac{\nabla u_i^++\nabla u_i^-}{2}\wedge[u]. \tag{2.18}$$

Since $(\operatorname{Curl} u)^{\pm}=0$, we have $\partial_j u_i^{\pm}=\partial_i u_j^{\pm}$, and thus

$$[\nabla u_i \wedge u]N_i = \|N\| \left[\frac{\partial u}{\partial n}\right] \wedge V + \frac{\|N\|}{2}\left(\frac{\partial u^+}{\partial n} + \frac{\partial u^-}{\partial n}\right) \wedge [u]. \tag{2.19}$$

Using the equality

$$\|N\| \left(\frac{\partial u}{\partial n}\right)^{\pm} = r_\mu \wedge \frac{\partial u^{\pm}}{\partial \lambda} - r_\lambda \wedge \frac{\partial u^{\pm}}{\partial \mu} \tag{2.20}$$

which results from $(\operatorname{div} u)^{\pm}=0$ and $(\operatorname{Curl} u)^{\pm}=0$ (see Appendix), Eq. (2.19) becomes

$$[\nabla u_i \wedge u]N_i = \left(\frac{\partial}{\partial \lambda}(r_\mu \wedge [u]) - \frac{\partial}{\partial \mu}(r_\lambda \wedge [u])\right) \wedge V + \left(r_\mu \wedge \frac{\partial V}{\partial \lambda} - r_\lambda \wedge \frac{\partial V}{\partial \mu}\right) \wedge [u]. \tag{2.21}$$

On the other hand

$$\begin{aligned} \frac{1}{\|N\|^2}(r_\mu \wedge N)_i [u_i u] \wedge N &= \frac{1}{\|N\|^2}(V, r_\mu, N)\Omega - \frac{1}{\|N\|^2}([u], r_\mu, N)(N \wedge V) \\ &= \frac{1}{\|N\|^2}(V, r_\mu, N)\Omega - ([u] \wedge r_\mu) \wedge V \end{aligned} \tag{2.22}$$

and a similar equation where λ and μ are exchanged. Substituting in Eq. (2.17b) and using the equality

$$\frac{\partial V}{\partial \lambda} \cdot r_\mu - \frac{\partial V}{\partial \mu} \cdot r_\lambda = 0 \tag{2.23}$$

which is a consequence of $(\operatorname{Curl} u)^{\pm}=0$ (see Appendix), we obtain

$$\begin{aligned} &\frac{\partial \Omega}{\partial t} - \frac{\partial}{\partial \lambda}\left\{\frac{\Omega}{\|N\|^2}(r_t, r_\mu, N)\right\} + \frac{\partial}{\partial \mu}\left\{\frac{\Omega}{\|N\|^2}(r_t, r_\lambda, N)\right\} \\ &+ \frac{\partial}{\partial \lambda}\left\{(V, r_\mu, N)\frac{\Omega}{\|N\|^2}\right\} - \frac{\partial}{\partial \mu}\left\{(V, r_\lambda, N)\frac{\Omega}{\|N\|^2}\right\} \\ &- ([u] \cdot r_\mu)\frac{\partial V}{\partial \lambda} + ([u] \cdot r_\lambda)\frac{\partial V}{\partial \mu} = 0. \end{aligned} \tag{2.24}$$

This leads to

Proposition 2.1. *If during a period of time, the vorticity of an incompressible three-dimensional flow remains concentrated on a smooth vortex sheet* $\mathscr{S}(t)=\{r=r(\lambda,\mu,t), (\lambda,\mu)\in\mathbb{R}^2\}$ *with a density* $\Omega(\lambda,\mu,t)$, *the following equations are satisfied:*

$$(r_t - V)\cdot N = 0, \tag{2.25a}$$

$$\begin{aligned} &\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \lambda}\left\{\frac{\Omega}{\|N\|^2}(V - r_t, r_\mu, N)\right\} - \frac{\partial}{\partial \mu}\left\{\frac{\Omega}{\|N\|^2}(V - r_t, r_\lambda, N)\right\} \\ &= \frac{1}{\|N\|^2}(r_\mu, N, \Omega)\frac{\partial V}{\partial \lambda} - \frac{1}{\|N\|^2}(r_\lambda, N, \Omega)\frac{\partial V}{\partial \mu}, \end{aligned} \tag{2.25b}$$

where

$$N = r_\lambda \wedge r_\mu \tag{2.25c}$$

and

$$V = -\frac{1}{4\pi} \int' \frac{r(\lambda,\mu,t) - r(\lambda',\mu',t)}{|r(\lambda,\mu,t) - r(\lambda',\mu',t)|^3} \wedge \Omega(\lambda',\mu',t)\, d\lambda' d\mu'. \tag{2.25d}$$

Remark 2.1. The two dimensional case often considered in the context of Kelvin-Helmholtz instability is recovered by taking $r(\lambda,\mu,t) = \{x_1(\lambda,t), \mu, x_3(\lambda,t)\}$ and Ω parallel to the x_2-direction and independent of μ. The motion in the (x_1, x_3)-plane is governed by

$$\begin{gathered} (\varrho_t - v)\cdot n = 0 \\ \Omega_t + \frac{\partial}{\partial\lambda}\left(\frac{\Omega}{\|n\|^2}(v - \varrho_t)\cdot\varrho_\lambda\right) = 0, \end{gathered} \tag{2.26}$$

where the components of ϱ and n in the (x_1, x_3) plane read

$$\varrho(\lambda,t) = (x_1(\lambda,t), x_3(\lambda,t)), \quad n(\lambda,t) = \left(-\frac{\partial x_3}{\partial\lambda}(\lambda,t), \frac{\partial x_1}{\partial\lambda}(\lambda,t)\right)$$

and

$$v(\lambda,t) = -\frac{1}{2\pi} \int \frac{\varrho(\lambda,t) - \varrho(\lambda',t)}{|\varrho(\lambda,t) - \varrho(\lambda',t)|^2} \wedge \Omega(\lambda',t)\, d\lambda'.$$

Remark 2.2. The above systems which are reversible in time, are equations of contact discontinuity without loss of energy.

3. Linear Stability of a Flat Vortex Sheet with Uniform Vorticity Density

When the vortex sheet can be resolved in the form $z = z(x,y,t)$, Eq. (2.25) reads

$$\begin{gathered} z_t = -z_x V_1 - z_y V_2 + V_3 \\ \Omega_t + \frac{\partial}{\partial x}(\Omega V_1) + \frac{\partial}{\partial y}(\Omega V_2) = \Omega_1 \frac{\partial V}{\partial x} + \Omega_2 \frac{\partial V}{\partial y}, \end{gathered} \tag{3.1}$$

where the subscripts 1, 2, 3 refer to the cartesian components of the vectors and

$$V\{\Omega, z\}(x,y,t) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{r(x,y,t) - r(x',y',t)}{|r(x,y,t) - r(x',y',t)|^3} \wedge \Omega(x',y',t)\, dx' dy'. \tag{3.2}$$

When linearized near the solution corresponding to a flat vortex sheet in the (x,y)-plane with a uniform vorticity density $\Omega^{(0)} = (\Omega_1^{(0)}, \Omega_2^{(0)}, 0)$, Eqs. (3.1) and (3.2) become

$$\begin{gathered} \tilde{z}_t = \tilde{V}_3 \\ \tilde{\Omega}_t + \frac{\partial}{\partial x}(\Omega^{(0)}\tilde{V}_1) + \frac{\partial}{\partial y}(\Omega^{(0)}\tilde{V}_2) = \Omega_1^{(0)}\frac{\partial \tilde{V}}{\partial x} + \Omega_2^{(0)}\frac{\partial \tilde{V}}{\partial y} \end{gathered} \tag{3.3}$$

(the subscript denotes infinitesimal perturbations) and

$$\tilde{V}_1 = -\frac{1}{4\pi}\left\{\mathrm{vp}\frac{y}{|\varrho|^3} * \left(\tilde{\Omega}_3 - \Omega_2^{(0)}\frac{\partial \tilde{z}}{\partial y}\right) - \mathrm{vp}\frac{x}{|\varrho|^3} * \Omega_2^{(0)}\frac{\partial \tilde{z}}{\partial x}\right\}$$
$$\tilde{V}_2 = -\frac{1}{4\pi}\left\{\mathrm{vp}\frac{x}{|\varrho|^3} * \left(\Omega_1^{(0)}\frac{\partial \tilde{z}}{\partial x} - \tilde{\Omega}_3\right) + \mathrm{vp}\frac{y}{|\varrho|^3} * \Omega_1^{(0)}\frac{\partial \tilde{z}}{\partial y}\right\} \tag{3.4}$$
$$\tilde{V}_3 = -\frac{1}{4\pi}\left\{\mathrm{vp}\frac{x}{|\varrho|^3} * \tilde{\Omega}_2 - \mathrm{vp}\frac{y}{|\varrho|^3} * \tilde{\Omega}_1\right\}$$

with $\varrho = (x, y)$. The two dimensional Fourier transform of $\mathrm{vp}\frac{\varrho}{|\varrho|^3}$ is $2i\pi\frac{k}{*k|} = 2i\pi(\cos\theta, \sin\theta)$, where θ is the polar angle of the wave vector k. The Fourier modes of the disturbances, denoted by a superscript $\hat{}$, thus satisfy

$$\frac{d}{dt}\begin{pmatrix}\hat{z}\\ \hat{\Omega}_1\\ \hat{\Omega}_2\\ \hat{\Omega}_3\end{pmatrix} = A\begin{pmatrix}\hat{z}\\ \hat{\Omega}_1\\ \hat{\Omega}_2\\ \hat{\Omega}_3\end{pmatrix} \tag{3.5}$$

with

$$A = \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0 \\ -\frac{i}{2}k^2|\Omega^{(0)}|^2\sin\theta & 0 & 0 & \frac{1}{2}(k\cdot\Omega^{(0)})\sin\theta \\ \frac{i}{2}k^2|\Omega^{(0)}|^2\cos\theta & 0 & 0 & -\frac{1}{2}(k\cdot\Omega^{(0)})\cos\theta \\ 0 & -\frac{1}{2}(k\cdot\Omega^{(0)})\sin\theta & \frac{1}{2}(k\cdot\Omega^{(0)})\cos\theta & 0 \end{pmatrix}.$$

The eigenvalues of the matrix A read

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -\alpha, \lambda_4 = \alpha \tag{3.7}$$

with

$$\alpha = \tfrac{1}{2}|k \wedge \Omega^{(0)}| = \tfrac{1}{2}|k \cdot [u]|. \tag{3.8}$$

Perturbations transverse to the direction of streaming are thus unaffected (Chandrasekhar, 1961, p. 484). The associated eigenvectors are given by

$$R_1 = (0, \cos\theta, \sin\theta, 0)$$
$$R_2 = ((\Omega^{(0)}\cdot k), 0, 0, ik^2|\Omega^{(0)}|^2)$$
$$R_3 = (i, -\sin\theta|k\wedge\Omega^{(0)}|, \cos\theta|k\wedge\Omega^{(0)}|, -(k\cdot\Omega^{(0)}))$$
$$R_4 = (i, \sin\theta|k\wedge\Omega^{(0)}|, -\cos\theta|k\wedge\Omega^{(0)}|, -(\Omega^{(0)}\cdot k)). \tag{3.9}$$

In two dimensions, the system (3.1) is replaced by

$$y_t = -y_x v_1 + v_2, \tag{3.10a}$$

$$\Omega_t + \frac{\partial}{\partial x}(\Omega v_1) = 0, \tag{3.10b}$$

where $x = x_1$, $y = x_3$; the components of v in the (x_1, x_3) plane read

$$v_1(x,t) = -\frac{1}{2\pi} \fint_{\mathbb{R}} \frac{y(x,t) - y(x',t)}{(x-x')^2 + (y(x,t) - y(x',t))^2} \Omega(x',t)\,dx'$$

$$v_2(x,t) = \frac{1}{2\pi} \fint_{\mathbb{R}} \frac{x - x'}{(x-x')^2 + (y(x,t) - y(x',t))^2} \Omega(x',t)\,dx'.$$

Note that (3.10b) is a continuity equation for the vorticity density. The system (3.10) also displays an instability at the rate $|k\Omega^{(0)}|$ when linearized near a flat interface with a uniform vorticity density $\Omega^{(0)}$.

Because of this instability known as the Kelvin-Helmholtz instability, analyticity of the initial data is required for the linear problem to be locally well set. Birkhoff (1962) conjectured that it is also the case for the nonlinear problem.

4. Short Time Analyticity of the Vortex Sheet

Local existence for the nonlinear problem both in two and three dimensions is based on an abstract nonlinear Cauchy Kowalewski theorem in the formulation of Nishida (1977) (see also Baouendi and Goulaouic, 1977) which improves a result of Ovsjannikov (1971) and Nirenberg (1972).

Theorem (Nishida and Baouendi and Goulaouic). *Let $\mathcal{S} = \{B_s\}_{s>0}$ be a scale of Banach spaces, and let all B_s for $s>0$ be linear subspaces of B_0. It is assumed by $B_s \subset B_{s'}$, $\|\cdot\|_{s'} \leqq \|\cdot\|_s$ for $s' \leqq s$, where $\|\cdot\|_s$ denotes the norm in B_s.*

Consider in $\mathcal{S}$ the initial value problem of the form

$$\frac{du}{dt} = F(u(t), t) \qquad |t| < \delta$$

$$u(0) = 0. \tag{E1}$$

Assume the following conditions on F:

(i) *For some numbers $R>0$, $\eta>0$, $s_0>0$ and every pair of numbers s, s' such that $0 \leqq s' < s < s_0$, $(u,t) \to F(u,t)$ is a continuous mapping of*

$$\{u \in B_s; \|u\|_s < R\} \times \{t; |t| < \eta\} \text{ into } B_{s'}. \tag{H1}$$

(ii) *For any $s' < s < s_0$ and all $u, v \in B_s$ with $\|u\|_s < R$, $\|v\|_s < R$ and for any t, $|t| < \eta$, F satisfies*

$$\|F(u,t) - F(v,t)\|_{s'} \leqq C\|u - v\|_s/(s - s') \tag{H2}$$

when C is a constant independent of t, u, v, s or s'.

(iii) *$F(0,t)$ is a continuous function of t, $|t|<\eta$, with values in B_s for every $s<s_0$ and satisfies, with a fixed constant K,*

$$\|F(0,t)\|_s \leqq K/(s_0-s) \qquad 0 \leqq s < s_0 . \tag{H3}$$

Under the preceding hypothesis there is a positive constant α such that there exists a unique function $u(t)$ which, for every positive $s<s_0$ and $|t|<\alpha(s_0-s)$, is a continuously differentiable function of t with values in B_s, $\|u\|_s<R$ and satisfies (E1).

If in addition to (H1) *and* (H2) *with t complex, F satisfies the following assumption: for $0<s'<s<1$ and u holomorphic for $t\in\mathbb{C}$, $|t|<\delta$ valued in B_s with $\sup_{|t|<\delta}\|u(t)\|_s<R$, $t\to F(t,u(t))$ is a holomorphic function for $|t|<\eta$ valued in $B_{s'}$, then u is an holomorphic function of t with values in B_s.*

For the use of Nishida theorem, the main quantity to deal with is the nonlinear integral operator ($n=2,3$)

$$\int \frac{r(x_1,\ldots,x_{n-1})-r(x_1',\ldots,x_{n-1}')}{|r(x_1,\ldots,x_{n-1})-r(x_1',\ldots,x_{n-1}')|^n} \wedge \Omega(x_1',\ldots,x_{n-1}')\,dx_1'\ldots dx_{n-1}'$$

which solves the equation

$$\nabla \wedge V = \Omega(r) \otimes \delta(r - r(x_1 \ldots x_{n-1})) .$$

In three dimensions, we shall study this operator following the general method for singular integral operators in the Hölder spaces $C^{k,\alpha}$. In two dimensions, it turns out that, due to the fact that $\{r=r(x)\}$ is a manifold of dimension 1, the study is simplified by using auxiliary complex variables and the classical relation between analytic functions of $z=x+iy$ and harmonic functions of (x,y). Therefore, we shall include different proofs in two and three dimensions.

A. Local Analyticity in Two Dimensions

As is generally the case in numerical calculations we assume that the interface is periodic in the x-direction. We look for solutions $(\Omega,y)(x,t)$ which would be the restriction of analytic functions $(\Omega,y)\,(x+i\zeta,t)$ defined in strips of complex plane. Since the problem is nonlinear, it is convenient to deal with Hölder spaces (defined below), which are stable by multiplication.

The scale of Banach spaces we use is therefore constituted by the spaces $B_s(s>0)$ of π-periodic functions, analytic in the strip $b_s=\{(x,\zeta),\ x\in\mathbb{R}/\pi\mathbb{Z},\ |\zeta|<s\}$ with the Hölder norm

$$\|u\|_s = |u|_s + \sup_{\substack{(x,\zeta)\in b_s\\(x',\zeta)\in b_s}} \frac{|u(x+i\zeta)-u(x'+i\zeta)|}{|x-x'|^\alpha}, \quad 0<\alpha<1 ,$$

wherein

$$|u|_s = \sup_{(x,\zeta)\in b_s} |u(x+i\zeta)| .$$

We use the same notation $|\cdot|_s$ in the case of functions of two variables defined on b_s

$$|\varphi|_s = \sup_{\substack{(x,\zeta)\in b_s \\ (x',\zeta)\in b_s}} |\varphi(x+i\zeta, x'+i\zeta)|\,.$$

When π-periodicity is assumed, the components v_1 and v_2 are easily computed using complex representation

$$\begin{aligned} z(x,t) &= x + iy(x,t) \\ \bar{v}(x,t) &= v_1 - iv_2\,, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \bar{v}\{\Omega, y\}(x,t) &= -\frac{1}{2\pi i} \fint_{-\infty}^{+\infty} \frac{\Omega(x',t)}{z(x,t)-z(x',t)}\,dx' \\ &= -\frac{1}{2\pi i} \fint_{x-\pi/2}^{x+\pi/2} \Omega(x',t)\cot(z(x,t)-z(x',t))\,dx'\,, \end{aligned} \tag{4.2}$$

which gives

$$v_i\{\Omega, y\}(x,t) = \frac{1}{2\pi} \fint_{x-\pi/2}^{x+\pi/2} \frac{1}{\sin(x-x')} g_i(x,x',t)\Omega(x',t)\,dx' \tag{4.3}$$

with

$$g_1(x,x',t) = \frac{1}{4\pi} \frac{\operatorname{sh} 2(y(x,t)-y(x',t))}{\sin(x-x')} \cdot \left(1 + \frac{\operatorname{sh}^2(y(x,t)-y(x',t))}{\sin^2(x-x')}\right)^{-1}, \tag{4.4a}$$

$$g_2(x,x',t) = -\frac{1}{2\pi}\cos(x-x')\left(1 + \frac{\operatorname{sh}^2(y(x,t)-y(x',t))}{\sin^2(x-x')}\right)^{-1}. \tag{4.4b}$$

$g_i(x,x,t)$ for $i=1,2$ are defined by continuity.

v_1 and v_2 are analytically continued in the strip b_s by

$$v_i\{\Omega, y\}(x+i\zeta, t) = \fint_{x-\pi/2}^{x+\pi/2} \frac{1}{\sin(x-x')} g_i(x+i\zeta, x'+i\zeta, t)\Omega(x'+i\zeta, t)\,dx'\,. \tag{4.5}$$

If f is an analytic function in b_s, we have for any $s' < s$

$$\left\|\frac{\partial f}{\partial x}\right\|_{s'} \leqq \frac{C}{s-s'} \|f\|_s\,,$$

and the main part of the proof consists in establishing the

Proposition 4.1. *For* $|\operatorname{Im} y_x|_s, |\operatorname{Im} \tilde{y}_x|_s$ *strictly smaller than* $1/2$ *and* $\|y_x\|_s$, $\|\tilde{y}_x\|_s$, $\|y_{xx}\|_s$, $\|\tilde{y}_{xx}\|_s$, $\|\Omega\|_s$, $\|\tilde{\Omega}\|_s$ *bounded, the analytic continuation of* v *to complex space variable satisfies* $(i=1,2)$

$$\|v_i\{\Omega, y\} - v_i\{\tilde{\Omega}, \tilde{y}\}\|_s \leqq C\{\|\Omega - \tilde{\Omega}\|_s + \|y_x - \tilde{y}_x\|_s + \|y_{xx} - \tilde{y}_{xx}\|_s\}\,. \tag{4.6}$$

Proof. We adapt the method of Ladyzenskaya and Uratlceva (1968) (Chap. 3, Sect. 2) for elliptic operators. We define

$$\delta\Omega=\Omega-\tilde{\Omega}\,;\delta g=g-\tilde{g}\,,$$

$$U\{\Omega,g\}(x+i\zeta)=\fint_{x-\pi/2}^{x+\pi/2}\frac{dq}{\sin(x-q)}\Omega(q+i\zeta)g(x+i\zeta,q+i\zeta)\,. \tag{4.7}$$

We have

$$U\{\Omega,g\}-U\{\tilde{\Omega},g\}=U\{\delta\Omega,g\}+U\{\tilde{\Omega},\delta g\}\,, \tag{4.8}$$

where (by symmetry)

$$\begin{aligned}U\{\delta\Omega,g\}(x+i\zeta)=\fint_{x-\pi/2}^{x+\pi/2}\frac{1}{\sin(x-q)}\{&\delta\Omega(q+i\zeta)g(x+i\zeta,q+i\zeta)\\&-\delta\Omega(x+i\zeta)g(x+i\zeta,x+i\zeta)\}\,dq\,.\end{aligned}$$

It follows that

$$|U\{\delta\Omega,g\}|_s\leqq C\|\delta\Omega\|_s\sup_{(x,\zeta)\in b_s}\|g(x+i\zeta,\cdot)\|_s\,. \tag{4.9}$$

We now turn to the quantity $U^{(1)}\{\delta\Omega,g\}(x+i\zeta)-U^{(1)}\{\delta\Omega,g)\}(x'+i\zeta)$. We assume $x<x'$. The case $|x-x'|>\pi/4$ is obvious; when $|x-x'|<\pi/4$, we write

$$\begin{aligned}&U^{(1)}\{\delta\Omega,g\}(x+i\zeta)-U^{(1)}\{\delta\Omega,g\}(x'+i\zeta)\\&=\int_{x-\pi/2}^{x'-\pi/2}dq\,\frac{\delta\Omega(q+i\sigma)g(x'+i\zeta,q+i\zeta)-\delta\Omega(x'+i\zeta)g(x'+i\zeta,x'+i\zeta)}{\sin(x'-q)}\\&\quad-\int_{x+\pi/2}^{x'+\pi/2}dq\,\frac{\delta\Omega(q+i\zeta)g(x'+i\zeta,q+i\zeta)-\delta\Omega(x'+i\zeta)g(x'+i\zeta,x'+i\zeta)}{\sin(x'-q)}\\&\quad+\int_{x-\pi/2}^{x+\pi/2}dq\left\{\frac{\delta\Omega(q+i\zeta)g(x+i\zeta,q+i\zeta)-\delta\Omega(x+i\zeta)g(x+i\zeta,x+i\zeta)}{\sin(x-q)}\right.\\&\quad\left.-\frac{\delta\Omega(q+i\zeta)g(x'+i\zeta,q+i\zeta)-\delta\Omega(x'+i\zeta)g(x'+i\zeta,x'+i\zeta)}{\sin(x'-q)}\right\}.\end{aligned} \tag{4.10}$$

The two first terms of (4.10) are both bounded from above by $C|\delta\Omega|_s|g|_s|x-x'|$. In the last one, referred to as δ, we split the integration domain into the ball $\Sigma=\{q,|x-q|<2|x-x'|\}$ and its complement. Let δ_Σ and $\delta_{\bar{\Sigma}}$ be the corresponding contributions to δ

$$|\delta_\Sigma|\leqq C\|\delta\Omega\|_s\sup_{(x,\zeta)\in b_s}\|g(x+i\zeta,\cdot)\|_s\int_{|x-q|<3|x-x'|}\frac{|x-q|^\alpha}{\sin|x-q|}dq\,. \tag{4.11}$$

Since $|x-x'|<\pi/4$, there exists C_1 such that $\sin|x-q|>C_1|x-q|$ and the integral is bounded by $|x-x'|^\alpha$.

$$\begin{aligned}\delta_{\bar{\Sigma}}=&\int_{\pi/2>|x-q|>2|x-x'|}dq\left(\frac{1}{\sin(x-q)}-\frac{1}{\sin(x'-q)}\right)\{\delta\Omega(q+i\zeta)g(x'+i\zeta,q+i\zeta)\\&-\delta\Omega(x'+i\zeta)g(x'+i\zeta,x'+i\zeta)\}\\&+\int_{\pi/2>|x-q|>2|x-x'|}\frac{dq}{\sin(x-q)}\delta\Omega(q+i\zeta)\{g(x+i\zeta,q+i\zeta)-g(x'+i\zeta,q+i\zeta)\}.\end{aligned} \tag{4.12}$$

The first term of $\delta_{\bar{\Sigma}}$ is bounded by

$$\|\delta\Omega\|_s \sup_{(x,\zeta)\in b_s} \|g(x+i\zeta,\cdot)\|_s \int_{|x-x'|<|x'-q|<3\pi/4} \frac{|x'-q|^\alpha |\sin(x-x')|}{|\sin\frac{2}{3}(x'-q)|\,|\sin(x'-q)|}\,dq,$$

where the integral is smaller than $C|x-x'|^\alpha$.

The second term is bounded by $|\delta\Omega|_s \left|\frac{\partial g}{\partial x}\right|_s |x-x'|^\alpha$. Thus

$$|\delta_{\bar{\Sigma}}| \leqq C\left\{\sup_{(x,\zeta)\in b_s} \|g(x+i\zeta,\cdot)\|_s + \left|\frac{\partial g}{\partial x}\right|_s\right\} \|\delta\Omega\|_s |x-x'|^\alpha \tag{4.13}$$

and

$$\|U\{\Omega,g\} - U\{\tilde{\Omega},\tilde{g}\}\|_s \leqq C\{\|\delta\Omega\|_s(|g|_s+|\nabla g|_s) + \|\tilde{\Omega}\|_s(|\delta g|_s+|\nabla\delta g|_s)\}. \tag{4.14}$$

The Proposition 4.1 follows from the Eq. (4.14) and the

Lemma 4.1. *If* $|\mathrm{Im}\, y_x|_s$ *and* $|\mathrm{Im}\,\tilde{y}_x|_s$ *are smaller than* 1/2 *and* $|\mathrm{Re}\, y_x|_s$, $|\mathrm{Re}\,\tilde{y}_x|_s$, $|y_{xx}|_s$, $|\tilde{y}_{xx}|_s$ *are bounded,* $|g_i|_s$ *and* $|\nabla g_i|_s (i=1,2)$ *are uniformly bounded and*

$$|g_i - \tilde{g}_i|_s \leqq C|y_x - \tilde{y}_x|_s \tag{4.15}$$

and

$$|\nabla g_i - \nabla\tilde{g}_i| \leqq C(|y_x - \tilde{y}_x|_s + |y_{xx} - \tilde{y}_{xx}|_s).$$

Proof. The denominator of g_i reads

$$\mathscr{D} = 1 + A^2,$$

where ($|x-x'|<\pi/2$)

$$A = \frac{\mathrm{sh}(a+ib)}{\sin(x-x')}$$

and a and b denote the real and imaginary parts of $y(x+i\zeta) - y(x'+i\zeta)$. $\mathscr{D}$ does not vanish provided

$$1 + (\mathrm{Re}\,A)^2 - (\mathrm{Im}\,A)^2 = 1 - \frac{\sin^2 b}{\sin^2(x-x')} + \frac{\mathrm{sh}^2 a}{\sin^2(x-x')}\cos 2b$$

be strictly positive. This is insured if $|\mathrm{Im}\, y_x|_s < 1/2$. Lemma 4.1 results from straightforward calculation.

Since in the estimation of $\|v\{\Omega,y\} - v\{\tilde{\Omega},\tilde{y}\}\|_s$ the first and second derivatives of y enter, we also have to deal with the equations satisfied by y_x and y_{xx}, and thus to estimate $\frac{\partial v}{\partial x}$.

Proposition 4.2. *For* $|\mathrm{Im}\, y_x|_s$, $|\mathrm{Im}\,\tilde{y}_x|_s$ *smaller than* 1/2, *and* $\|y_x\|_s$, $\|\tilde{y}_x\|_s$, $\|y_{xx}\|_s$, $\|\tilde{y}_{xx}\|_s$, $\|\Omega\|_s$, $\|\tilde{\Omega}\|_s$, $\|\Omega_x\|_s$, $\|\tilde{\Omega}_x\|_s$ *bounded,*

$$\left\|\frac{\partial v_i}{\partial x}\{\Omega,y\} - \frac{\partial v_i}{\partial x}\{\tilde{\Omega},\tilde{y}\}\right\|_s \leqq C(\|\Omega-\tilde{\Omega}\|_s + \|\Omega_x - \tilde{\Omega}_x\|_s + \|y_x - \tilde{y}_x\|_s + \|y_{xx} - \tilde{y}_{xx}\|_s). \tag{4.16}$$

Proof. $\frac{\partial v}{\partial x}\{\Omega, y\}(x+i\zeta, t)$ are easily obtained by computing the derivative of v in the real domain and then, analytically continuing the expression in the strip b_s. Using the notations (4.1) and (4.2) we have:

$$\frac{\partial \bar{v}}{\partial x}\{\Omega, y\}(x)=\frac{\partial z}{\partial x}(x)\bar{v}\left\{\frac{\partial}{\partial x}\left(\frac{\Omega}{\frac{\partial z}{\partial x}}\right), y\right\}(x). \tag{4.17}$$

The real and imaginary parts are then analytically continued by

$$\begin{aligned}
\frac{\partial v_1}{\partial x}\{\Omega, y\}(x+i\zeta)=&\, v_1\left\{\frac{\partial}{\partial x}\left(\frac{\Omega}{1+y_x^2}\right), y\right\}(x+i\zeta)\\
&+v_2\left\{\frac{\partial}{\partial x}\left(\frac{\Omega y_x}{1+y_x^2}\right), y\right\}(x+i\zeta)\\
&+y_x(x+i\zeta)v_2\left\{\frac{\partial}{\partial x}\left(\frac{\Omega}{1+y_x^2}, y\right)\right\}(x+i\zeta)\\
&+y_x(x+i\zeta)v_1\left\{\frac{\partial}{\partial x}\left(\frac{\Omega y_x}{1+y_x^2}, y\right)\right\}(x+i\zeta)\\
\frac{\partial v_2}{\partial x}\{\Omega, y\}(x+i\zeta)=&-y_x(x+i\zeta)v_1\left\{\frac{\partial}{\partial x}\left(\frac{\Omega}{1+y_x^2}\right), y\right\}(x+i\zeta)\\
&+y_x(x+i\zeta)v_2\left\{\frac{\partial}{\partial x}\left(\frac{\Omega y_x}{1+y_x^2}\right), y\right\}(x+i\zeta)\\
&+v_2\left\{\frac{\partial}{\partial x}\left(\frac{\Omega}{1+y_x^2}\right), y\right\}(x+i\zeta)\\
&+v_1\left\{\frac{\partial}{\partial x}\left(\frac{\Omega y_x}{1+y_x^2}\right), y\right\}(x+i\zeta).
\end{aligned} \tag{4.18}$$

We complete the proof by using Proposition 4.1.

Consequently, the system

$$\begin{aligned}
\frac{\partial \Omega}{\partial t}&=-\frac{\partial}{\partial x}(v_1\Omega)\\
\frac{\partial y}{\partial t}&=-(y_x v_1-v_2)\\
\frac{\partial y_x}{\partial t}&=-\frac{\partial}{\partial x}(y_x v_1-v_2)\\
\frac{\partial \Omega_x}{\partial t}&=-\frac{\partial}{\partial x}\left(\Omega_x v_1+\Omega\frac{\partial v_1}{\partial x}\right)\\
\frac{\partial}{\partial t}y_{xx}&=-\frac{\partial}{\partial x}\left(y_{xx}v_1+y_x\frac{\partial v_1}{\partial x}-\frac{\partial v_2}{\partial x}\right)
\end{aligned} \tag{4.19}$$

satisfies the hypothesis of the above nonlinear Cauchy-Kowalewski theorem. This leads to the

Theorem 4.1. *For initial conditions such that the analytic continuations of*

$$y_0, \frac{\partial y_0}{\partial x}, \frac{\partial^2 y_0}{\partial x^2}, \Omega_0, \frac{\partial \Omega_0}{\partial x} \text{ belong to } B_{s_0}, \text{ with } \left|\operatorname{Im}\frac{\partial y_0}{\partial x}\right|_{s_0} < 1/2,$$

there exists a constant α *such that for* $|t| < \alpha(s_0 - s)$, *the system* (3.10) *has a unique solution* (y, Ω) *which is an holomorphic function of* t *with value in* $(B_s)^2$.

Remark. The "classical" proof which will be described in the next section substitutes for the estimate (4.6) of Proposition 4.1 the refined one

$$\|v_i\{\Omega, y\} - v_i\{\tilde{\Omega}, \tilde{y}\}\|_s \leqq C(\|\Omega - \tilde{\Omega}\|_s + \|y_x - \tilde{y}_x\|_s).$$

This estimate, valid in two and three dimensions, is more complicated to prove but makes it possible to use only the three first equations of (4.19).

B. Local Analyticity in Three Dimensions

To study the three dimensional problem, it will be convenient to assume that the strength of the vortex sheet vanishes at infinity. Periodicity in x and y has some minor additional technical difficulties but should produce essentially the same results. For our purpose, it is suitable to use the scale of Banach spaces $B_s (s > 0)$ of functions which are analytic in the strip

$$b_s = \{(\varrho, \sigma), \varrho \in \mathbb{R}^2, |\sigma| < s\}$$

and belong to $L^2(\mathbb{R}^2 + i\sigma)$, $|\sigma| < s$. We equip B_s with the norm $|||\cdot|||_s$,

$$|||u|||_s = \|u\|_s + \|u\|_{L^2_s},$$

with

$$\|u\|_s = |u|_s + \sup_{\substack{(\varrho,\sigma)\in b_s \\ (\varrho',\sigma)\in b_s}} \frac{|u(\varrho + i\sigma) - u(\varrho' + i\sigma)|}{|\varrho - \varrho'|^\alpha}$$

$$|u|_s = \sup_{(\varrho,\sigma)\in b_s} |u(\varrho + i\sigma)|$$

$$\|u\|_{L^2_s} = \sup_{|\sigma| < s}\left(\left(\int_{\mathbb{R}^2} |u(\varrho + i\sigma)|^2 d\varrho\right)^{1/2}\right).$$

Dropping out the t-dependence, we rewrite V in the form

$$V\{\Omega, z\}(\varrho) = -\frac{1}{4\pi} \fint_{\mathbb{R}^2} \frac{r(\varrho, t) - r(q, t)}{|\varrho - q|^3} G(\varrho, q) \wedge \Omega(q) dq, \tag{4.20}$$

with

$$G(\varrho, q) = \left\{1 + \left(\frac{z(\varrho) - z(q)}{|\varrho - q|}\right)^2\right\}^{-3/2}. \tag{4.21}$$

For $|\mathrm{Im}\,\nabla z|_s < 1$, V can be analytically continued in the strip b_s as

$$V\{\Omega, z\}(\varrho+i\sigma) = -\frac{1}{4\pi} \oint_{\mathbb{R}^2} \frac{r(\varrho+i\sigma)-r(q+i\sigma)}{|\varrho-q|^3} G(\varrho+i\sigma, q+i\sigma) \wedge \Omega(q+i\sigma)\,dq\,. \tag{4.22}$$

Proposition 4.3. *If* (Ω, z) *satisfies* $|\mathrm{Im}\,\nabla z|_s < 1$, $|||\Omega|||_s$, $\|\nabla z\|_s$ *bounded, and similar conditions for* $(\tilde{\Omega}, \tilde{z})$, *the analytic continuation of* V *to complex space variables satisfies*

$$\| V\{\Omega, z\} - V\{\tilde{\Omega}, \tilde{z}\} \|_s \leqq C(|||\Omega - \tilde{\Omega}|||_s + \|\nabla(z-\tilde{z})\|_s)\,. \tag{4.23}$$

Proof. To simplify the writing, we define

$$U\{\Omega, G, z\}(\varrho+i\sigma) = -\frac{1}{4\pi} \oint_{\mathbb{R}^2} \frac{r(\varrho+i\sigma)-r(q+i\sigma)}{|\varrho-q|^3} G(\varrho+i\sigma, q+i\sigma) \wedge \Omega(q+i\sigma)\,dq\,.$$

Then

$$V\{\Omega, z\} - V\{\tilde{\Omega}, \tilde{z}\} = U\{\delta\Omega, G, z\} + U\{\tilde{\Omega}, \delta G, z\} + U\{\tilde{\Omega}, \tilde{G}, \delta z\}\,, \tag{4.24}$$

with $\delta\Omega = \Omega - \tilde{\Omega}$, $\delta G = G - \tilde{G}$, $\delta z = z - \tilde{z}$ and $\tilde{G}$ obtained by replacing z by $\tilde{z}$ in Eq. (4.21). We separate the contributions from origin and infinity in the above integrals by introducing an even, smooth function θ_1 with compact support which equals one in a neighborhood of the origin. We also define $\theta_2 = 1 - \theta_1$. Then

$$U\{\delta\Omega, G, z\} = U^{(1)}\{\delta\Omega, G, z\} + U^{(2)}\{\delta\Omega, G, z\}\,, \tag{4.25}$$

with

$$U_i^{(k)}\{\delta\Omega, G, z\}(\varrho+i\sigma) = -\frac{\varepsilon_{ijk}}{4\pi} \oint_{\mathbb{R}^2} \frac{\theta_k(|\varrho-q|)}{|\varrho-q|^3}(r_j(\varrho+i\sigma) - r_j(q+i\sigma)) \cdot G(\varrho+i\sigma, q+i\sigma)\delta\Omega_\ell(q+i\sigma)\,dq\,. \tag{4.26}$$

Proposition 4.3 follows from the estimates of $\|U^{(\ell)}(\delta\Omega, G, z)\|_s$, $\|U^{(\ell)}(\tilde{\Omega}, \delta G, z\|_s$, $\|U^{(\ell)}(\tilde{\Omega}, \tilde{G}, \delta z)\|_s$ given in Lemmas 4.2–4.4.

Lemma 4.2. *If* z *satisfies* $|\mathrm{Im}\,\nabla z|_s < 1$ *and* $\|\nabla z\|_s$ *bounded, we have*

$$\|U^{(1)}\{\delta\Omega, G, z\}\|_s \leqq C\|\delta\Omega\|_s\,, \tag{4.27}$$

$$\|U^{(2)}\{\delta\Omega, G, z\}\|_s \leqq C\|\delta\Omega\|_{L^2_s}\,. \tag{4.28}$$

Proof. By symmetry we have $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$

$$\int_{\mathbb{R}^2} \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}(\varrho-q) \cdot \nabla r_j(\varrho+i\sigma) g(\varrho+i\sigma, q+i\sigma)\delta\Omega_\ell(\varrho+i\sigma)\,dq = 0\,,$$

where

$$g(\varrho, q) = \left\{1 + \left(\frac{(\varrho-q)}{|\varrho-q|} \cdot \nabla z(\varrho)\right)^2\right\}^{-3/2}\,. \tag{4.29}$$

Thus

$$U_i^{(1)}\{\delta\Omega, G, z\}(\varrho+i\sigma) = -\frac{\varepsilon_{ijk}}{4\pi}\int_{\mathbb{R}^2}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}\psi_{j\ell}\{\delta\Omega, G, g, z\}(\varrho+i\sigma, q+i\sigma)dq, \tag{4.30}$$

with

$$\begin{aligned}\psi_{j\ell}\{\delta\Omega, G, g, z\}(\varrho+i\sigma, q+i\sigma) &= (r_j(\varrho+i\sigma)-r_j(q+i\sigma))G(\varrho+i\sigma, q+i\sigma)\delta\Omega_\ell(q+i\sigma)\\ &\quad -(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)g(\varrho+i\sigma, q+i\sigma)\delta\Omega_\ell(\varrho+i\sigma)\\ &= (\delta\Omega_\ell(q+i\sigma)-\delta\Omega_\ell(\varrho+i\sigma))(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)g(\varrho+i\sigma, q+i\sigma)\\ &\quad +\{G(\varrho+i\sigma, q+i\sigma)-g(\varrho+i\sigma, q+i\sigma)\}\{(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)\}\delta\Omega_\ell(q+i\sigma)\\ &\quad +\{r_j(\varrho+i\sigma)-r_j(q+i\sigma)\\ &\quad -(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)\}G(\varrho+i\sigma, q+i\sigma)\delta\Omega_\ell(q+i\sigma).\end{aligned} \tag{4.31}$$

We then notice that

$$z(\varrho+i\sigma)-z(q+i\sigma)-(\varrho-q)\cdot\nabla z(\varrho+i\sigma) = (\varrho-q)\cdot\int_0^1\{\nabla z(\lambda(\varrho+i\sigma)+(1-\lambda)(q+i\sigma))-\nabla z(\varrho+i\sigma)\}d\lambda. \tag{4.32}$$

Consequently

$$|z(\varrho+i\sigma)-z(q+i\sigma)-(\varrho-q)\cdot\nabla z(\varrho+i\sigma)| \leqq |\varrho-q|^{1+\alpha}\|\nabla z\|_s. \tag{4.33}$$

Thus, if the hypotheses of Proposition 4.3 are satisfied

$$|\psi_{j\ell}\{\delta\Omega, G, g, z\}(\varrho+i\sigma, q+i\sigma)| \leqq C|\varrho-q|^{1+\alpha}\|\delta\Omega\|_s \tag{4.34}$$

and

$$|U^{(1)}\{\delta\Omega, G, z\}|_s \leqq C\|\delta\Omega\|_s. \tag{4.35}$$

To estimate

$$\begin{aligned}\delta_i &\equiv U_i^{(1)}\{\delta\Omega, G, z\}(\varrho+i\sigma)-U_i^{(1)}\{\delta\Omega, G, z\}(\varrho'+i\sigma)\\ &= -\frac{\varepsilon_{ijk}}{4\pi}\int_{\mathbb{R}^2}\left(\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}\psi_{jk}(\varrho+i\sigma, q+i\sigma)-\frac{\theta_1(|\varrho'-q|)}{|\varrho'-q|^3}\psi_{jk}(\varrho'+i\sigma, q+i\sigma)\right)dq,\end{aligned} \tag{4.36}$$

we split the integration domain into the ball $\Sigma\{q/|\varrho-q|\leqq 2|\varrho-\varrho'|\}$ and its complement $\bar{\Sigma}=\mathbb{R}^2\backslash\Sigma$. Let $\delta_{i\Sigma}$ and $\delta_{i\bar{\Sigma}}$ be the corresponding contribution to δ_i. Using (4.34) and the inequality

$$|\varrho'-q| \leqq |\varrho-\varrho'|+|\varrho-q| \leqq 3|\varrho-\varrho'|,$$

we have

$$|\delta_{i\Sigma}| \leqq C|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s. \tag{4.37}$$

We rewrite $\delta_{i\bar{\Sigma}}$ in the form

$$\delta_{i\bar{\Sigma}}=-\frac{\varepsilon_{ij\ell}}{4\pi}\int_{|\varrho-q|>2|\varrho-\varrho'|}\left(\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}-\frac{\theta_1(|\varrho'-q|)}{|\varrho'-q|^3}\right)\psi_{ij}(\varrho'+i\sigma,q+i\sigma)dq$$
$$+\frac{\varepsilon_{ij\ell}}{4\pi}\int_{|\varrho-q|>2|\varrho-\varrho'|}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}(\psi_{ij}(\varrho+i\sigma,q+i\sigma)-\psi_{ij}(\varrho'+i\sigma,q+i\sigma))dq. \quad (4.38)$$

The first term of the right hand side of (4.38) is bounded by $|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s$. In the second term,

$$\psi_{j\ell}(\varrho+i\sigma,q+i\sigma)-\psi_{j\ell}(\varrho'+i\sigma,q+i\sigma)=A_{j\ell}+B_{j\ell}+C_{j\ell}+D_{j\ell}+E_{j\ell}+F_{j\ell}+H_{j\ell} \quad (4.39)$$

with

$$A_{j\ell}=(\delta\Omega_\ell(\varrho'+i\sigma)-\delta\Omega_\ell(\varrho+i\sigma))\{(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)\}g(\varrho+i\sigma,q+i\sigma)$$
$$B_{j\ell}=\{(\varrho-\varrho')\cdot\nabla r_j(\varrho+i\sigma)+(\varrho'-q)\cdot(\nabla r_j(\varrho+i\sigma)-\nabla r_j(\varrho'+i\sigma))\}(\delta\Omega_\ell(q+i\sigma)$$
$$-\delta\Omega_\ell(\varrho'+i\sigma))g(\varrho+i\sigma,q+i\sigma)$$
$$C_{j\ell}=(\delta\Omega_\ell(q+i\sigma)-\delta\Omega_\ell(\varrho'+i\sigma))(\varrho'-q)\cdot\nabla r_j(\varrho'+i\sigma)(g(\varrho+i\sigma,q+i\sigma)$$
$$-g(\varrho'+i\sigma,q+i\sigma))$$
$$D_{j\ell}=\delta\Omega_\ell(q+i\sigma)\{(\varrho-\varrho')\cdot\nabla r_j(\varrho+i\sigma)+(\varrho'-q)\cdot(\nabla r_j(\varrho+i\sigma)$$
$$-\nabla r_j(\varrho'+i\sigma)\}(G(\varrho'+i\sigma,q+i\sigma)-g(\varrho'+i\sigma,q+i\sigma))$$
$$E_{j\ell}=\delta\Omega_\ell(q+i\sigma)\{G(\varrho+i\sigma,q+i\sigma)-g(\varrho+i\sigma,q+i\sigma)-G(\varrho'+i\sigma,q+i\sigma)$$
$$+g(\varrho'+i\sigma,q+i\sigma)\}(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)$$
$$F_{j\ell}=\delta\Omega_\ell(q+i\sigma)\{r_j(\varrho+i\sigma)-r_j(\varrho'+i\sigma)-(\varrho-\varrho')\cdot\nabla r_j(\varrho'+i\sigma)-(\varrho-q)\cdot(\nabla r_j(\varrho+i\sigma)$$
$$-\nabla r_j(\varrho'+i\sigma))\}G(\varrho+i\sigma,q+i\sigma)$$
$$H_{j\ell}=\delta\Omega_\ell(q+i\sigma)\{r_j(\varrho'+i\sigma)-r_j(q+i\sigma)-(\varrho'-q)\cdot\nabla r_j(\varrho'+i\sigma)\}(G(\varrho+i\sigma,q+i\sigma)$$
$$-G(\varrho'+i\sigma,q+i\sigma)).$$

$A_{j\ell}$ does not contribute to $\delta_{i\bar{\Sigma}}$,

$$B_{j\ell}\leqq C\{|\varrho-\varrho'|\,|\varrho'-q|^\alpha+|\varrho'-q|^{1+\alpha}|\varrho-\varrho'|\}\,\|\delta\Omega\|_s$$

and its contribution to $\delta_{i\bar{\Sigma}}$ is bounded by $C|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s$. In $C_{j\ell}$, we notice that

$$|g(\varrho+i\sigma,q+i\sigma)-g(\varrho'+i\sigma,q+i\sigma)|\leqq C\left|\frac{\varrho-q}{|\varrho-q|}\cdot\nabla z(\varrho+i\sigma)-\frac{\varrho'-q}{|\varrho'-q|}\cdot\nabla z(\varrho'+i\sigma)\right|$$
$$\leqq C\left|\frac{\varrho-q}{|\varrho-q|}\cdot(\nabla z(\varrho+i\sigma)-\nabla z(\varrho'+i\sigma))\right.$$
$$\left.+\nabla z(\varrho'+i\sigma)\left(\frac{\varrho-q}{|\varrho-q|}-\frac{\varrho'-q}{|\varrho'-q|}\right)\right|$$
$$\leqq C\left(|\varrho-\varrho'|^\alpha+\frac{|\varrho-\varrho'|}{|\varrho_1-q|}\right) \quad (4.40)$$

with $\varrho_1=\lambda\varrho+(1-\lambda)\varrho' \; 0\leqq\lambda\leqq 1$. Since

$$|\varrho_1-q|>|\varrho-q|-|\varrho-\varrho_1|\geqq \tfrac{1}{2}|\varrho-q|,$$

$$|g(\varrho+i\sigma,q+i\sigma)-g(\varrho'+i\sigma,q+i\sigma)|\leqq C\left(|\varrho-\varrho'|^\alpha+\frac{|\varrho-\varrho'|}{|\varrho-q|}\right). \tag{4.41}$$

The contribution of $C_{j\ell}$ to $\delta_{i\Sigma}$ is thus bounded by $C|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s$. From (4.33) the contribution of $D_{j\ell}$ is bounded by $C|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s$. To estimate the contribution of $E_{j\ell}$ we define

$$a(\varrho+i\sigma,q+i\sigma)=\frac{z(\varrho+i\sigma)-z(q+i\sigma)}{|\varrho-q|}, \tag{4.42}$$

$$\alpha(\varrho+i\sigma,q+i\sigma)=\frac{(\varrho-q)}{|\varrho-q|}\cdot\nabla z(\varrho+i\sigma), \tag{4.43}$$

and write

$$G-g=(\alpha-a)\mathscr{A}\{\alpha,a\}, \tag{4.44}$$

with

$$\mathscr{A}\{\alpha,a\}=\frac{\{(2+a^2+\alpha^2)+(1+a^2)^{1/2}(1+\alpha^2)^{1/2}\}(\alpha+a)}{(1+a^2)^{3/2}(1+\alpha^2)^{3/2}\{(1+a^2)^{1/2}+(1+\alpha^2)^{1/2}\}}. \tag{4.45}$$

Thus

$$\begin{aligned}&(G-g)(\varrho+i\sigma,q+i\sigma)-(G-g)(\varrho'+i\sigma,q+i\sigma)\\&\quad=(\alpha(\varrho'+i\sigma,q+i\sigma)-a(\varrho'+i\sigma,q+i\sigma))\\&\qquad\cdot\{\mathscr{A}(\alpha,a)(\varrho+i\sigma,q+i\sigma)-\mathscr{A}(\alpha,a)(\varrho'+i\sigma,q+i\sigma)\}\\&\qquad+\mathscr{A}(\alpha,a)(\varrho+i\sigma,q+i\sigma)\{\alpha(\varrho+i\sigma,q+i\sigma)\\&\qquad-a(\varrho+i\sigma,q+i\sigma)-\alpha(\varrho'+i\sigma)+a(\varrho'+i\sigma,q+i\sigma)\},\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}&\mathscr{A}(\alpha,a)(\varrho+i\sigma,q+i\sigma)-\mathscr{A}(\alpha,a)(\varrho'+i\sigma,q+i\sigma)\\&\quad=\mathscr{B}(a(\varrho+i\sigma,q+i\sigma)-a(\varrho'+i\sigma,q+i\sigma))+\mathscr{C}(\alpha(\varrho+i\sigma,q+i\sigma)-\alpha(\varrho'+i\sigma,q+i\sigma))\end{aligned} \tag{4.47}$$

with $\mathscr{B}$ and $\mathscr{C}$ bounded.

The first term of the right hand side of (4.46) is bounded by

$$|\varrho'-q|^\alpha\left\{\frac{|\varrho-\varrho'|}{|\varrho-q|}+|\varrho-\varrho'|^\alpha\right\};$$

when substituted in $E_{j\ell}$, the resulting contribution to $\delta_{i\Sigma}$ is bounded by $|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s$. The second term gives

$$\begin{aligned}&\int_\Sigma\frac{dq\,\theta_1(|\varrho-q|)}{|\varrho-q|^3}\,\delta\Omega_\ell(q+i\sigma)(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)\mathscr{A}\{\alpha,a\}(\varrho+i\sigma,q+i\sigma)\\&\quad\cdot\{a(\varrho+i\sigma,q+i\sigma)-\alpha(\varrho+i\sigma,q+i\sigma)-a(\varrho'+i\sigma,q+i\sigma)+\alpha(\varrho'+i\sigma,q+i\sigma)\},\end{aligned} \tag{4.48}$$

with

$$a(\varrho+i\sigma,q+i\sigma)-\alpha(\varrho+i\sigma,q+i\sigma)-a(\varrho'+i\sigma,q+i\sigma)+\alpha(\varrho'+i\sigma,q+i\sigma)$$
$$=B_1+B_2+B_3 \tag{4.49}$$

and

$$B_1=\left(\frac{1}{|\varrho-q|}-\frac{1}{|\varrho'-q|}\right)(z(\varrho'+i\sigma)-z(q+i\sigma)-(\varrho'-q)\cdot\nabla z(\varrho'+i\sigma)), \tag{4.50}$$

$$B_2=\frac{1}{|\varrho-q|}\{z(\varrho+i\sigma)-z(\varrho'+i\sigma)-(\varrho-\varrho')\cdot\nabla z(\varrho'+i\sigma)\}, \tag{4.51}$$

$$B_3=-\frac{\varrho-q}{|\varrho-q|}\cdot(\nabla z(\varrho+i\sigma)-\nabla z(\varrho'+i\sigma)). \tag{4.52}$$

Clearly

$$\left|\int_{\Sigma}\frac{dq\,\theta_1(|\varrho-q|)}{|\varrho-q|^3}\delta\Omega_\ell(q+i\sigma)((\varrho-q)\cdot\nabla r_j(\varrho+i\sigma))\mathscr{A}\{\alpha,a\}(\varrho+i\sigma,q+i\sigma)(B_1+B_2)\right|$$
$$\leqq C|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s. \tag{4.53}$$

For the last term, we write

$$K=\int_{\Sigma}\frac{dq\,\theta_1(|\varrho-q|)}{|\varrho-q|^3}(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)\delta\Omega_\ell(q+i\sigma)\mathscr{A}\{\alpha,a\}(\varrho+i\sigma,q+i\sigma)B_3$$
$$=\int_{\Sigma}\frac{dq\,\theta_1(|\varrho-q|)}{|\varrho-q|^3}(\varrho-q)\cdot\nabla r_j(\varrho+i\sigma)B_3\{\delta\Omega_\ell(q+i\sigma)\mathscr{A}\{\alpha,a\}(\varrho+i\sigma,q+i\sigma)$$
$$-\delta\Omega_\ell(\varrho+i\sigma)\mathscr{A}\{\alpha,\alpha\}(\varrho+i\sigma,q+i\sigma)\}. \tag{4.54}$$

In (4.54),

$$|\delta\Omega_\ell(q+i\sigma)\mathscr{A}\{\alpha,a\}(\varrho+i\sigma,q+i\sigma)-\delta\Omega_\ell(\varrho+i\sigma)\mathscr{A}\{\alpha,\alpha\}(\varrho+i\sigma,q+i\sigma)|$$
$$\leqq C|\delta\Omega(q+i\sigma)-\delta\Omega_\ell(\varrho+i\sigma)|+C|\delta\Omega|_s|a(\varrho+i\sigma,q+i\sigma)-\alpha(\varrho+i\sigma,q+i\sigma)|$$
$$\leqq C|\varrho-q|^\alpha\|\delta\Omega\|_s. \tag{4.55}$$

Consequently

$$K\leqq C|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s\int_{|\varrho-q|>2|\varrho-\varrho'|}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}dq$$
$$\leqq C|\varrho-\varrho'|^\alpha\|\delta\Omega\|_s. \tag{4.56}$$

To estimate the contribution of $F_{j\ell}$ to $\delta_{i\Sigma}$, we make the separation

$$F_{ij}=D_1-D_2, \tag{4.57}$$

with

$$D_1=\delta\Omega_\ell(q+i\sigma)\{r_j(\varrho+i\sigma)-r_j(\varrho'+i\sigma)-(\varrho-\varrho')\cdot\nabla r_j(\varrho'+i\sigma)\}G(\varrho+i\sigma,q+i\sigma), \tag{4.58}$$

$$D_2=(\varrho-q)\cdot(\nabla r_j(\varrho+i\sigma)-\nabla r_j(\varrho'+i\sigma))\delta\Omega_\ell(q+i\sigma)G(\varrho+i\sigma,q+i\sigma). \tag{4.59}$$

We have

$$|D_1| \leqq C|\varrho - \varrho'|^{1+\alpha} \|\delta\Omega\|_s \tag{4.60}$$

and the contribution of D_1 to $\delta_{i\Sigma}$ is bounded by $C|\varrho-\varrho'|^\alpha \|\delta\Omega\|_s$. The contribution of D_2 reads

$$\begin{aligned} \left|\int_\Sigma \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3} D_2 dq\right| &= \left|\int_\Sigma \frac{\varrho-q}{|\varrho-q|^3} \cdot (\nabla r_j(\varrho+i\sigma) - \nabla r_j(\varrho'+i\sigma)) \right. \\ &\quad \left. \cdot \{\delta\Omega(q+i\sigma) G(\varrho+i\sigma, q+i\sigma) - \delta\Omega(\varrho+i\sigma) g(\varrho+i\sigma, q+i\sigma)\}\right| \\ &\leqq C|\varrho-\varrho'|^\alpha \|\delta\Omega\|_s \int_{|\varrho-q|>2|\varrho-\varrho'|} \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}} dq \leqq C|\varrho-\varrho'|^\alpha \|\delta\Omega\|_s. \end{aligned} \tag{4.61}$$

Finally

$$|H_{ij}| \leqq \|\delta\Omega\|_s |\varrho'-q|^{1+\alpha} \frac{|\varrho-\varrho'|}{|\varrho-q|} \tag{4.62}$$

and its contribution to $\delta_{i\Sigma}$ is bounded by $C|\varrho-\varrho'|^\alpha \|\delta\Omega\|_s$. This completes the proof of estimate (4.27).

To estimate $\|U^{(2)}\{\delta\Omega, G, z\}\|_s$ we use the Cauchy-Schwarz inequality, which under the hypothesis of Lemma 4.3, leads to

$$|U^{(2)}\{\delta\Omega, G, z\}|_s \leqq C|\delta\Omega|_{L^2_s} \tag{4.63}$$

and

$$|\nabla U^{(2)}\{\delta\Omega, G, z\}|_s \leqq C|\delta\Omega|_{L^2_s}. \tag{4.64}$$

Lemma 4.3. *If $z, \tilde{z}$ satisfy the hypothesis of Lemma 4.2 with $|||\tilde{\Omega}|||_s$ bounded, we have* ($\delta z = z - \tilde{z}$)

$$\|U^{(1)}\{\tilde{\Omega}, \delta G, z\}\|_s \leqq C\|\nabla\delta z\|_s, \tag{4.65}$$

$$\|U^{(2)}\{\tilde{\Omega}, \delta G, z\}\|_s \leqq C\|\nabla\delta z\|_s. \tag{4.66}$$

Proof. Replacing G by δG and g by δg, the proof is similar to that of Lemma 4.2, provided that estimates of a few quantities which enter in $\psi_{j\ell}\{\tilde{\Omega}, \delta G, \delta g, z\}$ are established. We have

$$|\delta g(\varrho+i\sigma, q+i\sigma)| \leqq C|\nabla\delta z|_s, \tag{4.67}$$

$$|\delta G(\varrho+i\sigma, q+i\sigma)| \leqq C|\nabla\delta z|_s, \tag{4.68}$$

$$|\delta G(\varrho+i\sigma, q+i\sigma) - \delta g(\varrho+i\sigma, q+i\sigma)| \leqq |\varrho-q|^\alpha \|\nabla\delta z\|_s. \tag{4.69}$$

Proving (4.67) and (4.68) is straightforward. To obtain (4.69), we use (4.42)–(4.45) and similar notations for the tilded quantities and write

$$\begin{aligned} \delta G - \delta g &= \frac{1}{(1+a^2)^{3/2}} - \frac{1}{(1+\tilde{a}^2)^{3/2}} - \frac{1}{(1+\alpha^2)^{3/2}} + \frac{1}{(1+\tilde{\alpha}^2)^{3/2}} \\ &= (a-\tilde{a})\mathscr{A}(a,\tilde{a}) - (\alpha-\tilde{\alpha})\mathscr{A}(\alpha,\tilde{\alpha}) \\ &= (\delta a - \delta\alpha)\mathscr{A}(a,\tilde{a}) + \delta\alpha(\mathscr{A}(a,\tilde{a}) - \mathscr{A}(\alpha,\tilde{\alpha})), \end{aligned} \tag{4.70}$$

which readily leads to (4.69). As a consequence,

$$|\psi_{j\ell}\{\tilde{\Omega}, \delta G, \delta g, z\}| \leqq C|\varrho - q|^{1+\alpha} \|\nabla\delta z\|_s \tag{4.71}$$

and

$$|U^{(1)}\{\tilde{\Omega}, \delta G, z\}| \leqq C \|\nabla\delta z\|_s . \tag{4.72}$$

For $|\varrho - q| > 2|\varrho - \varrho'|$, we have

$$|\delta g(\varrho + i\sigma, q + i\sigma) - \delta g(\varrho' + i\sigma, q + i\sigma)| \leqq C\left(|\varrho - \varrho'|^\alpha + \frac{|\varrho - \varrho'|}{|\varrho - q|}\right) \|\nabla\delta z\|_s , \tag{4.73}$$

$$|\delta G(\varrho + i\sigma, q + i\sigma) - \delta G(\varrho' + i\sigma, q + i\sigma)| \leqq C \frac{|\varrho - \varrho'|}{|\varrho - q|} \|\nabla\delta z\|_s . \tag{4.74}$$

$$|\delta G(\varrho + i\sigma, q + i\sigma) - \delta g(\varrho + i\sigma, q + i\sigma) - \delta G(\varrho' + i\sigma, q + i\sigma) + \delta g(\varrho' + i\sigma, q + i\sigma)| \leqq \|\nabla\delta_z\|_s |\varrho' - q|^\alpha \left(\frac{|\varrho - \varrho'|}{|\varrho - q|} + |\varrho - \varrho'|^\alpha\right). \tag{4.75}$$

To prove (4.73), we write

$$\begin{aligned} \delta g(\varrho + i\sigma, q + i\sigma) - \delta g(\varrho' + i\sigma, q + i\sigma) &= \{\delta\alpha(\varrho + i\sigma, q + i\sigma) \\ &\quad - \delta\alpha(\varrho' + i\sigma, q + i\sigma)\} \mathscr{A}(\alpha, \tilde{\alpha})(\varrho + i\sigma, q + i\sigma) \\ &\quad + \delta\alpha(\varrho' + i\sigma, q + i\sigma)\{\mathscr{A}(\alpha, \tilde{\alpha})(\varrho + i\sigma, q + i\sigma) - \mathscr{A}(\alpha, \tilde{\alpha})(\varrho' + i\sigma, q + i\sigma)\} \end{aligned} \tag{4.76}$$

and use $|\delta\alpha(\varrho + i\sigma, q + i\sigma) - \delta\alpha(\varrho' + i\sigma, q + i\sigma)| \leqq C\left(\frac{|\varrho - \varrho'|}{|\varrho - q|} + |\varrho - \varrho'|^\alpha\right) \|\nabla\delta z\|_s$; (4.77)

$$|\mathscr{A}\{\alpha, \alpha\}(\varrho + i\sigma, q + i\sigma) - \mathscr{A}\{\alpha, \alpha\}(\varrho' + i\sigma, q + i\sigma)| \leqq C\left(|\varrho - \varrho'|^\alpha + \frac{|\varrho - \varrho'|}{|\varrho - q|}\right), \tag{4.78}$$

and

$$|\delta\alpha(\varrho' + i\sigma, q + i\sigma)| \leqq |\nabla\delta z|_s . \tag{4.79}$$

The proof of (4.74) is analogous. For (4.75) we simplify the notations and characterize by a prime the function taken at $(\varrho' + i\sigma, q + i\sigma)$

$$\begin{aligned} \delta G - \delta g - \delta G' + \delta g' &= (\delta a' - \delta\alpha')\{\mathscr{A}\{\alpha, a\} - \mathscr{A}\{\alpha', a'\}\} \\ &\quad + \mathscr{A}(\alpha, a)\{\delta a - \delta\alpha - \delta a' + \delta\alpha'\} + (\tilde{a} - \tilde{\alpha} - \tilde{a}' + \tilde{\alpha}')\{\mathscr{A}\{\alpha, a\} - \mathscr{A}\{\tilde{\alpha}, \tilde{a}\}\} \\ &\quad + (\tilde{a}' - \tilde{\alpha}')(\mathscr{A}\{\alpha, a\} - \mathscr{A}\{\tilde{\alpha}, \tilde{a}\} - \mathscr{A}\{\alpha', a'\} + \mathscr{A}\{\tilde{\alpha}', \tilde{a}'\}). \end{aligned} \tag{4.80}$$

From (4.47) the first term of the right hand side of (4.80) is bounded by

$$\|\nabla\delta z\|_s |\varrho' - q|^\alpha \left(\frac{|\varrho - \varrho'|}{|\varrho - q|} + |\varrho - \varrho'|^\alpha\right)$$

and its contribution to $|U\{\tilde{\Omega}, \delta G, z\}(\varrho + i\sigma) - U\{\tilde{\Omega}, \delta G, z\}(\varrho' + i\sigma)|$ is bounded by $C|\varrho - \varrho'|^\alpha \|\nabla\delta z\|_s$. By (4.49)–(4.56), the contribution of the second term has the same upper bound. To estimate the contribution of the third term, we proceed as in (4.49)–(4.53) and replace (4.54) by

$$\begin{aligned}&\int_{\Sigma} dq \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}(\varrho-q)\cdot \nabla r_j(\varrho+i\sigma)\frac{\varrho-q}{|\varrho-q|}\cdot(\nabla\tilde{z}(\varrho+i\sigma)-\nabla\tilde{z}(\varrho'+i\sigma))\tilde{\Omega}_\ell(q+i\sigma)\\ &\quad\cdot\{\mathscr{A}\{\alpha,a\}(\varrho+i\sigma,q+i\sigma)-\mathscr{A}\{\tilde{\alpha},\tilde{a}\}(\varrho+i\sigma,q+i\sigma)\}\\ &=\int_{\Sigma} dq \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}\{(\varrho-q)\cdot \nabla r_j(\varrho+i\sigma))\frac{\varrho-q}{|\varrho-q|}\cdot(\nabla\tilde{z}(\varrho+i\sigma)-\nabla\tilde{z}(\varrho'+i\sigma))\\ &\quad\cdot\{\tilde{\Omega}_\ell(q+i\sigma)(\mathscr{A}\{\alpha,a\}-\mathscr{A}\{\tilde{\alpha},\tilde{a}\})(\varrho+i\sigma,q+i\sigma)\\ &\quad-\tilde{\Omega}_\ell(\varrho+i\sigma)(\mathscr{A}\{\alpha,\alpha\}-\mathscr{A}\{\tilde{\alpha},\tilde{\alpha}\})(\varrho+i\sigma,q+i\sigma)\}\,.\end{aligned}\tag{4.81}$$

We then use

$$\begin{aligned}&\mathscr{A}\{\alpha,a\}-\mathscr{A}\{\alpha,\alpha\}-\mathscr{A}\{\tilde{a},\tilde{\alpha}\}+\mathscr{A}\{\tilde{\alpha},\tilde{\alpha}\}\\ &\quad=\mathscr{E}\{\alpha,a\}(\delta a-\delta\alpha)+(\mathscr{E}\{\alpha,a\}-\mathscr{E}\{\tilde{\alpha},\tilde{a}\})(\tilde{a}-\tilde{\alpha}),\end{aligned}\tag{4.82}$$

where $\mathscr{E}$ is defined by:

$$\mathscr{A}\{\alpha,a\}-\mathscr{A}\{\alpha,\alpha\}=(a-\alpha)\mathscr{E}\{\alpha,a\}\,.\tag{4.83}$$

$\mathscr{E}$ is bounded and satisfies

$$\mathscr{E}\{\alpha,a\}-\mathscr{E}\{\tilde{\alpha},\tilde{a}\}=(\alpha-\tilde{\alpha})\mathscr{F}+(a-\tilde{a})\mathscr{G}\,,\tag{4.84}$$

with $\mathscr{F}$ and $\mathscr{G}$ bounded. Consequently

$$|(\mathscr{A}\{\alpha,a\}-\mathscr{A}\{\alpha,\alpha\}-\mathscr{A}\{\tilde{\alpha},\tilde{a}\}+\mathscr{A}\{\tilde{\alpha},\tilde{\alpha}\})(\varrho+i\sigma,q+i\sigma)|\leqq|\varrho-q|^\alpha\|\nabla\delta z\|_s\,.\tag{4.85}$$

The integrals of (4.81) are thus bounded by $C|\varrho-\varrho'|^\alpha\|\nabla\delta z\|_s$. We finally turn to the last term of the right hand side of (4.80)

$$|\tilde{a}'-\tilde{\alpha}'|\leqq|\varrho'-q|^\alpha\|\nabla\delta z\|_s\,.\tag{4.86}$$

Defining

$$\begin{aligned}&A_1\{\alpha,a\}=2+\alpha^2+a^2+(1+\alpha^2)^{1/2}(1+a^2)^{1/2}\\ &A_2\{\alpha,a\}=\alpha+a\\ &A_3\{\alpha,a\}=(1+\alpha^2)^{-3/2}(1+a^2)^{-3/2}\\ &A_4\{\alpha,a\}=\{(1+\alpha^2)^{1/2}+(1+a^2)^{1/2}\}^{-1}\,.\end{aligned}\tag{4.87}$$

We write

$$\begin{aligned}&\mathscr{A}\{\alpha,a\}-\mathscr{A}\{\alpha,a\}-\mathscr{A}\{\tilde{\alpha},\tilde{a}\}+\mathscr{A}\{\tilde{\alpha}',\tilde{a}'\}\\ &\quad=\sum_{\substack{i,j\\ i\neq j}}\mathscr{B}_i(A_i-\tilde{A}_i-A_i'+\tilde{A}_i')+\mathscr{C}_{ij}(A_i-\tilde{A}_i)(A_j-A_j')+\mathscr{D}_{ij}(A_i-\tilde{A}_i)(\tilde{A}_j-\tilde{A}_j'),\end{aligned}\tag{4.88}$$

where $\mathscr{B}_i$, $\mathscr{C}_{ij}$, and $\mathscr{D}_{ij}$ are bounded. We have

$$|A_i-\tilde{A}_i|\leqq C(|\alpha-\tilde{\alpha}|+|a-\tilde{a}|)\leqq C|\nabla\delta z|_s\,,$$
$$|A_j-A_j'|\leqq C(|\alpha-\alpha'|+|a-a'|)\leqq C\left(\frac{|\varrho-\varrho'|}{|\varrho-q|}+|\varrho-\varrho'|^\alpha\right), \tag{4.89}$$
$$|\tilde{A}_j-\tilde{A}_j'|\leqq C(|\tilde{\alpha}-\tilde{\alpha}'|+|\tilde{a}-\tilde{a}'|)\leqq C\left(\frac{|\varrho-\varrho'|}{|\varrho-q|}+|\varrho-\varrho'|^\alpha\right).$$

Finally,

$$|A_i-\tilde{A}_i-A_i'+\tilde{A}_i'|\leqq C\{|\delta\alpha-\delta\alpha'|+|\delta a-\delta a'|\,,$$
$$(|\alpha-\alpha'|+|\tilde{\alpha}-\tilde{\alpha}'|)|\delta\alpha'|+(|a-a'|+|\tilde{a}-\tilde{a}'|)|\delta a'|\}\leqq C\left(\frac{|\varrho-\varrho'|}{|\varrho-q|}+|\varrho-\varrho'|^\alpha\right)\|\nabla\delta z\|_s\,. \tag{4.90}$$

The contribution to $|U\{\tilde{\Omega},\delta G,z\}(\varrho+i\sigma)-U\{\tilde{\Omega},\delta G,z\}(\varrho'+i\sigma)|$ of the last term of the right hand side of (4.80) is thus bounded by $C|\varrho-\varrho'|^\alpha\|\nabla\delta z\|_s$. This completes the proof of estimate (4.65). Proceeding as in Lemma 4.2 and using (4.68), we readily obtain the estimate (4.66).

Lemma 4.4. *If z and $\tilde{z}$ satisfy the hypothesis of Lemma 4.2 with $|||\tilde{\Omega}|||_s$ bounded, we have:*

$$\|U^{(1)}\{\tilde{\Omega},\tilde{G},\delta z\}\|_s\leqq C\|\nabla\delta z\|_s\,, \tag{4.91}$$
$$\|U^{(2)}\{\tilde{\Omega},\tilde{G},\delta z\}\|_s\leqq C\|\nabla\delta z\|_s\,. \tag{4.92}$$

The proof is identical to that of Lemma 4.2.

Proposition 4.4. *If (Ω,z) satisfy $|\mathrm{Im}\,\nabla z|_s<1$, and $|||\Omega|||_s$, $|||\nabla z|||_s$, $|z|_s$ are bounded, and if similar conditions hold for $(\tilde{\Omega},\tilde{z})$, the analytic continuation of V to complex space variables satisfies*

$$|V\{\Omega,z\}-V\{\tilde{\Omega},\tilde{z}\}|_{L^2_s}\leqq C\{|||\Omega-\tilde{\Omega}|||_s+|||\nabla(z-\tilde{z})|||_s+|z-\tilde{z}|_s\}\,. \tag{4.93}$$

Proof. Using the decomposition (4.24), and defining

$$\check{G}=G-1\,,$$

we distinguish the contribution to $V\{\Omega,z\}-V\{\tilde{\Omega},\tilde{z}\}$ of

$$W_1(\varrho+i\sigma)=\fint_{\mathbb{R}^2}\frac{\varrho_j-q_j}{|\varrho-q|^3}\delta\Omega_\ell(q+i\sigma)dq\,, \tag{4.94}$$

$$W_2^{(k)}(\varrho+i\sigma)=\fint_{\mathbb{R}^2}\frac{\theta_k(|\varrho-q|)}{|\varrho-q|^3}(\varrho_j-q_j)\check{G}(\varrho+i\sigma,q+i\sigma)\delta\Omega_\ell(q+i\sigma)dq\,, \tag{4.95}$$

$$W_3^{(k)}(\varrho+i\sigma)=\fint_{\mathbb{R}^2}\frac{\theta_k(|\varrho-q|)}{|\varrho-q|^3}(z(\varrho+i\sigma)-z(q+i\sigma))G(\varrho+i\sigma,q+i\sigma)\delta\Omega_\ell(q+i\sigma)dq\,, \tag{4.96}$$

$$W_4^{(k)}(\varrho+i\sigma)= \fint_{\mathbb{R}^2} \theta_k(|\varrho-q|)\frac{r_j(\varrho+i\sigma)-r_j(q+i\sigma)}{|\varrho-q|^3}\delta G(\varrho+i\sigma,q+i\sigma)\tilde{\Omega}_\ell(q+i\sigma)dq, \tag{4.97}$$

$$W_5^{(k)}(\varrho+i\sigma)= \fint_{\mathbb{R}^2} \theta_k(|\varrho-q|)\frac{\delta z(\varrho+i\sigma)-\delta z(q+i\sigma)}{|\varrho-q|^3}\tilde{G}(\varrho+i\sigma,q+i\sigma)\tilde{\Omega}_\ell(q+i\sigma)dq. \tag{4.98}$$

We readily have

$$\|W_1\|_{L_s^2} \leqq C\|\delta\Omega\|_{L_s^2}. \tag{4.99}$$

Defining

$$\check{g}=g-1$$

we rewrite $W_2^{(1)}(\varrho+i\sigma)$ on the form

$$W_2^{(1)}(\varrho+i\sigma)= \int_{\mathbb{R}^2} \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3}\psi_{j\ell}\{\delta\Omega,\check{G},\check{g},z\}(\varrho+i\sigma,q+i\sigma)dq$$
$$(j=1,2;\ell=1,2,3), \tag{4.100}$$

with [see (4.31)]

$$\begin{aligned}\psi_{j\ell}\{\delta\Omega,\check{G},\check{g},z\} &=(\delta\Omega_\ell(q+i\sigma)-\delta\Omega_\ell(\varrho+i\sigma))(\varrho_j-q_j)\check{g}(\varrho+i\sigma,q+i\sigma)\\ &\quad+(\check{G}(\varrho+i\sigma,q+i\sigma)-\check{g}(\varrho+i\sigma,q+i\sigma))(\varrho_j-q_j)\delta\Omega_\ell(q+i\sigma).\end{aligned} \tag{4.101}$$

If the hypotheses of Proposition 4.4 are satisfied, we have

$$\begin{aligned}|\check{g}(\varrho+i\sigma,q+i\sigma)| &\leqq C|\nabla z(\varrho+i\sigma)|\\ |\check{G}(\varrho+i\sigma,q+i\sigma)-\check{g}(\varrho+i\sigma,q+i\sigma)| &\leqq C|\varrho-q|^\alpha.\end{aligned} \tag{4.102}$$

Consequently

$$\begin{aligned}\|W_2^{(1)}\|_{L_s^2}^2 \leqq C\|\delta\Omega\|_s^2 &\int_{\mathbb{R}^2} d\varrho|\nabla z(\varrho+i\sigma)|^2\left(\int_{\mathbb{R}^2}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}dq\right)^2\\ &+\int_{\mathbb{R}^2} d\varrho\left(\int_{\mathbb{R}^2}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}|\delta\Omega_\ell(q+i\sigma)|dq\right)^2.\end{aligned} \tag{4.103}$$

In the last integral we use the Cauchy-Schwarz inequality in the form

$$\begin{aligned}&\left(\int_{\mathbb{R}^2}\left(\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}\right)^{1/2}\left\{\left(\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}\right)^{1/2}|\delta\Omega_\ell(q+i\sigma)|\right\}dq\right)^2\\ &\leqq\left(\int_{\mathbb{R}^2}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}dq\right)\left(\int_{\mathbb{R}^2}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}|\delta\Omega_\ell(q+i\sigma)|^2dq\right)\\ &\leqq\int_{\mathbb{R}^2}\frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}|\delta\Omega_\ell(q+i\sigma)|^2dq.\end{aligned} \tag{4.104}$$

Substituting in (4.103) and using the Fubini theorem, we finally obtain

$$\|W_2^{(1)}\|_{L_s^2} \leqq C|||\delta\Omega|||_s. \tag{4.105}$$

To estimate $\|W_2^{(2)}\|_{L_s^2}$, we notice that

$$|\check{G}(\varrho+i\sigma, q+i\sigma)| \leqq C\frac{|z(\varrho+i\sigma)-z(q+i\sigma)|^2}{|\varrho-q|^2} \leqq \frac{C|z|^2}{|\varrho-q|^2}. \tag{4.106}$$

Thus

$$\begin{aligned}\|W_2^{(2)}\|_{L_s^2}^2 &\leqq \int_{\mathbb{R}^2} d\varrho \left(\int_{\mathbb{R}^2} dq \frac{\theta_2(|\varrho-q|)}{|\varrho-q|^4} |\delta\Omega_\ell(q+i\sigma)| dq\right)^2 \\ &\leqq \int_{\mathbb{R}^2} d\varrho \left(\int_{\mathbb{R}^2} \frac{\theta_2(|\varrho-q|)}{|\varrho-q|^4} dq\right)\left(\int_{\mathbb{R}^2} \frac{\theta_2(|\varrho-q|)}{|\varrho-q|^4} |\delta\Omega_\ell(q+i\sigma)|^2 dq\right) \leqq \mathscr{C}\|\delta\Omega\|_{L^2}^2. \end{aligned} \tag{4.107}$$

Using (4.31), $W_3^{(1)}$ is rewritten

$$W_3^{(1)}(\varrho+i\sigma) = \int \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3} \psi_{3\ell}\{\delta\Omega, G, g, z\}(\varrho+i\sigma, q+i\sigma) dq, \tag{4.108}$$

with

$$|\psi_{3\ell}\{\delta\Omega, G, g, z\}(\varrho+i\sigma, q+i\sigma)| \leqq C|\varrho-q|^{1+\alpha}(|\nabla z(\varrho+i\sigma)|\ \|\delta\Omega\|_s + |\delta\Omega(q+i\sigma)|. \tag{4.109}$$

Thus

$$\begin{aligned}\|W_3^{(1)}\|_{L_s^2}^2 \leqq \int_{\mathbb{R}^2} d\varrho &\left\{|\nabla z(\varrho+i\sigma)|^2 \|\delta\Omega\|_s^2 \left(\int \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}}\right)^2\right. \\ &\left.+\left(\int \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^{2-\alpha}} |\delta\Omega(q+i\sigma)| dq\right)^2\right\} \leqq C|||\delta\Omega|||_s^2. \end{aligned} \tag{4.110}$$

$\|W_3^{(2)}\|_{L_s^2}$ is readily bounded:

$$\|W_3^{(2)}\|_{L_s^2}^2 \leqq \int_{\mathbb{R}^2} d\varrho \left(\int_{\mathbb{R}^2} dq \frac{\theta_2(|\varrho-q|)}{|\varrho-q|^3} |\delta\Omega_\ell(q+i\sigma)| dq\right)^2 \leqq \|\delta\Omega\|_s^2 . \tag{4.111}$$

$W_4^{(1)}(\varrho+i\sigma)$ reads

$$W_4^{(1)}(\varrho+i\sigma) = \int_{\mathbb{R}^2} \frac{\theta_1(|\varrho-q|)}{|\varrho-q|^3} \psi_{j\ell}\{\tilde{\Omega}, \delta G, \delta g, z\}(\varrho+i\sigma, q+i\sigma) dq, \tag{4.112}$$

with

$$|\psi_{j\ell}\{\tilde{\Omega}, \delta G, \delta g, z\}(\varrho+i\sigma, q+i\sigma)| \leqq C|\varrho-q|^{1+\alpha}\{|\nabla\delta z(\varrho+i\sigma)| + \|\nabla\delta z\|_s |\tilde{\Omega}(q+i\sigma)|\}. \tag{4.113}$$

Thus

$$\|W_4^{(1)}\|_{L_s^2} \leqq C|||\nabla\delta z|||_s. \tag{4.114}$$

$$\|W_4^{(2)}\|_{L_s^2}^2 \leqq \int_{\mathbb{R}^2} d\varrho \left(\int dq \frac{\theta_2(|\varrho - q|)}{|\varrho - q|^3} |\delta z|_s |\tilde{\Omega}(q + i\sigma)| \right)^2$$
$$\leqq C|\delta z|_s^2 . \tag{4.115}$$

Finally, proceeding as in the estimation of $\|W_3^{(k)}\|_{L_s^2}$, we obtain

$$\|W_5^{(k)}\|_{L_s^2} \leqq C(|||\nabla\delta z|||_s + |\delta z|_s) . \tag{4.116}$$

This completes the proof of Proposition 4.4.

We thus have obtained that, if Ω, $\tilde{\Omega}$, z, $\tilde{z}$, ∇z, $\nabla\tilde{z}$ belong to B_s with $|\mathrm{Im}\,\nabla z|_s < 1$ and $|\mathrm{Im}\,\nabla\tilde{z}|_s < 1$, we have:

$$|||V\{\Omega, z\} - V\{\tilde{\Omega}, \tilde{z}\}|||_s \leqq C\{|||\Omega - \tilde{\Omega}|||_s + |||z - \tilde{z}|||_s + |||\nabla(z - \tilde{z})|||_s\} . \tag{4.117}$$

So, we can apply Nishida's theorem to the system

$$\begin{aligned} &\frac{\partial z}{\partial t} = -z_x V_1 - z_y V_2 + V_3 \\ &\frac{\partial}{\partial t} \nabla z = -\nabla(z_x V_1 + z_y V_2 - V_3) \\ &\frac{\partial}{\partial t} \Omega = -\frac{\partial}{\partial x}(\Omega V_1) - \frac{\partial}{\partial y}(\Omega V_2) + \Omega_1 \frac{\partial V_1}{\partial x} + \Omega_2 \frac{\partial V_2}{\partial x} \end{aligned} \tag{4.118}$$

and obtain the

Theorem 4.2. *For initial conditions such that the analytic continuation of* z_0, ∇z_0, Ω_0 *belong to* B_{s_0} *with* $|\mathrm{Im}\,\nabla z_0|_{s_0} \leqq C < 1$, *there exists a constant* α *such that for* $|t| < \alpha(s - s_0)$ *the system* (3.1) *and* (3.2) *has a unique solution* (z, Ω) *which is a holomorphic function of* t *with value in* $(B_s)^4$.

5. Existence in the Large for a Two-Dimensional Flow with Initial Discontinuous Velocity

In two dimensions, it is possible to give a meaning in the weak sense (for all time) to the Euler equation with initial condition u_0 which is irrotational in each of the two domains of $\mathbb{R}^2$ separated by an analytic line $\{r = r(x) = (x, f(x))\}$ and discontinuous across this line. Using the notations of Sect. 2, we have

$$\begin{aligned} &[u_0] \cdot n = 0 \\ &[u_0] \wedge n = \Omega_0 . \end{aligned} \tag{5.1}$$

We assume u_0 bounded in $L^2(\mathbb{R}^2)^2$ and Ω_0 bounded in $L^1(\mathbb{R})$.

We consider the regularized problem with initial condition

$$u_0^\varepsilon = \varrho^\varepsilon * u_0 , \tag{5.2}$$

where the regularizing kernel ϱ^ε is given by

$$\varrho^\varepsilon(r) = C\varepsilon^2 \exp\left(\left\{ \left(\frac{|r|}{\varepsilon} \right)^2 - 1 \right\}^{-1} \right) \tag{5.3}$$

with C such that $\int_{\mathbb{R}^2} \varrho^\varepsilon(r)dr = 1$. The initial vorticity reads

$$\omega_0^\varepsilon = \text{Curl}\,\varrho^\varepsilon * u_0 = \int_{\mathbb{R}} \Omega_0(x)\varrho^\varepsilon(r - r(x))dx. \tag{5.4}$$

This problem has a unique smooth solution $(u^\varepsilon, p^\varepsilon)$ (Wolibner, 1933; Kato, 1967) such that u^ε and $\omega^\varepsilon = \text{Curl}\,u^\varepsilon$ remain in bounded sets of $L^\infty(0, T; L^2(\mathbb{R}^2)^2)$ and $L^\infty(0, T; L^1(\mathbb{R}^2))$, respectively. In addition

$$u^\varepsilon(r,t) = -\frac{1}{2\pi}\frac{r}{|r|^2} * \omega^\varepsilon(r,t) \equiv -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{r - r'}{|r - r|^2} \wedge \omega^\varepsilon(r', t)dr'. \tag{5.5}$$

Lemma 5. *If ω^ε is in a bounded set of $L^\infty([0, T[, L^1(\mathbb{R}^2))$, there exists a subsequence u^η of u^ε which converges in $L^1_{\text{loc}}([0, T[\times \mathbb{R}^2)$ and thus almost everywhere in* **r** *and t, when $\eta \to 0$.*

Proof. ω^ε is in a bounded set of $L^\infty(0, T, L^1(\mathbb{R}^2)) \subset L^\infty([0, T[, W^{-1,3/2}(\mathbb{R}^2))$. Thus a subsequence ω^η exists which converges in $L^\infty([0, T[, W^{-1,3/2}(\mathbb{R}^2))$ weak star.

Let $\theta_\alpha(|r|)$ be a smooth function with support $\{r/|r| \leqq 1\}$, bounded by α, and let K be a compact set of $\mathbb{R}^2$

$$\int_0^T \int_K |u^\eta(r,t) - u^{\eta'}(r,t)|drdt \leqq \int_0^T \int_K \left|\theta_\alpha(|r|)\frac{r}{|r|^2} * (\omega^\eta(r,t) - \omega^{\eta'}(r,t))\right| drdt$$
$$+ \int_0^T \int_K \left|(1 - \theta_\alpha(|r|)\frac{r}{|r|^2} * (\omega^\eta(r,t) - \omega^{\eta'}(r,t))\right| drdt. \tag{5.6}$$

The first term of Eq. (5.6) is bounded by $4\pi\alpha T|\Omega_0|_{L^1(\mathbb{R})}$ and is thus made arbitrarily small by a convenient choice of α. In the second term $\omega^\eta(r,t)$ is a Cauchy sequence in the $L^\infty([0, T[, W^{-1,3/2}(\mathbb{R}^2))$ weak star topology. Since for fixed r, $(1 - \theta_\alpha(|r - r'|))\frac{r - r'}{|r - r'|^2}$ belongs to $W^{1,3}(\mathbb{R}^2)$,

$$\varphi^\eta(r,t) = (1 - \theta_\alpha)\frac{r}{|r|^2} * \omega^\eta(r,t) \tag{5.7}$$

is a Cauchy sequence for fixed r and t. Thus φ^η converges almost everywhere in r and t. In addition

$$|\varphi^\eta(r,t)| \leqq C|\Omega_0|_{L^1(\mathbb{R}^2)}. \tag{5.8}$$

It follows that φ^η converges in $L^1([0, T[\times K)$ and the second term of (5.6) can be made arbitrarily small by taking η and η' small enough.

Using the lemma, we can pass to the limit in the Euler equation written in the sense of distribution and obtain the

Theorem 5. *For initial velocity bounded in $L^2(\mathbb{R}^2)^2$ irrotationnal in each of the two domains of $\mathbb{R}^2$ separated by a smooth line, and discontinuous along this line with vorticity density bounded in $L^1(\mathbb{R})$, the two dimensional Euler equation has a weak solution in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)^2)$.*

Remark. We do not know if this solution is unique: uniqueness of solutions to the two-dimensional Euler equation is established only if the initial vorticity belongs to L^∞ (Yudovich, 1963; Bardos, 1972).

6. Extension of Birkhoff Equation to Three Dimensional Kelvin Helmholtz Instability

To describe the development of the instability and the possible occurrence of singularities, it is of interest to get a representation of the vortex sheet which permits rolling up of the interface. It follows from Eq. (2.25a) that there exist two functions $f(\lambda, \mu, t)$ and $g(\lambda, \mu, t)$ such that

$$V - r_t = f r_\lambda + g r_\mu \tag{6.1}$$

Let $\tilde{\lambda}(\alpha, \beta, t)$ and $\tilde{\mu}(\alpha, \beta, t)$ be solutions of

$$\begin{aligned} \tilde{\lambda}_t &= f(\tilde{\lambda}, \tilde{\mu}, t) \\ \tilde{\mu}_t &= g(\tilde{\lambda}, \tilde{\mu}, t) \\ \tilde{\lambda}(\alpha, \beta, 0) &= \alpha \\ \tilde{\mu}(\alpha, \beta, 0) &= \beta . \end{aligned} \tag{6.2}$$

We consider the point $M(\alpha, \beta, t)$ which satisfies:

$$\begin{aligned} &\frac{\partial M}{\partial t}(\alpha, \beta, t) = v(\alpha, \beta, t) \equiv V(\tilde{\lambda}(\alpha, \beta, t), \tilde{\mu}(\alpha, \beta, t), t) \\ &M(\alpha, \beta, 0) = r(\alpha, \beta, 0) . \end{aligned} \tag{6.3}$$

It thus follows, from (6.1) and (6.2) that

$$\frac{d}{dt}\{M(\alpha, \beta, t) - r(\tilde{\lambda}(\alpha, \beta, t), \tilde{\mu}(\alpha, \beta, t), t)\} = 0 . \tag{6.4}$$

So, the point $M(\alpha, \beta, t)$ remains on the vortex sheet

$$M(\alpha, \beta, t) = r(\tilde{\lambda}(\alpha, \beta, t), \tilde{\mu}(\alpha, \beta, t), t) \tag{6.5}$$

and V identifies with the vortex sheet velocity. Using (6.3), (6.4), and (2.25c), Eq. (2.25b) is rewritten:

$$\begin{aligned} \Omega_t + \tilde{\lambda}_t \Omega_\lambda + \tilde{\mu}_t \Omega_\mu + \Omega(f_\lambda + g_\mu) = &\frac{1}{\|N\|^2} \frac{\partial V}{\partial \lambda} \{|r_\mu|^2 (\Omega \cdot r_\lambda) - (r_\lambda \cdot r_\mu)(\Omega \cdot r_\mu)\} \\ &+ \frac{1}{\|N\|^2} \frac{\partial V}{\partial \mu} \{|r_\lambda|^2 (\Omega \cdot r_\mu) - (r_\lambda \cdot r_\mu)(\Omega \cdot r_\lambda)\} . \end{aligned} \tag{6.6}$$

To estimate $(f_\lambda + g_\mu)$ we write [from (6.2)]

$$\frac{\partial}{\partial \alpha} \tilde{\lambda}_t = f_\lambda \frac{\partial \tilde{\lambda}}{\partial \alpha} + f_\mu \frac{\partial \tilde{\mu}}{\partial \alpha} , \tag{6.7}$$

and similar equalities for $\frac{\partial}{\partial \alpha} \tilde{\mu}_t, \frac{\partial}{\partial \beta} \tilde{\lambda}_t, \frac{\partial}{\partial \beta} \tilde{\mu}_t$.

After linear combinations of these equalities, we obtain

$$J(f_\lambda + g_\mu) = \frac{dJ}{dt} , \tag{6.8}$$

where

$$J=\frac{\partial\tilde{\lambda}}{\partial\alpha}\frac{\partial\tilde{\mu}}{\partial\beta}-\frac{\partial\tilde{\lambda}}{\partial\beta}\frac{\partial\tilde{\mu}}{\partial\alpha} \tag{6.9}$$

is the Jacobian of the transformation $(\alpha,\beta)\mapsto(\tilde{\lambda},\tilde{\mu})$. Thus, the left hand side of (6.6) identifies with $\frac{1}{J}\frac{d}{dt}(J\Omega)$. In the right hand side, we express $\frac{\partial V}{\partial\lambda}$ and $\frac{\partial V}{\partial\mu}$ in terms of $\frac{\partial v}{\partial\alpha}$ and $\frac{\partial v}{\partial\beta}$ and r_λ and r_μ in terms of $\frac{\partial M}{\partial\alpha}$ and $\frac{\partial M}{\partial\beta}$. Defining

$$\Xi(\alpha,\beta,t)=J(\alpha,\beta,t)\Omega(\tilde{\lambda}(\alpha,\beta,t),\tilde{\mu}(\alpha,\beta,t),t) \tag{6.10}$$

and noticing that

$$\frac{\partial M}{\partial\alpha}\wedge\frac{\partial M}{\partial\beta}=JN, \tag{6.11}$$

we finally obtain

$$\begin{aligned}\frac{d\Xi}{dt}=\frac{1}{\left\|\frac{\partial M}{\partial\alpha}\wedge\frac{\partial M}{\partial\beta}\right\|}&\left\{\left(\frac{\partial M}{\partial\alpha}\cdot\Xi\right)\left(\frac{\partial v}{\partial\alpha}\left|\frac{\partial M}{\partial\beta}\right|^2-\frac{\partial v}{\partial\beta}\left(\frac{\partial M}{\partial\alpha}\cdot\frac{\partial M}{\partial\beta}\right)\right)\right.\\&\left.-\left(\frac{\partial M}{\partial\beta}\cdot\Xi\right)\left(-\frac{\partial v}{\partial\beta}\left|\frac{\partial M}{\partial\alpha}\right|^2+\frac{\partial v}{\partial\alpha}\left(\frac{\partial M}{\partial\alpha}\cdot\frac{\partial M}{\partial\beta}\right)\right)\right\}.\end{aligned} \tag{6.12}$$

Proposition 6. *Using a Lagrangian parametrization of the interface with initially $M(\alpha,\beta,0)=M_0(\alpha,\beta)$, the equations which govern the motion of a vortex sheet in a genuine three-dimensional ideal fluid read*

$$\begin{aligned}&\frac{\partial M}{\partial t}(\alpha,\beta,t)=v(\alpha,\beta,t)\\&\frac{\partial}{\partial t}\Xi(\alpha,\beta,t)=\frac{1}{\left\|\frac{\partial M}{\partial\alpha}\wedge\frac{\partial M}{\partial\beta}\right\|}\left\{\frac{\partial v}{\partial\alpha}\left(\frac{\partial M}{\partial\beta},\Xi,\frac{\partial M}{\partial\beta}\wedge\frac{\partial M}{\partial\alpha}\right)+\frac{\partial v}{\partial\beta}\left(\frac{\partial M}{\partial\alpha},\Xi,\frac{\partial M}{\partial\alpha}\wedge\frac{\partial M}{\partial\beta}\right)\right\},\end{aligned} \tag{6.13}$$

where

$$v(\alpha,\beta,t)=-\frac{1}{4\pi}\int\frac{M(\alpha,\beta,t)-M(\alpha',\beta',t)}{|M(\alpha,\beta,t)-M(\alpha',\beta',t)|^3}\wedge\Xi(\alpha',\beta',t)\,d\alpha'd\beta'$$

is the velocity of the current point of the interface and Ξ is the vorticity density defined by $(\varphi\in(\mathscr{D}(\mathbb{R}^3))^3)$

$$\langle\omega,\varphi\rangle=\int\Xi(\alpha,\beta,t)\cdot\varphi(M(\alpha,\beta,t))\,d\alpha d\beta\,.$$

In two dimensions, the Lagrangian description of the Kelvin-Helmholtz instability simplifies considerably. We start from system (2.26) and proceed as above; we obtain that v identifies with the velocity of the vortex line and that the vorticity density relative to the Langrangian coordinate α, $J\Omega$ is conserved along the trajectories ($J = \partial\lambda/\partial\alpha$). It follows that the current point $m(\alpha, t)$ of the interface satisfies

$$\frac{\partial m}{\partial t}(\alpha, t) = -\frac{1}{2\pi} \fint_{-\infty}^{+\infty} \frac{m(\alpha, t) - m(\alpha', t)}{|m(\alpha, t) - m(\alpha', t)|^2} \wedge \Omega_0(\alpha') d\alpha'.$$

Or using the complex representation (overbar denotes complex conjugate)

$$Z = m_1(\alpha, t) + i m_2(\alpha, t)$$

$$\frac{\partial \bar{Z}}{\partial t}(\alpha, t) = -\frac{1}{2i\pi} \fint_{-\infty}^{+\infty} \frac{\Omega_0(\alpha')}{Z(\alpha, t) - Z(\alpha', t)} d\alpha', \tag{6.14}$$

which is the equation proposed by Birkhoff (1955, 1962) for the two dimensional Kelvin-Helmoltz instability.

An asymptotic study of the Birkhoff equation in the case of an initially sinusoidal corrugation of the interface of small amplitude ε has been done by Moore (1979) who obtained a singularity at a time $t_* \sim \ln\frac{1}{\varepsilon}$. As noted by the author himself, this analysis is not rigorous because the asymptotic expansion does not remain valid in the neighborhood of the singularity. Nevertheless, an interesting property of Moore's solution is that at the instant where the singularity forms, the interface is still slightly distorted. There is no sign of rolling up as suggested by numerical calculations based on a representation of the vortex line by discrete point vortices, a method which is poorly adapted to study the occurrence of singularities. Recently, the Birkhoff equation for a periodic vorticity density concentrated on a straight line has been numerically investigated by computing the Fourier components of the interface corrugation as the sum of the temporal Taylor series (Meiron et al., 1980; Morf et al., 1980). Occurrence of a singularity is obtained after a finite time, in qualitative agreement with Moore's asymptotic result. Extension of the above asymptotic and numerical analysis to the three dimensional problem is under investigation.

Appendix

In this appendix, we derive from the equalities $(\nabla \cdot u)^{+} = 0$ and $(\operatorname{curl} u)^{\pm} = 0$, the relations

$$\frac{\partial u^{\pm}}{\partial \lambda} \cdot r_\mu - \frac{\partial u^{\pm}}{\partial \mu} \cdot r_\lambda = 0, \tag{A1}$$

$$\|N\| \left(\frac{\partial u}{\partial n}\right)^{\pm} = r_\mu \wedge \frac{\partial u^{\pm}}{\partial \lambda} - r_\lambda \wedge \frac{\partial u^{\pm}}{\partial \mu}. \tag{A2}$$

Using Eq. (2.14), we have,

$$0 = \|N\|^2 (\nabla \cdot u)^{\pm} = \|N\| \, N \cdot \left(\frac{\partial u}{\partial n}\right)^{\pm} + \left(r_\mu, N, \frac{\partial u^{\pm}}{\partial \lambda}\right) - \left(r_\lambda, N, \frac{\partial u^{\pm}}{\partial \mu}\right), \tag{A3}$$

$$0=\|N\|^2(\operatorname{Curl} u)^\pm=\|N\|\,N\wedge\left(\frac{\partial u}{\partial n}\right)^\pm+(r_\mu\wedge N)\wedge\frac{\partial u^\pm}{\partial\lambda}-(r_\lambda\wedge N)\wedge\frac{\partial u^\pm}{\partial\mu}$$

$$=\|N\|\,N\wedge\left(\frac{\partial u}{\partial n}\right)^\pm-\left(\frac{\partial u^\pm}{\partial\lambda}\cdot N\right)r_\mu+\left(\frac{\partial u^\pm}{\partial\mu}\cdot N\right)r_\lambda+N\left(\frac{\partial u^\pm}{\partial\lambda}\cdot r_\mu-\frac{\partial u^\pm}{\partial\mu}\cdot r_\lambda\right), \tag{A4}$$

and Eq. (A1) follows by writing that the normal component in Eq. (A4) vanishes.

Substituting Eqs. (A3) and (A4) in the identity

$$\|N\|^3\left(\frac{\partial u}{\partial n}\right)^\pm=\|N\|\left(N\cdot\left(\frac{\partial u}{\partial n}\right)^\pm\right)N-\|N\|\,N\wedge\left(N\wedge\left(\frac{\partial u}{\partial n}\right)^\pm\right), \tag{A5}$$

we get

$$\|N\|^3\left(\frac{\partial u}{\partial n}\right)^\pm=\left\{\left(r_\lambda,N,\frac{\partial u^\pm}{\partial\mu}\right)-\left(r_\mu,N,\frac{\partial u^\pm}{\partial\lambda}\right)\right\}N-N\wedge\left\{\left(\frac{\partial u^\pm}{\partial\lambda}\cdot N\right)r_\mu-\left(\frac{\partial u^\pm}{\partial\mu}\cdot N\right)r_\lambda\right\}, \tag{A6}$$

which we rewrite

$$\|N\|^3\left(\frac{\partial u}{\partial n}\right)^\pm=(A\cdot N)N-N\wedge(N\wedge A), \tag{A7}$$

with

$$A=\frac{\partial u^\pm}{\partial\mu}\wedge r_\lambda-\frac{\partial u^\pm}{\partial\lambda}\wedge r_\mu.$$

We thus obtain Eq. (A2).

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