Finite-time blow-up of solutions of an aggregation equation in R^n

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Abstract

We consider the aggregation equation $u_t + \nabla \cdot (u \nabla K * u) = 0$ in R^n, $n \geq 2$, where K is a rotationally symmetric, nonnegative decaying kernel with a Lipschitz point at the origin, e.g. $K(x) = e^{-|x|}$. We prove finite-time blow-up of solutions from specific smooth initial data, for which the problem is known to have short time existence of smooth solutions.

1 Introduction

The aggregation equation

$$u_t + \nabla \cdot (u \nabla K * u) = 0 \tag{1}$$

arises in a number of context of recent interest in the physics and biology literature. In biology, a swarming mechanism, in which individual organisms sense others at a distance, and move towards regions in which they sense the presence of others, involves a complex neurological process at the individual level. These local individual interactions lead to large scale patterns in nature for which it is desirable to have a tractable mathematical model. The model also arises in the context of self-assembly of nanoparticles (see e.g. Holm and Putkaradze [15, 16] and references therein).

The equation (1) with different classes of potentials and with additional regularizing terms appears in a number of recent and older papers. Topaz and Bertozzi [27] derive the model as multi-dimensional generalization of one-dimensional aggregation behavior discussed in the biology literature [13, 23]. Bodnar and Velazquez [2] consider well-posedness on R for different types of kernels. Burger and collaborators [5, 4] consider well-posedness of the model with an additional 'porous media' type smoothing term. This class of models, with diffusion, is also derived by Topaz et al [28] who cite some earlier rigorous work from the 1980s in one space dimension. We specifically consider the case where the kernel K has a Lipschitz point at the origin, as in the case of $e^{-|x|}$ which arises in both the biological and nanoscience applications.

As a mathematics problem, equation (1) is an *active scalar problem* [10] in which a quantity is transported by a vector field obtained by a nonlocal operator applied to the scalar field. Such problems commonly arise in fluid dynamics, for example, the two dimensional vorticity equation

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = 0, \quad v = \nabla^{\perp} N_e * \omega$$

where N_e is the Newtonian potential in two space dimensions. These active scalar equations sometimes serve as model problems for the study of finite time blowup of solutions of the 3D Euler equations in which the vorticity vector satisfies

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{v} \cdot \nabla \vec{\omega} = \omega \cdot \nabla v, \quad \vec{v} = \vec{K}_3 * \omega \tag{2}$$

where K_3 is the 3D Biot-Savart kernel, homogeneous of degree -2 in 3D. Stretching of vorticity is accomplished by the right hand side of of (2) in which the nonlocal amplification of vorticity occurs. For this problem, finite time blowup of solutions is an open problem. The 2D-quasi-geostrophic equations also pose an interesting family of active scalar equations [9, 17]. An important distinction, between our problem and that of the Euler-related problems, is that our transport field $\vec{v}$ is a gradient flow whereas the $\vec{v}$ in fluid dynamics is divergence free. Nevertheless it is interesting to note this analogy which also includes the well-known one dimensional model problem studied by Constantin, Lax, and Majda [8] $u_t = H(u)u$ where H denotes the Hilbert transform. There is no transport in this problem and the equation is known to have solutions that blow up in finite time.

By differentiation, our problem can be written as

$$\frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u = (-\Delta K * u)u, \tag{3}$$

an advection-reaction equation in which the solution u is amplified by the nonlocal operator $(-\Delta K * u)$. As in the Euler examples, there is a conserved quantity, namely the L^1 norm of the solution.

In one space dimension, (3) takes on a particularly simple form for which it is easy to show solutions become singular in finite time for pointy potentials such as $e^{-|x|}$. That is, the $\Delta K * u$ operator splits into convolution with a Dirac mass plus convolution with a bounded kernel. Thus, for a smooth solution one can apply a maximum principle argument to obtain an estimate of the form $(u_m)_t \geq C u_m^2 - D u_m$ (where u_m is the maximum value of u) which implies finite time blowup of the solution provided a suitable continuation theorem holds. This argument is described in a nonrigorous fashion in [15]. Bodnar and Velazquez consider the one-dimensional problem in [2] and prove finite time blowup by direct comparison with a Burgers-like dynamics. In higher space dimensions one does not obtain such a straightforward blowup result. This is because ΔK, for $n \geq 2$, in (3) does not have a Dirac mass, instead its singular part is of the form $\frac{1}{|x|}$. As a convolution operator, ΔK is increasingly less singularly, as the dimension of space, n, increases. Note for example that the Newtonian potential, which has the form of $\frac{c_n}{|x|^{2-n}}$ in R^n, $n \geq 3$, introduces a gain of two derivatives, as a convolution operator. Indeed, if K has bounded second derivative, there is no finite time blowup, so the type of kernel considered here poses an interesting problem for nonlocal active scalar equations.

Another related problem of biological relevance is the chemotaxis model

$$\rho_t + \nabla \cdot (\rho \cdot \nabla c) = \Delta \rho, \quad -\Delta c = \rho, \tag{4}$$

where ρ is a mass density of bacteria and c is concentration of a chemoattractant [3]. The model is related to the much-studied Keller-Segel model [18] and it also arises as an overdamped version of the Chandrasekhar equation for the gravitational equilibrium of isothermal stars [6]. In (4) the left had side has the same structure as (1) where the kernel K is now the Newtonian potential, which is significantly more singular than the aggregation kernels considered in this paper. Finite time singularities for (4) are known to occur even with the linear diffusion on the right hand side. The paper [3] shows that the behavior of the blowup depends strongly on the dimension of space. See [3] for a comprehensive literature on this problem.

Finally we mention a body of literature addressing both discrete and dynamic models for aggregation [7, 11, 12, 14, 21, 24, 26, 29]. In discrete models, individuals appear as points rather than as a continuum density. The analogue of (1) for the discrete problem is a kinematic gradient flow model for particles interacting via the pairwise potential K. As pointed out in [20], a pointy potential has a discontinuity in the flow field which can result in finite time aggregation of a finite number of particles. For smooth potentials, the aggregation occurs in infinite time since the velocity of individuals approaches zero as they amass. This manuscript considers the same issue for the continuum problem.

In our paper we prove, in dimension two and higher, that equation (1) with a pointy potential, such as $e^{-|x|}$, has solutions that blowup in finite time from smooth initial data. The proof involves two important steps. The first step, outlined in the next section, is to obtain a continuation theorem for the problem, that smooth solutions can be continued in time provided a low norm (in our case we use L^q) is bounded. This step is, for example, the analogue of the Beale-Kato-Majda result for 3D Euler [1]. The second step is to prove that there exist initial data for which the solution can not be continued past some finite time. Putting these results together, we obtain Cauchy data for which the unique smooth solution blows up in finite time in the L^q norm. Since we take our data to have compact support with radial symmetry, the support remains uniformly bounded on the time interval of existence, and thus we also obtain blowup in the L^∞ norm (and L^p for all $p \geq q$.) There remain a number of interesting and open problem regarding the details of the blowup. We discuss these in the last section of the paper.

2 Continuation of smooth solutions with H^s initial data

In [19, 20] one of the authors proves local existence of solutions with data in H^s and global existence in the case of a sufficiently smooth kernel. In this section we show that the local existence has a continuation result with control in an L^q norm. For simplicity we consider solutions with compact support as we are interested in localized blowup associated with aggregation. To prove the continuation result, we reconstruct the local-in-time H^s solution by a simple approximating problem involving smoothing the kernel K and smoothing the initial data. For the continuation result, we need only some very general properties for the potential K as described below.

Definition 1 *The potential K on R^n, $n \geq 2$ is* acceptable *if $\nabla K \in L^2(R^n)$ and $\Delta K \in L^p(R^n)$ for some $p \in [p^*, 2]$, where $\frac{1}{p^*} = \frac{1}{2} + \frac{1}{n}$.*

Note that the 'pointy' potentials of interest satisfy the above definition.

Consider $u_0 \in H_0^s(R^n)$ (i.e. H^s) with compactly supported initial data. Define the approximating problem as follows: $u^\epsilon(x,t)$ is the smooth solution of the problem

$$\begin{aligned} u_t + \nabla \cdot (u \nabla K_\epsilon * u) &= 0, \\ u^\epsilon(x,0) &= J_\epsilon u_0. \end{aligned} \tag{5}$$

Here J_ϵ denotes convolution with a standard mollifier $\rho_\epsilon = \frac{1}{\epsilon^n}\rho(x/\epsilon)$ where ρ has compact support and mass one. $K_\epsilon = J_\epsilon * K$ is a smoothed kernel. Then from [19, 20] we have the following result:

Theorem 1 *The approximating problem (5) above has a unique smooth solution for all time.*

The smoothness of the kernel K is the key to obtaining smooth solutions globally in time. We now show that in passing to the limit one obtains $u \in C[0,T;H^s]$, and thus one can continue the solution as long as the H^s norm is controlled. Moreover, we derive an a priori bound for this norm which shows that it is controlled by L^q for any $q > 2$.

We also need the following Lemma (2.4 from [20]) where we denote by T^α the trilinear form

$$T^\alpha(u,v,w) = \int_{R^n} (D^\alpha u)\ D^\alpha \nabla \cdot [v\,(w * \nabla K)]\ dx. \tag{6}$$

Lemma 1 *Assume K is an acceptable potential and $s \in Z^+$.*

(i) If α is a multiindex of order $s \geq 1$, then

$$|T^\alpha(u,u,w)| \leq C\,\|u\|_{H^s}^2\,\|w\|_{H^s} \qquad \forall u \in H^{s+1}(R^n) \text{ and } \forall w \in H^s(R^n).$$

The constant $C > 0$ depends only on α, K and n.

(ii) If $\alpha = 0$ then

$$\left|T^0(u,u,w)\right| \leq C\,\|u\|_{L^2}^2\,\|w\|_{H^1} \qquad \forall u,w \in H^1(R^n), \tag{7}$$

$$\left|T^0(u,v,w)\right| \leq C\,\|u\|_{L^2}\,\|v\|_{H^1}\,\|w\|_{L^2} \ \ \forall u,w \in L^2(R^n) \text{ and } \forall v \in H^1(R^n). \tag{8}$$

The constant $C > 0$ depends only on K and n.

The following argument is closely based on the arguments in chapter 3 of [22] proving continuation of solutions of the 3D Euler equation $C[0,T;H^s]$. The reader can review that material for some additional details. First we prove the following

Theorem 2 Local in time existence of solutions to the aggregation equation *Given K satisfying Definition 1, an initial condition $u_0 \in H^s$, and $s \geq 2$ is a positive integer, then*
i) There exists a time T with the rough upper bound

$$T \leq \frac{1}{c_s\|u_0\|_{H^s}}, \tag{9}$$

such that there exists the unique solution $u \in C([0,T];C^2(R^n)) \cap C^1([0,T];C(R^n))$ to the aggregation equation (1). The solution u is the limit of a subsequence of approximate solutions, u^ϵ, of (5).
ii) The approximate solutions u^ϵ and the limit u satisfy the higher order energy estimate

$$\sup_{0\leq t\leq T} \|u^\epsilon\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - c_sT\|u_0\|_{H^s}} \ . \tag{10}$$

iii) The approximate solutions and the limit u are uniformly bounded in the spaces $L^\infty([0,T],H^s(R^n))$, $Lip([0,T];H^{s-1}(R^n))$, $C_W([0,T];H^s(R^n))$.

Definition 2 *The space $C_W([0,T];H^s(R^n))$ denotes continuity on the interval $[0,T]$ with values in the* **weak topology** *of H^s, that is for any fixed $\phi \in H^s$, $(\phi, u(t))_s$ is a continuous scalar function on $[0,T]$, where*

$$(u,v)_s = \sum_{\alpha\leq s} \int_{R^n} D^\alpha u \cdot D^\alpha v dx.$$

We note that most of the above theorem is already proved in [20]. In particular uniqueness of the H^s solution is proved in section 2.2 of [20] so we do not rederive that here. However we rederive the short term existence result with a different approximating problem (5) in order to concisely derive a continuation result in the high norm and with a necessary condition for blowup involving the L^q norm.

The strategy for the local existence proof, that we implement below, is to first prove the bound (10) in the high norm, then show that we actually have a contraction in the L^2 norm. We then apply an interpolation inequality to prove convergence as $\epsilon \to 0$. Following the local existence proof, we establish Theorem 3 below, that the solution u is actually continuous in time in the highest norm H^s, and can be continued in time provided that its H^s norm remains bounded. We need this fact in order to discuss the link between the existence of these solutions globally in time and blowup in a lower norm. To prove Theorem 2 we need a priori higher derivative estimates that are also independent of ϵ.

Proposition 1 The H^s energy estimate *Let $u_0 \in H^s$ with integer $s \geq 2$. Then the unique regularized solution $u^\epsilon \in C^1([0,\infty);H^s)$ satisfies*

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|^2_{H^s} \leq c_s\|u^\epsilon\|_{H^{s-1}}\|u^\epsilon\|^2_{H^s}. \tag{11}$$

Proof.

To derive this estimate we need the following slightly improved version of Lemma 1: if α is a multiindex of order $s \geq 2$, then

$$|T^{\alpha}(u,u,u)| \leq C_{\alpha}(\|\nabla K\|_{L^2} + \|\Delta K\|_{L^p}) \|u\|_{H^s}^2 \|u\|_{H^{s-1}} \tag{12}$$

for all $u \in H^{s+1}$. Here p can be chosen to be 2 if we are in dimension $N \geq 3$. If we are in dimension $N = 2$, p can be chosen to be $3/2$. The constant C_{α} is a positive constant depending only on α. The proof of (12) is very similar to the one of Lemma 2.4 from [20]) and we refer the reader there for details. To prove (11), if α is a multiindex of order s then

$$\begin{aligned}(D^{\alpha}u^{\epsilon}, D^{\alpha}u_t^{\epsilon}) &= -T^{\alpha}(u^{\epsilon}, u^{\epsilon}, u^{\epsilon}) & (13)\\ &\leq C_{\alpha}(\|\nabla K^{\epsilon}\|_{L^2} + \|\Delta K^{\epsilon}\|_{L^p}) \|u^{\epsilon}\|_{H^s}^2 \|u^{\epsilon}\|_{H^{s-1}} & (14)\\ &\leq C_{\alpha}(\|\nabla K\|_{L^2} + \|\Delta K\|_{L^p}) \|u^{\epsilon}\|_{H^s}^2 \|u^{\epsilon}\|_{H^{s-1}}\,. & (15)\end{aligned}$$

The estimate of derivatives of lower order are simpler and left as an exercise for the reader. We just need to use Lemma 1 and proceed as above.

We can now complete the proof of Theorem 2.

First we show that the family (u^{ϵ}) of regularized solutions is uniformly bounded in H^s. The H^s energy estimate implies the time derivative of $\|u^{\epsilon}\|_{H^s}$ can be bounded by a quadratic function of $\|u^{\epsilon}\|_{H^s}$ independent of ϵ,

$$\sup_{0\leq t\leq T} \|u^{\epsilon}\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - c_s T\|u_0\|_{H^s}} = \|u_0\|_{H^s} + \frac{\|u_0\|_{H^s}^2 c_s T}{1 - c_s T\|u_0\|_{H^s}}\,. \tag{16}$$

Thus the family (u^{ϵ}) *is uniformly bounded in* $C([0,T];H^s)$, provided that $T < (c_s\|u_0\|_{H^s})^{-1}$. Furthermore, the family of time derivatives $(\frac{du^{\epsilon}}{dt})$ is uniformly bounded in H^{s-1}

$$\begin{aligned}\|\frac{du^{\epsilon}}{dt}\|_{H^{s-1}} &\leq \|\nabla\cdot(u_{\epsilon}\nabla K^{\epsilon} * u_{\epsilon})\|_{H^{s-1}} \leq \|u_{\epsilon}\nabla K_{\epsilon} * u_{\epsilon}\|_{H^s}\\ &\leq \|u_{\epsilon}\|_{H^s}\|\nabla K_{\epsilon} * u_{\epsilon}\|_{W^{s,\infty}} \leq \|u_{\epsilon}\|_{H^s}^2\|\nabla K\|_{L^2}.\end{aligned}$$

We now show that the solutions u^{ϵ} to the regularized equation (5) form a contraction in the low norm $C([0,T];L^2(R^n))$. Specifically we prove

Lemma 2 *The family u^{ϵ} forms a Cauchy sequence in $C([0,T];L^2(R^n))$. In particular, there exists a constant C depending only on $\|u_0\|_1$ and the time T so that for all ϵ and ϵ',*

$$\sup_{0<t<T} \|u^{\epsilon} - u^{\epsilon'}\|_{L^2} \leq C\max(\epsilon, \epsilon').$$

Proof: We have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|u^{\epsilon} - u^{\epsilon'}\|_{L^2}^2 &= \left(u^{\epsilon} - u^{\epsilon'}, \nabla\cdot(u^{\epsilon} - u^{\epsilon'})\nabla K^{\epsilon}u^{\epsilon}\right) + \left(u^{\epsilon} - u^{\epsilon'}, \nabla\cdot u^{\epsilon'}\nabla K^{\epsilon}(u^{\epsilon} - u^{\epsilon'})\right)\\ &+ \left(u^{\epsilon} - u^{\epsilon'}, \nabla\cdot u^{\epsilon'}\nabla K(J_{\epsilon} - J_{\epsilon'})u^{\epsilon'}\right)\\ &= T_1 + T_2 + T_3.\end{aligned}$$

Using Lemma 1 we have the following estimates:

Applying (7) to T_1 we have

$$|T_1| \leq C\|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 \|J_\epsilon u^\epsilon\|_{H^1}.$$

Applying (8) to T_2 and T_3 we have

$$|T_2| \leq C\|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 \|u^{\epsilon'}\|_{H^1},$$

and

$$|T_3| \leq C\|u^\epsilon - u^{\epsilon'}\|_{L^2} \max(\epsilon, \epsilon')\|u^{\epsilon'}\|_{H^1}^2.$$

Putting this all together gives

$$\frac{d}{dt}\|u^\epsilon - u^{\epsilon'}\|_{L^2} \leq C(M)(\max(\epsilon, \epsilon') + \|u^\epsilon - u^{\epsilon'}\|_{L^2})$$

where M is an upper bound, from (16) for the $\|u^\epsilon\|_{H^1}$ on $[0, T]$. Integrating this yields

$$\sup_{0<t<T} \|u^\epsilon - u^{\epsilon'}\|_{L^2} \leq C(M, T)\max(\epsilon, \epsilon'), \tag{17}$$

where the final inequality is established by recalling that $u_0^\epsilon - u_0^{\epsilon'}$ is bounded in L^2 by $\max(\epsilon, \epsilon')$ times the H^1 norm of u_0. Thus u^ϵ is a Cauchy sequence in $C([0, T]; L^2(R^n))$ so that it converges strongly to a value $u \in C([0, T]; L^2(R^n))$.

We have just proved the existence of a u such that

$$\sup_{0\leq t\leq T} \|u^\epsilon - u\|_{L^2} \leq C\epsilon. \tag{18}$$

We now use the fact that the u^ϵ are uniformly bounded in a high norm to show that we have strong convergence in all the intermediate norms. In order to do this, we need the following well-known interpolation lemma for the Sobolev spaces (see [22] and references therein):

Lemma 3 *Given $s > 0$, there exists a constant C_s so that for all $v \in H^s(R^N)$, and $0 < s' < s$,*

$$\|v\|_{s'} \leq C_s \|v\|_{L^2}^{1-s'/s} \|v\|_{H^s}^{s'/s}. \tag{19}$$

We now apply the interpolation lemma to the difference $u^\epsilon - u$. Using (16) and (18) gives

$$\sup_{0\leq t\leq T} \|u^\epsilon - u\|_{H^{s'}} \leq C(\|u_0\|_{H^s}, T)\epsilon^{1-s'/s}.$$

Hence for all $s' < s$ we have strong convergence in $C([0, T]; H^{s'}(R^n))$.

A bounded sequence $\|u_\epsilon\|_{H^s} \leq C$ in $H^s(R^N)$ has a subsequence that converges *weakly* to some limit in H^s, $u_\epsilon \rightharpoonup u$. The preceding arguments show that

$$\sup_{0\leq t\leq T} \|u^\epsilon\|_{H^s} \leq M \qquad \text{and} \tag{20}$$

$$\sup_{0\leq t\leq T} \|\frac{\partial u^\epsilon}{\partial t}\|_{H^{s-1}} \leq M_1. \tag{21}$$

Hence, u^ϵ is uniformly bounded in the Hilbert space $L^2([0,T];H^s(R^n))$ so there exists a subsequence that converges weakly to

$$u \in L^2([0,T];H^s(R^n)). \tag{22}$$

Moreover, if we fix $t \in [0,T]$, the sequence $u^\epsilon(\cdot,t)$ is uniformly bounded in H^s, so that it also has a subsequence that converges weakly to $u(t) \in H^s$. Thus we see that for each t, $\|u\|_s$ is bounded. This combined with (22) implies that $u \in L^\infty([0,T];H^s)$. A similar argument, applied to the estimate (21) shows that $u \in Lip([0,T];H^{m-1}(R^n))$.

Now we conclude that u is continuous in the weak topology of H^s. To prove that $u \in C_W([0,T];H^s(R^n))$, let $[\phi,u]$, $\phi \in H^{-s}$ denote the dual pairing of H^{-s} and H^s through the L^2 inner product. Estimates (20) and (21), the uniform compact support of u^e, and the Lions-Aubin lemma 10.4 from [22] imply $u^\epsilon \to u$ in $C([0,T];H^{s'})$; thus it follows that $[\phi,u^\epsilon(\cdot,t)] \to [\phi,u(\cdot,t)]$ uniformly on $[0,T]$ for any $\phi \in H^{-s'}$. Using (20) and the fact that $H^{-s'}$ is dense in H^{-s} for $s' < s$, via an $\epsilon/2$ argument using (20), we have $[\phi,u^\epsilon(\cdot,t)] \to [\phi,u(\cdot,t)]$ uniformly on $[0,T]$ for any $\phi \in H^{-s}$. This fact implies that $u \in C_W([0,T];H^s)$.

We now show that the limit u is a distribution solution of the original equation. Recall that the sequence of approximate u^ϵ satisfies

$$u^\epsilon \in C([0,\infty);H^s) \cap C^1([0,\infty);H^{s-1}) \quad \text{for all } s, \tag{23}$$

$$u^\epsilon_t + \nabla\cdot(u^\epsilon\nabla K_\epsilon * u^\epsilon) = 0, \tag{24}$$

$$u^\epsilon(0) = J_\epsilon u_0. \tag{25}$$

Since $u_0 \in H^2$, then we know that there exists a time T_1 such that u^ϵ is Cauchy sequence in $C([0,T_1];H^1)$. Therefore there exists a function u such that

$$u^\epsilon \to u \qquad \text{in } C([0,T_1];H^1). \tag{26}$$

Lemma 4 *The function u satisfies*

$$u \in C([0,T_1];H^1) \cap C^1([0,T_1];L^2), \tag{27}$$

$$u_t + \nabla\cdot(u\nabla K * u) = 0, \tag{28}$$

$$u(0) = u_0, \tag{29}$$

with the dynamic equation in the sense of distributions.

Proof.

From (26) one can easily obtain

$$\nabla\cdot(u^\epsilon\nabla K_\epsilon * u^\epsilon) \to \nabla\cdot(u\nabla K * u), \qquad \text{in } C([0,T_1];L^2). \tag{30}$$

We omit the details since they are elementary. Let $v = \nabla\cdot(u\nabla K * u)$. The proof will be completed once we establish that v is the distributional derivative of u, i.e.:

$$\int_0^{T_1} u(t,x)\phi'(t)dt = -\int_0^{T_1} v(t,x)\phi(t)dt \tag{31}$$

for all test function $\phi \in C_0^\infty(0,T_1)$. Since $u^\epsilon \in C^1([0,\infty);L^2)$ it is clear that

$$\int_0^{T_1} u^\epsilon(t,x)\phi'(t)dt = -\int_0^{T_1} u_t^\epsilon(t,x)\phi(t)dt. \tag{32}$$

Also note that (30) can be written

$$u_t^\epsilon \to v \qquad \text{in } C([0,T_1];L^2).$$

This convergence, together with (26), allows us to pass to the limit in (32), thus proving (31).

Now we show that u is continuous in time with values in the highest H^s norm.

Theorem 3 Continuity in the high norm *Let u be the solution described in Theorem 2. Then*

$$u \in C([0,T);H^s) \cap C^1([0,T);H^{s-1}).$$

Proof: The argument follows the same reasoning as in [22, 25] for the Euler equations. By virtue of the equation it is sufficient to show that $u \in C([0,T];H^s)$. It is important to obtain this sharper result of continuity in time with values in H^s in order to prove the blowup result of the following section. By virtue of the fact that $u \in C_W([0,T];H^s(R^n))$, it suffices to show that the norm $\|u(t)\|_s$ is a continuous function of time. Passing to the limit in this equation and using the fact that for fixed t, $\limsup_{\epsilon\to 0}\|u^\epsilon\|_{H^s} \geq \|u\|_{H^s}$ we obtain

$$\sup_{0\leq t\leq T}\|u\|_{H^s} - \|u_0\|_{H^s} \leq \frac{\|u_0\|_{H^s}^2 c_s T}{1-c_sT\|u_0\|_{H^s}}\ . \tag{33}$$

From the fact that $u \in C_W([0,T];H^s(R^n))$, we have $\liminf_{t\to 0^+}|u(\cdot,t)|_{H^s} \geq \|u_0\|_{H^s}$. The estimate (33) gives $\limsup_{t\to 0^+}|u(\cdot,t)|_{H^s} \leq \|u_0\|_{H^s}$. In particular, $\lim_{t\to 0^+}|u(\cdot,t)|_{H^s} = \|u_0\|_{H^s}$. This gives us strong right continuity at $t=0$. Since the analysis is time reversible, we could likewise show strong left continuity at $t=0$.

It remains to prove continuity of the $\|\cdot\|_{H^s}$ norm of the solution at times other than the initial time. Consider a time $T_0 \in [0,T]$, and the solution $v(\cdot,T_0)$. At this fixed time, $v(\cdot,T_0) \equiv v_0^{T_0} \in H^s(R^n)$ and from (16)

$$|u_0^{T_0}|_{H^s} \leq \|u_0\|_{H^s} + \frac{\|u_0\|_{H^s}^2 c_s T_0}{1-c_sT_0\|u_0\|_{H^s}}. \tag{34}$$

So we can take $u_0^{T_0}$ as initial data and construct a forward and backward in time solution as above by solving the regularized equation (5). We obtain approximate solutions $u_{T_0}^\epsilon(\cdot,t)$ that satisfy

$$\frac{d}{dt}\|u_{T_0}^\epsilon\|_{H^s} \leq c_s\|u_{T_0}^\epsilon\|_{H^s}^2\ , \tag{35}$$

we can pass to a limit in $u_{T_0}^\epsilon$ as before and find a solution $\tilde{u}$ to the Euler equation on a time interval $[T_0-T', T_0+T']$ with initial data u_{T_0}. Following the same estimates as above, we obtain that the time T' satisfies the constraint

$$0 < T' < \frac{1}{c_s\|u_0\|_{H^s}} - T_0.$$

Furthermore, this solution $\tilde{u}$ must agree with u on $[T_0 - T', T_0 + T'] \cap [0, T]$ by virtue of uniqueness of H^s solutions and the fact that u and $\tilde{u}$ agree at $t = T_0 \in [0, T]$. Following the argument above used to show that $\|u\|_{H^s}$ is continuous at $t = 0$, we conclude that $|\tilde{u}|_{H^s}$ is continuous at T_0, hence $|u|_{H^s}$ itself must be continuous at T_0. Since $T_0 \in [0, T]$ was arbitrary, we have just showed that $|u|_{H^s}$ is a continuous function on $[0, T]$ and hence by the fact that $u \in C_W([0, T]; H^s(R^n))$, we obtain $u \in C([0, T]; H^s(R^n))$.

We obtain the following corollary.

Corollary 1 *Given K satisfying Definition 1 and an initial condition $u_0 \in H^s$, $s \geq 2$ is a positive integer, then there exists a maximal time of existence $T^* \in (0, \infty]$ and a unique solution $u \in C([0, T^*); H^s) \cup C^1([0, T^*); H^{s-1})$ to (1). Moreover if $T^* < \infty$ then necessarily $\lim_{t \to T^*} \|u(\cdot, t)\|_{H^s} = \infty$.*

To continue solutions with control in a lower norm, we first show that the H^{s-1} control is inherited by the limit u and follow with an a priori bound in terms of the L^q norm of the solution.

Recall that $u^\epsilon \in C^1([0, \infty); H^s)$ for every s and satisfies

$$\frac{d}{dt} \|u^\epsilon\|_{H^s}^2 \leq C \|u^\epsilon\|_{H^{s-1}} \|u^\epsilon\|_{H^s}^2 . \tag{36}$$

Of course this implies

$$\frac{d}{dt} \|u^\epsilon\|_{H^s}^2 \leq C \|u^\epsilon\|_{H^s}^3 . \tag{37}$$

Since $u_0 \in H^s$, (37) implies that there exists a time T_1 such that

$$|u^\epsilon|_{L^\infty(0, T_1; H^s)} \leq C \tag{38}$$

where C is independent of ϵ. Moreover, we know that on this time interval $[0, T_1]$ we have the following convergence:

$$u^\epsilon \to u \qquad \text{in } C([0, T_1]; H^{s-1}) \text{ strong,} \tag{39}$$

$$u^\epsilon \rightharpoonup u \qquad \text{in } L^\infty(0, T_1; H^s) \text{ weak-star.} \tag{40}$$

Using Gronwall inequality we deduce from (36) that

$$|u^\epsilon|_{L^\infty(0, T_1; H^s)} \leq e^{C \int_0^T \|u^\epsilon(s)\|_{H^{s-1}} ds} . \|u_0\|_{H^s} \tag{41}$$

Fix now a time $T \leq T_1$. From (39) it is clear that

$$\int_0^T \|u^\epsilon(s)\|_{H^{s-1}} \, ds \to \int_0^T \|u(s)\|_{H^{k-1}} \, ds,$$

and therefore

$$\text{lim-inf}_{\epsilon \to 0} \|u^\epsilon\|_{L^\infty(0, T_1; H^s)} \leq e^{C \int_0^T \|u(s)\|_{H^{k-1}} ds} \|u_0\|_{H^k} . \tag{42}$$

Finally, because of (40) we get

$$\|u\|_{L^\infty(0, T_1; H^s)} \leq e^{C \int_0^T \|u(s)\|_{H^{s-1}} ds} \|u_0\|_{H^s} . \tag{43}$$

Since this inequality is true for every $T \leq T_1$, and since $u \in C([0, T_1]; H^s)$, we have just proven the following lemma:

Lemma 5 *For every $t \in [0, T_1]$,*

$$\|u(t)\|_{H^s} \leq e^{C\int_0^t \|u(s)\|_{H^{s-1}} ds} \|u_0\|_{H^s} . \tag{44}$$

We have proven existence of a continuous solution with value in H^s up to time T_1. By iterating the argument, we can continue the solution on $[T_1, T_2]$, then $[T_2, T_3]$, ...

We have also proven that

$$\|u(t)\|_{H^s} \leq e^{C\int_0^t \|u(s)\|_{H^{s-1}} ds} \|u_0\|_{H^s} \qquad \text{on } [0, T_1]. \tag{45}$$

The same proof show that

$$\|u(t)\|_{H^s} \leq e^{C\int_{T_1}^t \|u(s)\|_{H^{s-1}} ds} \|u(T_1)\|_{H^s} \qquad \text{on } [T_1, T_2]. \tag{46}$$

Combining (45) and (46), we obtain that, for $t \in [T_1, T_2]$,

$$\begin{aligned} \|u(t)\|_{H^s} &\leq e^{C\int_{T_1}^t \|u(s)\|_{H^{s-1}} ds} e^{C\int_0^{T_1} \|u(s)\|_{H^{s-1}} ds} \|u_0\|_{H^s} \\ &\leq e^{C\int_0^t \|u(s)\|_{H^{s-1}} ds} \|u_0\|_{H^s} . \end{aligned}$$

And therefore one can easily iterate the argument to find that, as long as the solution exists, it must satisfies the estimate

$$\|u(t)\|_{H^s} \leq e^{C\int_0^t \|u(s)\|_{H^{s-1}} ds} \|u_0\|_{H^s} . \tag{47}$$

To prove L^q control of blowup we need the following version of the Young inequality.

Lemma 6 *Suppose $1 \leq p \leq 2$, and q is the conjugate of p. If $\phi_1 \in L^2(R^n)$, $\phi_2 \in L^q(R^n)$, $\phi_3 \in L^2(R^n)$ and $\phi_4 \in L^p(R^n)$, then*

$$\left| \int_{R^n} \phi_1 \ \phi_2 \ (\phi_3 * \phi_4) \ dx \right| \leq \|\phi_1\|_{L^2} \|\phi_2\|_{L^q} \|\phi_3\|_{L^2} \|\phi_4\|_{L^p} .$$

Proof: Using Young's inequality, we get $\|\phi_3 * \phi_4\|_{L^r} \leq \|\phi_3\|_{L^2} \|\phi_4\|_{L^p}$ for r defined by $1/r = 1/2 + 1/p - 1$. Note that the condition $p \leq 2$ ensure that $1/r$ is nonnegative. One can then check that $1/r + 1/q = 1/2$. This allow us to use Hölder's inequality to show that $\|\phi_2 (\phi_3 * \phi_4)\|_{L^2} \leq \|\phi_2\|_{L^q} \|\phi_3 * \phi_4\|_{L^r}$. Then the Schwarz inequality concludes the proof.

Proposition 2 *Suppose $n \geq 3$. Then, for every $q \in [2, +\infty]$*

$$\frac{d}{dt} \|u\|_{H^1}^2 \leq C \left\|D^2 K\right\|_{L^p} \|u\|_{L^q} \|u\|_{H^1}^2 , \tag{48}$$

where p is the conjugate of q and $\left\|D^2 K\right\|_{L^p} = \sup_{i,j} \left\|K_{x_i,x_j}\right\|_{H^p}$. The constant C depends only on n.

If $n = 2$, then (2) holds for every $q \in (2, +\infty]$.

Note that in dimension $n = 2$, $\left\|D^2 K\right\|_{L^2} = \infty$. This is why, when $n = 2$, we exclude the case $p = q = 2$. In dimension $n = 3$, $\left\|D^2 K\right\|_{L^2} < \infty$ and we do not need to exclude the case $p = q = 2$.

Proof: Integrating by part twice we easily obtain

$$\frac{d}{dt}\|u\|_{L^2}^2 = \int \nabla K * u \cdot \nabla(u^2) = -\int u^2 u * \Delta K \tag{49}$$

$$\leq \|u\|_{L^2}^2 \|u * \Delta K\|_{L^\infty} \leq \|u\|_{L^2}^2 \|u\|_{L^q} \|\Delta K\|_{L^p} . \tag{50}$$

Differentiating the PDE we obtain

$$\begin{aligned} u_{t,x_i} = & - \nabla u_{x_i} \cdot (u * \nabla K) - \nabla u_{x_i}(u * \Delta K) \\ & - \nabla u \cdot (u * \nabla K_{x_i}) - u(u_{x_i} * \Delta K), \end{aligned}$$

and then, after some integration by parts

$$\begin{aligned} \frac{d}{dt}\|u_{x_i}\|_{L^2}^2 &= -\int u_{x_i}^2 u * \Delta K - 2\int u_{x_i} \nabla u \cdot u * \nabla K_{x_i} - 2\int u_{x_i} u \cdot u * \Delta K_{x_i} \\ &= -A - 2B - 2C. \end{aligned}$$

It is clear that

$$|A|, |B| \leq \left\|D^2 K\right\|_{L^p} \|u\|_{L^q} \|\nabla u\|_{L^2}^2 .$$

and, using Lemma 6 we get

$$\begin{aligned} |C| &\leq \|u_{x_i}\|_{L^2} \|u\|_{L^q} \|u_{x_i}\|_{L^2} \|\Delta K\|_{L^p} \\ &\leq \left\|D^2 K\right\|_{L^p} \|u\|_{L^q} \|\nabla u\|_{L^2}^2 . \end{aligned}$$

Note that in order to apply Lemma 6 it is necessary that $p \leq 2$ and therefore $q \geq 2$. From the estimate of $|A|$, $|B|$ and $|C|$ we easily obtain

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq C \left\|D^2 K\right\|_{L^p} \|u\|_{L^q} \|\nabla u\|_{L^2}^2 .$$

This together with (50) conclude the proof. Note that in order to estimate $\frac{d}{dt}\|u\|_{H^1}^2$ it was necessary for $u(t)$ to be in H^2 (if $u(t)$ was just in H^1 then the integration by parts would not have been justified).

The final result of this section is the continuation theorem:

Theorem 4 Continuation Theorem *Given initial data $u_0 \in H^s(R^n)$, $n \geq 2$, for positive integer $s \geq 2$, there exists a unique solution $u(x,t)$ of (1) and a maximal time interval of existence $[0, T^*)$ such that either $T^* = \infty$ or $\lim_{t \to T^*} \sup_{0 \leq s \leq t} \|u(\cdot, s)\|_{L^q} = \infty$. Here $q \geq 2$ for $n > 2$ and $q > 2$ for $n = 2$.*

We use this result in the following section to prove that for some specific initial data, the L^q norm blows up in finite time.

3 An energy estimate proving finite time blowup

In this section we consider the specific potential $K = e^{-|x|}$ and prove finite time blowup for specific smooth initial data. The argument extends to slightly more general potentials. The main feature is that $\nabla K = -N(x)$ at the origin. They key idea is to use an energy, which was first derived in [28] for biological aggregation with additional 'porous media' dissipation. See [15, 16] for a discussion of energies as they related to nanoparticle applications. We consider

$$E(u) = \int_{R^n} uK * udx, \tag{51}$$

for which integration by parts gives the following rate of increase of $E(u)$:

$$\frac{dE}{dt} = 2\int_{R^n} u|\nabla K * u|^2 dx. \tag{52}$$

We now show that E has an a priori uniform bound for all $u \in L^1(R^n)$ while at the same time $\frac{dE}{dt}$ has a positive lower bound for certain initial data. Thus the smooth solution can not be continued past some critical finite time.

We have the following Lemma:

Lemma 7 *For all $u \in L^1(R^n)$, we have*

$$E(u) \leq |u|^2_{L^1}.$$

Proof:

$$E(u) = \int_{R^n} uK * udx \leq |K * u|_{L^\infty}|u|_{L^1} \leq |K|_{L^\infty}|u|^2_{L^1} = |u|^2_{L^1}.$$

Before proceeding with the proof of blowup we need the following result from [20] Section 3.

Theorem 5 *Let u be the solution described in the preceding section. Assume that $u_0 \in W^{1,1}(R^n)$ and u_0 is nonnegative. Then for each t in the domain of existence, the solution $u(x,t)$ is nonnegative and moreover $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$.*

Using the above facts, the main idea of this section is to show that, for a specific class of functions, the right hand side of (52) has a positive pointwise lower bound. To do this we consider special smooth initial data close to a delta-distribution. Consider a *radially symmetric* C_0^∞ function $\rho \geq 0$ with compact support in $B_1(0)$, $\int_{R^n} \rho dx = 1$. Define

$$u_\delta(x) = \frac{1}{\delta^n}\rho(\frac{x}{\delta}) \tag{53}$$

to be a rescaling of the smooth function ρ to approximate a delta mass at the origin.

We now prove the following kinematic estimate associated with all functions that satisfy the above geometric constraint.

Proposition 3 *There exists a constant $C > 0$ such that for all δ sufficiently small, we have, for any radially symmetric L^1 function u_δ with support inside a ball of radius δ,*

$$\int_{R^n} u_\delta |\nabla K * u_\delta|^2 dx \geq C. \tag{54}$$

The main idea is that for data with small support, $\nabla K * u_\epsilon$ is approximately $-N * u_\epsilon$ where N is the kernel, homogeneous of degree zero, $\frac{\vec{x}}{|x|}$. Thus we first prove the bound for this kernel and then show that for data with small support, the support remains small and the correction to the kernel results in a small perturbation of the constant. The key lemma for N is described below.

Lemma 8 *Let u be a radially symmetric, nonnegative function, with compact support. Then*

$$v(|x|) := (N * u(x)) \cdot N(x)$$

is a nonnegative, nondecreasing function of $|x|$.

Proof:

Since u is radially symmetric, it suffices to prove the result for u a delta ring concentrated on radius 1. The reason is that space can be rescaled to any radius. Moreover the geometry of the kernel, and the symmetry of u, imply that $N * u$ is a scalar function times N, where the scalar function is a linear functional of u. Thus a general radially symmetric function u can be thought of as a linear superposition of infinitesimal ring elements at different radii and the resulting composite functional will inherit the nonnegativity and monotone properties of the functional evaluated on the ring.

For the delta ring (uniform distribution) of unit mass on $\partial B_1(0)$, we have

$$v_{\delta(1)}(x) := \frac{1}{\omega(n)} N(x) \cdot \int_{\partial B(1)} N(x-y) dS. \tag{55}$$

where $\omega(n)$ is the area of the unit ball in R^n. Note that the integral is a sum of unit vectors from a point in space to the boundary of the unit ball therefore the size of the convolution depends on the degree of cancellation for vectors in different directions.

Let us first prove that $v_{\delta(1)}(x)$ is nonnegative. We include a figure in 2D illustrating the notation and ideas. Without loss of generality, since u is radially symmetric, we take x on the x_1 axis and positive: $x = (x_1, \dots, x_n)$. If $x_1 \geq 1$ then the quantity $N(x) \cdot N(x-y)$ is clearly nonnegative for every y in on the domain of integration and therefore $v_{\delta(1)}(x)$ is nonnegative. If $0 \leq x_1 < 1$ we divide the domain of integration into regions where the integrand $N(x) \cdot N(x-y)$ is positive and negative. It is negative for $y_1 > x_1$, and positive for $-1 \leq y_1 < x_1$. Note that a larger mass of the delta-ring is on the positive part and moreover there is less cancellation of vectors for these values because y is farther away from x and thus $y - x$ has a direction more in line with the $\vec{x}$ vector. More precisely, for $y_1 > x_1 > 0$, compare a point $y := (y_1, y_2, \dots, y_n)$ on $\partial B_1(0)$ with its reflection $\tilde{y} := (-y_1, y_2, \dots, y_n)$ and note that $|N(x) \cdot N(x - \tilde{y})|$ is always greater than $|N(x) \cdot N(x-y)|$ with equality when $x = 0$. Thus $v_{\delta(1)}(x)$ is still nonnegative when x is inside the ball.

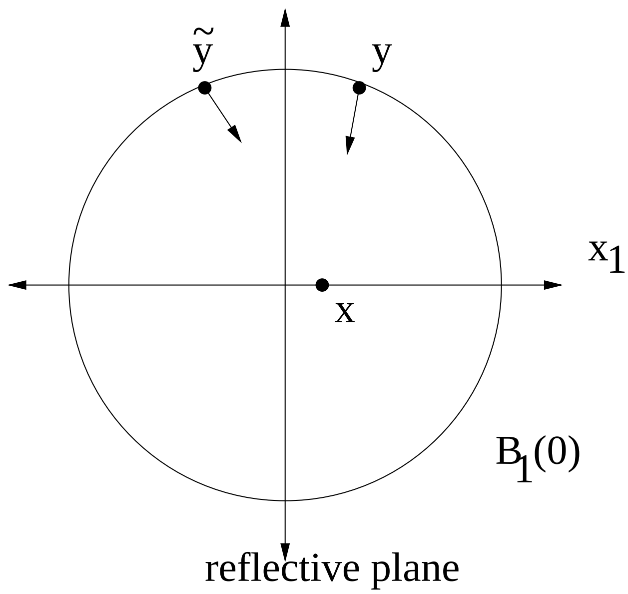

Figure 1: Figure for Lemma 8

To show that $v_{\delta(1)}$ is nondecreasing as a function of radius, note that for each y, the integrand in (55) increases monotonically as x_1 increases. Thus the integral inherits this monotonicity in x_1.

We have the following corollary.

Corollary 2 *For any decomposition $u = u_1 + u_2$, where u, u_1, and u_2 are nonnegative radially symmetric functions, let v, v_1, and v_2 be the respective associated functionals as in (55). Then v, v_i are all nonnegative and moreover $v = v_1 + v_2$ and thus $v_i \leq v$.*

Lemma 9 *Given a nonnegative radial function u with support inside $B_1(0)$ and unit mass, for $|x| \geq 1$, the associated v satisfies $v(x) \geq C_1$ where*

$$C_1 = e_1 \cdot \frac{1}{\omega(n)} \int_{\partial B_1(0)} N(e_1 - y) ds > 0, \tag{56}$$

where e_1 is the unit vector in the x_1 direction.

Proof: Since u has support inside the unit ball, following the arguments in Lemma 8, we see that v is nondecreasing for $|x|$ increasing. Thus it suffices to compute an estimate for $|x| = 1$. Now we consider for what radially symmetric u does one minimize $v(1)$. By the geometry of the problem, $v(1)$ decreases by moving more of the mass of u to the edge of the support. Thus the minimum occurs for u as a delta ring concentrated on the unit ball, yielding formula (56).

Proof of Proposition 3.

First we prove the proposition for the kernel $K = -N$ where u is not required to have small support. Given a radially symmetric u of unit mass, decompose u into two nonnegative, radially symmetric parts, u_1, u_2, defined as

$$u_1 = \chi_{B_R(0)} u, \tag{57}$$
$$u_2 = [1 - \chi_{B_R(0)}] u, \tag{58}$$

where R is defined to be the radius such that $\int_{B_R(0)} u dx = 1/2$. Define v, v_1 and v_2 to be the respective functionals as in (55). Note that

$$\int_{B_R(0)} N(x-y) u_1(y) dy = \int_{B_1(0)} N(R\hat{x} - R\hat{y}) u_1(R\hat{y}) R^n d\hat{y} = \int_{B_1(0)} \hat{u}_1(\hat{y}) d\hat{y}$$

where $\hat{y} = y/R$, $\hat{x} = x/R$, $\hat{u}_1(\hat{y}) = R^n u_1(R\hat{y})$, and we use the fact that N is homogeneous of degree zero. Since $\hat{u}_1$ has total mass $1/2$ when integrated in the $\hat{y}$ variable, then $v \geq v_1$ and moreover $v_1(x) \geq C_1$ for $|x| > R$. Now we compute

$$\int_{R^n} u |N * u|^2 dx \geq \int_{x>R} u |N * u|^2 dx \geq C_1^2 \int_{x>R} u dx = \frac{1}{2} C_1^2.$$

In the above we use the pointwise lower bound for $N * u$ outside of $B_R(0)$.

To finish the proof of the proposition, we consider the general kernel K and smooth initial data of unit mass and support inside $B_\delta(0)$. Note that such data can be written as u_δ defined in (53). We write $\int_{R^n} u_\delta |\nabla K * u_\delta|^2 dx$ and rescale space as $\hat{x} = x/\delta$, $\hat{y} = y/\delta$. Thus

$$\nabla K * u_\delta(x) = \int_{R^n} \nabla K(x-y) \frac{1}{\delta^n} \rho(\frac{y}{\delta}) dy = \int_{R^n} \nabla K(x - \delta\hat{y}) \rho(\hat{y}) d\hat{y}.$$

And

$$\int_{R^n} u_\delta(x) |\nabla K * u_\delta(x)|^2 dx = \int_{R^n} \rho(\hat{x}) |\nabla K * u_\delta(\delta\hat{x})|^2 d\hat{x}$$

where

$$\nabla K * u_\delta(\delta\hat{x}) = \int_{B_1(0)} \nabla K(\delta\hat{x} - \delta\hat{y}) \rho(\hat{y}) d\hat{y}.$$

We now use the fact that $\nabla K(x) = -N(x) + S(x)$ where the vector field $S(x)$ is Lipschitz continuous. We use this decomposition to estimate the integral in (54) from below. We have

$$\int_{B_\delta(0)} |\nabla K * u_\delta|^2 u_\delta \geq \int_{B_\delta(0)} u_\delta |N * u_\delta|^2 - 2 \int_{B_\delta(0)} |N * u_\delta||S * u_\delta| u_\delta dx \geq C - 2 \int_{B_\delta(0)} |N * u_\delta||S * u_\delta| u_\delta dx. \tag{59}$$

In the above we use the pointwise lower bound for $|N_\delta|$ as described above.

It remains to show the the last term in the above expression is small for δ small.

$$\int_{B_\delta(0)} |N * u_\delta||S * u_\delta| u_\delta dx \leq |N * u_\delta|_{L^\infty} |S * u_\delta|_{L^\infty(B_\delta(0))}$$

where we assume u_δ has unit mass. Note that

$$|N * u_\delta|_{L^\infty} \leq |N|_{L^\infty} = 1.$$

Using the above change of variables, we have

$$S * u_\delta = \int_{B_1(0)} S(\delta\hat{x} - \delta\hat{y})\rho(\hat{y})d\hat{y} \leq \delta|S|_{Lip} \int_{B_1(0)} |\hat{x} - \hat{y}|\rho(\hat{y})d\hat{y} \leq \delta|S|_{Lip}.$$

Thus

$$|S * u_\delta|_{L^\infty(B_\delta(0))} \leq 2\delta|S|_{Lip}.$$

Here we use that $\sup_{B_1(0)\times B_1(0)} |\hat{x} - \hat{y}| = 2$ and ρ has unit mass. Thus the last term in (59) is bounded by $2\delta|S|_{Lip}$, where we use the fact that u_δ has mass one. Thus we have finished the proof of Proposition 3.

Putting things together, we have the following theorem.

Theorem 6 Blowup Theorem *Consider equation (1) on R^n for $n \geq 2$ with H^s initial data, radially symmetric, nonnegative, with compact support inside a ball of radius δ as defined by Proposition 3. Then there exists a finite time T^* and a unique solution $u(x,t) \in C([0,T^*); H^s) \cap C^1([0,T^*); H^{s-1})$ such that for all $q \geq 2$ ($q > 2$ for $n = 2$) $\sup_{0\leq s\leq t} |u(\cdot, s)|_{L^q} \to \infty$ as $t \to T^*$.*

Proof: Note that since u_0 has compact support and since the vector field $\nabla K * u$ points inward, the support of the solution u, by method of characteristics, can only shrink over time. Thus the solution at later times, which we know to exist for at least a finite time, has support contained inside the support of the initial data. The symmetry of the kernel and the equations (along with uniqueness of solutions) guarantees that the solution retains radial symmetry as long as it exists. Lastly, Theorem 5 guarantees that the L^1 norm is preserved and the solution remains nonnegative. Thus the solution will satisfy Proposition 3 as long as it remains in H^s, which implies a lower bound on the rate of increase of the energy. Since the energy has a finite upper bound for all L^1 functions, the only choice is for the solution to cease to exist as an H^s solution after some finite time. The continuation theorem of the previous section tells us that the L^q norm must blowup as t approaches the blowup time.

Note that for $s > n/2$ the solution is pointwise bounded up to the blowup time. Thus the initial blowup involves the solution going to infinity in finite time in such a way that the L^q norm blows up for all $q \geq 2$ ($q > 2$ for $n = 2$).

4 Discussion

We have proved that (1) on R^n, $n \geq 2$, with H^s initial data ($s \geq 2$), with sufficiently small compact support, and nonnegative and radially symmetric, has a unique solution that blows up in finite time. The blowup result involves a singularity in the $L^q(R^n)$ norm for $q \geq 2$, if $n \geq 3$, and $q > 2$ for $n = 2$. The blowup proof uses an energy estimate for the solution that makes careful use of the fact that ∇K has a discontinuity at the origin which approximates the homogeneous kernel (of degree zero) $N(x) = \vec{x}/|x|$. Without this discontinuity, in particular when ΔK is bounded, the finite time blowup result is false and one can obtain an a priori bound for the L^∞ norm of the solution. This bound was originally derived in [27] and global existence of smooth solutions with such kernels was established in [20]. This paper settles the question, in dimensions two and higher, of whether finite time blowup is possible for less regular kernels, in particular the commonly used biological kernel $K(x) = e^{-|x|}$.

However a number of open questions remain regarding the nature of the blowup. Bodnar and Velazquez [2] prove the shape of the blowup for the one-dimensional problem, by analogy to asymptotic theory for shock waves. In multi-dimensions, one very important question is whether the blowup results in a concentration of mass at the blowup time. The energy estimate is highly suggestive of this, but does not constitute a proof. In fact, many PDEs exist for which an energy argument proves blowup, yet the actual nature of the blowup is not as suggested by the energy, but rather involves a more subtle form of singularity at a time preceding the blowup time predicted by the a priori energy estimate. The rigorous results of this paper do not preclude the possibility of blowup happening at a point, with a singularity weaker than Dirac delta formation, but strongly enough to give L^q blowup as described above. The precise nature of the blowup could be important for modelling if one wants to include diffusion effects on small lengthscales, which would desingularize the singularity. To our knowledge, there are no careful computational results in dimension $n \geq 2$ that address blowup of these kinds of problems. We also mention that it would be interesting to know about local and global well-posedness of the problem for weaker classes of initial data, such as $L^\infty \cap L^1$, $L^p \cap L^1$, and the space of nonnegative measures.

Another point of further study is to link the discussed phenomena to the problem of dynamic models for aggregation (see e.g. [7, 11, 12, 14, 21, 26, 29]). Two recent papers [7, 11] show a connection between the scaling properties of dynamic discrete swarms and the notion of H-stability of the interaction potential (from classical statistical physics). Dynamic aggregation, defined by models in which the velocity of motion is not determined by a kinematic rule but rather by a separate momentum equation for the motion, exhibits much richer dynamics than its kinematic cousin. It would be interesting to know how the singular part of the potential affects solutions of dynamic swarms, in particular in regards to the scaling issues discussed in the recent literature.

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References

[1] J. T. Beale, T. Kato, and A. J. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94:61–66, 1984.

[2] M. Bodnar and J. J. L. Velazquez. An integro-differential equation arising as a limit of individual cell-based models. *J. Differential Equations*, 222(2):341–380, 2006.

[3] M. P. Brenner, P. Constantin, L. P. Kadanoff, A. Schenkel, and S. C. Venkataramani. Diffusion, attraction, and collapse. *Nonlinearity*, 12:1071–1098, 1999.

[4] M. Burger, V. Capasso, and D. Morale. On an aggregation model with long and short range interactions, 2003.

[5] Martin Burger and Marco Di Francesco. Large time behavior of nonlocal aggregation models with nonlinear diffusion, 2006.

[6] S. Chandrasekhar. *An introduction to the study of stellar structure*. Dover, New York, 1967.

[7] Y.-L. Chuang, M. R. D'Orsogna, D. Marthaler, A. L. Bertozzi, and L. Chayes. State transitions and the continuum limit for a 2D interacting, self-propelled particle system, 2006.

[8] P. Constantin, P. D. Lax, and A. Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Commun. Pure. Appl. Math.*, 38:715–724, 1985.

[9] P. Constantin, A. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. *Nonlinearity*, 7:1495–1533, 1994.

[10] Peter Constantin. Geometric statistics in turbulence. *SIAM Review*, 36(1):73–98, 1994.

[11] Maria R. D'Orsogna, Yao-Li Chuang, Andrea L. Bertozzi, and Lincoln Chayes. Self-propelled particles with soft-core interactions: patterns, stability, and collapse. *Phys. Rev. Lett.*, 96, 2006.

[12] W. Ebeling and U. Erdmann. Nonequilibrium statistical mechanics of swarms of driven particles. *Complexity*, 8:23–30, 2003.

[13] L. Edelstein-Keshet, J. Watmough, and D. Grunbaum. Do travelling band solutions describe cohesive swarms? An investigation for migratory locusts. *J. Math. Bio.*, 36:515–549, 1998.

[14] G. Flierl, D. Grünbaum, S. Levin, and D. Olson. From individuals to aggregations: the interplay between behavior and physics. *J. Theor. Biol.*, 196, 1999.

[15] Darryl D. Holm and Vakhtang Putkaradze. Aggregation of finite size particles with variable mobility. *Phys. Rev. Lett.*, 95:226106, 2005.

[16] Darryl D. Holm and Vakhtang Putkaradze. Clumps and patches in self-aggregation of finite size particles, 2006.

[17] Ning Ju. Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space. *Commun. Math. Phys.*, 251:365–376, 2004.

[18] E. F. Keller and L. A. Segel. *J. Theor. Biol.*, 26:399–415, 1970.

[19] Thomas Laurent. PhD thesis, Duke University, Department of Mathematics, 2006.

[20] Thomas Laurent. Local and global existence for an aggregation equation, 2006.

[21] H. Levine, W.J. Rappel, and I. Cohen. Self-organization in systems of self-propelled particles. *Phys. Rev. E*, 63, 2000.

[22] A. J. Majda and A. L. Bertozzi. *Vorticity and Incompressible Flow*. Cambridge University Press, 2002.

[23] A. Mogilner and L. Edelstein-Keshet. A non-local model for a swarm. *J. Math. Bio.*, 38:534–570, 1999.

[24] J. Parrish and L. Edelstein-Keshet. Complexity, pattern, and evolutionary trade-offs in animal aggregation. *Science*, 294:99–101, 1999.

[25] M. E. Taylor. *Partial Differential Equations*, volume 3. Springer-Verlag, New York, 1996. Nonlinear Equations.

[26] J. Toner and Y. Tu. Long-range order in a two-dimensional dynamical xy model: how birds fly together. *Phys. Rev. Lett.*, 75:4326–4329, 1995.

[27] Chad M. Topaz and Andrea L. Bertozzi. Swarming patterns in a two-dimensional kinematic model for biological groups. *SIAM J. Appl. Math.*, 65(1):152–174 (electronic), 2004.

[28] C.M. Topaz, A.L. Bertozzi, and M.A. Lewis. A nonlocal continuum model for biological aggregation. *Bulletin of Mathematical Biology*, 2006. published in electronic form.

[29] T. Vicsek, A. Czirk, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.*, 75:1226–1229, 1995.