Differential and Integral Equations Volume 16, Number 4, April 2003, Pages 427–452

FINITE TIME BLOW-UP OF THE SOLUTION FOR A NONLINEAR PARABOLIC EQUATION OF DRIFT-DIFFUSION TYPE

Masaki Kurokiba
Department of Applied Mathematics, Fukuoka University
Fukuoka, 814-0180, Japan

Takayoshi Ogawa
Graduate School of Mathematics, Kyushu University
Fukuoka 812-8581, Japan

(Submitted by: Y. Giga)

Dedicated to Professor Masayasu Mimura on the occasion of his sixties birthday

Abstract. We discuss the existence of the blow-up solution for some nonlinear parabolic system called attractive drift-diffusion equation in two space dimensions. We show that if the initial data satisfies a threshold condition, the corresponding solution blows up in a finite time. This is a system case for the blow-up result of the chemotactic equation proved by Nagai [28] and Nagai-Senba-Suzuki [30] and gravitational interaction of particles by Biler-Nadzieja [7], [8].

1. Introduction

As a model of self interacting particles, the drift-diffusion equation with attractive sign has been considered. In connection with the result of the Keller-Segel equation appearing in the chemotactic theory, the following equation is expected to show the instability of a solution, namely a finite time blow-up of solution:

$$\begin{cases} \partial_t n - \Delta n + \nabla(n\nabla\psi) = 0, & t>0, x\in\mathbb{R}^n, \\ \partial_t p - \Delta p - \nabla(p\nabla\psi) = 0, & t>0, x\in\mathbb{R}^n, \\ -\Delta\psi = \alpha(p-n)+g, \quad x\in\mathbb{R}^n, & \alpha=\pm1, \\ \quad n(0,x)=n_0(x), & p(0,x)=p_0(x), \end{cases} \tag{1.1}$$

where $n(t,x)$ and $p(t,x)$ denote the particle density of negative and positive electric charge, $g(x)$ is the background charge density and ψ denotes their electric potential.

Accepted for publication: May 2002.
AMS Subject Classifications: 35K15, 35K55, 35Q60; 78A35.

In our previous work [23], we have shown the local and global well-posedness for the drift-diffusion equation in L^p when $\alpha = 1$ with given functions f and g,

$$\begin{cases} \partial_t n - \Delta n + \nabla(n\nabla\psi) = f, \qquad t>0, x\in\mathbb{R}^n, \\ \partial_t p - \Delta p - \nabla(p\nabla\psi) = f, \qquad t>0, x\in\mathbb{R}^n, \\ -\Delta\psi = p-n-g, \quad x\in\mathbb{R}^n, \\ \qquad n(0,x)=n_0(x), \qquad p(0,x)=p_0(x). \end{cases} \tag{1.2}$$

The above equation ($\alpha = 1$) is the typical and simplest model for the semiconductor device simulation ([4] [11], [12], [14], [15], [20], [23], [24], [32]). On the other hand, when $\alpha = -1$ and $n = 2$, the equation is strongly relevant to the model of self-interacting particles with different mass (see Biler-Nadzieja [8], see also [3], [6]) and the model of the aggregation of mold. In view of the recent mathematical results on the Keller-Segel system [22], it is suggested that the solution that has a proper regularity (and integrability) may have an instability of the solution, namely under a certain condition on the initial data, the solution may blow up in a finite time. In the case of a single kind of self-interacting particles, it is also shown that the solution blows up in a finite time as one can find in [3], [6]. This property is naturally inherited to the system (1.1). The argument in the Keller-Segel model (cf. (1.10) below) systematically used that the solution is in $L^1(\mathbb{R}^2)$ ([25], [27], [28]), moreover if the solution is non-negative, $\|n(t)\|_1$ and $\|p(t)\|_1$ are preserved in time. To show the blow-up result for (1.1), we should consider the solution belonging to $L^1(\mathbb{R}^2)$. However, it is not quite obvious to find the solution in the space $L^1(\mathbb{R}^2)$ since the equation involves the singular integral operator and unfortunately, our previous result [23] (based on the method in [16] and [19]) does not cover the existence result in $L^1(\mathbb{R}^n)$. To avoid this difficulty, we introduce the weighted L^2 space which is defined as follows. For $s>0$,

$$L^2_s(\mathbb{R}^2) = \{f \in L^1_{loc}(\mathbb{R}^2); \langle x\rangle^s f(x) \in L^2(\mathbb{R}^2)\},$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. Noting that $L^2_s(\mathbb{R}^2)\subset L^1(\mathbb{R}^2)$ if $s>1$ (see Lemma 2.1 in Section 2), we first show the existence and uniqueness of the solution of the two dimensional drift-diffusion type system in a subset of $L^1(\mathbb{R}^2)$.

Theorem 1.1. (Local well-posedness) *For any $s>1$, let*

$$(n_0,p_0)\in L^2_s(\mathbb{R}^2)\times L^2_s(\mathbb{R}^2) \quad \textit{and} \quad \nabla(-\Delta)^{-1}g \in L^\infty(\mathbb{R}^2).$$

Then there exists $T = T(\|n_0\|_{L^2_s}, \|p_0\|_{L^2_s}) > 0$ and a unique solution (n,p) of (1.1) with the initial data (n_0,p_0) such that $n,p \in C([0,T); L^2_s(\mathbb{R}^2)) \cap L^2(0,T;\dot H^1(\mathbb{R}^2)) \cap C^1((0,T);L^2(\mathbb{R}^2)) \cap C((0,T);\ \dot H^2(\mathbb{R}^2))$. Moreover the *solution is continuously dependent on the initial data. Furthermore, if the maximum existence time of the solution is finite, i.e.,* $T_m < \infty$*, then we have*

$$\|n(t)\|_{L^2_s(\mathbb{R}^2)},\ \|p(t)\|_{L^2_s(\mathbb{R}^2)} \to \infty, \quad \text{as } t \to T_m. \tag{1.3}$$

Theorem 1.1 is shown by considering the equation for $v(t,x) = n(t,x) + p(t,x)$ and $w(t,x) = n(t,x) - p(t,x)$ which is as follows;

$$\begin{cases} \partial_t v - \Delta v + \nabla(w\nabla\psi) = 0, & t>0, x \in \mathbb{R}^2, \\ \partial_t w - \Delta w + \nabla(v\nabla\psi) = 0, & t>0, x\in\mathbb{R}^2, \\ -\Delta\psi = -\alpha w + g, \quad x \in \mathbb{R}^2, \\ \qquad v(0,x) = p_0(x) + n_0(x), & w(0,x) = n_0(x) - p_0(x), \end{cases} \tag{1.4}$$

provided (n,p) is the solution to (1.1).

The solution obtained in Theorem 1.1 belongs to $L^1(\mathbb{R}^2)$ and hence it follows that the solution maintains some conservation laws and meaningful identities. First of all, if the initial data is non-negative, then the maximum principle for the parabolic equation assures that a weak solution preserves non-negative structure. This fact immediately gives the $L^1(\mathbb{R}^2)$ conservation laws for the weak solution:

Proposition 1.2. (Positivity and L^1 preserving) *Suppose that the initial data of* (1.1) *satisfies* $n(0,x) \geq 0$, $p(0,x) \geq 0$. *Then for any solution in* $C([0,T);C^2(\mathbb{R}^2)) \times C([0,T);C^2(\mathbb{R}^2))$*, we have*

$$n(t,x) \geq 0, \quad p(t,x) \geq 0 \tag{1.5}$$

and if moreover $n_0, p_0 \in L^1(\mathbb{R}^2)$*, we have*

$$\|n(t)\|_1 = \|n_0\|_1, \quad \|p(t)\|_1 = \|p_0\|_1.$$

Second, we can find the inequality related to the entropy and the energy as follows:

Proposition 1.3. (A priori estimate) *Let* (n,p) *be a smooth solution to* (1.1) *in* $L^2(\mathbb{R}^2) \cap L\log L(\mathbb{R}^2)$*. Then*

(1) *If we set*

$$\begin{aligned} V(t) \equiv &\int_{\mathbb{R}^2} (1+n(t))\log(1+n(t))\,dx \\ &+ \int_{\mathbb{R}^2} (1+p(t))\log(1+p(t))\,dx + \frac{\alpha}{2}\|\nabla\psi(t)\|_2^2. \end{aligned} \tag{1.6}$$

Then it holds that

$$V(t) + \int_0^t \Big[\int_{\mathbb{R}^2} n(\tau)|\nabla(\log(1+n(\tau))) - \nabla\psi(\tau)|^2 dx$$

$$+\int_{\mathbb{R}^2} p(\tau)|\nabla(\log(1+p(\tau))) + \nabla\psi(\tau))|^2 dx\Big] d\tau \le CV(0)e^t. \qquad (1.7)$$

(2) *Similarly, we have*

$$\|n(t)\|_2^2 + \|p(t)\|_2^2 + 2\int_0^t \{\|\nabla n(\tau)\|_2^2 + \|\nabla p(\tau)\|_2^2\} d\tau \qquad (1.8)$$
$$+\alpha \int_0^t \int_{\mathbb{R}^2} (n(\tau)+p(\tau))|n(\tau)-p(\tau)|^2 dx d\tau \le (\|n_0\|_2^2 + \|p_0\|_2^2)e^{CT}.$$

According to those a priori estimates, we have a global existence result (cf. [23]) for a positive solution to the system (1.2) with any data if $\alpha = 1$.

Proposition 1.4. (Global existence) *Assume that* n_0 *and* p_0 *are nonnegative in* $L^2_s(\mathbb{R}^2)$ *with* $s > 1$. *If* $\alpha = 1$, *then the corresponding solution* (n, p) *obtained by Theorem* 1.1, *globally exists.*

In view of the results for the Keller-Segel model and self-interacting particles [7], [8], it is possible to show that when $\alpha = -1$, for some small constant $C_0 > 0$ and the initial data satisfies

$$\|n_0\|_1 + \|p_0\|_1 < C_0,$$

the corresponding solution exists globally in time. We will show this elsewhere [29].

On the other hand, when $\alpha = -1$ and the data is large enough, then the solution to (1.1) possibly blows up in a finite time. More specifically we have the following blow-up criterion.

Theorem 1.5. (Finite time blow-up) *For* $s > 1$, *let* n_0 *and* p_0 *be in* $L^2_s(\mathbb{R}^2)$ *with* $n_0, p_0 \ge 0$ *everywhere and satisfies*

$$\frac{\left(\int_{\mathbb{R}^2}(n_0 - p_0)dx\right)^2}{\int_{\mathbb{R}^2}(n_0 + p_0)dx} > 8\pi. \qquad (1.9)$$

Then the solution to (1.1) *with* $\alpha = -1$ *and* $g \equiv 0$ *blows up in a finite time.*

Remark. Theorem 1.5 states that if the difference of the total density of n_0 and p_0 is large enough, then the finite time blow-up occurs. Here the finite time blow-up stands for (1.3) in the local well posedness result. We expect that if the data satisfies the counter assumption, then the solution exists globally.

The drift-diffusion system equation has a strong analogy to the model of chemotactic attraction. The dynamics of the density of mucus is governed by the following system of parabolic equations called the Keller-Segel system (see Keller-Segel [22], Herrero-Velázquez [17],[18], Nagai [25], Nagai-Senba-Yoshida [31], Nagai-Senba-Suzuki [30] and Senba-Suzuki [37]),

$$\begin{cases} \partial_t u - \nu\Delta u + \chi\nabla(u\nabla\psi) = 0, \qquad t>0, x\in\Omega\subset\mathbb{R}^2, \\ \mu\partial_t\psi - \nu\Delta\psi + \gamma\psi = \alpha u, \qquad t>0, x\in\Omega, \\ \qquad u(0,x) = u_0(x), \qquad \psi(0,x) = \psi_0(x), \\ \qquad \frac{\partial u}{\partial n} = \frac{\partial \psi}{\partial n} = 0, \qquad x\in\partial\Omega, \end{cases} \tag{1.10}$$

where $\mu, \nu, \chi, \gamma, \alpha$ are positive constants. When the parameter $\mu = 0$, the system is elliptic-parabolic and very close looking to the drift-diffusion model. It is known that the existence of the blow-up solution corresponding to the concentration of mucus has been proved (Herrero-Velázquez [17], [18], Nagai [25]). However the appearance of positive constant $\gamma > 0$ makes the L^1 local well-posedness for (1.10) rather easier since it does not involve the singular integral operator. On the other hand for the global existence, it is observed that the solution blows up in a finite time in the case $\mu \geq 0$.

Theorem 1.5 shows that if the difference of the densities of initial data n_0 and p_0 are sufficiently large compare to the total density, then the solution cannot exists globally in time. Our result is an extension to the result on the equation of self-interacting particles. Biler-Nadzieja [7], [8], Biler-Hilhorst-Nadzieja [6] and Biler [3] showed that the solution of

$$\begin{cases} \partial_t u - \Delta u + \nabla(u\nabla\psi) = 0, \qquad t>0, x\in\Omega, \\ -\Delta\psi = u, \quad x\in\Omega, \\ \quad u(0,x) = u_0(x), \quad t=0, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.11}$$

blows up in a finite time provided the initial data satisfies a sufficient condition and Ω is either ball or radially symmetric. If we set either $n_0 \equiv 0$ or $p_0 \equiv 0$, then the condition (1.9) implies the L^1 of either of those initial data should be larger than 8π. This is also corresponding to the result for the model of chemotaxis.

To show the blow up of the solution, it is sufficient to derive a contradiction with all time existence since if the maximal existence time T_m is finite, then the solution must satisfy

$$\|n(t)\|_\infty \to \infty, \quad \|p(t)\|_\infty \to \infty \tag{1.12}$$

as $t \to T_m$. To show this result, we follow the argument found in Nagai [28] and Biler-Nadzieja [8] (see also Biler-Hilhorst-Nadzieja [6] and Biler [3]). We consider the equivalent system; we let $v(t) = n(t)+p(t)$ and $w(t) = n(t)-p(t)$

then (v,w) solves

$$\begin{cases} \partial_t v - \Delta v + \nabla(w\nabla\psi) = 0, \qquad t>0, x\in\mathbb{R}^2, \\ \partial_t w - \Delta w + \nabla(v\nabla\psi) = 0, \qquad t>0, x\in\mathbb{R}^2, \\ -\Delta\psi = w, \quad x\in\mathbb{R}^2, \\ \qquad v(0,x) = n_0(x)+p_0(x), \qquad v(0,x) = n_0(x)-p_0(x). \end{cases} \tag{1.13}$$

Then we show that the second order localized moment of v, namely

$$\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx,$$

where $\Phi(x) \simeq |x|^2$ around $|x|\simeq 0$ and $|\Phi(x)|\le C$ satisfies some differential inequality. This differential inequality shows that the moment reaches zero in a finite time. This contradicts the fact that the solution preserves the positivity and hence the moment itself, provided the initial data is nonnegative.

2. Preliminary lemmas

Before proving the existence theorem, we show the auxiliary inequalities which will be used in the following sections.

Lemma 2.1. *For $s>1$ and $f\in L^2(\mathbb{R}^2)\cap L^2_s(\mathbb{R}^2)$, there exists a constant $C=C(s)>0$ such that*

$$\|f\|_1 \le C\|f\|_2^{1-1/s}\||x|^s f\|_2^{1/s}. \tag{2.1}$$

Proof. It suffices to show the lemma for $f\in C_0^\infty(\mathbb{R}^2)$. For $R>0$, we have by the Hölder inequality that

$$\int_{\mathbb{R}^2}|f(x)|dx = \int_{B_R(0)}|f(x)|dx + \int_{B_R^c(0)}|f(x)|dx \tag{2.2}$$

$$\le |B_R|^{\frac12}\Big(\int_{B_R(0)}|f(x)|^2dx\Big)^{\frac12} + \Big(\int_{B_R^c(0)}|x|^{-2s}dx\Big)^{\frac12}\Big(\int_{B_R^c(0)}||x|^s f(x)|^2dx\Big)^{\frac12}$$

$$\le \pi^{1/2}R\|f\|_2 + \big(\frac{2\pi}{2s-2}\big)^{1/2}R^{1-s}\||x|^2 f\|_2.$$

By choosing $R=(\frac{\||x|^s f\|_2}{\|f\|_2})^{1/s}$, we conclude that

$$\|f\|_1 \le C\|f\|_2^{1-\frac1s}\||x|^s f\|_2^{\frac1s}. \qquad \square$$

The following Sobolev type inequality is a variant of the Brezis-Gallouet inequality [9].

Lemma 2.2. *For $f \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, we have*

$$\|\nabla(-\Delta)^{-1} f\|_\infty \le C\|f\|_2 \Big\{ \log \Big(e + \frac{\|\nabla f\|_2 + \|f\|_1}{\|f\|_2} \Big) \Big\}^{1/2}. \tag{2.3}$$

Especially,

$$\|\nabla(-\Delta)^{-1} f\|_\infty \le C \left\{ 1 + \|f\|_2 \left(\log \left(\|\nabla f\|_2 + \|f\|_1 \right) \right)^{1/2} \right\}. \tag{2.4}$$

Remark. If we apply the original Brezis-Gallouet inequality to $\nabla(-\Delta)^{-1} f \in H^1(\mathbb{R}^2)$, we have

$$\|\nabla(-\Delta)^{-1} f\|_\infty \le C \left(1 + \|\nabla(-\Delta)^{-1} f\|_2 + \|f\|_2 \left(\log(e + \|\nabla f\|_2) \right)^{1/2} \right).$$

However, simple use of the Sobolev inequality $\|g\|_2 \le C\|\nabla g\|_1$ does not imply the inequality (2.3).

Proof of Lemma 2.2. Let $\dot{B}^0_{p,\sigma}(\mathbb{R}^n)$ be the homogeneous Besov space defined by the following norm;

$$\|f\|_{\dot{B}^0_{p,\sigma}(\mathbb{R}^n)} \equiv \begin{cases} \Big(\sum_{j=-\infty}^{\infty} 2^{js\sigma} \|\phi_j * f\|_p^\sigma \Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \sup_{j \in \mathbb{N}} 2^{js} \|\phi_j * f\|_p, & \sigma = \infty, \end{cases}$$

where $\{\phi_j\}_{j\in\mathbb{N}}$ is the Littlewood-Paley dyadic decomposition of unity such that $\widehat{\phi}_j(\xi) = \hat{\phi}(2^{-j}|\xi|)$ with

$$\hat{\phi}(\xi) = \begin{cases} \text{smooth, positive}, & \xi \in B_2 \setminus \bar{B}_{1/2}, \\ 0, & \xi \in B_2^c \cup \bar{B}_{1/2} \end{cases}$$

and $\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) \equiv 1$ $(\xi \ne 0)$ (see for the details, Bergh-Löfström [1] and Triebel [39]). Let $\psi(x)$ be a smooth function such that

$$\hat{\psi}(\xi) = \begin{cases} 1 & |\xi| \le 1/2 \\ 0 & |\xi| \ge 1 \end{cases}$$

and set $\widehat{\psi}_j(\xi) = \widehat{\psi}(\xi/2^j)$. Then by the L^1-L^∞ boundedness of the inverse Fourier transform, we have

$$\begin{aligned} \|\psi_{-M} * \nabla(-\Delta)^{-1} f\|_\infty &\le C \Big\| \widehat{\psi}_{-M} \frac{i\xi \hat{f}}{|\xi|^2} \Big\|_1 \le C \int_{B_{2^{-M}}} \frac{|\hat{f}(\xi)|}{|\xi|} d\xi \\ &\le C\|\hat{f}\|_\infty \int_{B_{2^{-M}}} |\xi|^{-1} d\xi \le C 2^{-M} \|f\|_1 \to 0 \end{aligned} \tag{2.5}$$

as $M \to \infty$. Hence, for sufficiently large M, we have

$$\|\psi_{-M} * \nabla(-\Delta)^{-1} f\|_\infty \le \|f\|_{\dot{B}^1_{2,2}}$$

and it suffices to estimate

$$\sum_{j \ge -M} \phi_j * \nabla(-\Delta)^{-1} f.$$

By the inequality shown by Ogawa-Taniuchi [33], we have

$$\|g\|_{\dot{B}^0_{\infty,1}} \le C\|g\|_{\dot{B}^1_{2,2}} \Big\{ \log\Big(e + \frac{\|g\|_{\dot{B}^2_{2,2}} + \|g\|_{\dot{B}^0_{2,\infty}}}{\|g\|_{\dot{B}^1_{2,2}}} \Big) \Big\}^{1/2} \tag{2.6}$$

for $g \in \dot{B}^2_{2,2} \cap \dot{B}^0_{2,\infty}$. For the lemma we apply the above inequality to $\nabla(-\Delta)^{-1} f$. We have $\|\nabla(-\Delta)^{-1} f\|_{\dot{B}^1_{2,2}} \le C\|f\|_2$ and since $n = 2$ and

$$\|\nabla(-\Delta)^{-1} f\|_{\dot{B}^0_{2,\infty}} \le \|\nabla(-\Delta)^{-1} f\|_{\dot{B}^1_{1,\infty}} \le \|f\|_{\dot{B}^0_{1,\infty}}. \tag{2.7}$$

Note that the above inequality holds for the singular integral operator, since we employ the homogeneous Besov norm. By (2.7) and Lemma 2.1 and $\|\phi_j * f\|_1 \le C(\phi)\|f\|_1$,

$$\begin{aligned}
\|\nabla(-\Delta)^{-1} f\|_\infty &\le \|\psi_{-M} * \nabla(-\Delta)^{-1} f\|_\infty + \sum_{j=-M}^{\infty} \|\phi_j * \nabla(-\Delta)^{-1} f\|_\infty \\
&\le C\|f\|_2 \Big\{ \log\Big(e + \frac{\|\nabla f\|_2 + \|\nabla^2(-\Delta)^{-1} f\|_{\dot{B}^0_{1,\infty}}}{\|f\|_{\dot{B}^0_{2,2}}} \Big) \Big\}^{1/2} \\
&\le C\|f\|_2 \Big\{ \log\Big(e + \frac{\|\nabla f\|_2 + \|f\|_{\dot{B}^0_{1,\infty}}}{\|f\|_2} \Big) \Big\}^{1/2} \\
&\le C\|f\|_2 \Big\{ \log\Big(e + \frac{\|\nabla f\|_2 + \|f\|_1}{\|f\|_2} \Big) \Big\}^{1/2}.
\end{aligned}$$

The inequality (2.4) can be obtained by a simple modification of (2.3). □

3. Proof of existence

Proof of Theorem 1.1. Let $s > 1$. For the given initial data $(u_0, v_0) \in L^2_s \times L^2_s$, we choose $M^2 = 4(\|\langle x\rangle^s u_0\|_2^2 + \|\langle x\rangle^s v_0\|_2^2) \vee (\|f\|^2_{L^2(I;L^2_s)} + G^2)$ and $G = \|\nabla(-\Delta)^{-1} g\|_\infty$. The solution can be constructed in the complete metric space

$$X_T = \Big\{ (\phi, \psi) \in \big\{ C([0,T); L^2_s) \cap L^2(0,T; \dot{H}^1) \big\}^2; \|\phi\|_X^2 + \|\psi\|_X^2 \le M^2 \Big\},$$

where

$$\|\phi\|_X \equiv \Big(\sup_{t\in[0,T)}(\|\phi(t)\|_2^2+\||x|^s\phi(t)\|_2^2)+\int_0^T(\|\nabla\phi(\tau)\|_2^2+\||x|^s\nabla\phi(\tau)\|_2^2)d\tau\Big)^{\frac{1}{2}}.$$

For $\tilde{w}\in X_T$, define a map $\Phi:\tilde{w}\to v$, $\Psi:\tilde{w}\to w$ such that (v,w) solve the following linearized system:

$$\begin{cases} \partial_t v-\Delta v+\nabla(w\nabla\bar{\psi})=0, & t>0, x\in\mathbb{R}^2,\\ \partial_t w-\Delta w+\nabla(v\nabla\bar{\psi})=0, & t>0, x\in\mathbb{R}^2,\\ -\Delta\bar{\psi}=-\alpha\tilde{w}+g, \quad x\in\mathbb{R}^2, \\ \quad v(0,x)=v_0(x)\equiv p_0(x)+n_0(x), \\ \quad w(0,x)=w_0(x)\equiv n_0(x)-p_0(x). \end{cases} \tag{3.1}$$

The following proposition shows that the map (Φ,Ψ) is a contraction in X_T.

Proposition 3.1. *Under the same condition as Theorem* 1.1, *there exist a small constant* $T>0$ *and* $(u,v)\in X_T$ *such that*

$$\|\Phi(\tilde{w})\|_X+\|\Psi(\tilde{w})\|_X\le M, \tag{3.2}$$

$$\|\Phi(\tilde{w}_1)-\Phi(\tilde{w}_2)\|_X+\|\Psi(\tilde{w}_1)-\Psi(\tilde{w}_2)\|_X\le\frac{1}{2}\|\tilde{w}_1-\tilde{w}_2\|_X. \tag{3.3}$$

Proof of Proposition 3.1. Let $\bar{w}\in X_T$ and we construct a solution to the following linearized parabolic system:

$$\begin{cases} \partial_t v-\Delta v+\nabla(w\nabla(-\Delta)^{-1}(-\alpha\tilde{w}+g))=0, & t>0, x\in\mathbb{R}^2,\\ \partial_t w-\Delta w+\nabla(v\nabla(-\Delta)^{-1}(-\alpha\bar{w}+g))=0, & t>0, x\in\mathbb{R}^2,\\ \quad v(0,x)=v_0(x), \qquad w(0,x)=w_0(x). \end{cases} \tag{3.4}$$

The existence of a smooth solution of the above system is well known. Now by multiplying the first equation of (1.4) by v and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2+\|\nabla v(t)\|_2^2-\int_{\mathbb{R}^2}w(t)\nabla v(t)\cdot\nabla(-\Delta)^{-1}(-\alpha\tilde{w}(t)+g)dx=0. \tag{3.5}$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2+\|\nabla w(t)\|_2^2-\int_{\mathbb{R}^2}v(t)\nabla w(t)\cdot\nabla(-\Delta)^{-1}(-\alpha\tilde{w}(t)+g)dx=0. \tag{3.6}$$

Adding (3.5) to (3.6) and integrating it over $[0,T]$, we have

$$\|v(t)\|_2^2+\|w(t)\|_2^2+2\int_0^t\{\|\nabla v(\tau)\|_2^2+\|\nabla w(\tau)\|_2^2\}d\tau \tag{3.7}$$

$$= \|v_0\|_2^2 + \|w_0\|_2^2 + 2\int_0^t \int_{\mathbb{R}^2} \nabla(v(\tau)w(\tau))\nabla(-\Delta)^{-1}(\alpha\tilde{w}(\tau) - g)dxd\tau.$$

By Lemma 2.1 and the Brezis-Gallouet type inequality (2.4) in Lemma 2.2,

$$\begin{aligned} &\Big|\int_{\mathbb{R}^2} \nabla(vw)\cdot\nabla(-\Delta)^{-1}(\alpha\tilde{w}+g)dx\Big| \qquad (3.8)\\ &\le (\|\nabla v\|_2\|w\|_2 + \|\nabla w\|_2\|v\|_2)\left(\|\nabla(-\Delta)^{-1}\tilde{w}\|_\infty + \|\nabla(-\Delta)^{-1}g\|_\infty\right)\\ &\le C(\|\nabla v\|_2\|w\|_2 + \|\nabla w\|_2\|v\|_2)\\ &\times\left(1 + \|\tilde{w}\|_2\sqrt{\log(e + \|\nabla\tilde{w}\|_2 + \|\langle x\rangle^s\tilde{w}\|_2)} + G\right).\end{aligned}$$

For some constant $C > 0$, we note that $\sqrt{\log(x+e)} \le C(1 + x^{1/2})$, we have,

$$\begin{aligned} &\|v(t)\|_2^2 + \|w(t)\|_2^2 + 2\int_0^t \left(\|\nabla v(\tau)\|_2^2 + \|\nabla w(\tau)\|_2^2\right)d\tau \le \|v_0\|_2^2 + \|w_0\|_2^2\\ &+ C(1+G)\sup_{\tau\in[0,T]}(\|v(\tau)\|_2 + \|w(\tau)\|_2)\int_0^t (\|\nabla v(\tau)\|_2 + \|\nabla w(\tau)\|_2)\,d\tau\\ &+ C\sup_{\tau\in[0,T]}(\|v(\tau)\|_2 + \|w(\tau)\|_2) \qquad (3.9)\\ &\quad\times\int_0^t (\|\nabla v(\tau)\|_2 + \|\nabla w(\tau)\|_2)\,\|\tilde{w}\|_2\left(\|\nabla\tilde{w}(\tau)\|_2^{1/2} + \|\langle x\rangle^s\tilde{w}\|_2^{1/2}\right)d\tau\\ \le&\|v_0\|_2^2 + \|w_0\|_2^2\\ &+ C(1+G)\sup_{\tau\in[0,T]}(\|v(\tau)\|_2 + \|w(\tau)\|_2)\int_0^T (\|\nabla v(\tau)\|_2 + \|\nabla w(\tau)\|_2)\,d\tau\\ &+ CM\sup_{\tau\in[0,T]}(\|v(\tau)\|_2 + \|w(\tau)\|_2)\\ &\quad\times\left(\int_0^T\left(\|\nabla v(\tau)\|_2^{\frac{4}{3}} + \|\nabla w(\tau)\|_2^{\frac{4}{3}}\right)d\tau\right)^{\frac{4}{3}}\left(\int_0^T\left(\|\nabla\tilde{w}(\tau)\|_2^2 + \|\langle x\rangle^s\tilde{w}\|_2^2\right)d\tau\right)^{\frac{1}{4}}\\ \le&\|v_0\|_2^2 + \|w_0\|_2^2\\ &+ \varepsilon C((1+G)^2 + (1+T^{1/2})M^3)\sup_{\tau\in[0,T]}(\|v(\tau)\|_2^2 + \|w(\tau)\|_2^2)\\ &+ (\varepsilon^{-1}T + T^{1/3})\int_0^T\left(\|\nabla v(\tau)\|_2^2 + \|\nabla w(\tau)\|_2^2\right)d\tau.\end{aligned}$$

Without loosing generality, we may assume that $T < 1$. Then choose $\varepsilon > 0$ small such that

$$\varepsilon C((1+G)^2 + 2M^3) < \frac{1}{2}, \qquad (3.10)$$

then we take $T < 1$ small such that

$$\varepsilon^{-1}T + T^{1/3} < 1 \qquad (3.11)$$

if necessary. Then we conclude that

$$\sup_t\{\|v(t)\|_2^2+\|w(t)\|_2^2\}+\int_0^T\left(\|\nabla v(\tau)\|_2^2+\|\nabla w(\tau)\|_2^2\right)d\tau$$
$$\leq 2(\|v_0\|_2^2+\|w_0\|_2^2)\leq \frac{1}{2}M^2. \tag{3.12}$$

Next we derive the weighted estimates. Multiplying the first equation of (1.4) by $|x|^{2s}v$ and integrating by parts, we have by setting $\tilde{\psi}=(-\Delta)^{-1}(-\alpha\tilde{w}+g)$,

$$\frac{1}{2}\frac{d}{dt}\||x|^s v(t)\|_2^2+\||x|^s\nabla v(t)\|_2^2+2s\int_{\mathbb{R}^2}|x|^{2s-2}x\cdot\nabla v(t)v(t)dx \tag{3.13}$$
$$-\int_{\mathbb{R}^2}|x|^{2s}w(t)\nabla v(t)\cdot\nabla\bar{\psi}(t)dx-2s\int_{\mathbb{R}^2}|x|^{2s-2}w(t)v(t)x\cdot\nabla\bar{\psi}(t)dx=0.$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\||x|^s w(t)\|_2^2+\||x|^s\nabla w(t)\|_2^2+2s\int_{\mathbb{R}^2}|x|^{2s-2}x\cdot\nabla w(t)w(t)dx \tag{3.14}$$
$$-\int_{\mathbb{R}^2}|x|^{2s}v(t)\nabla w(t)\cdot\nabla\bar{\psi}(t)dx-2s\int_{\mathbb{R}^2}|x|^{2s-2}w(t)v(t)x\cdot\nabla\bar{\psi}(t)dx=0.$$

Summing up (3.13) and (3.14)

$$\frac{1}{2}\frac{d}{dt}\{\||x|^s v(t)\|_2^2+\||x|^s w(t)\|_2^2\}+\||x|^s\nabla v(t)\|_2^2+\||x|^s\nabla w(t)\|_2^2$$
$$=-s\int_{\mathbb{R}^2}|x|^{2s-2}x\cdot\nabla(v^2(t)+w^2(t))dx \tag{3.15}$$
$$+\int_{\mathbb{R}^2}|x|^{2s}\nabla(vw)(t)\cdot\nabla(-\Delta)^{-1}(-\alpha\tilde{w}(t)+g)dx$$
$$+4s\int_{\mathbb{R}^2}|x|^{2s-2}v(t)w(t)x\cdot\nabla(-\Delta)^{-1}(-\alpha\tilde{w}(t)+g)dx\equiv I+II+III.$$

The first term on the right hand side of (3.15) is bounded as follows:

$$|I|\leq C\int_{\mathbb{R}^2}|x|^{2s-2}(v^2(t)+w^2(t))dx$$
$$\leq C(\|v(t)\|_2^2+\|w(t)\|_2^2)+(\||x|^s v(t)\|_2^2+\||x|^s w(t)\|_2^2).$$

And also

$$|II+III|\leq(\||x|^s v(t)\|_2\||x|^s\nabla w(t)\|_2+\||x|^s w(t)\|_2\||x|^s\nabla v(t)\|_2)$$
$$\times\left(\|\nabla(-\Delta)^{-1}\tilde{w}(t)\|_\infty+\|\nabla(-\Delta)^{-1}g\|_\infty\right)$$
$$+\||x|^s v(t)\|_2\||x|^{s-1}w(t)\|_2\left(\|\nabla(-\Delta)^{-1}\tilde{w}(t)\|_\infty+\|\nabla(-\Delta)^{-1}g\|_\infty\right)$$

$$\leq \Big(\||x|^s v(t)\|_2 \||x|^s \nabla w(t)\|_2 + \||x|^s w(t)\|_2 \||x|^s \nabla v(t)\|_2$$
$$+ \||x|^s v(t)\|_2 \||x|^{s-1} w(t)\|_2\Big)$$
$$\times \Big(1 + \|\tilde{w}(t)\|_2 \sqrt{\log(e + \|\nabla \tilde{w}(t)\|_2 + \|\langle x\rangle^s \tilde{w}(t)\|_2)} + G\Big).$$

Thus, we have

$$\||x|^s v(t)\|_2^2 + \||x|^s w(t)\|_2^2 + 2\int_0^t \{\||x|^s \nabla v(\tau)\|_2^2 + \||x|^s \nabla w(\tau)\|_2^2\} d\tau \tag{3.16}$$
$$\leq \||x|^s v_0\|_2^2 + \||x|^s w_0\|_2^2 + CT \sup_{t\in[0,T]} (M^2 + \||x|^s v(t)\|_2^2 + \||x|^s w(t)\|_2^2)$$
$$+ 2\int_0^t \Big(\||x|^s v(\tau)\|_2 \||x|^s \nabla w(\tau)\|_2 + \||x|^s w(\tau)\|_2 \||x|^s \nabla v(\tau)\|_2$$
$$+ \||x|^s v(\tau)\|_2 \||x|^{s-1} w(\tau)\|_2\Big)$$
$$\times \Big(1 + G + \|\tilde{w}\|_2 \sqrt{\log(e + \|\nabla \tilde{w}(\tau)\|_2 + \|\langle x\rangle^s \tilde{w}(\tau)\|_2)}\Big) d\tau.$$

We continue the estimates as in (3.9), then

$$\||x|^s v(t)\|_2^2 + \||x|^s w(t)\|_2^2 + 2\int_0^t \{\||x|^s \nabla v(\tau)\|_2^2 + \||x|^s \nabla w(\tau)\|_2^2\} d\tau \tag{3.17}$$
$$\leq \||x|^s v_0\|_2^2 + \||x|^s w_0\|_2^2 + CT(M^2 + \||x|^s v(t)\|_2^2 + \||x|^s w(t)\|_2^2)$$
$$+ \varepsilon C((1+G)^2 + 2M^3) \sup_{\tau\in[0,T]} \big(\||x|^s v(\tau)\|_2^2 + \||x|^s v(\tau)\|_2^2\big)$$
$$+ \Big(\varepsilon^{-1} T + T^{1/3}\Big) \varepsilon \int_0^t (\||x|^s \nabla v(\tau)\|_2 + \||x|^s \nabla w(\tau)\|_2)\, d\tau.$$

Again by choosing $\varepsilon > 0$ and T as in (3.10) and (3.11) with $CT(1+M) < \frac{1}{2}$, we have

$$\||x|^s v(t)\|_2^2 + \||x|^s w(t)\|_2^2 + 2\int_0^T \{\||x|^s \nabla v(t)\|_2^2 + \||x|^s \nabla w(t)\|_2^2\} dt \leq M^2. \tag{3.18}$$

Combining (3.12) and (3.18), we obtain (3.2).

Next for $\tilde{w}_1$ and $\tilde{w}_2$, we consider the solution of equations with $\tilde{w} = \tilde{w}_i$ in (1.4). Let (v_i, w_i) $(i = 1, 2)$ be the pair of the corresponding solution. We define the differences $V(t) = v_1(t) - v_2(t)$, $W(t) = w_1(t) - w_2(t)$ and $\tilde{W}(t) = \tilde{w}_1(t) - \tilde{w}_2(t)$. Then by a similar argument as in (3.8),

$$\frac{d}{dt}\{\|V(t)\|_2^2 + \|W(t)\|_2^2\} + 2\|\nabla V(t)\|_2^2 + \|\nabla W(t)\|_2^2 \tag{3.19}$$

$$\leq \int_{\mathbb{R}^2} \nabla(VW)(t)\cdot\nabla(-\Delta)^{-1}(-\alpha\tilde{w}_1(t)+g)dx$$
$$+\int_{\mathbb{R}^2}(w_2(t)\nabla V(t)+v_2(t)\nabla W(t))\cdot\nabla(-\Delta)^{-1}\tilde{W}(t)dx \equiv \quad I_1(t)+I_2(t).$$

The nonlinear term can be dominated in a similar way. Noting that $\sqrt{\log x} \leq (\tilde{\varepsilon}^{-1}+\tilde{\varepsilon}x)$ for any small $\tilde{\varepsilon}>0$ and Lemma 2.2 (2.4), we have for the time integral of the right hand side in (3.19):

$$\Big|\int_0^t I_1(\tau)d\tau\Big| \leq \int_0^t \Big|\int_{\mathbb{R}^2}\nabla(VW)(\tau)\cdot\nabla(-\Delta)^{-1}(-\tilde{w}_1(\tau)-g)dx\Big|d\tau$$
$$\leq \int_0^t (\|V(\tau)\|_2\|\nabla W(\tau)\|_2+\|W(\tau)\|_2\|\nabla V(\tau)\|_2)$$
$$\times\left(\|\nabla(-\Delta)^{-1}\tilde{w}_1(\tau)\|_\infty+G\right)d\tau$$
$$\leq C\sup_{t\in[0,T]}(\|V(t)\|_2+\|W(t)\|_2)\int_0^t(\|\nabla W(\tau)\|_2+\|\nabla V(\tau)\|_2)$$
$$\times\left(1+\|\tilde{w}\|_2\sqrt{\log\left(e+\|\nabla\tilde{w}_1(\tau)\|_2+\|\langle x\rangle^s\tilde{w}_1(\tau)\|_2\right)}+G\right)d\tau$$
$$\leq C\sup_{t\in[0,T]}(\|V(t)\|_2+\|W(t)\|_2)$$
$$\times\Big\{(1+\tilde{\varepsilon}^{-1}M+G)\int_0^t(\|\nabla W(\tau)\|_2+\|\nabla V(\tau)\|_2)\,dt$$
$$+\tilde{\varepsilon}M\int_0^t(\|\nabla W(\tau)\|_2+\|\nabla V(\tau)\|_2)\,(\|\nabla\tilde{w}_1(\tau)\|_2+\|\langle x\rangle^s\tilde{w}_1(\tau)\|_2)d\tau\Big\}.$$

By Young's inequality, it follows for small $\varepsilon_1>0$,

$$\Big|\int_0^t I_1(t)dt\Big| \leq C\varepsilon_1\sup_{t\in[0,T]}\left(\|V(t)\|_2^2+\|W(t)\|_2^2\right) \tag{3.20}$$
$$+(1+\tilde{\varepsilon}^{-1}M+G)\varepsilon_1^{-1}T\int_0^t\left(\|\nabla W(\tau)\|_2^2+\|\nabla V(\tau)\|_2^2\right)d\tau$$
$$+\varepsilon_1^{-1}\tilde{\varepsilon}(M+TM)\int_0^t\left(\|\nabla W(\tau)\|_2^2+\|\nabla V(\tau)\|_2^2\right)d\tau$$

and from (2.3), for $\varepsilon_2>0$,

$$\Big|\int_0^t I_2(t)dt\Big| \leq C\int_0^t(\|w_2(\tau)\|_2\|\nabla V(\tau)\|_2+\|v_2(\tau)\|_2\|\nabla W(\tau)\|_2)$$
$$\times\|\tilde{W}(\tau)\|_2\sqrt{\log\left(e+\frac{\|\nabla\tilde{W}(\tau)\|_2+\|\langle x\rangle^s\tilde{W}(\tau)\|_2}{\|\tilde{W}(\tau)\|_2}\right)}dt$$

$$
\begin{aligned}
&\quad (\text{by noting } \log(e+x) \le (\tilde\varepsilon^{-1} + \tilde\varepsilon x)^2) \\
&\le \tilde\varepsilon^{-1} CM \int_0^t (\|\nabla V(\tau)\|_2 + \|\nabla W(\tau)\|_2)\, \|\tilde W(\tau)\|_2 d\tau \\
&+ \tilde\varepsilon CM \int_0^t (\|\nabla V(\tau)\|_2 + \|\nabla W(\tau)\|_2) \Big(\|\nabla \tilde W(\tau)\|_2 + \|\langle x\rangle^s \tilde W(\tau)\|_2\Big)\, d\tau \\
&\le \tilde\varepsilon^{-1} CMT^{1/2} \sup_{t\in[0,T]} \|\tilde W(t)\|_2 \left(\int_0^t \left(\|\nabla V(\tau)\|_2^2 + \|\nabla W(\tau)\|_2^2\right) dt\right)^{1/2} \\
&\le \tilde\varepsilon^{-1} CM^2 T^{1/2} \sup_{t\in[0,T]} \|\tilde W(t)\|_2^2 \\
&+ \tilde\varepsilon CM^2 \int_0^t \Big(\|\nabla \tilde W(\tau)\|_2^2 + \|\langle x\rangle^s \tilde W(\tau)\|_2^2\Big)\, dt. \qquad (3.21)
\end{aligned}
$$

Gathering (3.20) and (3.21) we have by integrating (3.19) over $[0,T]$ that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left(\|V(t)\|_2^2 + \|W(t)\|_2^2\right) + 2\int_0^T \left(\|\nabla V(t)\|_2^2 + \|\nabla W(t)\|_2^2\right) dt \\
&\le C\varepsilon_1 \sup_{t\in[0,T]} \left(\|V(t)\|_2^2 + \|W(t)\|_2^2\right) \\
&+ \left((1+\tilde\varepsilon^{-1}M + G)\varepsilon_1^{-1} T + \tilde\varepsilon M(1+T)\varepsilon_1^{-1}\right) \int_0^T \left(\|\nabla W(t)\|_2^2 + \|\nabla V(t)\|_2^2\right) dt \\
&+ \tilde\varepsilon^{-1} CM^2 T^{1/2} \sup_{t\in[0,T]} \|\tilde W(t)\|_2^2 + \tilde\varepsilon CM^2 \int_0^T \Big(\|\nabla \tilde W(t)\|_2^2 + \|\langle x\rangle^s \tilde W(t)\|_2^2\Big)\, dt.
\end{aligned}
$$

Then choosing ε_1, ε_2 and $\tilde\varepsilon$ small such that $C\varepsilon_1 < \frac12$, $2\tilde\varepsilon M\varepsilon_1^{-1} < \frac12$, $\tilde\varepsilon CM^2 < \frac{1}{32}$, and taking T small enough such that

$$
(1+\tilde\varepsilon^{-1}M + G)\varepsilon_1^{-1} T < \frac12, \quad \tilde\varepsilon^{-1} CM^2 T^{1/2} < \frac{1}{32},
$$

we conclude that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left(\|V(t)\|_2^2 + \|W(t)\|_2^2\right) + \int_0^T \left(\|\nabla V(t)\|_2^2 + \|\nabla W(t)\|_2^2\right) dt \qquad (3.22) \\
&\le \frac{1}{16}\Big(\sup_{t\in[0,T]} \left(\|\tilde W(t)\|_2^2 + \|\langle x\rangle^s \tilde W(t)\|_2^2\right) + \int_0^t \|\nabla \tilde W(t)\|_2^2 dt\Big).
\end{aligned}
$$

Similarly, to (3.5) and (3.22), we have

$$
\frac12 \frac{d}{dt}\{\||x|^s V(t)\|_2^2 + \||x|^s W(t)\|_2^2\} + \||x|^s \nabla V(t)\|_2^2 + \||x|^s \nabla W(t)\|_2^2
$$

$$
\begin{aligned}
&= 2s^2 \int_{\mathbb{R}^2} |x|^{2s-2}(V^2+W^2)dx \\
&+ \int_{\mathbb{R}^2} |x|^{2s}(W(t)\nabla V(t) + W(t)\nabla W(t)) \cdot \nabla(-\Delta)^{-1}(-\alpha \tilde{w}_1(t) + g)dx \\
&+ 4s \int_{\mathbb{R}^2} |x|^{2s-2} W(t)V(t)x \cdot \nabla(-\Delta)^{-1}(-\alpha \tilde{w}_1(t) + g)dx \\
&+ \alpha \int_{\mathbb{R}^2} |x|^{2s}(w_2(t)\nabla V(t) + v_2(t)\nabla W(t)) \cdot \nabla(-\Delta)^{-1}\tilde{W}(t)dx \\
&+ 2s\alpha \int_{\mathbb{R}^2} |x|^{2s-2}(w_2(t)V(t) + v_2(t)W(t))x \cdot \nabla(-\Delta)^{-1}\tilde{W}(t)dx.
\end{aligned}
$$

By a similar argument, we have

$$
\begin{aligned}
&\frac{d}{dt}\{\||x|^s V(t)\|_2^2 + \||x|^s W(t)\|_2^2\} + 2\||x|^s \nabla V\|_2^2 + \||x|^s \nabla W\|_2^2 \qquad (3.23)\\
&\le 4s^2 \left(\||x|^{s-1}V(t)\|_2^2 + \||x|^{s-1}W(t)\|_2^2\right) + 2\Big(\||x|^s V(t)\|_2 \||x|^s \nabla W(t)\|_2 \\
&+ \||x|^s W(t)\|_2 \||x|^s \nabla V(t)\|_2 + 4s \int |x|^{2s-2} x V(t) W(t) dx\Big) \\
&\quad \times \left(\|\nabla(-\Delta)^{-1}\tilde{w}_1(t)\|_\infty + G\right) + 2\Big(\int |x|^{2s}|w_2(t)\nabla V(t) + v_2(t)\nabla W(t)|dx \\
&+ 2s \int |x|^{2s-2}|x w_2(t) V + x v_2(t) W(t)|dx\Big)\|\nabla(-\Delta)^{-1}\tilde{W}(t)\|_\infty \\
&\equiv J_1(t) + J_2(t) + J_3(t).
\end{aligned}
$$

Integrating over $[0,T]$ we see from (3.23) and (3.22) that for small $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\tilde{\varepsilon} > 0$,

$$
\begin{aligned}
&\int_0^T J_1(t)dt \le C\varepsilon_1 \sup_{t\in[0,T]} \left(\||x|^s V(t)\|_2^2 + \||x|^s W(t)\|_2^2\right) \qquad (3.24)\\
&+ \varepsilon_1^{-1} \int_0^T \left(\|V(t)\|_2^2 + \|W(t)\|_2^2\right) dt \\
&\le C\varepsilon_1 \sup_{t\in[0,T]} \left(\||x|^s V(t)\|_2^2 + \||x|^s W(t)\|_2^2\right) + \varepsilon_1^{-1} T \|W\|_{X_T}^2,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T J_2(t)dt \le \int_0^T \big(\||x|^s V(t)\|_2 \||x|^s \nabla W(t)\|_2 + \||x|^s W(t)\|_2 \||x|^s \nabla V(t)\|_2 \\
&+ 8\||x|^s V(t)\|_2 \||x|^{s-1} W(t)\|_2\big) \qquad (3.25)\\
&\qquad \times \left(\|\tilde{w}_1(t)\|_2 \sqrt{\log\left(e + \frac{\|\nabla \tilde{w}_1(t)\|_2 + \|\langle x\rangle^s \tilde{w}_1(t)\|_2}{\|\tilde{w}_1(t)\|_2}\right)} + G\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&\le C\varepsilon_2 \sup_{t\in[0,T]} \left(\||x|^s V(t)\|_2^2 + \||x|^s W(t)\|_2^2\right) \\
&+ C\varepsilon_2^{-1}\tilde{\varepsilon}^{-1} M^2 T \int_0^T \left(\||x|^s \nabla W(t)\|_2^2 + \||x|^s \nabla V(t)\|_2^2\right) dt \\
&+ CM\varepsilon_2^{-1}\tilde{\varepsilon} \int_0^T \left(\||x|^s \nabla W(t)\|_2^2 + \||x|^s \nabla V(t)\|_2^2\right) dt,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T J_3(t)dt &\le C\varepsilon_3 \sup_{t\in[0,T]} \left(\||x|^s V(t)\|_2^2 + \||x|^s W(t)\|_2^2\right) \qquad (3.26)\\
&+ C\varepsilon_3 \tilde{\varepsilon}^{-1} M \int_0^T \left(\||x|^s \nabla W(t)\|_2^2 + \||x|^s \nabla V(t)\|_2^2\right) dt \\
&+ \varepsilon_3^{-1} T \sup_{t\in[0,T]} \left(\|\tilde{W}(t)\|_2^2 + \||x|^s \tilde{W}(t)\|_2^2\right) + \tilde{\varepsilon} \int_0^T \|\nabla \tilde{W}(t)\|_2^2 dt.
\end{aligned}
$$

Now choosing ε_1, ε_2, ε_3, $\tilde{\varepsilon}$ and T properly small, it follows from (3.23), (3.24) and (3.26) that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left(\||x|^s V(t)\|_2^2 + \||x|^s W(t)\|_2^2\right) + \int_0^T \left(\||x|^s \nabla V(t)\|_2^2 + \||x|^s \nabla W(t)\|_2^2\right) dt \\
&\le \frac{1}{16}\Big(\sup_{t\in[0,T]} \left(\|\tilde{W}(t)\|_2^2 + \||x|^s \tilde{W}(t)\|_2^2\right) + \int_0^t \|\nabla \tilde{W}(t)\|_2^2 dt\Big). \qquad (3.27)
\end{aligned}
$$

Thus, combining (3.22) and (3.27), we proved the second desired estimate (3.3). □

Proof of Theorem 1.1, concluded. By Proposition 3.1, it follows that the solution map $\Phi \times \Psi; \tilde{w} \longmapsto (v, w)$ is a contraction mapping from X_T to X_T. Therefore, the Banach fixed point theorem yields that there exists a unique solution of $(v, w) = (\Phi(w), \Psi(w))$. From the formulation (3.1), this is a unique weak solution to (1.4). Standard parabolic regularity argument gives that the solution becomes regular after $t > 0$. The continuous dependence on the initial data and blowing up of solution as $t \to T_m$ if $T_m < \infty$ naturally follows the properties of the map (Φ, Ψ). This completes the proof of Theorem 1.1. □

Here we give the proof of the a priori estimates.

Proof of Proposition 1.3. (1) Multiplying the equations in (1.13) by $\log(1 + n(t))$ and $\log(1 + p(t))$ respectively, integrating by parts, we have

$$
\frac{d}{dt}\int_{\mathbb{R}^2} (1 + n(t)) \log(1 + n(t))\, dx + \int_{\mathbb{R}^2} \frac{1}{1+n} |\nabla(n(t) + 1)|^2 dx
$$

$$-\int_{\mathbb{R}^2} \nabla\psi(t)\cdot\nabla n(t)dx + \int_{\mathbb{R}^2} \nabla\psi\cdot\nabla\log(1+n)dx = 0, \tag{3.28}$$
$$\frac{d}{dt}\int_{\mathbb{R}^2}(1+p(t))\log\left(1+p(t)\right)dx + \int_{\mathbb{R}^2}\frac{1}{1+p(t)}|\nabla(p(t)+1)|^2 dx$$
$$+\int_{\mathbb{R}^2}\nabla\psi(t)\cdot\nabla p(t)dx - \int_{\mathbb{R}^2}\nabla\psi(t)\cdot\nabla\log(1+p(t))dx = 0$$

and

$$\frac{d}{dt}\Big(\frac{\alpha}{2}\|\nabla\psi(t)\|_2^2\Big) - \int_{\mathbb{R}^2}\nabla n(t)\cdot\nabla\psi(t)dx + \int_{\mathbb{R}^2}\nabla p(t)\cdot\nabla\psi(t)dx \tag{3.29}$$
$$+\int_{\mathbb{R}^2} n(t)|\nabla\psi(t)|^2 dx + \int_{\mathbb{R}^2} p(t)|\nabla\psi(t)|^2 dx = 0.$$

Gathering (3.28) and (3.29),

$$\frac{d}{dt}\Big\{\int_{\mathbb{R}^2}(1+n(t))\log\left(1+n(t)\right)dx + \int_{\mathbb{R}^2}(1+p(t))\log\left(1+p(t)\right)dx$$
$$+\frac{\alpha}{2}\|\nabla\psi(t)\|_2^2\Big\} + \int_{\mathbb{R}^2}(n(t)+1)|\nabla(\log(1+n(t))) - \nabla\psi(t))|^2 dx$$
$$+\int_{\mathbb{R}^2}(p(t)+1)|\nabla(\log(1+p(t))) + \nabla\psi(t))|^2 dx$$
$$-2\int_{\mathbb{R}^2}|\nabla\psi(t)|^2 dx + \int_{\mathbb{R}^2}\nabla\psi(t)\cdot(\nabla\log(1+n(t)) - \log(1+p(t))) = 0.$$

Thus, we have shown that the smooth L^1 solution (n,p) satisfies

$$V(t) \equiv \int_{\mathbb{R}^2}(1+n(t))\log\left(1+n(t)\right)dx + \int_{\mathbb{R}^2}(1+p(t))\log(1+p(t))dx$$
$$+\frac{\alpha}{2}\|\nabla\psi(t)\|_2^2,$$

$$\frac{d}{dt}V(t) + \int_{\mathbb{R}^2} n(t)|\nabla(\log(1+n(t))) - \nabla\psi(t))|^2 dx$$
$$+\int_{\mathbb{R}^2} p(t)|\nabla(\log(1+p(t))) + \nabla\psi(t))|^2 dx \le V(t) \tag{3.30}$$

and hence

$$V(t) + \int_0^t\Big[\int_{\mathbb{R}^2} n(\tau)|\nabla(\log(1+n(\tau))) - \nabla\psi(\tau))|^2 dx \tag{3.31}$$
$$+\int_{\mathbb{R}^2} p(\tau)|\nabla(\log(1+p(\tau))) + \nabla\psi(\tau))|^2 dx\Big]d\tau \le CV(0)e^t.$$

(2) The energy estimate follows from the standard argument by integration by parts. □

Proof of Proposition 1.4. By the weak maximum principle, the local solution obtained in Theorem 1.1 is positive for all times, i.e., $n(t,x) \geq 0$ and $p(t,x) \geq 0$.

The case $\alpha = 1$. By the a priori estimate, we have

$$\|n(t)\|_2^2 + \|p(t)\|_2^2 + 2\int_0^t \{\|\nabla n(t)\|_2^2 + \|\nabla p(t)\|_2^2\} d\tau \leq (\|n_0\|_2^2 + \|p_0\|_2^2)e^{CT}. \tag{3.32}$$

Also similar to the estimate (3.15), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\||x|^s v(t)\|_2^2 + \||x|^s w(t)\|_2^2\} + \||x|^s \nabla v(t)\|_2^2 + \||x|^s \nabla w(t)\|_2^2 \\
&= 2s\int_{\mathbb{R}^2} |x|^{2s-2} x (v^2(t) + w^2(t)) dx \\
&- \alpha \int_{\mathbb{R}^2} |x|^{2s} v(t) w^2(t) dx + \int_{\mathbb{R}^2} |x|^{2s} v(t) w(t) g dx \\
&\leq \||x|^s v\|_2^2 + \||x|^s w\|_2^2 + \|g\|_\infty \||x|^s v\|_2 \||x|^s w(t)\|_2.
\end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33), we have the a priori estimate and the solution cannot blow up in a finite time. This concludes the theorem. □

Remark. When $\alpha = -1$, by the same argument found in the literature, it is possible to show that the solution can be continued globally in time. In fact, we expect the counter assumption of the blow-up holds, then the solution can be continued globally. We refer to same analogous result for the slightly different system in [26], [27] and [8].

4. ESTIMATE FOR THE LOCAL MOMENT

We define the localized weight function. Let $\phi(r)$ be a smooth function such that

$$\phi(r) = \begin{cases} r^2, & \text{if } 0 \leq r \leq 1, \\ 2 - (r-2)^2, & \text{if } 1 \leq r \leq 2, \\ 2, & \text{if } 2 \leq r. \end{cases} \tag{4.1}$$

Setting $\Phi(x) = \phi(|x|)$, we see that $|\nabla\Phi| \leq 2(\Phi(x))^{1/2}$, $\Delta\Phi \leq 4$ and support of $\nabla\Phi$ and $\Delta\Phi$ are in B_2. Here and hereafter, B_r denotes the open disk centered at the origin with radius $r > 0$. The following lemma is used in [34] and [28].

Lemma 4.1. *We have for* $(x, y) \in B_1 \times B_1$,

$$(\nabla\Phi(x) - \nabla\Phi(y)) \cdot \nabla G(x - y) = -\frac{1}{\pi}, \tag{4.2}$$

and for $(x, y) \in (B_1 \times B_1)^c$,

$$(\nabla\Phi(x) - \nabla\Phi(y)) \cdot \nabla G(x - y) \leq \frac{1}{\pi}. \tag{4.3}$$

Proof of Lemma 4.1. We follow the argument of [28]. It is easy to see that (4.2) holds, since $\nabla\Phi(x) = 2x$ on B_1 and

$$\nabla G(x - y) = -\frac{1}{2\pi}\frac{x - y}{|x - y|^2}.$$

To obtain (4.3), it suffices to show that for $(x, y) \in (B_1 \times B_1)^c$,

$$(\nabla\Phi(x) - \nabla\Phi(y)) \cdot (y - x) \leq 2|x - y|^2. \tag{4.4}$$

For $(x, y) \in B_1 \times (B_2 \setminus B_1)$, using (4.4) and

$$\begin{cases} 2x \cdot (y - x) = (|y|^2 - |x|^2) - |x - y|^2, \\ 2y \cdot (y - x) = (|y|^2 - |x|^2) - |x + y|^2 \end{cases} \tag{4.5}$$

and noting $|y| - 1 \leq |y| - |x| \leq |y - x|$, we have

$$\{\nabla\Phi(x) - \nabla\Phi(y)\}) \cdot (y - x) = \{2x - 2(2 - |y|)\frac{y}{|y|}\} \cdot (y - x)$$
$$\leq \frac{2(|y| + |x|)}{|y|}(|y| - |x|)^2 - \left\{\frac{2(|y| + |x|)}{|y|} - 2\right\}|y - x|^2 \leq 2|y - x|^2.$$

For $(x, y) \in (\mathbb{R}^2 \setminus B_2) \times B_1$, noting $|x - y| \geq 1$, we have

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot (y - x) = 2x \cdot (y - x) \leq 2|x||x - y|$$
$$\leq \frac{2|x|}{|x - y|}|y - x|^2 \leq 2|y - x|^2.$$

For $(x, y) \in (B_2 \setminus B_1) \times (B_2 \setminus B_1)$, by (4.1) and (4.5), we have

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot (y - x) = \left\{\frac{2 - |x|}{|x|}2x \cdot (y - x) - \frac{2 - |y|}{|y|}2y \cdot (y - x)\right\}$$
$$\leq \left\{\frac{2(|y| + |x|)}{|x||y|}|y - x|^2 - \left(\frac{2(|x| + |y|)}{|x||y|} - 2\right)|x - y|^2\right\} = 2|x - y|^2.$$

For $(x, y) \in (B_2 \setminus B_1) \times (\mathbb{R}^2 \setminus B_2)$, since $|y| \geq 2$, we have

$$\{\nabla\Phi(x) - \nabla\Phi(y)\}) \cdot (y - x) = 2(2 - |x|)\frac{x}{|x|} \cdot (y - x)$$
$$\leq 2(|y| - |x|)|y - x| \leq 2|y - x|^2.$$

For $(x,y) \in (\mathbb{R}^2 \setminus B_2) \times (\mathbb{R}^2 \setminus B_2)$, since $\nabla\Phi(x) = \nabla\Phi(y) = 0$, we have (4.4). Other cases are reduced to the cases above, since

$$\{\nabla\Phi(x) - \nabla\Phi(y)\} \cdot (y-x) = \{\nabla\Phi(y) - \nabla\Phi(x)\} \cdot (x-y).$$

Hence, we have completed the proof of Lemma 2.1. □

The following lemma is a key tool which is essentially due to Nagai [28] (see also Nagai-Senba-Suzuki [30]).

Lemma 4.2. *Let* (v,w) *be a smooth nonnegative* L^1 *solution to* (1.13). *Then the inequality*

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx \le 4\int_{\mathbb{R}^2} v(t,x)dx - \frac{1}{2\pi}\Big(\int_{\mathbb{R}^2} w(t,x)dx\Big)^2 \\ + C_1(\|v\|_1)\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx + C_2(\|v\|_1)\Big(\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx\Big)^{1/2} \end{aligned} \tag{4.6}$$

holds for some positive constants C_1 *and* C_2 *depending only on* $\|v_0\|_1$.

Proof of Lemma 4.2. Without loosing the generality, we may assume that (v,w) is a smooth solution of (1.13) belonging to $L^1(\mathbb{R}^2)$. The general case can be shown by the approximation argument and the well-posedness of the solution in Theorem 1.1. Multiply the first equation of (1.13) by $\Phi(x)v(x)$ and integrate by parts, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx = \int_{\mathbb{R}^2} v(t,x)\Delta\Phi(x)dx + \int_{\mathbb{R}^2} w(t,x)\nabla\psi(t,x)\cdot\nabla\Phi(x)dx. \tag{4.7}$$

Since $\Delta\Phi(x) \le 4$, the first term of the right hand side is bounded by $4\int_{\mathbb{R}^2} v(t,x)dx$. Let $G(x-y)$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^2$. Then ψ can be expressed as

$$\psi(x) = \int_{\mathbb{R}^2} G(x-y)w(t,y)dy.$$

Then noting that the support of $\nabla\Phi$ is restricted in $\bar{B_2} \setminus B_1$ and $\eta \equiv 1$ on this support, the second term in (4.7),

$$\begin{aligned} &\int_{B_2} w(t,x)\nabla\psi(t,x)\cdot\nabla\Phi(x)dx \\ &= \int_{B_2} w(t,x)\nabla_x\Big(\int_{B_3} G(x-y)w(t,y)dy\Big)\cdot\nabla\Phi(x)dx \\ &+ \int_{B_2} w(t,x)\nabla_x\Big(\int_{B_3^c} G(x-y)w(t,y)dy\Big)\cdot\nabla\Phi(x)dx \equiv I + II. \end{aligned} \tag{4.8}$$

By

$$
\begin{aligned}
I &= -\frac{1}{2\pi}\iint_{B_2\times B_3} \frac{x-y}{|x-y|^2}\cdot\nabla\Phi(x)w(t,x)w(t,y)dxdy \\
&\qquad (\text{since } \nabla\Phi(x)=0 \text{ on } B_3\setminus B_2\ ) \qquad (4.9)\\
&= -\frac{1}{2\pi}\iint_{B_3\times B_3} \frac{x-y}{|x-y|^2}\cdot\nabla\Phi(x)w(t,x)w(t,y)dxdy
\end{aligned}
$$

(here the each integration region is corresponding to the order of integration variable $dxdy$).

Observing the symmetry properties of the integral kernel and the integrant, the first term I can be expressed as

$$
\begin{aligned}
I &= -\frac{1}{4\pi}\iint_{B_3\times B_3} \frac{x-y}{|x-y|^2}\cdot(\nabla\Phi(x)-\nabla\Phi(y))w(t,x)w(t,y)dxdy \\
&= -\frac{1}{2\pi}\iint_{B_1\times B_1} w(t,x)w(t,y)dxdy \qquad (4.10)\\
&\quad -\frac{1}{4\pi}\iint_{(B_3\setminus B_2)^2} \frac{x-y}{|x-y|^2}\cdot(\nabla\Phi(x)-\nabla\Phi(y))w(t,x)w(t,y)dxdy.
\end{aligned}
$$

Then it follows

$$
\begin{aligned}
I &\le -\frac{1}{2\pi}\Big(\int_{B_1} w(t,x)dx\Big)^2 + \frac{1}{2\pi}\iint_{(B_3\setminus B_1)^2}|w(t,x)||w(t,y)|dxdy \qquad (4.11)\\
&\le -\frac{1}{2\pi}\Big(\int_{\mathbb{R}^2} w(t,x)dx\Big)^2 + \frac{1}{\pi}\Big(\int_{B_1} w(t,x)dx\Big)\Big(\int_{B_1^c} w(t,x)dx\Big)\\
&\quad + \frac{1}{2\pi}\Big(\int_{B_3\setminus B_1}|w(t,x)|dx\Big)\Big(\int_{B_3\setminus B_1}\Phi(x)|w(t,x)|dx\Big)\\
&\le -\frac{1}{2\pi}\Big(\int_{\mathbb{R}^2} w(t,x)dx\Big)^2 + \frac{1}{\pi}\Big(\int_{B_1}|w(t,x)|dx\Big)\Big(\int_{\mathbb{R}^2}\Phi(x)|w(t,x)|dx\Big)\\
&\quad + \frac{1}{2\pi}\Big(\int_{B_3\setminus B_1}|w(t,x)|dx\Big)\Big(\int_{B_3\setminus B_1}\Phi(x)|w(t,x)|dx\Big)\\
&\le -\frac{1}{2\pi}\Big(\int_{\mathbb{R}^2} w(t,x)dx\Big)^2 + \frac{1}{\pi}\Big(\int_{B_3}|w(t,x)|dx\Big)\Big(\int_{\mathbb{R}^2}\Phi(x)|w(t,x)|dx\Big).
\end{aligned}
$$

The second term in (4.8) is estimated using $\eta(x)\le 1$, $|\nabla\Phi|\le 2\Phi^{1/2}$, $|x-y|\ge 1$, then

$$
II \le \frac{1}{2\pi}\iint_{B_2\times K\setminus B_3}\frac{1}{|x-y|}|\nabla\Phi(x)||w(t,x)||w(t,y)|dxdy
$$

$$\leq \frac{1}{\pi}\Big(\int_{B_2} \Phi(x)^{1/2}|w(t,x)|dx\Big)\Big(\int_{B_3^c} |w(t,y)|dy\Big) \tag{4.12}$$

(by using Hölder's inequality)

$$\leq \frac{1}{\pi}\Big(\int_{B_2} \Phi(x)|w(t,x)|dx\Big)^{1/2}\Big(\int_{B_2} |w(t,y)|dy\Big)^{1/2}\Big(\int_{B_3^c} |w(t,y)|dy\Big).$$

Hence, we obtain

$$\begin{aligned} I + II \leq & -\frac{1}{2\pi}\Big(\int_{\mathbb{R}^2} w(t,x)dx\Big)^2 \\ & + \frac{1}{\pi}\Big(\int_{B_3} |w(t,x)|dx\Big)\Big(\int_{\mathbb{R}^2} \Phi(x)|w(t,x)|dx\Big) \\ & + \frac{1}{\pi}\Big(\int_{B_3} |w(t,y)|dy\Big)^{3/2}\Big(\int_{B_2} \Phi(x)|w(t,x)|dx\Big)^{1/2}. \end{aligned} \tag{4.13}$$

Now we note that $|w(t,x)| \leq |n(t,x)| + |p(t,x)| = v(t,x)$ by the the positivity of $n(x)$ and $p(x)$. Therefore, we obtain the desired estimate;

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx \leq & \, 4\int_{\mathbb{R}^2} v(t,x)dx - \frac{1}{2\pi}\Big(\int_{\mathbb{R}^2} w(t,x)dx\Big)^2 \\ & + C_1\Big(\int_{B_3} v(t,x)dx\Big)\Big(\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx\Big) \\ & + C_2\Big(\int_{B_3} v(t,x)dx\Big)^{3/2}\Big(\int_{B_2} \Phi(x)v(t,x)dx\Big)^{1/2}. \end{aligned} \tag{4.14}$$

5. Proof of blow-up

In this section, we give a proof of the blow-up. We first show that some restricted initial data develops finite time blow-up.

Proposition 5.1. *Let the initial data $v_0 = n_0 + p_0$ and $w_0 = n_0 - p_0$ in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $n_0, p_0 \geq 0$ everywhere and satisfy*

$$\eta \equiv \frac{1}{2\pi}\Big(\Big(\int_{\mathbb{R}^2} w_0 dx\Big)^2 - 8\pi\int_{\mathbb{R}^2} v_0 dx\Big) > 0. \tag{5.1}$$

Define an auxiliary function $H(f) = -\eta + C_1 f + C_2 f^{1/2}$, where $C_1, C_2 > 0$ are the constants in Lemma 4.2. Then if we assume that $\int_{B_{r_1}} \Phi(x)(n_0(x) + p_0(x))dx$ is small enough so that

$$H\Big(\int_{\mathbb{R}^2} \Phi(x)v_0(x)dx\Big) < 0, \tag{5.2}$$

then the corresponding solution of (1.13) blows up in a finite time.

Proof of Proposition 5.1. Assuming the corresponding solution(v, w) to (1.13) exists globally, we derive a contradiction. By the local existence result, we may assume that the solution is smooth and everywhere positive.

By the assumption on the initial data, we see that for $\eta > 0$,

$$H\Big(\int_{\mathbb{R}^2} \Phi(x)v_0(x)dx\Big) < 0.$$

Let

$$T_b = \sup\Big\{t: \text{ for all } s \in (0,t), \int_{\mathbb{R}^2} \Phi(x)v(s,x)dx \le \int_{\mathbb{R}^2} \Phi(x)v_0(x)dx\Big\}.$$

We have $T_b > 0$. If $T_b < \infty$, then we have

$$\int_{\mathbb{R}^2} \Phi(x)v(T_b,x)dx = \int_{\mathbb{R}^2} \Phi(x)v_0(x)dx$$

by the definition of T_b. From Lemma 4.2,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx\Big|_{t=T_b} &\le H\Big(\int_{\mathbb{R}^2} \Phi(x)v(T_b,x)dx\Big)\\ &= H\Big(\int_{\mathbb{R}^2} \Phi(x)v_0(x)dx\Big) < 0.\end{aligned}$$

Hence, it follows that there exists $T' > T_b$ such that for all $s \in [T_b, T')$,

$$\int_{\mathbb{R}^2} \Phi(x)v(s,x)dx \le \int_{\mathbb{R}^2} \Phi(x)v_0(x)dx. \tag{5.3}$$

This contradicts the definition of T_b and hence $T_b = \infty$. Since $H(f)$ is strictly increasing function of f and $H(0) < 0$, there exists a unique zero point, $\alpha > 0$ and $H(f) < 0$ for $0 < f < \alpha$. Now by assumption again, we may assume that

$$\int_{\mathbb{R}^2} \Phi(x)v_0(x)dx \le \frac{1}{2}\alpha.$$

Then by (5.3), we have for all existence time $t \in (0,\infty)$,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx \le H\Big(\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx\Big) \le H(\frac{\alpha}{2}) < 0. \tag{5.4}$$

Which conclude that

$$\int_{\mathbb{R}^2} \Phi(x)v(t,x)dx \le H(\alpha/2)t + \int_{\mathbb{R}^2} \Phi(x)v_0(x)dx$$

and the left hand side meets zero in a finite time which contradicts the left hand side is nonnegative definite. Therefore, in view of Theorem 1.1, the solution blows up in a finite time. □

Now we are ready for proving Theorem 1.5.

Proof of Theorem 1.5. It is clear that the following scaling leaves the system invariant; for $\lambda > 0$,

$$n_\lambda(t,x) = \lambda^2 n(\lambda^2 t, \lambda x), \quad p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x). \tag{5.5}$$

This scaling also preserves the $L^1(\mathbb{R}^2)$ norm.

Let $v(x) = v(0,x) = n_0(x) + p(x) \in L^1$ and $\varepsilon > 0$ be arbitrary fixed constant. Then one can choose $R > 0$ sufficiently large such that

$$\int_{|x|>R} v(x)dx < \frac{\varepsilon}{4}.$$

Then we see that for large $\lambda > R$ and $v_\lambda(x) = \lambda^2 v(\lambda x)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \Phi(x) v_\lambda(x)dx &\le \int_{B_{\lambda^{-1}R}(0)} |x|^2 v_\lambda(x)dx + \int_{B_{\lambda^{-1}R}(0)^c} \Phi(x)v_\lambda(x)dx \\ &= \lambda^{-2}\int_{B_R(0)} |x|^2 v(x)dx + \int_{B_R(0)^c} \Phi(x)v(x)dx \\ &\le \lambda^{-2}\int_{B_R(0)} |x|^2 v(x)dx + 2\int_{B_R(0)^c} v(x)dx \\ &\le \lambda^{-2}R^2\int_{B_R(0)} v(x)dx + \frac{\varepsilon}{2}.\end{aligned} \tag{5.6}$$

Thus, by choosing $\lambda > 0$ sufficiently large, we have

$$\int_{\mathbb{R}^2} \Phi(x)v_\lambda(x)dx < \varepsilon.$$

We choose ε sufficiently small such that $v_\lambda(x)$ satisfies the condition (5.2) and for fixed λ. Noting that the scaling conserves the L^1 norm, the condition (5.1) still holds for the scaled solutions.

Then by Proposition 5.1, the solution does not exist for all time. This completes the proof of theorem. □

Acknowledgement. The authors would like to thank Professor Toshitaka Nagai for advices and encouragement. They are also grateful for referee's useful comment. Work of T. Ogawa is partially supported by JSPS Grant-in-aid for Scientific Research #11440057.

References

[1] J. Bergh and J. Löfström, "Interpolation Spaces; an Introduction," Springer-Verlag, Berlin, 1976.

[2] P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl., 8 (1998), 715–743.

[3] P. Biler, *Existence and nonexistence of solutions for a model of gravitational interaction of particles,* I, Colloq. Math., 68 (1995), 229–239.

[4] P. Biler and J. Dolbeault, *Long time behavior of solutions to Nernst-Planck and Debye-Hünkel drift-diffusion systems*, Ann. Henri Poincaré, 1 (2000), 461–472.
[5] P. Biler, W. Hebisch, and T. Nadzieja, *The Debye system: existence and large time behavior of solutions*, Nonlinear Anal. T.M.A., 23 (1994), 1189–1209.
[6] P. Biler, D. Hilhorst, and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles,* $\mathbb{I}$, Colloq. Math., 67 (1994), 297–308.
[7] P. Biler and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interactions of particles I*, Colloq. Math., 66 (1994), 319–334.
[8] P. Biler and T. Nadzieja, *A nonlolcal singular parabolic problem modelling gravitational interaction of particles*, Adv. Diff. Equations, 3 (1998), 177–197.
[9] H. Brezis and T. Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. T.M.A., 4 (1980), 677-681.
[10] T. Cazenave and A. Haraux, "An Introduction to Semilinear Evolution Equations," Oxford Science Publ. 1998.
[11] W. Fang and K. Ito, *Global solutions of the time-dependent drift-diffusion semiconductor equations*, J. Diff. Equations, 123 (1995), 523-566.
[12] W. Fang and K. Ito, *Solutions to a nonlinear drift-diffusion model for semiconductors*, Electron. J. Differential Equations, 15 (1999), 38 pages (electronic).
[13] H. Fujita and T. Kato, *On Navier-Stokes initial value problem 1*, Arch. Rat. Mech. Anal., 46 (1964), 269-315.
[14] H. Gajewski, *On the uniqueness of solutions to the drift-diffusion model of semiconductor devices*, Math. Model. Meth. Appl Sci., 4 (1994), 121-133.
[15] H. Gajewski and K. Gröger, *On the basic equations for carrier transport in semiconductors*, J. Math. Anal. Appl., 113 (1986), 12-35.
[16] Y. Giga, *Solutions for semilinear Parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations, 61 (1986), 186-212.
[17] M. A. Herrero and J. J. L. Velázquez, *Chemotaxis collapse for the Keller-Segel model.* J. Math. Biol., 35 (1996), 177-194.
[18] M. A. Herrero and J. J. L. Velázquez, *Singularity patterns in a chemotaxis model,* Math. Ann., 306 (1996), 583-623.
[19] H. Hoshino and Y. Yamada, *Solvability and smoothing effect for semilinear parabolic equations,* Funkcial. Ekvac., 34 (1991), 475-494.
[20] A. Jüngel, *Qualitative behavior of solutions of a degenerate nonlinear drift-diffusion model for semiconductors*, Math. Model. Meth. Appl. Sci., 5 (1995), 497-518.
[21] T. Kato, *Strong L^p - solution of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions,* Math. Z., 187 (1984) , 471-480.
[22] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., 26 (1970), 399-415.
[23] M. Kurokiba and T. Ogawa, L^p well posedness of the for the drift-diffusion system arising from the semiconductor device simulation, preprint, Fukuoka University.
[24] M. S. Mock, *An initial value problem from semiconductor device theory*, SIAM, J. Math., 5 (1974), 597-612.
[25] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci.Appl., 5 (1995), 581-601.
[26] T. Nagai, *Global existence of solutions to a parabolic system for chemotaxis in two space dimensions*, Nonlinear Anal. T.M.A., 30 (1997), 5381-5388.
[27] T. Nagai, *Global existence and blowup of solutions to a chemotaxis system*, preprint.

[28] T. Nagai, *Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains*, to appear in J. Inequal. Appl.

[29] T. Nagai, M. Kurokiba, and T. Ogawa, *Brezis-Merle inequality of parabolic type and application to the global existence of the Keller-Segel equations*, work in preparation.

[30] T. Nagai, T. Senba, and T. Suzuki, *Chemotactic collapse in a parabolic system of mathematical biology*, preprint.

[31] T. Nagai, T. Senba, and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., 40 (1997), 411–433.

[32] S. Odanaka, *Mathematical modeling of theory of semiconductor devices*, Lecture Note at Osaka University, preprint (Japanese).

[33] T. Ogawa and Y. Taniuchi, *Critical Sobolev inequality and uniqueness problem to the Navier-Stokes equations*, preprint, Kyushu University.

[34] T. Ogawa and Y. Tsutsumi, *Blow-up of H^1 solutions for the nonlinear Schrödinger equation*, J. Diff. Equations, 92 (1991), 317-330.

[35] T. Ogawa and Y. Tsutsumi, *Blow-up of H^1 solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc., 111 (1991), 487-496.

[36] S. Selberherr, "Analysis and Simulation of Semiconductor Devices," Springer Verlag 1983.

[37] T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology*, Adv. Differential Equations, 6 (2001), 21-50.

[38] E. Stein, "Singular Integrals and Differentiability Properties of Functions," Prinston Univ. Press, Prinston, New Jersey 1970.

[39] H. Triebel, "Theory of Function Spaces," Birkhäuser, Basel, 1983.

[40] F. Weissler, *Existence and non-existence of global solutions for a semilinear heat equation*, Israel J. Math., 38 (1981), 29-40.

[41] A. Yagi, *Norm behavior of solutions to a parabolic system of chemotaxis*, Math. Japon., 45 (1997), 241–265.

[42] W. P. Ziemer, "Weakly Differentiable Functions," Graduate Texts in Math. 120, Springer-Verlag, 1989.