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Research Article

Finite-Time Boundedness and H_∞ Control for Affine Switched Systems

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For affine switched systems, the existence of multiple equilibria is related to subsystems owing to the affine terms, which makes asymptotic and finite-time stability analysis nontrivial. In this paper, the problems of finite-time boundedness (FTB) analysis and stabilization are addressed for affine switched systems, and several definitions and sufficient conditions are proposed to study FTB and H_∞ performance. At first, the definition of FTB for affine switched systems is improved concerning the affine terms and multiple equilibria. Based on the FTB definition, sufficient conditions ensuring finite-time boundedness for affine switched systems under a prespecified state boundary are given. Then the results are extended to solve H_∞ finite-time boundedness problem, in which the H_∞ controllers are designed to guarantee the finite-time boundedness of affine switched system with H_∞ performance. In our investigation, average dwell-time approach is employed to study the time-dependent constrained switching case. Finally, several numerical examples are given to illustrate the effectiveness of the proposed results.

1. Introduction

Switched systems are distinctive subclass of hybrid systems. They are composed of a family of continuous-time or discrete-time subsystems with a criterion that rules the switching among them. This switching rule can be classified as time-dependent, state-dependent, or time-state-dependent [1]. Since many physical processes possess switching nature, and many real-world applications resort to switching strategy to improve the control performance, the theory and application of switched systems have received a great attention during the recent decades. For more details on the recent results about the basic problems in stability and stabilization for switched systems, readers are referred to surveys [2–4] and books [1, 5] and the references cited therein.

The issue of stability analysis and stabilization is an important topic for switched dynamical systems [6–10]. Finding sufficient conditions ensuring the Lyapunov asymptotic stability dealing with infinite time interval has been the major concern for switched systems. Numerous published results discussed the asymptotic stability analysis and stabilization employing different variations of Lyapunov function [7, 11, 12]. Average dwell-time approach [13, 14] and Lie-algebraic condition technique [15, 16] are effective tools for analysis of switched systems. On the contrary, the finite-time behavior of dynamical systems is also of interest in many practical applications. It concerns that the states do not exceed a certain bound during a fixed time interval, e.g., to avoid saturations or excitation. The theory of finite-time stability (FTS) and finite-time boundedness (FTB) focuses on the transient response of dynamical systems over a finite-time interval, while asymptotic behavior is for infinite time. In the survey of recent development of this innovative theory, some necessary and sufficient conditions for finite-time stability and stabilization of continuous-time systems or discrete-time systems have been provided in [17, 18]. Based upon it, necessary and sufficient conditions for finite-time stability of systems with impulsive effects were obtained in [19, 20]. The authors [21, 22] applied FTS/FTB conceptions to switched systems and compared the conservativeness among different conditions. In [23], the mixed H_∞/finite-time stability control problem was discussed. For quadratic input-output finite-time stability with an H_∞ bound, [23] provided a necessary and sufficient condition. Then the method was extended to robust H_∞ controller and filter design for switched system with exogenous noise [24, 25]. It should be noted that finite-time stability and Lyapunov asymptotic stability are independent concepts: a Lyapunov asymptotic stable system may not fulfill FTS/FTB criteria since the transient response of a system may exceed the bound, and vice versa [26]. In many practical applications, switching is likely to occur in some short-time intervals, whereas for remaining long time no switching occurs. Since Lyapunov stability concerns with infinite time, it may not be influenced by such short-time switching. However, the boundedness of state may be affected by the switching. Hence, FTB criteria are needed to be considered for designing controller and switching laws during such applications.

Most of the existing literatures on stability issues of switched systems are based on the premise that all subsystems share a common equilibrium (typically the origin). On the other hand, for affine switched system, subsystems have different equilibria, so complex and interesting phenomena emerge. Almost all the practical hybrid systems can be modeled as affine switched systems. Many results like [27–30] analyzed interesting behaviors similar to those of asymptotically stable systems near an equilibrium for affine switched systems and depicted their real-world applications. Many extensions of the conventional stability concepts have been obtained for affine switched systems. S-Procedure method with the extensional state vector has been proposed in [31, 32] to analyze the asymptotic stability for continuous affine switched system. The relative results were extended to discrete affine switched systems in [33]. In [34], a method for designing switching rules driving the state of affine switched system to a desired equilibrium was investigated. Almost all the existing literatures on stability analysis of affine switched systems focused on the asymptotic stability. However, the boundedness of state for affine switched systems under constrained dwell-time switching is also of significant interest for affine switched systems. In FTB analysis, we also need to deal with affine terms leading to multiple equilibria for affine switched systems, but the investigation of this problem lacks researchers' interest previously. Potential of affine switched systems theory and importance of finite-time transient behavior from the perspective of real-world applications are the major motivations for this investigation presented in this paper.

The main objective in this paper is to find sufficient conditions ensuring the FTB of affine switched systems by switching signal and feedback controllers design and to drive the state of affine switched system to the prescribed neighborhood of a desired equilibrium during a finite-time interval. Taking into account the influence of affine terms on FTB for affine switched system, we propose an innovative FTB concept. Based on this definition, sufficient conditions ensuring the affine switched system finite-time bounded are proposed. Specifically, with the prespecified state boundary, average dwell time and state-feedback controllers for each subsystem are determined to guarantee the finite-time boundedness. The paper [22] points out that the more information about switching signal we know, the less conservative results can be derived. We extend this idea to switched affine systems to further reduce the conservatism. Classifying subsystems into asymptotically stable and unstable systems, we get the less conservative results of finite-time boundedness for affine switched system with the help of additional information of switching signal. Then, results are extended to solve the FTB problem for H_∞ controller design.

The rest of this paper is organized as follows. In Section 2, definitions of finite-time boundedness and H_∞ finite-time boundedness for affine switched system are revisited. Based on these definitions, finite-time boundedness analysis and finite-time stabilization are presented in Section 3. Then in presence of exogenous signals, H_∞ finite-time boundedness and the controllers design are investigated in Section 4. In Section 5, several numerical examples are presented to validate the proposed results. Conclusions are given in Section 6.

2. Preliminaries and Problem Formulation

For our investigation, we consider continuous-time affine switched system described as

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t) + b_i, \quad x(0) = x_0 \\ y(t) &= C_i x(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the measurement output, A_i, B_i, and C_i are system matrices with appropriate dimensions, constants b_i are affine terms, and $i(t) : \mathbb{R}^+ \longrightarrow I = \{1, \cdots, m\}$ is switching signal. For notational simplicity, we use i in place of $i(t)$.

Matrix variables A_i, B_i, and b_i give rise to an equilibrium (stable or unstable) for each subsystem; assuming all A_i to be nonsingular, we consider a given reference x_r as the required equilibrium for the whole system, referred to as *switched equilibrium*. Without loss of generality, it is assumed that the desirable equilibrium is different from all the equilibria of subsystems. Now although the asymptotic stability of affine switched system may be achieved by other types of switching strategy such as min-switching and sliding method, the state will not exactly converge to x_r under dwell-time constrained switching. The reason is that there always exist time interval (dwell time is always greater than zero) in which state must diverge from x_r. In our FTB investigation, we provide solution for boundedness of error state under dwell-time switching, which depicts the importance and innovation of our approach.

Here first we will extend the FTS and FTB concepts for affine switched systems keeping in view prescribed equilibrium x_r. In absence of control input, system (1) can be stated as

$$\dot{x}(t) = A_i x(t) + b_i, \quad x(0) = x_0 \tag{2}$$

Definition 1. Autonomous affine switched system (2) is said to be *finite-time bounded* with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$ if the following inequalities hold:

$$\begin{gathered} \left(x_0 - x_r\right)^T R_e \left(x_0 - x_r\right) \le \delta_e^2 \\ k_{\max}^T R_\omega k_{\max} \le \delta_\omega^2 \\ t = 0 \end{gathered} \tag{3}$$

$$\left(x(t) - x_r\right)^T R_e \left(x(t) - x_r\right) \le \varepsilon^2 \quad 0 < t \le T$$

where $k_{\max} = \operatorname{argmax}_{i=1,\cdots,m}\{k_i^T R_\omega k_i\}$, $k_i = A_i x_r + b_i$, $0 \le \delta_e < \varepsilon$, $\delta_\omega \ge 0$, $R_e > 0$, $R_\omega > 0$, and $T \in \mathbb{R}^+$.

Remark 2. Given equilibrium x_r and system (2), its tracking error system can be written as

$$\dot{e}(t) = A_i e(t) + k_i \tag{4}$$

where $e(t) = x(t) - x_r$, $k_i = b_i + A_i x_r$. According to *Definition 1*, we can conclude that affine switched system (2) is finite-time bounded with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$ if $e(t)^T R_e e(t) \le \varepsilon^2$ whenever $e_0^T R_e e_0 \le \delta_e^2$ and $k_{\max}^T R_\omega k_{\max} \le \delta_\omega^2$. The FTB criteria of affine switched systems ensure the state tracking the desired equilibrium x_r within the boundary ε. In other words, it guarantees the error state $e(t)$ tracking the origin in finite-time interval. Therefore, our study about FTB of affine switched systems can be turned into analyzing its corresponding tracking error system. Moreover, it is worth noting that FTB theory for general switched systems is related to initial state x_0 [35, 36]; whereas for affine switched systems, we are concerned with x_0 as well as the desired equilibrium x_r and affine terms b_i. Thus, in the *Definition 1*, the premise constraint conditions are extended to both initial state x_0 and k_i to analyze the FTB of affine switched systems, where k_i is related to the desired equilibrium x_r and affine terms b_i.

Remark 3. With the state-feedback controller $u(t) = K_i x(t)$, $i \in I$, affine switched system (1) can be rewritten into the following closed-loop system:

$$\dot{x}(t) = \overline{A}_i x(t) + b_i, \quad x(0) = x_0 \tag{5}$$

where $\overline{A}_i = A_i + B_i K_i$ and the FTB analysis method can be used directly. Similar to the significant impact of switching laws on asymptotic stability, the switching signals affect the finite-time boundedness of affine switched systems property significantly. Therefore, both switching signals and robust controllers should be designed during the FTB analysis of affine switched systems.

On the other hand, external disturbances are inevitable to dynamical systems. We can state affine switched system with time-varying disturbance $\omega(t)$ as

$$\dot{x}(t) = A_i x(t) + B_i u(t) + G_i \omega(t) + b_i, \quad x(0) = x_0 \tag{6}$$

$\omega(t)$ is assumed to be energy-bounded and hence for some scalar $d > 0$ it satisfies the inequality $\int_0^T \omega^T(t)\omega(t)dt \le d^2$. For simplifying FTB analysis, following *Definition 1* we can transform affine switched system (6) to its error tracking switched system as

$$\begin{aligned} \dot{e}(t) &= A_i e(t) + B_i u(t) + G_i \omega(t) + k_i \quad, x(0) = x_0 \\ z(t) &= C_i e(t) + D_{1i} u(t) + D_{2i} \omega(t) \end{aligned} \tag{7}$$

where $e(t) = x(t) - x_r$, $k_i = A_i x_r + b_i$, x_r is the desirable reference point, $z(t) \in \mathbb{R}^q$ is the controlled output, and the switched equilibrium of system is moved to the origin accordingly. Considering state-feedback controller $u(t) = K_i x(t)$, we derive the following closed-loop switched system:

$$\begin{aligned} \dot{e}(t) &= \overline{A}_i e(t) + G_i \omega(t) + k_i, \quad x(0) = x_0 \\ z(t) &= \overline{C}_i e(t) + D_{2i} \omega(t) \end{aligned} \tag{8}$$

where $\overline{A}_i = A_i + B_i K_i$, $\overline{C}_i = C_i + D_{1i} K_i$. Now we are able to state the following definition.

Definition 4. For affine switched system (7), considering state-feedback controller $u(t) = K_i x(t)$ and H_∞ performance index $\gamma > 0$, if the following two conditions are satisfied:

(1) the closed-loop error tracking switched system (8) is finite-time bounded;

(2) under zero-initial condition, the controlled output z satisfies the inequality

$$\begin{aligned} \int_0^T z^T(t) z(t)\, dt &< \gamma^2 \int_0^T \tilde{\omega}^T(t) \tilde{\omega}(t)\, dt \\ &< \gamma^2 \int_0^T \left(\omega^T(t)\omega(t) + k_{\max}^T k_{\max}\right) dt \end{aligned} \tag{9}$$

where $\tilde{\omega}(t) = \left[\omega^T(t) \;\; k_i^T\right]^T$, $k_{\max} = \operatorname{argmax}_{i=1,\cdots,m}\{k_i^T R_\omega k_i\}$, then $u(t)$ is called '*finite-time H_∞ controller*'.

Assuming $u(t) = 0$, $k_i = 0$ system (7) is expressed as

$$\begin{aligned} \dot{e}(t) &= A_i e(t) + G_i \omega(t), \quad x(0) = x_0 \\ z(t) &= C_i e(t) + D_{2i} \omega(t) \end{aligned} \tag{10}$$

Now *Definition 4* can be reduced to the following form. Switched system (10) is said to be H_∞ finite-time bounded with performance index γ, if

(1) the error tracking switched system (10) is FTB;

(2) under zero-initial condition, the controlled output z satisfies

$$\int_0^T z^T(t) z(t)\, dt < \gamma^2 \int_0^T \omega^T(t) \omega(t)\, dt \tag{11}$$

Based upon the above preliminaries we will focus on how to find sufficient conditions to ensure the finite-time boundedness of affine switched systems and address the H_∞ analysis and synthesis of piecewise linear state-feedback controllers resorting to LMI-based algorithms. The main problems we concern in this paper can be stated as follows.

Problem 5 (finite-time boundedness for affine switched systems). Given affine switched system (2), find sufficient conditions ensuring the finite-time boundedness with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$.

Problem 6 (state-feedback stabilization under FTB). Given affine switched system (1), find set of static state-feedback controllers $u(t) = K_i x(t)$ to ensure that the closed-loop system (5) is finite-time bounded with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$.

Problem 7 (H_∞ performance and controller design). Given affine switched system (8), analyze the H_∞ performance and design set of H_∞ controllers defined in *Definition 4* to ensure the finite-time boundedness with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$ and reduce the effect of the exogenous signal ω and k_i on the controlled output z to a prescribed level γ.

3. Finite-Time Boundedness and State-Feedback Stabilization

In this section, Problems 5 and 6 are taken into consideration. Our main aim is to find sufficient conditions and state-feedback controllers to ensure the finite-time boundedness of affine switched system in the form of (2). For a finite-time interval $[0, T]$, we consider finite switchings $k_{[0,T]}$. Each subsystem has an (stable or unstable) equilibrium point $x_{ri} = -A_i^{-1} b_i$. Regarding reference point x_r as an equilibrium point for the whole system called *switched equilibrium* and taking into account average dwell time, we will derive sufficient conditions ensuring finite-time boundedness.

Theorem 8. *Affine switched system (2) is finite-time bounded with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$, if there exist positive definite matrices P_i, scalars $\alpha, \beta > 0$, $\xi \geq 0$, such that*

$$\alpha R_e < P_i < \beta R_e \tag{12a}$$

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i \\ P_i & 0 \end{bmatrix} < \xi \begin{bmatrix} R_e & 0 \\ 0 & R_\omega \end{bmatrix} \tag{12b}$$

$$e^{(\xi/\alpha)T + k_{[0,T]} \ln(\beta/\alpha)} \left(\beta \delta_e^2 + T \xi \delta_\omega^2\right) - \alpha \varepsilon^2 < 0 \tag{12c}$$

Proof. Consider the error tracking switched system (4), let $\mathscr{R} = \operatorname{dig}(R_e, R_\omega)$, $\eta_i = \begin{bmatrix} e^T(t) & k_i^T \end{bmatrix}^T$. We choose piecewise Lyapunov function $V_i(t) = e^T(t) P_i e(t)$. From condition (12b) we have

$$\begin{aligned} \dot{V}_i(t) &= \dot{e}^T(t) P_i e(t) + e^T(t) P_i \dot{e}(t) \\ &= \begin{bmatrix} e(t) \\ k \end{bmatrix}^T \begin{bmatrix} A_i^T P_i + P_i A_i & P_i \\ P_i & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ k_i \end{bmatrix} < \xi \eta_i^T \mathscr{R} \eta_i \end{aligned} \tag{13}$$

Employing (12a) we derive

$$\begin{aligned} \dot{V}_i(t) &< \xi \alpha^{-1} \left(e^T(t) P_i e(t) + \alpha k_i^T R_\omega k_i\right) \\ &\leq \xi \alpha^{-1} V_i + \xi \delta_\omega^2 \end{aligned} \tag{14}$$

Let $\forall t > 0$, $t_0 < t_1 < \cdots < t_k$ be the switching instant of switched system. For overall system we can write $V(t) = \sum_{i \in I} \theta_i V_i(t), \theta_i \in \{0, 1\}$. Now from inequality (14),

$$V(t) < \phi(t, t_k) V(t_k^+) + \xi \delta_\omega^2 \int_{t_k}^{t} \phi(t, \tau)\, d\tau \tag{15}$$

where $\phi(t, \tau) = \exp(\xi \alpha^{-1}(t - \tau)) < \exp(\xi \alpha^{-1} T)$, T denotes the finite-time interval. Accordingly, the Lyapunov inequality in single step satisfies

$$\begin{aligned} V(t_{k+1}) &< \phi(t_{k+1}, t_k) \frac{V(t_k^+)}{V(t_k^-)} V(t_k^-) \\ &\quad + \xi \delta_\omega^2 \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \tau)\, d\tau \end{aligned} \tag{16}$$

Suppose system switches from mode i to j at some instant t_k; then from condition (12a),

$$\begin{aligned} \frac{V(t_k^+)}{V(t_k^-)} &= \frac{V_j(t_k)}{V_i(t_k)} = \frac{e^T(t_k) P_j e(t_k)}{e^T(t_k) P_i e(t_k)} \\ &< \frac{\beta e^T(t_k) R_e e(t_k)}{\alpha e^T(t_k) R_e e(t_k)} = \frac{\beta}{\alpha} \end{aligned} \tag{17}$$

It is evident that $\beta/\alpha > 1$ and following (16) iteratively we can derive easily that

$$\begin{aligned} V(t_k) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \Bigg[\phi(t_k, t_0) V(t_0) \\ &\quad + \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{-n} \xi \delta_\omega^2 \phi(t_k, t_n) \int_{t_{n-1}}^{t_n} \phi(t_n, \tau)\, d\tau \Bigg] \end{aligned} \tag{18}$$

Applying (15) and (18) we deduce

$$\begin{aligned} V(t) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \Bigg[\phi(t, t_0) V(t_0) \\ &\quad + \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{-n} \xi \delta_\omega^2 \phi(t, t_n) \int_{t_{n-1}}^{t_n} \phi(t_n, \tau)\, d\tau \Bigg] \\ &\quad + \xi \delta_\omega^2 \int_{t_k}^{t} \phi(t, \tau)\, d\tau < \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T} V(t_0) \\ &\quad + \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]} - n} \xi \delta_\omega^2 \int_{t_{n-1}}^{t_n} \phi(t, \tau)\, d\tau + \xi \delta_\omega^2 e^{(\xi/\alpha)T} (t \\ &\quad - t_k) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T} V(t_0) \\ &\quad + \xi \delta_\omega^2 e^{(\xi/\alpha)T} \left[\sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} (t_n - t_{n-1}) + (t - t_k)\right] \\ &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T} V(t_0) + T \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \xi \delta_\omega^2 e^{(\xi/\alpha)T} \end{aligned} \tag{19}$$

On the other hand, from condition (12a), we have

$$V(t) = e^T(t) P_i e(t) > \alpha e^T(t) R_e e(t) \tag{20}$$

Using the fact that $V(t_0) = e_0^T P_i e_0 < \beta e_0^T R_e e_0 \le \beta \delta_e^2$, in order to ensure the finite-time boundedness of switched system (4), i.e., $e^T(t) R_e e(t) \le \varepsilon^2$, the following condition should be satisfied:

$$\begin{aligned} \alpha e^T(t) R_e e(t) < {} & \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T} \beta \delta_e^2 \\ & + T \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \xi \delta_\omega^2 e^{(\xi/\alpha)T} < \alpha \varepsilon^2 \end{aligned} \tag{21}$$

which can be rewritten as condition (12c). Therefore, we get $(x(t) - x_r)^T R_e (x(t) - x_r) \le \varepsilon^2$ and we conclude that the affine switched system (2) is finite-time bounded which completes the proof. □

Remark 9. When other parameters are fixed, condition (12c) can be described by average dwell time as [37]

$$\tau_a \ge \tau_a^* = T \ln \frac{\beta}{\alpha} \left(\ln \frac{\alpha \varepsilon^2}{\beta \delta_e^2 + T \xi \delta_\omega^2} - \frac{\xi}{\alpha} T \right)^{-1} \tag{22}$$

where $\tau_a^* = T / k_{[0,T]}$. In other words, the average dwell time τ_a should be chosen large enough to ensure that inequality (22) is satisfied, which is necessary to guarantee the finite-time boundedness of affine switched system (2). Moreover, assuming $R_e = I$, from (12a) and (19) we deduce

$$\begin{aligned} \sqrt{\alpha} \| e(t) \| < {} & \sqrt{e^T(t) P_i e(t)} \\ < {} & \sqrt{\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T} \beta} \, \| e(t_0) \| \\ & + \sqrt{T \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \xi \delta_\omega^2 e^{(\xi/\alpha)T}} \end{aligned} \tag{23}$$

When $t \longrightarrow \infty$, $T \longrightarrow \infty$ and the term $\sqrt{T (\beta/\alpha)^{k_{[0,T]}} \xi \delta_\omega^2 e^{(\xi/\alpha)T}}$ on the right side of (23) will become infinite, which explains that the affine switched system (2) is not ultimately bounded, which illustrates FTB and ultimately boundedness are independent concepts.

Remark 10. Once the state bound ε is not ascertained, the minimum value $\varepsilon_{\min}$ is of interest, which can be found through optimization problem $\min (\beta/\alpha)^{k_{[0,T]}} e^{(\xi/\alpha)T} (\beta \delta_e^2 + T \xi \delta_\omega^2) \alpha^{-1}$ subject to (12a) and (12b). If we fix the parameter ξ and let $\alpha = 1$, $\beta = \theta \alpha$, the optimization problem becomes

$$\begin{aligned} & \min_{\theta \ge 1} \quad \theta \\ & \text{s.t.} \quad R_e < P_i < \theta R_e \\ & \qquad \begin{bmatrix} A_i^T P_i + P_i A_i & P_i \\ P_i & 0 \end{bmatrix} < \xi \begin{bmatrix} R_e & 0 \\ 0 & R_\omega \end{bmatrix} \end{aligned} \tag{24}$$

Then $\varepsilon_{\min} = \sqrt{\theta^{k_{[0,T]}} e^{\xi T} (\theta \delta_e^2 + T \xi \delta_\omega^2)}$ can be derived with the optimized value θ.

It is evident that smaller value of ε gives rise to less conservative FTB conditions. In *Theorem 8*, the parameter ξ indicates the asymptotic stability property of each subsystem. It is well known that when $\xi = 0$ in condition (12b), this condition can be regarded as Lyapunov function condition which ensures each subsystem to be asymptotic stable; whereas when $\xi > 0$, the condition that $\dot{V}(t)$ must be negative is relaxed in FTB sense, and $\dot{V}(t)$ just should be no greater than $\dot{V}_i(t) < \xi \alpha^{-1} V_i + \xi \delta_\omega^2$ to guarantee the boundedness of state in finite-time interval $[0, T]$. The parameter $\xi \ge 0$ in condition (12b) covers both the asymptotic stable and unstable subsystems. Now let subsystems $A_1, \cdots, A_r$ be asymptotic stable and $A_{r+1}, \cdots, A_m$ are unstable, and T^-, T^+ denote the total activation time for stable and unstable subsystems during $[0, T]$. Then the less conservative results about FTB of affine switched system can be obtained in the following corollary.

Corollary 11. *Switched system (2) is finite-time bounded (FTB) with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$, if there exist a set of positive definite symmetric matrices P_i, $i \in I$, scalars $\alpha > 0$, $\beta > 0$, and $\xi^+ \ge 0$ such that the following conditions are satisfied:*

$$\alpha R_e < P_i < \beta R_e \tag{25a}$$

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i \\ P_i & 0 \end{bmatrix} \le \begin{cases} 0 & i \le r \\ \xi^+ \begin{bmatrix} R_e & 0 \\ 0 & R_\omega \end{bmatrix} & i > r \end{cases} \tag{25b}$$

$$e^{(\xi^+/\alpha) T^+ + k_{[0,T]} \ln(\beta/\alpha)} \left(\beta \delta_e^2 + T^+ \xi^+ \delta_\omega^2 \right) - \alpha \varepsilon^2 < 0 \tag{25c}$$

Proof. Consider the error tracking switched system (4), let $\mathscr{R} = \operatorname{dig}(R_e, R_\omega)$, $\eta_i = \left[e^T(t) \;\; k_i^T \right]^T$, $i \in I$; we choose piecewise Lyapunov function $V_i(t) = e^T(t) P_i e(t)$.

From condition (25b), we get

$$\begin{aligned} \dot{V}_i(t) &= \dot{e}^T(t) P_i e(t) + e^T(t) P_i \dot{e}(t) \\ &= \begin{bmatrix} e(t) \\ k_i \end{bmatrix}^T \begin{bmatrix} A_i^T P_i + P_i A_i & P_i \\ P_i & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ k_i \end{bmatrix} \\ &< \begin{cases} 0 & i \le r \\ \xi^+ \eta_i^T \mathscr{R} \eta_i & i > r \end{cases} \Longrightarrow \\ \dot{V}_i(t) &< \begin{cases} 0 & i \le r \\ \dfrac{\xi^+}{\alpha} V_i + \xi^+ \delta_\omega^2 & i > r \end{cases} \end{aligned} \tag{26}$$

Let $\forall t > 0$, $t_0 < \cdots < t_k$ be the switching instant of switched system, and $V(t) = \sum_{i\in I}\theta_i V_i(t), \theta_i \in \{0,1\}$. From inequalities (25a) and (26), we have

$$V(t) < \begin{cases} \beta\alpha^{-1}V(t_k^-) & i(t_k^+) \le r \\ \beta\alpha^{-1}\phi(t,t_k)V(t_k^-) + \xi^+\delta_\omega^2\int_{t_k}^{t}\phi(t,\tau)\,d\tau & i(t_k^+) > r \end{cases} \tag{27}$$

where $\phi(t,\tau) = \exp(\xi^+\alpha^{-1}(t-\tau)) < \exp(\xi^+\alpha^{-1}T^+)$. Following (27) iteratively

$$\begin{aligned} V(t) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi^+/\alpha)T^+}V(t_0) \\ &\quad + T^+\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\xi^+\delta_\omega^2 e^{(\xi^+/\alpha)T^+} \end{aligned} \tag{28}$$

By the same proof line in Theorem 8, we know that in order to ensure the finite-time boundedness of switched system (4), i.e., $e^T(t)R_e e(t) \le \varepsilon^2$, the following condition should be satisfied:

$$\begin{aligned} \alpha e^T(t)R_e e(t) &< V(t) \\ &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi^+/\alpha)T^+}V(t_0) \\ &\quad + T^+\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\xi^+\delta_\omega^2 e^{(\xi^+/\alpha)T^+} < \alpha\varepsilon^2 \end{aligned} \tag{29}$$

Since $V(t_0) = e_0^T P_i e_0 < \beta e_0^T R_e e_0 \le \beta\delta_e^2$, we have

$$\begin{aligned} &\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi^+/\alpha)T^+}\beta\delta_e^2 + T^+\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\xi^+\delta_\omega^2 e^{(\xi^+/\alpha)T^+} \\ &\quad < \alpha\varepsilon^2 \end{aligned} \tag{30}$$

which can be rewritten as (25c). Hence, $(x(t)-x_r)^T R_e(x(t)-x_r) \le \varepsilon^2$ and proof is complete. □

Remark 12. Similar to the optimization problem of state bound ε described in Remark 10, the optimal value $\varepsilon_{\min}$ can be found according to $\min(\beta/\alpha)^{k_{[0,T]}}e^{\xi^+\alpha^{-1}T^+}(\beta\delta_e^2 + T^+\xi^+\delta_\omega^2)\alpha^{-1}$ subject to (25a) and (25b). We fix the parameter ξ^+ and let $\alpha = 1$, $\beta = \theta\alpha$, the optimization problem becomes

$$\begin{aligned} &\min_{\theta\ge 1} \quad \theta \\ &s.t. \quad R_e < P_i < \theta R_e \\ &\qquad \begin{bmatrix} A_i^T P_i + P_i A_i & P_i \\ P_i & 0 \end{bmatrix} < \xi^+\begin{bmatrix} R_e & 0 \\ 0 & R_\omega \end{bmatrix} \end{aligned} \tag{31}$$

Then the minimum $\varepsilon_{\min} = \sqrt{\theta^{k_{[0,T]}}e^{\xi^+T^+}(\theta\delta_e^2 + T^+\xi^+\delta_\omega^2)}$ can be derived with the optimized value θ. Since $T^+ \le T$, comparing the value of the optimal state bound $\varepsilon_{\min}$ in *Theorem 8* and *Corollary 11*, we know that, by classifying subsystems into asymptotically stable and unstable, the FTB conditions derived in *Corollary 11* are less conservative than that in *Theorem 8*.

Constituting state-feedback controller of the form $u(t) = K_i x(t)$, affine switched system (1) can be transformed into the closed-loop form of (5) and *Definition 1* of FTB can be used directly. Now we will consider problem-2 to provide sufficient conditions for finite-time state-feedback stabilization.

Theorem 13. *For affine switched system (1) holding* Definition 1 *, if there exist state-feedback controllers $u(t) = K_i x(t)$, positive definite matrices Q_i, matrices X_i, and scalars $\theta \ge 1$, $\xi \ge 0$ such that*

$$\begin{bmatrix} -Q_i & Q_i \\ * & -R_e^{-1} \end{bmatrix} < 0, \qquad \begin{bmatrix} -\theta R_e & I \\ * & -Q_i \end{bmatrix} < 0 \tag{32a}$$

$$\begin{bmatrix} A_iQ_i + Q_iA_i^T + B_iX_i + X_i^TB_i^T - \xi Q_i & I \\ I & -\xi R_\omega \end{bmatrix} < 0 \tag{32b}$$

$$e^{\xi T + k_{[0,T]}\ln\theta}\left(\theta\delta_e^2 + T\xi\delta_\omega^2\right) - \varepsilon^2 < 0 \tag{32c}$$

then closed-loop system (5) is FTB with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$ with $K_i = X_iQ_i^{-1}$.

Proof. Assume x_r is the switched equilibrium point of affine switched system (1). Applying coordinate transformation we can get its corresponding error tracking switched system as

$$\dot{e}(t) = A_ie(t) + B_iu(t) + k_i \tag{33}$$

where $k_i = A_ix_r + b_i$ and $e(t) = x(t) - x_r$. Then under the state-feedback controllers $u(t) = K_ie(t)$, and the closed-loop error system can be written as

$$\dot{e}(t) = \overline{A}_ie(t) + k_i \tag{34}$$

where $\overline{A}_i = A_i + B_iK_i$. From Remark 2, we know that FTB analysis and finite-time control can be realized employing tracking error system. Hence, we consider the closed-loop error system (34) here to design the controllers stabilizing the system (1) in finite-time interval.

Let $\mathscr{R} = \mathrm{dig}(R_e, R_\omega)$, $\eta_i = \left[e^T(t)\ k_i^T\right]^T$, we choose piecewise Lyapunov function $V_i(t) = e^T(t)P_ie(t)$ for each subsystem; then the derivative of V_i along the solution of system (34) is described as

$$\begin{aligned}\dot{V}_i(t) &= \dot{e}^T(t) P_i e(t) + e^T(t) P_i \dot{e}(t) = \begin{bmatrix} e(t) \\ k_i \end{bmatrix}^T \\ &\cdot \begin{bmatrix} \overline{A}_i^T P_i + P_i \overline{A}_i & P_i \\ P_i & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ k_i \end{bmatrix} = \begin{bmatrix} e(t) \\ k_i \end{bmatrix}^T \\ &\cdot \begin{bmatrix} A_i^T P_i + K_i^T B_i^T P_i + P_i A_i + P_i B_i K_i & P_i \\ P_i & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ k_i \end{bmatrix}\end{aligned} \tag{35}$$

Letting $Q_i = P_i^{-1}$, pre- and postmultiplying (32b) by $\mathrm{diag}(P_i, I)$ we get

$$\begin{bmatrix} P_i A_i + A_i^T P_i + P_i B_i K_i + K_i^T B_i^T P_i - \xi P_i & P_i \\ P_i & -\xi R_\omega \end{bmatrix} < 0 \tag{36}$$

Due to condition (32a) and Schur's complement formula [38], we deduce

$$\begin{gathered}\begin{bmatrix} -Q_i & Q_i \\ * & -R_e^{-1} \end{bmatrix} < 0 \Longrightarrow \\ R_e < P_i, \\ \begin{bmatrix} -\theta R_e & I \\ * & -Q_i \end{bmatrix} < 0 \Longrightarrow \\ P_i < \theta R_e\end{gathered} \tag{37}$$

Now using (36), from (35) we can derive

$$\begin{aligned}\dot{V}_i(t) &< \eta_i^T \begin{bmatrix} \xi P_i & 0 \\ 0 & \xi R_\omega \end{bmatrix} \eta_i = \xi e^T(t) P_i e(t) + \xi k_i^T R_\omega k_i \\ &< \xi V_i + \xi \delta_\omega^2\end{aligned} \tag{38}$$

By the same proof guidelines of Theorem 8, FTB conditions (32a) and (32c) of closed-loop error system (34) can be derived. Accordingly we get $(x(t) - x_r)^T R_e (x(t) - x_r) \le \varepsilon^2$, which proves that the affine switched system (1) is finite-time bounded under state-feedback controllers $u(t) = K_i e(t)$. □

4. H_∞ Performance Analysis and Controller Design of Affine Switched Systems

Based upon FTB investigation of previous section, our main aim now is to design a set of H_∞ controllers to solve Problem 7. As stated in Remark 2, finite-time H_∞ control can be realized through tracking error system, and this will be the main focus in this section. For the sake of simplicity, we firstly consider the autonomous error switched system in the form of (10) assuming that $k_i = 0$, $u(t) = 0$ and the corresponding theorem is stated as follows; then we will show how to remove the assumption and extend the results to the general affine switched system with exogenous signal input.

Theorem 14. *Given autonomous robust switched system (10), if there exist positive definite matrices P_i, scalars $\alpha > 0$, $\beta > 0$, and $\xi \ge 0$ such that*

$$\alpha R_e < P_i < \beta R_e \tag{39a}$$

$$\begin{bmatrix} A_i^T P_i + P_i A_i + C_i^T C_i & P_i G_i + C_i^T D_{2i} \\ * & -\gamma^2 I + D_{2i}^T D_{2i} \end{bmatrix} < \xi \begin{bmatrix} R_e & 0 \\ * & R_\omega \end{bmatrix} \tag{39b}$$

$$e^{(\xi/\alpha)T + k_{[0,T]} \ln(\beta/\alpha)} \left(\beta \delta_e^2 + T \xi \delta_\omega^2 + \gamma^2 d^2\right) - \alpha \varepsilon^2 < 0 \tag{39c}$$

then this system is finite-time bounded with H_∞ performance γ with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$.

Proof. Let $\mathscr{R} = \mathrm{dig}(R_e, R_\omega)$, $\eta_i = \left[e^T(t) \;\; \omega^T(t)\right]^T$, and we opt Lyapunov function $V_i(t) = e^T(t) P_i e(t)$ and

$$\begin{aligned}\dot{V}_i(t) &= \dot{e}^T(t) P_i e(t) + e^T(t) P_i \dot{e}(t) \\ &= \begin{bmatrix} e(t) \\ \omega(t) \end{bmatrix}^T \begin{bmatrix} A_i^T P_i + P_i A_i & P_i G_i \\ * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \omega(t) \end{bmatrix}\end{aligned} \tag{40}$$

Since $\left[\begin{smallmatrix} C_i^T C_i & C_i^T D_{2i} \\ * & D_{2i}^T D_{2i} \end{smallmatrix}\right] = \left[\begin{smallmatrix} C_i^T \\ D_{2i}^T \end{smallmatrix}\right] [C_i \;\; D_{2i}] \ge 0$, condition (39b) implies that

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i G_i \\ * & -\gamma^2 I \end{bmatrix} < \xi \begin{bmatrix} R_e & 0 \\ * & R_\omega \end{bmatrix} \tag{41}$$

From (40) and (41), $\dot{V}_i(t) < \xi e^T(t) R_e e(t) + \xi \delta_\omega^2 + \gamma^2 \omega^T(t) \omega(t)$ and together with condition (39a), we get

$$\begin{aligned}\dot{V}_i(t) &< \frac{\xi}{\alpha} e^T(t) P_i e(t) + \xi \delta_\omega^2 + \gamma^2 \omega^T(t) \omega(t) \\ &= \frac{\xi}{\alpha} V_i + \xi \delta_\omega^2 + \gamma^2 \omega^T(t) \omega(t)\end{aligned} \tag{42}$$

Let $\forall t > 0$, $t_0 < \cdots < t_k$ be the switching instants, and $V(t) = \sum_{i \in I} \theta_i V_i(t)$, $\theta_i \in \{0, 1\}$, where θ_i is the indication function for activated subsystem. From inequality (42), we have

$$\begin{aligned}V(t) &< \phi(t, t_k) V(t_k^+) \\ &\quad + \int_{t_k}^{t} \phi(t, \tau) \left[\xi \delta_\omega^2 + \gamma^2 \omega^T(\tau) \omega(\tau)\right] d\tau\end{aligned} \tag{43}$$

where $\phi(t, \tau) = \exp((\xi/\alpha)(t - \tau)) < \exp((\gamma/\alpha)T)$. Accordingly, the Lyapunov inequality in single step satisfies

$$\begin{aligned}&V(t_{k+1}) \\ &\quad < \phi(t_{k+1}, t_k) \frac{V(t_k^+)}{V(t_k^-)} V(t_k^-) \\ &\quad + \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \tau) \left[\xi \delta_\omega^2 + \gamma^2 \omega^T(\tau) \omega(\tau)\right] d\tau\end{aligned} \tag{44}$$

Let system switch from mode i to j at instant $t_k (0 < t_k < T)$; then condition (39a) implies that

$$\begin{aligned}\frac{V(t_k^+)}{V(t_k^-)} &= \frac{V_j(t_k)}{V_i(t_k)} = \frac{e^T(t_k)P_j e(t_k)}{e^T(t_k)P_i e(t_k)} \\ &< \frac{\beta e^T(t_k)R_e e(t_k)}{\alpha e^T(t_k)R_e e(t_k)} = \frac{\beta}{\alpha}\end{aligned} \tag{45}$$

Noting that $\beta/\alpha > 1$, following relation (44) iteratively, we can derive

$$\begin{aligned}V(t_k) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \Bigg\{\phi(t_k,t_0)V(t_0) \\ &+ \sum_{n=1}^{k}\left(\frac{\beta}{\alpha}\right)^{-n}\phi(t_k,t_n) \\ &\cdot \int_{t_{n-1}}^{t_n}\phi(t_n,\tau)\left[\xi\delta_\omega^2 + \gamma^2\omega^T(\tau)\omega(\tau)\right]d\tau\Bigg\}\end{aligned} \tag{46}$$

Applying (43) and (46), the following inequality is obtained:

$$\begin{aligned}V(t) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \Bigg\{\phi(t,t_0)V(t_0) \\ &+ \sum_{n=1}^{k}\left(\frac{\beta}{\alpha}\right)^{-n}\phi(t,t_n) \\ &\cdot \int_{t_{n-1}}^{t_n}\phi(t_n,\tau)\left[\xi\delta_\omega^2 + \gamma^2\omega^T(\tau)\omega(\tau)\right]d\tau\Bigg\} \\ &+ \int_{t_k}^{t}\phi(t,\tau)\left[\xi\delta_\omega^2 + \gamma^2\omega^T(\tau)\omega(\tau)\right]d\tau \\ &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T}V(t_0) + \sum_{n=1}^{k}\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}-n} \\ &\cdot \int_{t_{n-1}}^{t_n}\phi(t,\tau)\left[\xi\delta_\omega^2 + \gamma^2\omega^T(\tau)\omega(\tau)\right]d\tau \\ &+ e^{(\xi/\alpha)T}\int_{t_k}^{t}\xi\delta_\omega^2 + \gamma^2\omega^T(\tau)\omega(\tau)\,d\tau < \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \\ &\cdot e^{(\xi/\alpha)T}V(t_0) + \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T}\xi\delta_\omega^2 T + \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \\ &\cdot e^{(\xi/\alpha)T}\left[\gamma^2\int_{t_0}^{t}\omega^T(\tau)\omega(\tau)\,d\tau\right]\end{aligned} \tag{47}$$

Using the fact $\int_0^T \omega^T(\tau)\omega(\tau)d\tau \le d^2$, (47) can be rewritten as

$$\begin{aligned}V(t) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T}V(t_0) \\ &+ T\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\xi\delta_\omega^2 e^{(\xi/\alpha)T} \\ &+ \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\gamma^2 d^2 e^{(\xi/\alpha)T}\end{aligned} \tag{48}$$

On the other hand, from condition (39a), we have

$$V(t) = e^T(t)P_i e(t) > \alpha e^T(t)R_e e(t) \tag{49}$$

Since $V(t_0) = e^T(t_0)P_i e(t_0) < \beta e^T(t_0)R_e e(t_0) \le \beta\delta_e^2$ we conclude that in order to ensure FTB for system (10) such that $e^T(t)R_e e(t) \le \varepsilon^2$, the following condition should be satisfied:

$$\begin{aligned}\alpha e^T(t)R_e e(t) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T}\beta\delta_e^2 \\ &+ T\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\xi\delta_\omega^2 e^{(\xi/\alpha)T} \\ &+ \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\gamma^2 d^2 e^{(\xi/\alpha)T} < \alpha\varepsilon^2\end{aligned} \tag{50}$$

which can be rewritten as condition (39c). Hence, FTB analysis for system (10) is completed.

Considering $z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot V(t)$, from (39a) and (39b) we deduce

$$\begin{aligned}&z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot V(t) = \begin{bmatrix} e(t) \\ \omega(t)\end{bmatrix}^T \\ &\quad\cdot \begin{bmatrix} A_i^T P_i + P_i A_i + C_i^T C_i & P_i G_i + C_i^T D_{2i} \\ * & D_{2i}^T D_{2i} - \gamma^2 I\end{bmatrix}\begin{bmatrix} e(t) \\ \omega(t)\end{bmatrix} \\ &\quad< \xi\begin{bmatrix} e(t) \\ \omega(t)\end{bmatrix}^T\begin{bmatrix} R_e & 0 \\ * & R_\omega\end{bmatrix}\begin{bmatrix} e(t) \\ \omega(t)\end{bmatrix} < \frac{\xi}{\alpha}V(t) + \xi\delta_\omega^2 \Longrightarrow \\ &\dot V(t) < \xi\alpha^{-1}V(t) + \xi\delta_\omega^2 - z^T(t)z(t) + \gamma^2\omega^T(t)\omega(t)\end{aligned} \tag{51}$$

Integrating both sides of (51) and through iterations, we can deduce

$$\begin{aligned}V(t_k) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}}\Bigg[\phi(t_k,t_0)V(t_0) + \sum_{n=1}^{k}\left(\frac{\beta}{\alpha}\right)^{-n} \\ &\cdot \phi(t_k,t_n) \\ &\cdot \int_{t_{n-1}}^{t_n}\left[\xi\delta_\omega^2 + \gamma^2\omega^T(\tau)\omega(\tau) - z^T(\tau)z(\tau)\right] \\ &\cdot \phi(t_n,\tau)\,d\tau\Bigg]\end{aligned} \tag{52}$$

where $\phi(t,\tau) = \exp(\xi\alpha^{-1}(t-\tau)) < \exp(\xi\alpha^{-1}T)$. Then following the proof line of Theorem 8, we get

$$\begin{aligned} 0 \le V(t) &< \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T} V(t_0) + T\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \\ &\cdot \xi\delta_\omega^2 e^{(\xi/\alpha)T} + \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \\ &\cdot e^{(\xi/\alpha)T}\int_{t_0}^{t} \gamma^2\omega^T(\tau)\,\omega(\tau) - z^T(\tau)\,z(\tau)\,d\tau \end{aligned} \tag{53}$$

Since the zero-initial condition, we have $V(0) = 0$; thus,

$$\begin{aligned} &\left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T}\int_{t_0}^{t} \gamma^2\omega^T(\tau)\,\omega(\tau) - z^T(\tau)\,z(\tau)\,d\tau \\ &\quad > 0 \end{aligned} \tag{54}$$

Setting $t = T$, $t_0 = 0$, the H_∞ performance condition (11) is satisfied which completes the proof. □

Remark 15. The parameter γ is H_∞ performance index and its minimum value $\gamma_{\min}$ is often of interest from practical viewpoint; hence, we can state the optimization problem as

$$\begin{aligned} &\min \quad \gamma^2 \\ &s.t. \quad (39a), (39b), (39c) \end{aligned} \tag{55}$$

Similarly, fulfilling FTB criteria, minimum value of state bound $\varepsilon_{\min}$ is also desired, which can be found as the optimization problem: $\min(\beta/\alpha)^{k_{[0,T]}} e^{(\xi/\alpha)T}(\beta\delta_e^2 + T\xi\delta_\omega^2 + \gamma^2 d^2)\alpha^{-1}$ subject to (39a) and (39b). If we fix the parameter ξ and let $\alpha = 1$, $\beta = \theta\alpha$, then we can state optimization problem as

$$\begin{aligned} &\min_{\theta \ge 1} \quad \theta \\ &s.t. \quad R_e < P_i < \theta R_e \\ &\qquad \begin{bmatrix} A_i^T P_i + P_i A_i + C_i^T C_i & P_i G_i + C_i^T D_{2i} \\ * & -\gamma^2 I + D_{2i}^T D_{2i} \end{bmatrix} \\ &\qquad < \xi \begin{bmatrix} R_e & 0 \\ * & R_\omega \end{bmatrix} \end{aligned} \tag{56}$$

and $\varepsilon_{\min} = \sqrt{\theta^{k_{[0,T]}} e^{\xi T}(\theta\delta_e^2 + T\xi\delta_\omega^2 + \gamma^2 d^2)}$ is derived with the optimized value of θ. We can adopt a convex combination of $\gamma_{\min}$ and $\varepsilon_{\min}$ as $J(\rho) = \rho\gamma_{\min}^2 + (1-\rho)\varepsilon_{\min}^2$, $0 \le \rho \le 1$ and a more general convex optimization problem can be stated as

$$\begin{aligned} &\min \quad J(\rho) \\ &s.t. \quad R_e < P_i < \theta R_e \\ &\qquad \begin{bmatrix} A_i^T P_i + P_i A_i + C_i^T C_i & P_i G_i + C_i^T D_{2i} \\ * & -\gamma^2 I + D_{2i}^T D_{2i} \end{bmatrix} \\ &\qquad < \xi \begin{bmatrix} R_e & 0 \\ * & R_\omega \end{bmatrix} \\ &\qquad e^{(\xi/\alpha)T + k_{[0,T]}\ln(\beta/\alpha)}\left(\beta\delta_e^2 + T\xi\delta_\omega^2 + \gamma^2 d^2\right) - \alpha\varepsilon^2 \\ &\qquad < 0 \end{aligned} \tag{57}$$

Now we will extend the results to design the H_∞ controllers, ensuring FTB of the closed-loop affine switched system (8). Different equilibria for subsystems exist because of the affine terms k_i, and hence stability analysis and H_∞ control are not trivial. To solve this problem, a few results are available proposing extended state space method in [12, 31]. However, this approach seems conservative for system synthesis because the eigenvalues of the extended state matrices $\overline{A}_{iex}$ related to the affine terms are not exactly the same as for the original state matrices $\overline{A}_i$. For state-dependent affine switched system, S-procedure method can be used to reduce the conservatism. However, for time-dependent affine switched systems, there are only few effective results. In our investigation, we redefine exogenous signal $\omega(t)$ as

$$\widetilde{\omega}(t) = \begin{bmatrix} \omega(t) \\ k_i \end{bmatrix} \tag{58}$$

Hence, the closed-loop switched system (8) can be rewritten as

$$\begin{aligned} \dot{e}(t) &= \overline{A}_i e(t) + \widetilde{G}_i \widetilde{\omega}(t), \quad x(0) = x_0 \\ z(t) &= \overline{C}_i e(t) + \widetilde{D}_{2i}\widetilde{\omega}(t), \quad \widetilde{\omega}(0) = \widetilde{\omega}_0 \end{aligned} \tag{59}$$

where $\widetilde{G}_i = [G_i, I]$, $\widetilde{D}_{2i} = [D_{2i}, 0]$. Since in the H_∞ framework $\int_0^T \omega^T(t)\omega(t)dt \le d^2$ holds, the proposed extension of the disturbance input is reasonable and we can design the H_∞ controllers of the equivalent closed-loop error switched system (59) to ensure the finite-time H_∞ boundedness for original affine switched system (8).

Theorem 16. *The closed-loop switched system (59) is FTB with H_∞ performance γ regarding $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$, if there exist constant state-feedback controller $u(t) = K_i x(t)$, positive definite matrices Q_i, matrices W_i, scalars $\theta \ge 1$, $\xi \ge 0$ such that*

$$\begin{aligned} &\begin{bmatrix} -Q_i & Q_i \\ * & -R_e^{-1} \end{bmatrix} < 0, \\ &\begin{bmatrix} -\theta R_e & I \\ * & -Q_i \end{bmatrix} < 0 \end{aligned} \tag{60a}$$

$$\begin{bmatrix} A_iQ_i + Q_iA_i^T + B_iW_i + W_i^TB_i^T - \xi Q_i & \widetilde{G}_i & Q_iC_i^T + W_i^TD_{1i}^T \\ \widetilde{G}_i^T & -\gamma^2 I - \xi R_\omega & \widetilde{D}_{2i}^T \\ C_iQ_i + D_{1i}W_i & \widetilde{D}_{2i} & -I \end{bmatrix} < 0 \tag{60b}$$

$$e^{\xi T + k_{[0,T]}\ln\theta}\left(\theta\delta_e^2 + T\xi\delta_\omega^2 + \gamma^2 d^2\right) - \varepsilon^2 < 0 \tag{60c}$$

where $K_i = W_iQ_i^{-1}$.

Proof. Let $\mathscr{R} = \text{dig}(R_e, R_\omega)$ and $\eta_i = \left[e^T(t)\ \ \widetilde{\omega}^T(t)\right]^T$, $i \in I$. Defining $V_i(t) = e^T(t)P_ie(t)$ as before, for system (59) we can state that

$$\begin{aligned} \dot{V}_i(t) &= \dot{e}^T(t)P_ie(t) + e^T(t)P_i\dot{e}(t) \\ &= \begin{bmatrix} e(t) \\ \widetilde{\omega}(t) \end{bmatrix}^T \begin{bmatrix} \overline{A}_i^TP_i + P_i\overline{A}_i & P_i\widetilde{G}_i \\ * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \widetilde{\omega}(t) \end{bmatrix} \end{aligned} \tag{61}$$

Assuming $Q_i = P_i^{-1}$, pre- and postmultiplying (60b) by $\text{diag}(P_i, I, I)$,

$$\begin{bmatrix} \overline{A}_i^{\,T}P_i + P_i\overline{A}_i - \xi P_i & P_i\widetilde{G}_i & \overline{C}_i^T \\ \widetilde{G}_i^TP_i & -\gamma^2I - \xi R_\omega & \widetilde{D}_{2i}^T \\ \overline{C}_i & \widetilde{D}_{2i} & -I \end{bmatrix} < 0 \tag{62}$$

Using Schur lemma, (62) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} \overline{A}_i^{\,T}P_i + P_i\overline{A}_i - \xi P_i + \overline{C}_i^T\overline{C}_i & P_i\widetilde{G}_i + \overline{C}_i^T\widetilde{D}_{2i} \\ * & -\gamma^2I - \xi R_\omega + \widetilde{D}_{2i}^T\widetilde{D}_{2i} \end{bmatrix} \\ &\quad < 0 \end{aligned} \tag{63}$$

Since $\begin{bmatrix} \overline{C}_i^T\overline{C}_i & \overline{C}_i^T\widetilde{D}_{2i} \\ * & \widetilde{D}_{2i}^T\widetilde{D}_{2i} \end{bmatrix} = \begin{bmatrix} \overline{C}_i^T \\ \widetilde{D}_{2i}^T \end{bmatrix}\begin{bmatrix} \overline{C}_i & \widetilde{D}_{2i} \end{bmatrix} \geq 0$, we can get

$$\begin{bmatrix} \overline{A}_i^{\,T}P_i + P_i\overline{A}_i & P_i\widetilde{G}_i \\ * & -\gamma^2I \end{bmatrix} < \xi\begin{bmatrix} P_i & 0 \\ * & R_\omega \end{bmatrix} \tag{64}$$

which implies that

$$\begin{aligned} \dot{V}_i(t) &< \xi e^T(t)P_ie(t) + \xi k_i^TR_\omega k_i + \gamma^2\widetilde{\omega}^T(t)\widetilde{\omega}(t) \\ &= \xi V_i + \xi\delta_\omega^2 + \gamma^2\widetilde{\omega}^T(t)\widetilde{\omega}(t) \end{aligned} \tag{65}$$

Employing (60a) and using Schur complement formula,

$$\begin{gathered} \begin{bmatrix} -Q_i & Q_i \\ * & -R_e^{-1} \end{bmatrix} < 0 \Longrightarrow \\ R_e < P_i, \\ \begin{bmatrix} -\theta R_e & I \\ * & -Q_i \end{bmatrix} < 0 \Longrightarrow \\ P_i < \theta R_e \end{gathered} \tag{66}$$

Following the proof guidelines of Theorem 14, condition (60c) which guarantees the FTB of robust affine switched system can be obtained.

Now we need to prove condition (9) for H_∞ performance under zero-initial conditions. From (60b),

$$z^T(t)z(t) - \gamma^2\widetilde{\omega}^T(t)\widetilde{\omega}(t) + \dot{V}(t) < \xi V(t) + \xi\delta_\omega^2 \tag{67}$$

Applying integration and iterations, and setting $V(t_0) = 0$ under zero-initial conditions, we get

$$\begin{aligned} 0 &\leq V(t) \\ &< T\theta^{k_{[0,T]}}\xi\delta_\omega^2e^{\xi T} \\ &\quad + \theta^{k_{[0,T]}}e^{\xi T}\int_{t_0}^{t}\gamma^2\widetilde{\omega}^T(\tau)\widetilde{\omega}(\tau) - z^T(\tau)z(\tau)\,d\tau \end{aligned} \tag{68}$$

Then setting $t = T$, $t_0 = 0$, we obtain that

$$\begin{aligned} &\int_0^T\gamma^2\widetilde{\omega}^T(\tau)\widetilde{\omega}(\tau) - z^T(\tau)z(\tau)\,d\tau \\ &\quad = \int_0^T\gamma^2\left(\omega^T(\tau)\omega(\tau) + k_{\max}^Tk_{\max}\right) - z^T(\tau)z(\tau)\,d\tau \\ &\quad > 0 \end{aligned} \tag{69}$$

which illustrates that condition (9) is satisfied. We conclude that the affine switched system (59), and hence closed-loop affine switched system (8), is FTB with H_∞ performance γ. □

Remark 17. Unlike the normal switched system, the existence of multiple equilibria for affine switched systems is related to subsystems owing to the affine terms k_i, which makes asymptotic and finite-time stability analysis nontrivial. As for the finite-time H_∞ controller design, concerning with both the external disturbance $\omega(t)$ and the affine terms k_i, we redefined the conception of H_∞ controller for affine switched system in *Definition 4*, based on which the results in this section are obtained. It is worth noting that the H_∞ performance of affine switched system reduces to normal H_∞ performance when assuming $k_i = 0$.

5. Numerical Examples

Example 1. Consider the affine switched system (2) with two modes of operation:

$$\begin{gathered} A_1 = \begin{bmatrix} 0.01 & -2 \\ 1 & 0.02 \end{bmatrix}, \\ A_2 = \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix}; \\ b_1 = \begin{bmatrix} -3.98 \\ -1.16 \end{bmatrix}, \\ b_2 = \begin{bmatrix} -1.8 \\ -6.4 \end{bmatrix} \end{gathered} \tag{70}$$

A_1 is unstable, A_2 is Hurwitz stable, and eigenvalues $\lambda(A_1) = \{0.015 \pm 1.4142i\}$, $\lambda(A_2) = \{-0.1 \pm 1.732i\}$. Assuming desired reference $x_r = [2, -2]^T$, error tracking switched system will be

$$\begin{gathered} A_1 = \begin{bmatrix} 0.01 & -2 \\ 1 & 0.02 \end{bmatrix}, \\ A_2 = \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix}; \\ k_1 = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}, \\ k_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix} \end{gathered} \tag{71}$$

Evidently $k_{\max} = \left[\begin{smallmatrix} 0 \\ 0.8 \end{smallmatrix}\right]$. Let $\delta_e = 1$, $\delta_\omega = 0.8$, $\varepsilon = 5.885$, $R_e = R_\omega = I$, $\xi = 0.3$, and $T = 5s$. From the FTB condition (24), we get the average dwell time $\tau_a^* = 1.5s$ to ensure the finite-time boundedness with respect to ε, so that the switching signal S can be chosen as a periodical signal with $T_s = 1.5s$, which implies that $k_{[0,T]} = 3$ and $T^+ = 2s$, $T^- = 3s$ during the finite-time interval $[0, 5]$. Given the initial error state $e(0) = [0.5, 0.8]^T$, the conditions $e_0^T R_e e_0 \le \delta_e^2$ and $k_{\max}^T R_\omega k_{\max} \le \delta_\omega^2$ are separately satisfied, then the error state trajectory of error affine switched system and the value of $e^T R_e e$ under the switching signal S are shown in Figure 1.

It is easy to see in Figure 1 that subject system is FTB with conditions (12a), (12b), and (12c) satisfied. Moreover, assuming $\xi^+ = \xi = 1$ and using optimization process (24) and (31), the optimal value $\theta_{\min} = 1.0058$, $P_1 = \left[\begin{smallmatrix} 0.8425 & 0.1287 \\ 0.1287 & 1.2053 \end{smallmatrix}\right]$, $P_2 = \left[\begin{smallmatrix} 1.3718 & -0.0977 \\ -0.0977 & 1.0416 \end{smallmatrix}\right]$ can be obtained. Then substituting $\theta_{\min}$ into (12c) and (25c) separately, we get

$$\begin{gathered} \varepsilon_{1\min} = 3.014, \\ \varepsilon_{2\min} = 1.923 \end{gathered} \tag{72}$$

where $\varepsilon_{1\min}$, $\varepsilon_{2\min}$ denote the minimum bound of state derived by Theorem 8 and Corollary 11. It is obvious that Corollary 11 is less conservative than Theorem 8 since $\varepsilon_{2\min} > \varepsilon_{1\min}$.

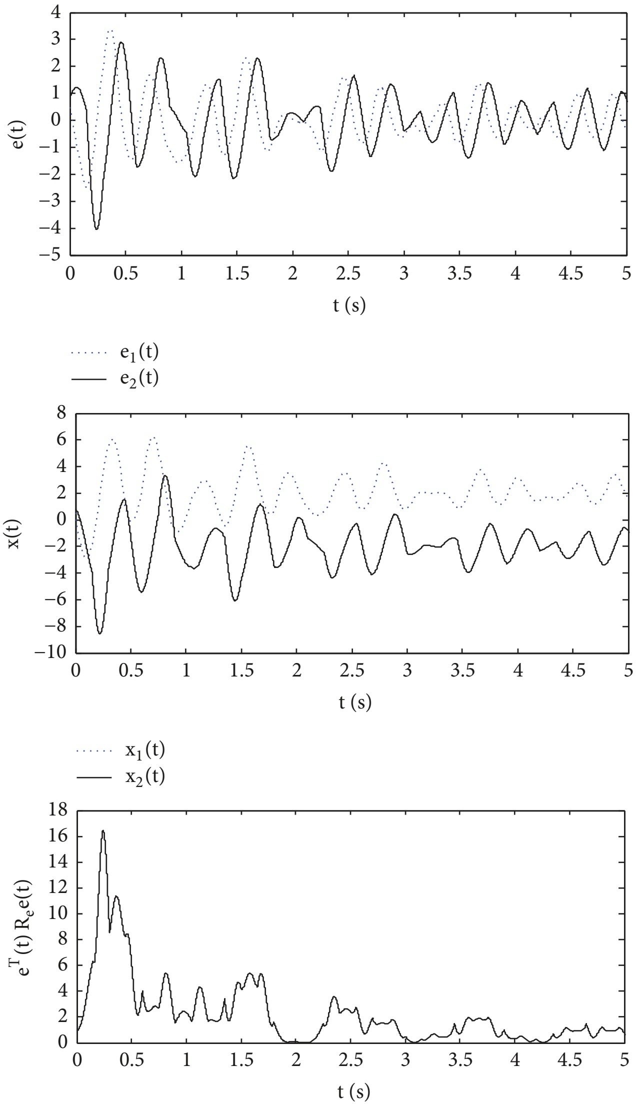

Figure 1: The state trajectories and the value of $e^T(t)R_e e(t)$ under switching signal S.

Example 2. Keeping in view autonomous error switched system (10), we consider this system:

$$\begin{gathered} A_1 = \begin{bmatrix} 0.01 & -2 \\ 1 & 0.02 \end{bmatrix}, \\ k_1 = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}, \\ G_1 = \begin{bmatrix} 0.25 \\ 0.01 \end{bmatrix}, \\ C_1 = [0.10 \quad 0.33], \end{gathered}$$

$$
\begin{aligned}
D_{21} &= 0.05\\
A_2 &= \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix},\\
k_2 &= \begin{bmatrix} 0 \\ -0.2 \end{bmatrix},\\
G_2 &= \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix},\\
C_2 &= \begin{bmatrix} 0.3 & 0.01 \end{bmatrix},\\
D_{22} &= 0.028
\end{aligned}
\tag{73}
$$

with disturbance signal:

$$
\omega(t) = \begin{cases} 8 & 0 \le t \le 5 \\ 0 & else \end{cases}
\tag{74}
$$

which satisfies$(\int_0^T \omega^T(t)\omega(t)dt)^{1/2} = (\int_0^5 \omega^T(t)\omega(t)dt)^{1/2} = 8\sqrt{5}$. Let $\delta_e = 1$, $\delta_\omega = 0.8$, $\varepsilon = 21.758$, $\gamma = 0.2$, and $\xi = 0.3$. From FTB condition (39c), we get the average dwell time $\tau_a^* = 1.5s$ to ensure FTB with respect to ε. Then for the finite-time H_∞ performance, we should have

$$
\left(\int_0^T z^T(t)\, z(t)\, dt\right)^{1/2} < \gamma \left(\int_0^T \omega^T(t)\, \omega(t)\, dt\right)^{1/2} \approx 3.57
\tag{75}
$$

The simulation results with initial error state $e(0) = [0.5, 0.8]^T$ are shown in Figure 2.

We observe in Figure 2 that the system is FTB, and the H_∞ performance satisfies

$$
\left(\int_0^T z^T(t)\, z(t)\, dt\right)^{1/2} \approx 2.18 < 3.57
\tag{76}
$$

Thus, according to *Definition 4*, the autonomous robust error switched system can be regarded as finite-time H_∞ bounded. Moreover, using optimization procedure (56) we get $\theta_{\min} = 1.932$, $P_{1,1} = \left[\begin{smallmatrix} 1.0165 & 0.0016 \\ 0.0016 & 1.9154 \end{smallmatrix}\right]$, $P_{2,1} = \left[\begin{smallmatrix} 2.2218 & -0.2047 \\ -0.2047 & 1.0425 \end{smallmatrix}\right]$. Putting in (39c), we get $\varepsilon_{\min} = 6.314$.

Example 3. Consider the affine error switched system (7) with two modes of operation:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.01 & -2 \\ 1 & 0.02 \end{bmatrix},\\
k_1 &= \begin{bmatrix} 0 \\ 0.8 \end{bmatrix},\\
\mathrm{B}_1 &= \begin{bmatrix} 0.1 \\ -1 \end{bmatrix},
\end{aligned}
$$

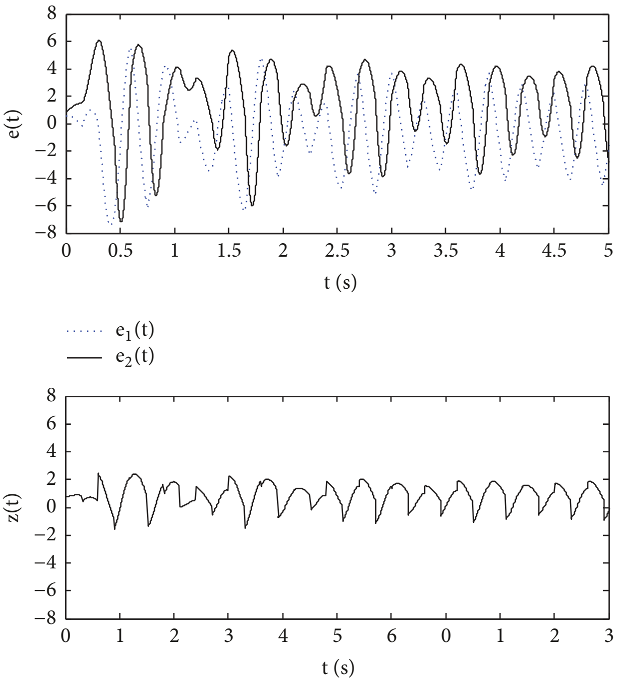

FIGURE 2: The state responds of e(t) and z(t) under switching signal S.

$$
\begin{aligned}
\mathrm{G}_1 &= \begin{bmatrix} 0.25 \\ 0.01 \end{bmatrix},\\
C_1 &= \begin{bmatrix} 0.10 & 0.33 \end{bmatrix},\\
\mathrm{D}_{11} &= 0.13,\\
D_{21} &= 0.05\\
A_2 &= \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix},\\
k_2 &= \begin{bmatrix} 0 \\ -0.2 \end{bmatrix},\\
\mathrm{B}_2 &= \begin{bmatrix} -1 \\ 0.5 \end{bmatrix},\\
G_2 &= \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix},\\
C_2 &= \begin{bmatrix} 0.3 & 0.01 \end{bmatrix},\\
\mathrm{D}_{12} &= 0.2,\\
D_{22} &= 0.028\\
\omega(t) &= \begin{cases} 8 & 0 \le t \le 5 \\ 0 & else \end{cases}
\end{aligned}
\tag{77}
$$

which implies that $(\int_0^T \omega^T(t)\omega(t)dt)^{1/2} = (\int_0^5 \omega^T(t)\omega(t)dt)^{1/2} = 8\sqrt{5}$ and $k_{\max} = [\begin{smallmatrix}0\\0.8\end{smallmatrix}]$.

The objective in this example is to design a set of robust H_∞ controllers ensuring finite-time H_∞ boundedness of closed-loop error switched system with respect to $(\delta_e, \delta_\omega, \varepsilon, R_e, R_\omega, T)$, where $\delta_e = 1$, $\delta_\omega = 0.8$, $\gamma = 0.2$, $\varepsilon = 4.536$, $R_e = R_\omega = I$, $\xi = 0.3$, and $T = 5s$. Setting switching signal S as a periodical signal with $T_s = 1.5s$, based on Theorem 16, we calculate

$$\begin{aligned} Q_1 &= \begin{bmatrix} 1.4665 & 0.0957 \\ 0.0957 & 1.3408 \end{bmatrix}, \\ W_1 &= \begin{bmatrix} -1.2110 & 0.4831 \end{bmatrix} \\ Q_2 &= \begin{bmatrix} 1.1617 & -0.2958 \\ -0.2958 & 1.7466 \end{bmatrix}, \\ W_2 &= \begin{bmatrix} 3.0403 & 0.9730 \end{bmatrix} \end{aligned} \tag{78}$$

Then we can get the set of H_∞ controllers for each subsystem as

$$\begin{aligned} K_1 &= W_1 Q_1^{-1} = [-0.8533, 0.4212], \\ K_2 &= W_2 Q_2^{-1} = [2.8834, 1.0454] \end{aligned} \tag{79}$$

Substitute controller gains into system (8), the closed-loop error switched system can be written as

$$\begin{aligned} \overline{A}_1 &= \begin{bmatrix} -0.0753 & -1.9579 \\ 1.8533 & -0.4012 \end{bmatrix}, \\ k_1 &= \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}, \\ G_1 &= \begin{bmatrix} 0.25 \\ 0.01 \end{bmatrix}, \\ \overline{C}_1 &= \begin{bmatrix} -0.0109 & 0.3848 \end{bmatrix}, \\ D_{21} &= 0.05 \\ \overline{A}_2 &= \begin{bmatrix} -2.9834 & -2.0454 \\ 4.4417 & 0.4227 \end{bmatrix}, \\ k_2 &= \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}, \\ G_2 &= \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, \\ \overline{C}_2 &= \begin{bmatrix} 0.8767 & 0.2191 \end{bmatrix}, \\ D_{22} &= 0.028 \end{aligned} \tag{80}$$

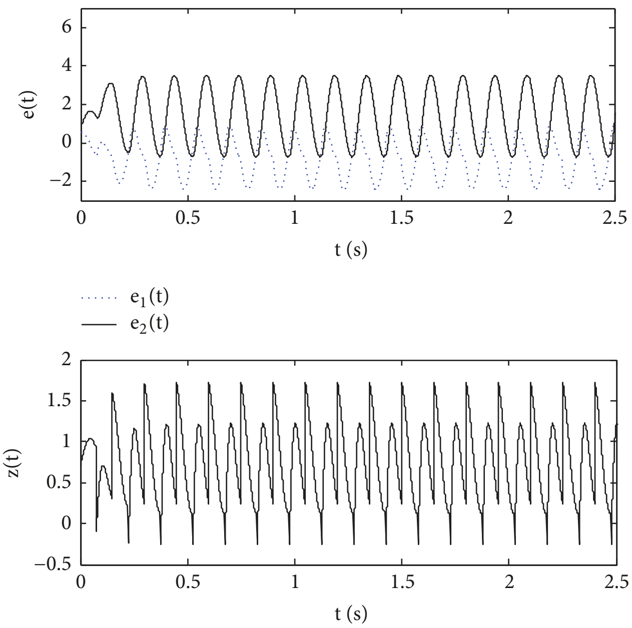

Figure 3: The state responds of $e(t)$ and $z(t)$ under switching signal S.

State responses under state-feedback controllers and switching signal S are shown in Figure 3. We can observe that the closed-loop system is FTB, and the H_∞ performance satisfies

$$\begin{aligned} &\left(\int_0^T z^T(t)\, z(t)\, dt\right)^{1/2} \approx 0.9366 \\ &\quad < \gamma \left(\int_0^T \omega^T(t)\, \omega(t) + k_{\max}^T k_{\max} dt\right)^{1/2} \approx 3.596 \end{aligned} \tag{81}$$

Thus, according to Definition 4, the given affine switched system can be regarded as finite-time H_∞ bounded under designed H_∞ controller gains.

6. Conclusion

In this paper, the problem of finite-time boundedness and finite-time H_∞ control for affine switched systems has been investigated. Several definitions and sufficient conditions for FTB and H_∞ performance are proposed. Based on the average dwell-time method, the FTB conditions of affine switched linear system with known state boundary are derived first in this investigation. To reduce the conservatism of FTB conditions, by classifying subsystems into asymptotically stable and unstable systems, we get the improved FTB conditions for affine switched system presented in Corollary 11. The conservatism of conditions under the two situations is compared. Then applying the finite-time boundedness analysis results, finite-time H_∞ performance is discussed. Finite-time H_∞ controllers are designed to ensure the corresponding closed-loop switched system FTB with H_∞ performance. Numerical examples are finally provided to validate our theoretical results. Many real-world systems concern with finite-time and transient behavior; meanwhile, many engineering applications can be modeled as affine switched systems. Therefore, our theoretical results about finite-time boundedness of affine switched systems are supposed to have great potential in the application of practical switched systems. Furthermore, the proposed results in this paper can be extended to the nonlinear affine switched systems which will be considered in future work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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