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Finite-Time Input-to-State Stability and Applications to Finite-Time Control ⋆

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Abstract: This paper extends the well-known concept, input-to-state stability (ISS), to finite-time control problems. In other words, a new concept, finite-time input-to-state stability (FTISS), along with related concepts such as finite-time input-output stability and finite-time small-gain theorems, is discussed, and then is applied to both the finite-time stability analysis and the finite-time stabilizing feedback design. With finite-time convergence, non-smoothness has to be considered, which poses serious technical challenges in the analysis and synthesis of closed-loop finite-time systems. It is found that FTISS plays a key role in the study of finite-time stability and stabilization of nonlinear systems.

1. INTRODUCTION

Systems analysis and control synthesis to deal with the stability and stabilization of nonlinear systems have become more and more important following various practical demands. Many nonlinear control approaches, including feedback linearization, backstepping, control Lyapunov functions, input-to-state stability, passivity-based control, and nonlinear small-gain techniques were proposed in the last few decades (Huang [2004], Isidori [1995], Khalil [2002], Krstic et al. [1995], Sontag [2000], Jiang et al. [1994], Teel [1996]) to tackle stability and stabilization of nonlinear systems.

Most of these nonlinear feedback tools focus on the design of smooth controllers for various classes of nonlinear systems. Among the nonlinear control techniques, Sontag's input-to-state stability (ISS) Sontag [2000] provides an effective way to tackle stabilization of nonlinear systems or their robust and adaptive control in the presence of various uncertainties arising from control engineering applications. On the other hand, non-smooth (including discontinuous and continuous but not Lipschitz continuous) control approaches have drawn increasing attention in nonlinear control system design. One of the main benefits of the non-smooth finite-time control strategy is that it can force a control system to reach a desirable target in finite time. This approach was first studied in the literature of optimal control. In recent years, different finite-time stabilizing feedback laws have been constructed for some classes of nonlinear systems (Bhat et al. [2000], Hong et al. [2001], Moulay et al. [2006]). In addition, some control designs have been proposed for specific classes of uncertain nonlinear systems using a backstepping-like procedure (Hong et al. [2006]). These finite-time controllers can also yield, in some sense, fast response and high tracking precision as well as disturbance-rejection properties because of their non-smoothness. Despite its potential application to practical problems, the study of finite-time stabilization is quite under-developed, partially because of the lack of effective tools in non-smooth analysis and synthesis.

The objective of this paper is to develop a framework for the design of non-smooth vs. smooth controllers with finite-time stability based on the finite-time variant of ISS, which we term as *finite-time ISS* (FTISS). Characterizations of FTISS are presented and its combination with non-smooth feedback is proposed to yield a new design tool for finite-time stabilization of nonlinear systems.

The paper is organized as follows. The problem formulation is introduced in Section 2. Then, in Section 3, some results on FTISS are reviewed, while issues on finite-time input-to-output stability (IOS) are addressed in Section 4. Following that, finite-time feedback design via FTISS is reported in Section 5. Finally, concluding remarks are given in Section 6.

2. CONCEPTS AND PRELIMINARIES

In this section, we will give some related preliminary knowledge to deal with the non-smoothness resulting from of finite-time stability of nonlinear systems for the following investigation.

First of all, we should not that the Dini derivative is important in the analysis of non-smooth dynamics. Consider a system in the following form

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$$\dot{x} = f(t, x), \quad f(t, 0) = 0, \quad x \in R^n \tag{1}$$

and let $x(t)$ be a generic solution. Further, let $V(t, x) : R \times R^n \to R$ be a continuous functions, satisfying a local Lipschitz condition for x, uniformly with respect to t. The detailed introduction of Dini derivative of $V(x, x(t))$ can be found in Rouch et al. [1977].

The following is a basic concept about finite-time control Bhat et al. [2000]: The equilibrium $x = 0$ of system (1) is (locally) finite-time stable if it is Lyapunov stable and (locally) finite-time convergent in a neighborhood U. By "finite-time convergence", we mean that, for any initial conditions $x(t_0) = x_0 \in U$, there is a settling-time function $T \geq 0$, which is a continuous function of x_0, such that every solution $x(t; t_0, x_0)$ of system (1) is defined with $x(t; t_0, x_0) \in U/\{0\}$ for $t \in [t_0, T)$ and satisfies

$$\lim_{t \to t_0 + T} x(t; t_0, x_0) = 0, \quad x(t; t_0, x_0) = 0, \ \forall t > t_0 + T.$$

When $U = R^n$, the origin is a globally finite-time stable equilibrium.

In what follows, we will focus on the finite-time stability analysis for autonomous systems. In this case, the considered systems with finite-time convergence must be nonsmooth (more precisely, non-Liptchitz).

A function $\gamma : R_+ \to R_+$ is said to be a generalized K-function if it is continuous with $\gamma(0) = 0$, and satisfies

$$\gamma(s) = \max\{0, \bar{\gamma}(s) - \bar{\gamma}(s_0)\}, \tag{2}$$

where $\bar{\gamma}$ is a K-function and s_0 is a given parameter. Note that the class of generalized K-functions includes as a special case the class of (conventional) K-functions, which are defined as continuous and strictly increasing functions with $\gamma(0) = 0$. As usual, a function γ is a K_∞-function if it is a (conventional) K-function and also $\gamma(s) \to \infty$ as $s \to \infty$; and it is a positive definite function if $\gamma(s) > 0$ for all $s > 0$ and $\gamma(0) = 0$. A function $\beta : R_+ \times R_+ \to R_+$ is a generalized KL-function if, for each fixed $t \geq 0$, the function $\beta(s, t)$ is a generalized K-function, and for each fixed $s \geq 0$ it decreases to zero as $t \to T$ for some $T \leq \infty$.

For simplicity, we still say "K-function" (or "KL-function") when we use generalized K-function (or generalized KL-function) in the sequel when there is no confusion.

Then, we introduce new concepts to combine finite-time control with ISS ideas. Consider system

$$\dot{z} = f(z, v), \quad f(0, 0) = 0, \quad z \in R^n, \ v \in R^m, \tag{3}$$

where f is continuous with respect to z and v.

Definition 1. System (3) is (locally) finite-time input-to-state stable (ISS) (with v as the input) in a neighborhood U, if, for any initial time $t_0 \geq 0$, initial state $z(t_0) = z_0 \in U$ and bounded input $v \in U_v$ (for some neighborhood U_v), we have $z(t) \in U$ and

$$||z(t)|| \leq \beta(||z_0||, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} ||v(\tau)||), \tag{4}$$

where γ is a K-function and β is a KL-function with $\beta(||z_0||, t - t_0) \equiv 0$ when $t \geq t_0 + T$ for some function T continuous with respect to z_0. When $U = R^n$ and $U_v = R^m$, the system is (globally) finite-time input-to-state stable.

In what follows, we write "finite-time input-to-state stable" instead of "globally finite-time input-to-state stable" if there is no confusion.

If a system is FTISS with $v \in R^m$ as the input, then it is FTISS with input $w = \chi(v)$ for a continuous function χ with $\chi(0) = 0$. Moreover, suppose χ is a homeomorphism between v and $w = \chi(v)$ and then a system is FTISS with $v \in R^m$ as the input if and only if it is FTISS with w as the input.

Obviously, finite-time ISS implies ISS. Note that the main difference between ISS and finite-time ISS is the finite-time convergence of β; that is, "$\beta(||z_0||, t - t_0) \equiv 0$ when $t \geq t_0 + T$.

Example 1. Consider the system, $\dot{z} = -sgn(z) - z^3 + v^2$, where $sgn(\cdot)$ is the sign function. If we take

$$\beta(||z_0||, t - t_0) = \begin{cases} ||z(t_0)|| - (t - t_0) & \text{if} \quad t_0 \leq t \\ & \leq ||z(t_0)|| + t_0 \ , \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma(s) = s^{2/3} \ ,$$

then we obtain that the system is FTISS. ∎

Example 2. FTISS implies the finite-time stability property when $v = 0$, but the converse may not be true, even in the bounded-input-bounded-state (BIBS) case. Consider

$$\dot{z} = -(1 + \sin v) z^{\frac{1}{3}}, \tag{5}$$

which is BIBS and finite-time stable when $v = 0$. However, taking $v = 3\pi/2$ makes the system not finite-time stable and therefore, (5) is not finite-time ISS. ∎

Definition 2. System (3) is robustly finite-time stable with a stability margin if there is a K_∞-function ρ (called a stability margin) and a KL-function β with $\beta(||z_0||, t) \equiv 0$ when $t \geq T$ for some function T continuous with respect to z_0, such that, for every feedback law $v(t, z)$ bounded by $\rho(||z||)$ it holds that

$$||z(t)|| \leq \beta(||z(0)||, t), \quad \forall t \geq 0 \tag{6}$$

for every solution of system

$$\dot{z} = f(z, v(t, z))$$

Fix any smooth positive definite and proper (or radially bounded) function φ. Then for any $d(t) \in M$, where

$$M = \{\text{all measurable functions from } R \text{ to } [-1, \ 1]^m\}, \tag{7}$$

we can rewrite (3) as

$$\dot{z} = f(z(t), d(t)\varphi(z(t))) = g(z, d), \ g(0, d) = 0. \tag{8}$$

In equation (8), $d(t)$ can be viewed as the disturbance input. System (8) is uniformly globally finite-time stable (UGFTS) if it is uniformly stable (that is, for some K_∞-function $\delta(\cdot)$, and for each $\varepsilon \geq 0$, the estimate $z(t, z_0, d) \leq \varepsilon$ holds for all $d \in M$, $||z_0|| \leq \delta$, and $t \geq 0$) and uniformly finite-time convergent (that is, for each $r > 0$, there is a $T > 0$ such that, $||z(t, z_0, d)|| = 0$ for every $d \in M$, $||z_0|| \leq r$, $t \geq T$). System (3) is *weakly* robustly finite-time stable if there is a smooth positive definite and proper function φ satisfying $||\bar{\varphi}(||z_0||) \leq ||\varphi(z_0)||$ for some K_∞-function $\bar{\varphi}$ so that the corresponding system (8) is UGFTS.

Then, we introduce some useful inequalities:

Lemma 3. (Jiang et al. [1994]) For any K-function γ, any K_∞-function ρ such that $\rho - Id$ is a K_∞ function, and any nonnegative real numbers a and b, we have

$$\gamma(a+b) \leq \gamma(\rho(a)) + \gamma(\rho \circ (\rho - Id)^{-1}(b)). \quad (9)$$

Lemma 4. For any continuous function $g(x,z)$, there are continuous nonnegative functions $g_1(x)$ and $g_2(z)$ such that

$$|g(x,z)| \leq g_1(x) + g_2(z). \quad (10)$$

Moreover, both functions $g_1(x)$ and $g_2(z)$ vanish at zero when $g(0,0) = 0$.

Note that the proof of the above lemma is quite easy and that the inequality has been used in the research work of others, too many to cite.

Lemma 5. (Young's inequality, Hardy et al. [1952]): Let $f(x)$ be a strictly increasing continuous function. Then

$$ab \leq \int_0^a f(x)\, dx + \int_0^b f^{-1}(x) dx, \quad (11)$$

where f^{-1} is the inverse function of f. Particularly, we have

$$ab \leq \frac{a^{1+c}}{1+c} + \frac{c\, b^{1+\frac{1}{c}}}{1+c}, \quad a \geq 0,\ b \geq 0,\ c > 0. \quad (12)$$

3. ISS-LYAPUNOV FUNCTION

In this section, we will review the relationship between FTISS and the existence of an ISS-Lyapunov function guaranteeing finite-time ISS. In fact, the main result of this section can be found in Hong et al. [2007].

A function V is called an ISS-Lyapunov function for system (3) if there exist K_∞-functions ϕ_1, ϕ_2, and K-functions ϕ_4, ϕ_3, such that

(A1)

$$\phi_1(||z||) \leq V(z) \leq \phi_2(||z||), \quad \forall z \in R^n; \quad (13)$$

(A2) For any z, v with $||z|| \geq \phi_0(||v||)$, we have

$$D^+V(z)f(z,v) \leq -\phi_3(||z||), \quad (14)$$

which is equivalent to asking that there are K_∞-functions ϕ_3, ϕ_4 such that, for any $z \in R^n$, $v \in R^m$,

$$D^+V(z)f(z,v) \leq -\phi_3(||z||) + \phi_4(||v||), \quad (15)$$

A function $V(z)$ is called a finite-time ISS-Lyapunov function for system (3) if it is an ISS-Lyapunov function with conditions (A1) and (A2), and $\phi_3(||z||)$ is $O(V(z)^a)$ (as $z \to 0$) for some constant a with $0 < a < 1$.

Consider a dynamic system

$$\dot{z} = g(z(t), d(t)), \quad g(0, d(t)) = 0 \quad (16)$$

where g is continuous with respect to z and d. (16) is called to be robustly finite-time stable if it is finite-time stable for any disturbance $d(t)$ taking values in a compact set of R^m.

Without loss of generality, we assume that $d \in M \subset R^m$ defined as in (7), and call functions d time-varying parameters.

If system (16) is finite-time stable for any fixed $d(t)$, then we define *the duration between the initial time and the settling time* as $T_{d(t)}(t_0, z) = \inf\{t - t_0 : \psi_d(t, t_0, z) = 0\}$ with initial condition $z(t_0) = z$ because the solution may not be unique. Here $T_{d(t)}(t_0, z)$ of system (16) is assumed to be continuous with respect to (t_0, z, d).

In our case, $d(t)$ is not fixed. Define a function

$$T_*(z) = \sup_{d \in M} T_d(t_0, z) \geq 0 \quad (17)$$

for system (16), where $\psi_d(t, t_0, z)$ represents the state at the fixed moment t with $z(t_0) = z$ as its initial state. Moreover, we have the following facts about T_*:

1) There is $d_*(t) \in M$ (for a given t_0) to make T_* become a settling time function T_{d_*}, that is, $T_*(z) = T_{d_*}(t_0, z)$, where $d_*(t)$ can be viewed as the "worst" disturbance to slow down the convergence rate for any fixed initial condition z. Therefore, T_* is still bounded and continuous with respect to z because $T_{d_*}(t_0, z)$ is so.
2) T_* is certainly a function of t_0, z and $d_*(t)$, expressed as $T_*(t_0, d_*, z)$, but we can prove that it is a function only depending on the initial condition z. In fact, we consider two cases: $z(t_0^1) = \bar{z}$ and $z(t_0^2) = \bar{z}$. Clearly, system (16) with $z(t_0^1) = \bar{z}$ and any given $d^1(t) \in M$ is equivalent to system (16) with $z(t_0^2) = \bar{z}$ and $d^2(t) = d^1(t + t_0^1 - t_0^2) \in M$. If we take the relative settling time $T_{d^i}(t_0^i, \bar{z})$ for the case of $z(t_0^i) = \bar{z}$, $(i = 1, 2)$, then $T_{d^1}(t_0^1, \bar{z}) = T_{d^2}(t_0^2, \bar{z})$. Therefore, $T_*(t_0^1, d_*^1, \bar{z}) = T_*(t_0^2, d_*^2, \bar{z})$, which implies that T_* of system (16) only depends on the value of initial state $\bar{z}$.
3) $T_*(\psi_{d_*}(t, t_0, z)) \geq T_d(t_0, \psi_d(t, t_0, z))$ at every point.

It was shown in Bhat et al. [2000] that $\dot{z} = f(z)$ is finite-time stable if and only if there are a continuous Lyapunov function V and constants $0 < c$, $0 < a < 1$ such that $\dot{V}(z) \leq -cV(z)^a$. In the following, we extend this result to a dynamical system with disturbances.

The equivalence relation between FTISS and the existence of finite-time ISS-Lyapunov function was shown in Hong et al. [2007].

Theorem 6. The following conditions are equivalent:

1. System (3) is finite-time ISS with v as input;

2. there is an ISS-Lyapunov function $V_0(z)$, which is a positive-definite and proper function such that

$$\dot{V}_0 := D^+V_0\dot{z} \leq -\gamma_3(V_0) + \gamma_4(||v||), \quad (18)$$

where $\gamma_3(V_0) = O(V_0^a)$ as $V_0 \to 0$ for some positive constant $a < 1$, and γ_i $(i = 3, 4)$ are K_∞-functions.

3. System (3) is robustly finite-time stable.

4. System (3) is weakly robustly finite-time stable.

Bhat et al. [2000] considered the sensitivity of a finite-time stable system to perturbations, where boundedness or (local) finite-time convergence were derived on basis of the existence of Lipschitz continuous Lyapunov function. In our work, finite-time robust stability is related to, but different from, their works.

Example 3. Going back to Example 1, it is easy to see that we can select a smooth finite-time ISS-Lyapunov function $V(z) = z^2$ with

$$\dot{V} \leq -2V^{\frac{1}{2}} + \frac{3}{2}v^{\frac{8}{3}}.$$

∎

4. FINITE-TIME INPUT-OUTPUT STABILITY

In this section, we will extend FTISS to finite-time input-output stability and obtain a related small-gain theorem.

In fact, the above discussion can be extended to the systems with measured variables. In other words, we can also consider finite-time input-output properties for nonlinear systems with output variables:

$$\begin{cases} \dot{x} = f(x,u), & x \in R^n,\ x(t_0) = x_0,\ u \in R^m \\ y = h(x,u), & y \in R^l,\ h(0,0) = 0,\ f(0,0) = 0 \end{cases} \tag{19}$$

where f and h are continuous.

System (19) is finite-time input-output stable (FTIOS) if there exist a KL-function β and a K-function γ such that, for each initial condition $x(t_0) = x_0$, each measurable essentially bounded control u on $[t_0, \infty)$, and each t in the right maximal interval of the definition of the corresponding solution of system (19), we have

$$||y(t)|| \le \beta(||x_0||, t - t_0) + \gamma(\sup_{t_0 \le \tau \le t} ||u(\tau)||), \tag{20}$$

with $\beta(||x_0||, t - t_0) \equiv 0$ when $t \ge T$ for some function T continuous with respect to x_0.

Moreover, system (19) is said to be finite-time zero-state detectable if for all $x \in R^n$, $u \equiv 0$, $y \equiv 0$, $\forall t \ge t_0$ will leads to $\lim_{t \to T} x(t) = 0$ and $x \equiv 0$, $\forall t \ge T$ for some finite time T. Moreover, system (19) is said to be finite-time strongly detectable (SD) if there exist a KL-function β^0 and a K-function γ^0 such that, for each measurable control $u(t)$ defined on $[t_0, T^0)$ with $t_0 \le T^0 \le \infty$, the solution $x(t)$ of (19) right maximally defined on $[t_0, T')$ ($t_0 \le T' \le T^0$) satisfies

$$||x(t)|| \le \beta^0(||x_0||, t - t_0) + \gamma^0(\sup_{t_0 \le \tau \le t} ||(u(\tau)^T, y(\tau)^T)^T||),$$

for $t \in [t_0, T')$ with $\beta^0(||x_0||, t - t_0) \equiv 0$ when $t \ge T$ for some $T \le T'$ depending on x_0.

Clearly, finite-time strong detectability implies finite-time zero-state detectability, which implies the conventional zero-state detectability.

Then we consider the system (19) with output variable y, and the following result shows the relationship between finite-time ISS and finite-time IOS.

Theorem 7. If system (19) is finite-time ISS (FTISS) with input u, then it is finite-time zero-state detectable and is finite-time IOS. Conversely, if system (19) is finite-time IOS and finite-time SD, then it is finite-time ISS.

Proof: Due to ISS of the x-system of system (19), we have

$$||x(t)|| \le \beta(||x_0||, t - t_0) + \gamma(||u||_{[t_0,t]})$$

with $\beta(||x_0||, t - t_0) = 0$, $t \ge T$ for some $T \ge t_0$, and then finite-time SD can be easily obtained.

Indeed, it is not hard to see that there are two K-functions γ_x and γ_u such that

$$||h(x,u)|| \le \gamma_x(||x||) + \gamma_u(||u||)$$

by Lemma 3. Therefore, FTIOS follows readily because we have

$$||y(t)|| \le \gamma_x \circ 2\beta(||x_0||, t - t_0) + (\gamma_x \circ 2\gamma + \gamma_u)(||u||_{[t_0,t]}).$$

Conversely, we have two KL-functions β and β^0 and two K-functions γ and γ^0 such that, for $t \ge t_0 \ge 0$,

$$||y(t)|| \le \beta(||x(t_0)||, t - t_0) + \gamma(||u||_{[t_0,t]}), \tag{21}$$

$$||x(t)|| \le \beta^0(||x(t_0)||, t - t_0) + \gamma^0(||(u,y)||_{[t_0,t]}), \tag{22}$$

where $\beta(||x(t_0)||, t - t_0) = 0$ for $t \ge T(t_0)$ and $\beta^0(||x(t_0)||, t - t_0) = 0$ for $t \ge T_0$, with $T(t_0)$ and $T_0(t_0)$ continuous with respect to $x(t_0)$.

By taking $t_0 = t/2$ in (22) and plugging (21) with $t_0 = 0$, we have

$$||x(t)|| \le \beta^0 \left(||x(\frac{t}{2})||, \frac{t}{2} \right) + \gamma^0(\cdot). \tag{23}$$

Note that, according to Lemma 3 for any K_∞-function ρ with $\rho - Id$ being K_∞-function, we have

$$\begin{aligned} \gamma^0(\cdot) &:= \gamma^0 \left(||u||_{[0,t]} + \beta(||x(0)||, \frac{t}{2}) + \gamma(||u||_{[0,t]}) \right) \\ &\le \gamma^0(\rho(\beta(||x(0)||, \frac{t}{2})) + \bar{\gamma}(||u||_{[0,t]}) \end{aligned}$$

where $\bar{\gamma} = \gamma^0 \left(\rho \circ (\rho - Id)^{-1} \circ (Id + \gamma) \right)$ is a K_∞-function.

Moreover,

$$||x(\frac{t}{2})|| \le \beta^0(||x(0)||, \frac{t}{2}) + \gamma^0(\rho(\beta(||x(0)||, 0))) + \bar{\gamma}(||u||_{[0,t]}).$$

Then, again by Lemma 3 and some manipulations, for all $t \ge 0$, we have

$$||x(t)|| \le \beta_*(||x(0)||, t) + \gamma_*(||u||_{[0,t]}) \tag{24}$$

with some KL-function $\beta_*(||x(0)||, t)$ and $\gamma_* = \bar{\gamma} + \hat{\gamma}$, which leads to ISS of system (19). Moreover, $\beta^0(||x(0)||, t) = 0$ and $\beta(||x(0)||, t) = 0$ when $t \ge T_0$ for some constant T_0 due to the finite-time convergence properties of β^0 and β, and then we will see that there is a T^0 such that $\beta_*(||x(0)||, t) = 0$ when $t \ge T^0$, which implies FTISS. ∎

The next result is a generalized small-gain theorem for finite-time ISS systems. Consider nonlinear interconnected systems in the following form:

$$\dot{x} = f(x, y_z, u),\ f(0,0,0) = 0,\ y_x = h_1((x, y_z, u) \tag{25}$$

$$\dot{z} = g(z, y_x, v),\ g(0,0,0) = 0,\ y_z = h_2(z, y_x, v), \tag{26}$$

where $x \in R^{n_x}$, $z \in R^{n_z}$, $u \in R^{n_u}$, $v \in R^{n_v}$, $y_x \in R^{m_x}$, $y_z \in R^{m_z}$, and $(y_z, y_z) = h(x,z,u,v)$ is the unique solution of

$$\begin{cases} y = h_1((x, h_2(z, y_x, v), u), \\ z = h_2(z, h_1(x, y_z, u), v) \end{cases}$$

Theorem 8. Suppose systems (25) and (26) are finite-time IOS with (y_z, u) and (y_x, v) as input, and y_x and y_z as output, respectively, satisfying:

$$\begin{cases} ||y_x(t)|| \le \beta_1(||x_0||, t) \\ \qquad + \gamma_1^y(\sup_{0 \le \tau \le t} ||y_z(\tau)||) + \gamma_1^u(\sup_{0 \le \tau \le t} ||u(\tau)||), \\ ||y_z(t)|| \le \beta_2(||x_0||, t) \\ \qquad + \gamma_2^y(\sup_{0 \le \tau \le t} ||y_x(\tau)||) + \gamma_2^u(\sup_{0 \le \tau \le t} ||v(\tau)||), \end{cases} \tag{27}$$

for suitable functions $\beta_i, \gamma_i^y,\ \gamma_i^u,\ i = 1,2$. Also suppose (25) and (26) are finite-time SD. If there are two K_∞-functions $\rho_i,\ 1,2$ and a nonnegative number c_l satisfying

$$(Id + \rho_2) \circ \gamma_2^y \circ (Id + \rho_1) \circ \gamma_1^y(s) \le s, \qquad s \ge c_l, \tag{28}$$

(or equivalently, $(Id + \rho_1) \circ \gamma_1^y \circ (Id + \rho_2) \circ \gamma_2^y(s) \le s$), then system (25)-(26) with $w = (u,v)$ as input, $y = (y_x, y_z)$ as output, and $\xi = (x,z)$ as state is finite-time IOS.

Proof: IOS can be obtained directly from Theorem 2.1 of Jiang et al. [1994]. Consider the construction of $\hat{\beta}_i$ based on β_i $(i = 1, 2)$ in the proof of Theorem 2.1 of Jiang et al. [1994] and then the finite-time convergence can also be obtained. ∎

Corollary 9. Suppose the ζ-subsystem of the system

$$\begin{cases} \dot{\xi} = q(\xi, v), & q(0,0) = 0 \\ \dot{\zeta} = g(\xi, \zeta, v), & g(0,0,0) = 0. \end{cases} \tag{29}$$

is finite-time ISS with (ξ, v) as input and the ξ-subsystem of system (29) is finite-time ISS with input v. Then system (29) is finite-time ISS with input v.

The conclusion can be obtained directly from Theorem 8 by taking $x = y_x = \xi$, $y_z = 0$, $u = v$, and $z = \zeta$.

5. FINITE-TIME ISS-STABILIZABILITY

In this section, we will see how these ISS concepts can be used in the finite-time control. Note that, for a finite-time stable system, we are not sure if it admits a smooth Lyapunov function (in fact, it may not be so in general, referring to Bhat et al. [2000]). However, most of the existing results on finite time stabilization exhibit the existence of smooth Lyapunov function. Thus, when we study how to apply finite time ISS in control design, we can consider a Lipschitz continuous (or even smooth) Lyapunov function.

At first, we introduce Jensen's inequality:

$$(\sum_{i=1}^{n} x_i^{c_2})^{1/c_2} \leq (\sum_{i=1}^{n} x_i^{c_1})^{1/c_1}, \quad 0 < c_1 < c_2, \tag{30}$$

with $x_i \geq 0,\ 1 \leq i \leq n$.

Next, we give a lemma to generalize the discussions on finite-time stability and finite-time ISS-Lyapunov function.

Lemma 10. Consider system (3). Suppose there are an ISS-Lyapunov function $V(z) = \sum_{i=1}^{n} V_i(z_i^0)$ with $z_i^0 = (0, ..., 0, z_i, 0, ..., 0)^T$, a positive definite function φ_1, and a K-function φ_2 such that

$$\dot{V}(z) \leq -\varphi_1(z) + \varphi_2(||v||), \tag{31}$$

where $\varphi_1(z) \sim \sum_{i=1}^{n} \bar{c}_i V_i(z_i^0)^{a_i}$, $\bar{c}_i > 0$, and $0 < a_i < 1$ for $i = 1, ..., n$. Then the system is finite-time ISS with v as the input.

Its proof is omitted for the space limitations.

Then, we consider the system of the form:

$$\begin{cases} \dot{z} = g(x, z), & g(0,0) = 0,\ f(0,0) = 0, \\ \dot{x} = f(x, z) + v, & (v, x, z) \in R \times R \times R^l \end{cases} \tag{32}$$

where f and g are smooth functions. It is not hard to see that, for any continuous function $\mu(z)$,

$$|f(x, z)| \leq f_1(\bar{h}) + f_2(z), \quad \bar{h} = x - \mu(z) \tag{33}$$

with continuous functions $f_1(\bar{h}) = O(f)$ for fixed $z \neq 0$ and $f_2(z) = O(f)$ for fixed $\bar{h} \neq 0$.

Theorem 11. Suppose the z-subsystem of (32) is FTISS with respect to the input function $h(x, z) = |x|^p sgn(x) - |\mu(z)|^p sgn(\mu(z))$ for some $p > 1$, where $|\mu(z)|^p sgn(\mu(z))$ is C^1 with respect to z and $\mu(0) = 0$. Moreover, the z-subsystem admits a Lipschitz continuous Lyapunov function V_z and there is a constant $q > p$ such that

$$\dot{V}_z(z) \leq -\gamma_1(V_z(z)) + \gamma_2(|h(x, z)|),\ \gamma_2(|h|) = O(|h|^{1+\frac{1}{q}}), \tag{34}$$

where γ_i $(i = 1, 2)$ are K-functions with $\gamma_1(V_z) \sim V_z^a$ (as $z \to 0$) for some constant $0 < a < 1$,

$$|f(x, z)| \leq \hat{f}(h) + \bar{f}(z),$$
$$\hat{f}(h) = O(|h|^{\frac{1}{p}}),\ \bar{f}(z) = O(\gamma_1(V_z(z))^{\frac{1}{1+q}}), \tag{35}$$

and

$$\left| \frac{\partial |\mu(z)|^p sgn(\mu(z))}{\partial z} g(x, z) \right| \leq \hat{g}_1(h) + \hat{g}_2(z), \tag{36}$$

with nonnegative functions $\hat{g}_1(h) = O(|h|)$ and $\hat{g}_2(z) = O(\gamma_1(V_z(z))^{\frac{p}{1+p}})$. Then there is a continuous feedback $\mu_*(h)$ with $\mu_*(0) = 0$ and $|\mu_*|^q sgn(\mu_*)$ of class C^1 such that system (32) is FTISS with w as the input by taking

$$v = \mu_*(h) + w.$$

Remark 1. It is worth noting that the z-subsystem is not assumed to be FTISS with respect to x, but with respect to the (virtual) input $h(x, z) = |x|^p sgn(x) - |\mu(z)|^p sgn(\mu(z))$ for $p > 1$. This represents one of the main differences between C^1 stabilizing control and non-smooth finite time stabilizing control. Clearly, when $p \geq 1$, $|x|^p sgn(x)$ is differentiable. Also it is not hard to show that $|x - \mu(z)| \leq 2|h(x, z)|^{\frac{1}{p}}$ and $|h(x, z)| \leq 2^p |x - \mu(z)|^p$. With these inequalities, the system $\dot{z} = g(x, z)$ is FTISS with $\bar{h} = x - \mu(z)$ as the input if and only if it is FTISS when h is considered as the input. Additionally, if a system is FTISS with h as the input, then it is also FTISS with the input $|h|^s$ for any constant $s > 0$.

Proof of Theorem 11: From (34), $\dot{z} = g(\mu(z), z)$ is finite-time stable (since $h = 0$).

Consider the following positive-definite Lyapunov function

$$V(x, z) = V_*(z) + W_*(x, z), \tag{37}$$

with

$$V_* = \int_0^{V_z(z)} \bar{\rho}(s) ds, \quad W_* = \int_{\mu(z)}^{x} h(s, z)\, ds$$

where $\bar{\rho} : R_+ \to R_+$ is an increasing and positive continuous function to be determined later. Note that $|\mu| sgn(\mu) = \mu$ and $|\mu|^p sgn(\mu) \cdot \mu = |\mu|^{p+1}$, and then

$$W_*(x, z) = \frac{|x|^{p+1} + p|\mu(z)|^{p+1}}{p+1} - x|\mu(z)|^p sgn(\mu(z)),$$

is positive for $h \neq 0$ and it is C^1 with respect to z because $|\mu|^p sgn(\mu)$ is C^1 given in the conditions. Then

$$\dot{V}(x, z) \leq -\bar{\rho}(V_z)\gamma_1(V_z) + \bar{\rho}(V_z)\gamma_2(|h|) + \dot{W}_*,$$

where

$$\dot{W}_* == h[f(x, z) + u] - \bar{h} \frac{\partial |\mu|^p\, sgn(\mu)}{\partial z} g(x, z),$$

with $\bar{h} = x - \mu(z)$. From Remark 1, $|\bar{h}| \leq 2h^{1/p}$.

According to (35), $q > p$, Remark 1, and Young's inequality,

$$hf(x, z) \leq |h|[\hat{f}(h) + \bar{f}(z)] \leq |h|^{1+\frac{1}{q}} \bar{f}_1(h) + \alpha_f(z), \tag{38}$$

for nonnegative continuous functions $\bar{f}_1$ and $\alpha_f(z) = O(\gamma_1(V_z(z)))$.

Moreover, by (36) and Remark 1, along with Lemma 4 and (30), we obtain

$$(x - \mu) \frac{\partial |\mu|^p sgn(\mu)}{\partial z} g(x, z) \leq |h|^{1+\frac{1}{q}} \bar{g}_2(h) + \alpha_g(z)$$

with suitable nonnegative functions $\bar{g}_2(h)$ and $\alpha_g(z) = O(\gamma_1(V_z(z)))$.

Select a function $\bar{\rho}$ such that $\bar{\rho}(V_z(z))\gamma_1(V_z(z)) > 2(\alpha_f + \alpha_g)(z)$ noting that $(\alpha_f + \alpha_g)(z) = O(\gamma_1(V_z(z)))$.

Moreover, by Young's inequality and (34), we have

$$\bar{\rho}(V_z)\gamma_2(|h|) \leq \frac{1}{4}\bar{\rho}(V_z)\gamma_1(V_z) + |h|^{1+\frac{1}{q}}\bar{g}_1(h).$$

Take the control law:

$$\mu_*(h) = -|h|^{\frac{1}{q}} sgn(h)\Phi(h), \quad (39)$$

where Φ is C^1 and dominates $1 + \bar{g}_1(h) + \bar{g}_2(h) + \bar{f}_1(h)$, or equivalently,

$$h\mu_*(h) + |h|^{1+\frac{1}{q}}[\bar{f}_1(h) + \sum_{i=1}^{2}\bar{g}_i(h)] \leq -|h|^{1+\frac{1}{q}}.$$

Therefore, according to Young's inequality, we have:

$$\dot{V}(x,z) \leq -\frac{1}{4}\bar{\rho}(V_z)\gamma_1(V_z(z)) - |h|^{1+\frac{1}{q}} + |w|^{1+q},$$

and it is easy to see that $|\mu_*(h)|^q sgn(\mu_*(h))$ is C^1.

Since $\bar{\rho}(0) > 0$ and $q > p$,

$$\bar{\rho}(V_z)\gamma_1(V_z(z)) \sim V_z(z)^a \sim V_*(z)^a, \quad \text{as} \quad z \to 0,$$

and $|h|^{1+1/p} \sim W_*$, or equivalently

$$|h|^{1+\frac{1}{q}} \sim W_*^{a_0}, \quad a_0 = \frac{p(1+q)}{q(1+p)} < 1, \quad \text{as} \quad h \to 0.$$

Then, by Lemma 10, the conclusion follows. ∎

For illustration, we can revisit a system that was studied in Hong et al. [2006] from the FTISS viewpoint:

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(x_1, ..., x_i), \quad 1 \leq i \leq n-1 \\ \dot{x}_n = u + f_n(x_1, ..., x_n) \end{cases} \quad (40)$$

where $u \in R$ is the control input; $x := (x_1, \ldots, x_n)^T \in R^n$ is the state; and f_i is C^1 with respect to $(x_1, ..., x_i)$ with $f_i(0, \ldots, 0) = 0$ $(i = 1, ..., n)$.

Then we give a recursive procedure to design finite-time stabilizing controller by applying Theorem 11 and Corollary 9 repeatedly. For space limitations, the details are omitted here (referring to Hong et al. [2006]).

6. CONCLUSIONS

In this paper, the finite-time ISS for nonlinear systems was investigated and the relationship between FTISS and other basic finite-time concepts are revealed. Moreover, under assumptions related to FTISS, finite-time feedback design was given. In fact, the systematic application of FTISS alleviates the mathematical technicality and complexity associated with non-smooth feedback approaches. We believe that FTISS will play a role in finite-time control, as important as what the conventional ISS has played in asymptotic stability analysis and stabilization control.

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