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Finite Time Stability of Fractional Order Systems of Neutral Type

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Abstract: This work deals with a new finite time stability (FTS) of neutral fractional order systems with time delay (NFOTSs). In light of this, FTSs of NFOTSs are demonstrated in the literature using the Gronwall inequality. The innovative aspect of our proposed study is the application of fixed point theory to show the FTS of NFOTSs. Finally, using two examples, the theoretical contributions are confirmed and substantiated.

Keywords: fractional calculus; neutral systems; fixed-point theory

Citation: Ben Makhlouf, A.; Baleanu, D. Finite Time Stability of Fractional Order Systems of Neutral Type. *Fractal Fract.* **2022**, *6*, 289. https://doi.org/10.3390/fractalfract6060289

Academic Editors: Ricardo Almeida and Yongguang Yu

Received: 23 March 2022
Accepted: 24 May 2022
Published: 26 May 2022

1. Introduction

The Fractional Order System (FOS) is a nonlinear system presented with a non-integer derivative. It is well established that mathematical models can be used to describe physical systems. These mathematical models are used to operate such systems in a variety of ways, including controlling, observing, and detecting. The faults and errors of modelization may affect the system quality and performance. Therefore, the use of Fractional derivatives can approach such a mathematical model to physical reality. This fact is proved in many real physical systems, see for example [1]. Recently, the fractional calculus has attracted the attention of many researchers and numerous works have been published in this context [2–11]. In fact, by using quantum calculus, the work in [6] deals with the extension of a hybrid fractional differential operator. Utilizing the local fractional Laplace variational iteration methods and the local fractional reduced differential transform, authors in [7] have obtained an approximation of the solutions for coupled Korteweg De Vries Equations. The application of these FOSs is numerous in different domain applications, whether in electricity [10], thermal [5], chemistry [11], signal processing [12], biology [13,14] or control theory, such as fault estimation [15], stabilization [16], observer design [16,17], optimal control [18], and asymptotic stability [19,20].

The study of FTS for the Fractional Order Time Delay Systems (FOTDSs) has been largely studied in the literature in the case of continuous and discrete time [21–30]. In [30], H. Ye et al., have shown a Generalized Gronwall Inequality (GGI). After that, authors in [25] have used the GGI to study the FTS for FOTDSs. The stability of neutral fractional order time delay systems with Lipschitz nonlinearities in finite time has been investigated by F. Du et al. in [23]. The finite-time stability of a class of fractional delayed neural networks with commensurate order between 0 and 1 was studied by the authors in [28]. Additionally, the authors in [26] have provided an analytical method based on the Laplace transform and the 'inf-sup' approach for evaluating the finite-time stability of singular fractional-order switching systems with delay. The authors have proposed a constructive geometric design for switching laws based on the partitioning of the stability state regions in convex cones. The suggested technique allows for the development of novel delay-dependent adequate conditions for the system's regularity, impulse-free, and finite-time stability in terms of tractable matrix inequalities and Mittag–Leffler functions. A case study is offered to demonstrate the proposed method's efficacy. Using the Lyapunov method, Thanh et al. in [27] have investigated a novel FTS analysis of FOTDSs. By using Banach fixed point method, author in [21] has studied the FTS for FOTDSs. In the discrete case, one has the following references [22,24,29]. Indeed, authors in [24] have proposed a sufficient condition for ensuring the FTS for Nabla uncertain FOS. Furthermore, authors in [22] have established a new Gronwall Inequality and they have used it to study the FTS of a class of nonlinear fractional delay difference systems. Furthermore, in [29], the FTS of Caputo delta fractional difference equations is investigated. On a finite time domain, a generalized Gronwall inequality is given. For fractional differential equations, a finite-time stability condition is suggested. The concept is then generalized to discrete fractional cases. There are finite-time stable conditions for a linear fractional difference equation with constant delays. To support the theoretical result, one example is numerically shown.

Motivated by the above study, this article treats the FTS for FOS of neutral type by using a version of the Banach fixed point theorem and some properties of the Mittag–Leffler Function (MLF). The contribution of this work is summarized as follows:

- Knowing that, FTS of NFOTSs are proved in the literature based on the Gronwall inequality, see [23]. The novelty of our suggested work comes from the use of the fixed point theory to demonstrate the FTS of NFOTSs;
- A novel FTS result of FOS of neutral type is given;
- The theoretical contributions are confirmed and validated by two examples.

The rest of the paper is organized as follows. The second section deals with some preliminaries. Some basic results related to fractional calculus, fixed point theory, as well as finite time stability are shown. In regards to the third section, the stability analysis of the suggested system (2), in the case of $(\lambda_1 < \lambda_2)$ and $(\lambda_1 = \lambda_2)$, is investigated and described. Note that the fixed point approach is used to demonstrate the main results. The fourth section is concentrated to show the validity of the proposed results. Two examples are suggested to demonstrate the efficiency of the main results. Finally, to end the work, a conclusion is presented in the fifth section showing the principle fundamentals of the work.

2. Basic Results

Definition 1 ([31])**.** *Given $0 < \chi < 1$. The CFD is given by,*

$$ {}^{C}D_a^{\chi} g(s) = \frac{1}{\Gamma(1-\chi)} \frac{d}{ds} \int_a^s (s-\omega)^{-\chi} \big(g(\omega) - g(a)\big) d\omega. \tag{1} $$

Definition 2 ([31])**.** *The MLF is defined by :*

$$ E_{\chi}(s) = \sum_{q=0}^{+\infty} \frac{s^q}{\Gamma(q\chi + 1)}, $$

with $\chi > 0$, $s \in \mathbb{C}$.

Lemma 1 ([21])**.** *We have for $s \geq 0$*

$$ \frac{s^{\chi}}{E_{\chi}(\lambda s^{\chi})} \leq \frac{\Gamma(\chi+1)}{\lambda}, $$

where $0 < \chi < 1$ and $\lambda > 0$.

Remark 1. *The function $d(t) = E_{\chi}\big(b(t-\tau)^{\chi}\big)$ satisfies ${}^{C}D_a^{\chi} d(t) = bd(t)$, where $b \in \mathbb{R}^*$.*

Definition 3. *A mapping $\beta : B \times B \to [0, \infty]$ is called a generalized metric on a nonempty set B if:*

S1 $\beta(\omega_1, \omega_2) = 0$ *if, and only if,* $\omega_1 = \omega_2$*;*

S2 $\beta(\omega_1, \omega_2) = \beta(\omega_2, \omega_1)$ *for all* $\omega_1, \omega_2 \in B$*;*

S3 $\beta(\omega_1, \omega_3) \leq \beta(\omega_1, \omega_2) + \beta(\omega_2, \omega_3)$ *for all* $\omega_1, \omega_2, \omega_3 \in B$*.*

Theorem 1. *Let (B, β) be a generalized complete metric space. Suppose that $K : B \to B$ is contractive with $k < 1$. If there is an integer $k_0 \geq 0$, such that $\beta(K^{k_0+1}b_0, K^{k_0}b_0) < \infty$ for some $b_0 \in B$, so:*

(a) $\lim_{n \longrightarrow +\infty} K^n b_0 = b_1$ *with* $K(b_1) = b_1$*;*

(b) b_1 *is the unique fixed point of K in* $B^* := \{b_2 \in B : \beta(K^{k_0}b_0, b_2) < \infty\}$*;*

(c) *If* $b_2 \in B^*$*, then* $\beta(b_1, b_2) \leq \frac{1}{1-k}\beta(Kb_2, b_2)$*.*

We consider the following system:

$$\begin{aligned} {}^C D_0^{\lambda_2} x(t) - C\, {}^C D_0^{\lambda_1} x(t - \varsigma(t)) &= B_0 x(t) + B_1 x(t - \varsigma(t)) \\ &\quad + B_2 v(t) + F(t, x(t), x(t - \varsigma(t)), v(t)),\ t \geq 0, \end{aligned} \tag{2}$$

with the initial condition $x(s) = \zeta(s)$ for $-\varsigma \leq s \leq 0$, with $0 < \lambda_1 \leq \lambda_2 < 1$, $\varsigma(t)$ is continuous, $0 \leq \varsigma(t) \leq \varsigma$, $v(t) \in \mathbb{R}^p$ is the disturbance, $\zeta \in C^1([-\varsigma, 0], \mathbb{R}^q)$, $C \in \mathbb{R}^{q \times q}$, $B_0 \in \mathbb{R}^{q \times q}$ $B_1 \in \mathbb{R}^{q \times q}$, $B_2 \in \mathbb{R}^{q \times p}$.

The function F is continuous and satisfies:

$$\|F(\tau, \sigma_1, \sigma_2, \sigma_3) - F(\tau, \psi_1, \psi_2, \psi_3)\| \leq f(\tau)\big(\|\sigma_1 - \psi_1\| + \|\sigma_2 - \psi_2\| + \|\sigma_3 - \psi_3\|\big), \tag{3}$$

and $F(\tau, 0, 0, 0) = 0$, for all $(\tau, \sigma_1, \sigma_2, \sigma_3, \psi_1, \psi_2, \psi_3) \in \mathbb{R}_+ \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^p$ where f is a continuous function.

The function v is continuous and satisfies:

$$\exists \varrho > 0 : \quad v^T(t) v(t) \leq \varrho^2. \tag{4}$$

Definition 4. *The FOS (2) possesses FTS w.r.t.* $\{\gamma_1, \gamma_2, \varrho, T\}$, $\gamma_1 < \gamma_2$ *if*

$$\|\zeta\| \leq \gamma_1,$$

implies:

$$\|x(t)\| \leq \gamma_2,\ \forall t \in [0, T],$$

for all v satisfying (4), where $\|\zeta\| = \sup_{\tau \in [-\varsigma, 0]} \|\zeta(\tau)\|$.

3. Stability Analysis

This section is used to show our main results.

First, let us denote $b_i = \max_{r \in [0,T]} \Big(f(r) + \|B_i\|\Big)$ for $i = 0, 1, 2$ and $c = \|C\|$.

In the next subsections, we study the FTS of (2) when $\lambda_1 < \lambda_2$ and when $\lambda_1 = \lambda_2$.

3.1. The Case $\lambda_1 < \lambda_2$

From Theorem 1 in [23], we have the solution of the FOS (2) is the solution of the following system

$$\begin{aligned} x(t) &= \zeta(0) - C\zeta(-\varsigma(0))\frac{t^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)} + \frac{1}{\Gamma(\lambda_2-\lambda_1)}\int_0^t (t-s)^{\lambda_2-\lambda_1-1} Cx\big(s-\varsigma(s)\big)ds \\ &+ \frac{1}{\Gamma(\lambda_2)}\int_0^t (t-s)^{\lambda_2-1}\Big[B_0x(s) + B_1x(s-\varsigma(s)) \\ &+ B_2v(s) + F(s, x(s), x(s-\varsigma(s)), v(s))\Big]ds,\ 0 \le t \le T, \end{aligned}$$

$x(t) = \zeta(t), -\varsigma \le t \le 0.$

Theorem 2. *The FOS (2) is FTS w.r.t.* $\{\gamma_1, \gamma_2, \varrho, T\}$, $\gamma_1 < \gamma_2$ *if there exist* $\eta_1, \eta_2 > 0$, *such that*

$$G(\gamma_1, \varrho) \le \gamma_2, \tag{5}$$

where

$$\begin{aligned} G(\gamma_1, \varrho) &= \Big(\delta + c_1 E_{\lambda_2-\lambda_1}\big((c+\eta_1)T^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)T^{\lambda_2}\big)\Big)\gamma_1 \\ &+ c_2 E_{\lambda_2-\lambda_1}\big((c+\eta_1)T^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)T^{\lambda_2}\big)\varrho, \end{aligned} \tag{6}$$

$\delta = 1 + c\frac{T^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}, c_1 = \dfrac{1}{(1-\eta)}\Big(\dfrac{c\delta M_1}{\Gamma(\lambda_2-\lambda_1+1)} + \dfrac{b_0\delta M_2}{\Gamma(\lambda_2+1)} + \dfrac{b_1\delta M_2}{\Gamma(\lambda_2+1)}\Big),$
$c_2 = \dfrac{b_2M_2}{(1-\eta)\Gamma(\lambda_2+1)}, M_1 = \sup\limits_{\tau\in[0,T]}\Big(\dfrac{\tau^{\lambda_2-\lambda_1}}{E_{\lambda_2-\lambda_1}\big((c+\eta_1)\tau^{\lambda_2-\lambda_1}\big)}\Big),$
$M_2 = \sup\limits_{\tau\in[0,T]}\Big(\dfrac{\tau^{\lambda_2}}{E_{\lambda_2}\big((b_0+b_1+\eta_2)\tau^{\lambda_2}\big)}\Big)$ *and* $\eta = \big(\frac{c}{c+\eta_1} + \frac{b_0+b_1}{b_0+b_1+\eta_2}\big).$

Proof. Let $\zeta \in C^1\big([-\varsigma, 0], \mathbb{R}^q\big)$, such that $\|\zeta\| \le \gamma_1$.

Let $\mathcal{F} = C\big([-\varsigma, T], \mathbb{R}^q\big)$ and consider the metric β on $\mathcal{F}$ by

$$\beta(y_1, y_2) = \inf\Bigg\{r \in [0,\infty] : \|y_1(t) - y_2(t)\| \le rg(t), \forall t \in [-\varsigma, T]\Bigg\},$$

where g is given by $g(\tau) = E_{\lambda_2-\lambda_1}\big((c+\eta_1)\tau^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)\tau^{\lambda_2}\big)$ for $\tau \in [0,T]$ and $g(\tau) = 1$, for $\tau \in [-\varsigma, 0]$.

We consider the operator: $\mathcal{D} : \mathcal{F} \to \mathcal{F}$, such that

$$\begin{aligned} (\mathcal{D}X)(w) &= \zeta(0) - C\zeta(-\varsigma(0))\frac{w^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)} \\ &+\frac{1}{\Gamma(\lambda_2-\lambda_1)}\int_0^w (w-s)^{\lambda_2-\lambda_1-1} CX\big(s-\varsigma(s)\big)ds \\ &+\frac{1}{\Gamma(\lambda_2)}\int_0^w (w-s)^{\lambda_2-1}\Big[B_0X(s) + B_1X(s-\varsigma(s)) \\ &+B_2v(s) + F(s, X(s), X(s-\varsigma(s)), v(s))\Big]ds, \end{aligned} \tag{7}$$

for $w \in [0,T]$ and $(\mathcal{D}X)(w) = \zeta(w)$, for $w \in [-\varsigma, 0]$.

Note that, $\mathcal{D}$ is well defined, $(\mathcal{F}, \beta)$ is a generalized complete metric space, $\beta(\mathcal{D}X_0, X_0) < \infty$, and $\{X_1 \in \mathcal{F} : \beta(X_0, X_1) < \infty\} = \mathcal{F},\ \forall X_0 \in \mathcal{F}$.

Let $X_1, X_2 \in \mathcal{F}$, for $w \in [-\varsigma, 0]$, we get $(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w) = 0$.
For $w \in [0, T]$, we have

$$\begin{aligned}
&\Big\|(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w)\Big\| \\
&\leq \int_0^w \frac{(w-r)^{\lambda_2-\lambda_1-1}}{\Gamma(\lambda_2-\lambda_1)} c\|X_1(r-\varsigma(r)) - X_2(r-\varsigma(r))\|dr \\
&+ \int_0^w \frac{(w-r)^{\lambda_2-1}}{\Gamma(\lambda_2)} \Big[\Big(f(r) + \|B_0\|\Big)\|X_1(r) - X_2(r)\| \\
&+ \Big(f(r) + \|B_1\|\Big)\|X_1(r-\varsigma(r)) - X_2(r-\varsigma(r))\|\Big]dr \\
&\leq c\int_0^w \frac{(w-r)^{\lambda_2-\lambda_1-1}}{\Gamma(\lambda_2-\lambda_1)}\|X_1(r-\varsigma(r)) - X_2(r-\varsigma(r))\|dr \\
&+ b_0\int_0^w (w-r)^{\lambda_2-1}\, \frac{\|X_1(r) - X_2(r)\|}{\Gamma(\lambda_2)}dr \\
&+ b_1\int_0^w (w-r)^{\lambda_2-1}\, \frac{\|X_1(r-\varsigma(r)) - X_2(r-\varsigma(r))\|}{\Gamma(\lambda_2)}dr.
\end{aligned} \tag{8}$$

Then,

$$\begin{aligned}
&\Big\|(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w)\Big\| \\
&\leq c\int_0^w \frac{(w-r)^{\lambda_2-\lambda_1-1}}{\Gamma(\lambda_2-\lambda_1)}\, \frac{\|X_1(r-\varsigma(r)) - X_2(r-\varsigma(r))\|}{g(r-\varsigma(r))}g(r-\varsigma(r))dr \\
&+ \frac{b_0}{\Gamma(\lambda_2)}\int_0^w (w-r)^{\lambda_2-1}\, \frac{\|X_1(r) - X_2(r)\|}{g(r)}g(r)dr \\
&+ \frac{b_1}{\Gamma(\lambda_2)}\int_0^w (w-r)^{\lambda_2-1}\, \frac{\|X_1(r-\varsigma(r)) - X_2(r-\varsigma(r))\|}{g(r-\varsigma(r))}g(r-\varsigma(r))dr \\
&\leq c\beta(X_1, X_2)\int_0^w \frac{(w-r)^{\lambda_2-\lambda_1-1}}{\Gamma(\lambda_2-\lambda_1)}g(r-\varsigma(r))dr \\
&+ \frac{b_0\beta(X_1, X_2)}{\Gamma(\lambda_2)}\int_0^w (w-r)^{\lambda_2-1}\, g(r)dr \\
&+ \frac{b_1\beta(X_1, X_2)}{\Gamma(\lambda_2)}\int_0^w (w-r)^{\lambda_2-1}\, g(r-\varsigma(r))dr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Big\|(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w)\Big\| &\leq c\beta(X_1, X_2)\int_0^w \frac{(w-\tau)^{\lambda_2-\lambda_1-1}}{\Gamma(\lambda_2-\lambda_1)}g(\tau)d\tau \\
&+ \frac{(b_0+b_1)\beta(X_1, X_2)}{\Gamma(\lambda_2)}\int_0^w (w-\tau)^{\lambda_2-1}\, g(\tau)d\tau \\
&\leq c\beta(X_1, X_2)E_{\lambda_2}\big((b_0+b_1+\eta_2)w^{\lambda_2}\big) \\
&\times \int_0^w \frac{(w-\tau)^{\lambda_2-\lambda_1-1}}{\Gamma(\lambda_2-\lambda_1)}E_{\lambda_2-\lambda_1}\big((c+\eta_1)\tau^{\lambda_2-\lambda_1}\big)d\tau \\
&+ (b_0+b_1)\beta(X_1, X_2)E_{\lambda_2-\lambda_1}\big((c+\eta_1)w^{\lambda_2-\lambda_1}\big) \\
&\times \int_0^w \frac{(w-\tau)^{\lambda_2-1}}{\Gamma(\lambda_2)}E_{\lambda_2}\big((b_0+b_1+\eta_2)\tau^{\lambda_2}\big)d\tau.
\end{aligned}$$

Using Remark 1, we get

$$\begin{aligned}\Big\|(\mathcal{D}X_1)(w)-(\mathcal{D}X_2)(w)\Big\| &\leq \frac{c}{c+\eta_1}\beta(X_1,X_2)g(w)+\frac{b_0}{b_0+b_1+\eta_2}\beta(X_1,X_2)g(w)\\ &+ \frac{b_1}{b_0+b_1+\eta_2}\beta(X_1,X_2)g(w)\\ &\leq \Big(\frac{c}{c+\eta_1}+\frac{b_0+b_1}{b_0+b_1+\eta_2}\Big)\beta(X_1,X_2)g(w). \end{aligned} \tag{9}$$

Then,

$$\frac{\Big\|(\mathcal{D}X_1)(w)-(\mathcal{D}X_2)(w)\Big\|}{g(w)} \leq \Big(\frac{c}{c+\eta_1}+\frac{b_0+b_1}{b_0+b_1+\eta_2}\Big)\beta(X_1,X_2).$$

Thus,

$$\beta(\mathcal{D}X_1,\mathcal{D}X_2) \leq \Big(\frac{c}{c+\eta_1}+\frac{b_0+b_1}{b_0+b_1+\eta_2}\Big)\beta(X_1,X_2).$$

Therefore, $\mathcal{D}$ is contractive.

Let x_0 be the function given by $x_0(\tau)=\zeta(\tau)$, for $\tau\in[-\varsigma,0]$ and $x_0(\tau)=\zeta(0)-C\zeta\big(-\varsigma(0)\big)\frac{\tau^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}$ for $\tau\in[0,T]$.

Then, we have

$$\|x_0(\tau)\| \leq \Big(\|\zeta\|+c\|\zeta\|\frac{T^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}\Big),$$

for all $\tau\in[-\varsigma,T]$.

For $\tau\in[-\varsigma,0]$, we get $(\mathcal{D}x_0)(\tau)-x_0(\tau)=0$.

For $w\in[0,T]$, we have

$$\begin{aligned}\Big\|(\mathcal{D}x_0)(w)-x_0(w)\Big\| &\leq \int_0^w \frac{(w-s)^{\lambda_2-\lambda_1-1}}{\Gamma(\lambda_2-\lambda_1)}c\|x_0\big(s-\varsigma(s)\big)\|ds\\ &+ \frac{1}{\Gamma(\lambda_2)}\int_0^w (w-s)^{\lambda_2-1}\big[b_0\|x_0(s)\|+b_1\|x_0\big(s-\varsigma(s)\big)\|+b_2\varrho\big]ds\\ &\leq c\Big(\|\zeta\|+c\|\zeta\|\frac{T^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}\Big)\frac{w^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}\\ &+ \Big(b_0\big(\|\zeta\|+c\|\zeta\|\frac{T^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}\big)+b_1\big(\|\zeta\|\\ &+ c\|\zeta\|\frac{T^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}\big)+b_2\varrho\Big)\frac{w^{\lambda_2}}{\Gamma(\lambda_2+1)}\\ &\leq c\|\zeta\|\delta\frac{w^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2-\lambda_1+1)}\\ &+ \big(b_0\|\zeta\|\delta+b_1\|\zeta\|\delta+b_2\varrho\big)\frac{w^{\lambda_2}}{\Gamma(\lambda_2+1)}. \end{aligned} \tag{10}$$

Then

$$\begin{aligned}\frac{\Big\|(\mathcal{D}x_0)(w)-x_0(w)\Big\|}{g(w)} &\leq \frac{c\|\zeta\|\delta M_1}{\Gamma(\lambda_2-\lambda_1+1)}\\ &+ \big(b_0\|\zeta\|\delta+b_1\|\zeta\|\delta+b_2\varrho\big)\frac{M_2}{\Gamma(\lambda_2+1)}, \end{aligned} \tag{11}$$

for all $w\in[0,T]$.

Therefore,

$$\beta(\mathcal{D}x_0, x_0) \leq \frac{c\|\zeta\|\delta M_1}{\Gamma(\lambda_2-\lambda_1+1)} + (b_0\|\zeta\|\delta + b_1\|\zeta\|\delta + b_2\varrho)\frac{M_2}{\Gamma(\lambda_2+1)}. \tag{12}$$

It follows from Theorem 1 that there is a unique solution x of (2) with initial conditions of ζ, such that

$$\begin{aligned}\beta(x_0, x) &\leq \frac{1}{1-\eta}\Big[\frac{c\|\zeta\|\delta M_1}{\Gamma(\lambda_2-\lambda_1+1)} \\ &+ (b_0\|\zeta\|\delta + b_1\|\zeta\|\delta + b_2\varrho)\frac{M_2}{\Gamma(\lambda_2+1)}\Big] \\ &\leq c_1\gamma_1 + c_2\varrho.\end{aligned} \tag{13}$$

Therefore,

$$\|x_0(t) - x(t)\| \leq (c_1\gamma_1 + c_2\varrho)E_{\lambda_2-\lambda_1}\big((c+\eta_1)T^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)T^{\lambda_2}\big),$$

for every $t \in [0, T]$.

Then,

$$\begin{aligned}\|x(t)\| &\leq \|x_0(t)\| + \|x(t) - x_0(t)\| \\ &\leq \Big(\delta + c_1E_{\lambda_2-\lambda_1}\big((c+\eta_1)T^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)T^{\lambda_2}\big)\Big)\gamma_1 \\ &+ c_2E_{\lambda_2-\lambda_1}\big((c+\eta_1)T^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)T^{\lambda_2}\big)\varrho,\end{aligned} \tag{14}$$

for every $t \in [0, T]$.

Thus, $\|x(t)\| \leq \gamma_2$, for all $t \in [0, T]$, if (5) is satisfied. □

Remark 2. *Using Lemma 1, we get*

$$c_1 \leq \frac{1}{(1-\eta)}\Big(\frac{c\delta}{c+\eta_1} + \frac{b_0\delta}{b_0+b_1+\eta_2} + \frac{b_1\delta}{b_0+b_1+\eta_2}\Big)$$

and

$$c_2 \leq \frac{1}{(1-\eta)}\frac{b_2}{b_0+b_1+\eta_2}.$$

Let

$$\tilde{c}_1 = \frac{1}{(1-\eta)}\Big(\frac{c\delta}{c+\eta_1} + \frac{b_0\delta}{b_0+b_1+\eta_2} + \frac{b_1\delta}{b_0+b_1+\eta_2}\Big)$$

and

$$\tilde{c}_2 = \frac{1}{(1-\eta)}\frac{b_2}{b_0+b_1+\eta_2}.$$

Therefore, the condition (5) can be relaxed by:

$$\tilde{G}(\gamma_1, \varrho) \leq \gamma_2, \tag{15}$$

where

$$\begin{aligned}\tilde{G}(\gamma_1, \varrho) &= \Big(\delta + \tilde{c}_1E_{\lambda_2-\lambda_1}\big((c+\eta_1)T^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)T^{\lambda_2}\big)\Big)\gamma_1 \\ &+ \tilde{c}_2E_{\lambda_2-\lambda_1}\big((c+\eta_1)T^{\lambda_2-\lambda_1}\big)E_{\lambda_2}\big((b_0+b_1+\eta_2)T^{\lambda_2}\big)\varrho.\end{aligned} \tag{16}$$

3.2. The Case $\lambda_1 = \lambda_2$

The solution of the FOS (2) is the solution of

$$\begin{aligned} x(t) &= \zeta(0) + C\Big(x\big(t-\varsigma(t)\big) - \zeta(-\varsigma(0))\Big) + \frac{1}{\Gamma(\lambda_2)}\int_0^t (t-s)^{\lambda_2-1}\Big[B_0 x(s) + B_1 x(s-\varsigma(s)) \\ &+ B_2 v(s) + F(s, x(s), x(s-\varsigma(s)), v(s))\Big]ds,\ 0 \le t \le T, \end{aligned}$$

$$x(t) = \zeta(t), -\varsigma \le t \le 0.$$

Theorem 3. *The FOS (2) is FTS w.r.t.* $\{\gamma_1, \gamma_2, \varrho, T\}$, $\gamma_1 < \gamma_2$ *if there exist* $\theta > 0$, *such that*

$$\eta < 1,$$

and

$$K(\gamma_1, \varrho) \le \gamma_2, \tag{17}$$

where

$$\eta = (c + \frac{b_0 + b_1}{b_0 + b_1 + \theta}),$$

$$\begin{aligned} K(\gamma_1, \varrho) &= \Big(1 + c_1 E_{\lambda_2}\big((b_0 + b_1 + \theta)T^{\lambda_2}\big)\Big)\gamma_1 \\ &+ c_2 E_{\lambda_2}\big((b_0 + b_1 + \theta)T^{\lambda_2}\big)\Big)\varrho, \end{aligned} \tag{18}$$

$$c_1 = \frac{1}{(1-\eta)}\Big(2c + \frac{b_0 M}{\Gamma(\lambda_2+1)} + \frac{b_1 M}{\Gamma(\lambda_2+1)}\Big),\ c_2 = \frac{b_2 M}{(1-\eta)\Gamma(\lambda_2+1)} \textit{ and}$$
$$M = \sup_{\tau\in[0,T]} \Big(\frac{\tau^{\lambda_2}}{E_{\lambda_2}\big((b_0 + b_1 + \theta)\tau^{\lambda_2}\big)}\Big).$$

Proof. Let $\zeta \in C^1\big([-\varsigma, 0], \mathbb{R}^q\big)$, such that $\|\zeta\| \le \gamma_1$.

Let $\mathcal{F} = C\big([-\varsigma, T], \mathbb{R}^q\big)$ and consider the metric β on $\mathcal{F}$ by

$$\beta(y_1, y_2) = \inf\left\{ r \in [0, \infty] : \frac{\|y_1(l) - y_2(l)\|}{g(l)} \le r, \forall l \in [-\varsigma, T] \right\},$$

where g is given by $g(l) = 1$, for $l \in [-\varsigma, 0]$ and $g(l) = E_{\lambda_2}\big((b_0 + b_1 + \theta)l^{\lambda_2}\big)$ for $l \in [0, T]$.

We consider the operator: $\mathcal{D} : \mathcal{F} \to \mathcal{F}$, such that

$$\begin{aligned} (\mathcal{D}X)(w) &= \zeta(0) + C\Big(X\big(w - \varsigma(w)\big) - \zeta(-\varsigma(0))\Big) \\ &+ \frac{1}{\Gamma(\lambda_2)}\int_0^w (w-s)^{\lambda_2-1}\Big[B_0 X(s) + B_1 X(s-\varsigma(s)) \\ &+ B_2 v(s) + F(s, X(s), X(s-\varsigma(s)), v(s))\Big]ds, \end{aligned} \tag{19}$$

for $w \in [0, T]$ and $(\mathcal{D}X)(w) = \zeta(w)$, for $w \in [-\varsigma, 0]$.
Note that, $\mathcal{D}$ is well defined, $(\mathcal{F}, \beta)$ is a generalized complete metric space, $\beta(\mathcal{D}X_0, X_0) < \infty$, and $\{X_1 \in \mathcal{F} : \beta(X_0, X_1) < \infty\} = \mathcal{F}$, $\forall X_0 \in \mathcal{F}$.
Let X_1, $X_2 \in \mathcal{F}$, for $w \in [-\varsigma, 0]$, we get $(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w) = 0$.

For $w \in [0, T]$, we have

$$
\begin{aligned}
&\Big\|(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w)\Big\| \\
&\leq c\|X_1(w-\varsigma(w)) - X_2(w-\varsigma(w))\| \\
&+ \int_0^w \frac{(w-r)^{\lambda_2-1}}{\Gamma(\lambda_2)}\Big[\Big(f(r)+\|B_0\|\Big)\|X_1(r)-X_2(r)\| \\
&+\Big(f(r)+\|B_1\|\Big)\|X_1(r-\varsigma(r)) - X_2(r-\varsigma(r))\|\Big]dr \\
&\leq c\frac{\|X_1(w-\varsigma(w)) - X_2(w-\varsigma(w))\|}{g\big(w-\varsigma(w)\big)}g\big(w-\varsigma(w)\big) \\
&+b_0\int_0^w \frac{(w-u)^{\lambda_2-1}}{\Gamma(\lambda_2)}\,\frac{\|X_1(u)-X_2(u)\|}{g(u)}g(u)du \\
&+b_1\int_0^w \frac{(w-u)^{\lambda_2-1}}{\Gamma(\lambda_2)}\,\frac{\|X_1(u-\varsigma(u)) - X_2(u-\varsigma(u))\|}{g\big(u-\varsigma(u)\big)}g\big(u-\varsigma(u)\big)du \\
&\leq c\beta(X_1,X_2)g\big(w-\varsigma(w)\big) + \frac{b_0\beta(X_1,X_2)}{\Gamma(\lambda_2)}\int_0^w (w-u)^{\lambda_2-1}\,g(u)du \\
&+\frac{b_1\beta(X_1,X_2)}{\Gamma(\lambda_2)}\int_0^w (w-u)^{\lambda_2-1}\,g(u)du. \qquad (20)
\end{aligned}
$$

Using Remark 1, we get

$$
\begin{aligned}
\Big\|(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w)\Big\| &\leq c\beta(X_1,X_2)g(w) + \frac{b_0}{b_0+b_1+\theta}\beta(X_1,X_2)g(w) \\
&+ \frac{b_1}{b_0+b_1+\theta}\beta(X_1,X_2)g(w) \\
&\leq (c+\frac{b_0+b_1}{b_0+b_1+\theta})\beta(X_1,X_2)g(w). \qquad (21)
\end{aligned}
$$

Then,

$$
\frac{\Big\|(\mathcal{D}X_1)(w) - (\mathcal{D}X_2)(w)\Big\|}{g(w)} \leq (c+\frac{b_0+b_1}{b_0+b_1+\theta})\beta(X_1,X_2),
$$

Thus,

$$
\beta(\mathcal{D}X_1,\mathcal{D}X_2) \leq (c+\frac{b_0+b_1}{b_0+b_1+\theta})\beta(X_1,X_2).
$$

Therefore, $\mathcal{D}$ is contractive.

Let x_0 be the function given by $x_0(\tau) = \zeta(\tau)$, for $\tau \in [-\varsigma, 0]$ and $x_0(\tau) = \zeta(0)$ for $\tau \in [0, T]$.

Then, we have

$$
\|x_0(\tau)\| \leq \|\zeta\|,
$$

for all $t \in [-\varsigma, T]$.

For $\tau \in [-\varsigma, 0]$, we get $(\mathcal{D}x_0)(\tau) - x_0(\tau) = 0$.

For $w \in [0, T]$, we have

$$
\begin{aligned}
\Big\|(\mathcal{D}x_0)(w) - x_0(w)\Big\| &\leq 2c\|\zeta\| \\
&+ \frac{1}{\Gamma(\lambda_2)}\int_0^w (w-s)^{\lambda_2-1}\big[b_0\|x_0(s)\| + b_1\|x_0\big(s-\varsigma(s)\big)\| + b_2\varrho\big]ds \\
&\leq 2c\|\zeta\| + \frac{w^{\lambda_2}}{\Gamma(\lambda_2+1)}\Big(b_0\|\zeta\| + b_1\|\zeta\| + b_2\varrho\Big). \qquad (22)
\end{aligned}
$$

Then,

$$\frac{\left\|(\mathcal{D}x_0)(w) - x_0(w)\right\|}{g(w)} \leq 2c\|\zeta\| + \left(b_0\|\zeta\| + b_1\|\zeta\| + b_2\varrho\right)\frac{M}{\Gamma(\lambda_2+1)}, \tag{23}$$

for all $w \in [0,T]$.

Therefore,

$$\beta(\mathcal{D}x_0, x_0) \leq 2c\|\zeta\| + \left(b_0\|\zeta\| + b_1\|\zeta\| + b_2\varrho\right)\frac{M}{\Gamma(\lambda_2+1)}. \tag{24}$$

Theorem 1 implies that (2) has a unique solution x with initial conditions of ζ, such that

$$\begin{aligned}\beta(x_0, x) &\leq \frac{1}{1-\eta}\left[2c\|\zeta\| + \left(b_0\|\zeta\| + b_1\|\zeta\| + b_2\varrho\right)\frac{M}{\Gamma(\lambda_2+1)}\right] \\ &\leq c_1\gamma_1 + c_2\varrho.\end{aligned} \tag{25}$$

Therefore,

$$\|x_0(t) - x(t)\| \leq (c_1\gamma_1 + c_2\varrho)E_{\lambda_2}\left((b_0+b_1+\theta)T^{\lambda_2}\right),$$

for all $t \in [0,T]$.

Then,

$$\begin{aligned}\|x(t)\| &\leq \|(x-x_0)(t)\| + \|x_0(t)\| \\ &\leq \left(1 + c_1E_{\lambda_2}\left((b_0+b_1+\theta)T^{\lambda_2}\right)\right)\gamma_1 \\ &+ c_2E_{\lambda_2}\left((b_0+b_1+\theta)T^{\lambda_2}\right)\varrho.\end{aligned} \tag{26}$$

Thus, $\|x(t)\| \leq \gamma_2$, for all $t \in [0,T]$, if (17) is satisfied. □

Remark 3. *Using Lemma 1, we get*

$$c_1 \leq \frac{1}{(1-\eta)}\left(2c + \frac{b_0}{b_0+b_1+\theta} + \frac{b_1}{\theta+b_1+b_0}\right)$$

and

$$c_2 \leq \frac{1}{(1-\eta)}\frac{b_2}{b_0+b_1+\theta}.$$

Let us consider

$$\tilde{c}_1 = \frac{1}{(1-\eta)}\left(2c + \frac{b_0}{\theta+b_1+b_0} + \frac{b_1}{b_0+b_1+\theta}\right)$$

and

$$\tilde{c}_2 = \frac{1}{(1-\eta)}\frac{b_2}{\theta+b_1+b_0}.$$

Therefore, the condition (17) can be relaxed by:

$$\tilde{K}(\gamma_1, \varrho) \leq \gamma_2, \tag{27}$$

where

$$\tilde{K}(\gamma_1,\varrho) = \Big(1+\tilde{c}_1 E_{\lambda_2}\big((b_0+b_1+\theta)T^{\lambda_2}\big)\Big)\gamma_1 + \tilde{c}_2 E_{\lambda_2}\big((b_0+b_1+\theta)T^{\lambda_2}\big)\varrho. \tag{28}$$

Remark 4. *In the Theorem 3, $c<1$ it is a necessary condition.*

Remark 5. *In the case when $C=0$, we get the results in [21].*

4. Examples

Two examples are studied to prove the applicability of Theorems 2 and 3.

Example 1. *Consider the NFOTDSs (2), with $\lambda_2=0.7$, $\lambda_1=0.2$, $\varsigma(s)=0.1$,*

$$v(\tau)=\big(0.5,0\big)^T,\quad \zeta(\tau)=\big(0.05,0\big)^T,\ for\ \tau\in[-0.1,0],$$

$$F(s,x(s),x(s-\varsigma(s)),v(s)) = 0.01\Big(\sin\big(x_2(s-\varsigma(s))\big),\sin\big(x_1(s)\big)\Big)^T,$$

and

$$B_0=\begin{pmatrix}0 & 0.4\\ 0.1 & 0\end{pmatrix},\ B_1=\begin{pmatrix}-0.6 & 0\\ -0.2 & 0\end{pmatrix},\ B_2=\begin{pmatrix}0.3 & 0\\ 0.4 & 0\end{pmatrix},\ C=\begin{pmatrix}0.2 & 0\\ -0.1 & 0\end{pmatrix}.$$

We get $b_0=0.41$, $b_1=0.64$, $b_2=0.51$ and $c=0.2236$.

For $\eta_1=\eta_2=1$, $\varrho=1$, $\gamma_1=0.3$ and $\gamma_2=60$. Moreover, if we calculate δ, $\tilde{c}_1$ and $\tilde{c}_2$, then $\tilde{G}(\gamma_1,\varrho)\simeq 59<\gamma_2$, for $T=0.61$. Based on theorem 2 it is clear that the NFOTDSs is FTS w.r.t $\big(0.3,60,1,0.61\big)$.

Example 2. *Consider the NFOTDSs (2), with $\lambda_2=\lambda_1=0.6$, $\varsigma(s)=0.1$,*

$$v(\tau)=\big(0,0.5,0\big)^T,\quad \zeta(\tau)=\big(0.04,0,0.02\big)^T,\ for\ \tau\in[-0.1,0],$$

$$F(s,x(s),x(s-\varsigma(s)),v(s)) = 0.01\Big(\sin\big(x_2(s-\varsigma(s))\big),\sin\big(x_3(s-\varsigma(s)\big),\sin\big(x_1(s)\big)\Big)^T,$$

and

$$B_0=\begin{pmatrix}0.01 & -0.2 & 0.25\\ -0.02 & 0.05 & 0.1\\ 0.2 & -0.01 & 0.15\end{pmatrix},\ B_1=\begin{pmatrix}0.01 & -0.15 & 0.31\\ 0.25 & 0.12 & -0.14\\ 0.13 & -0.12 & 0.22\end{pmatrix},$$

$$B_2=\begin{pmatrix}0.08 & 0.07 & 0.2\\ 0.08 & -0.07 & -0.06\\ -0.12 & -0.03 & -0.14\end{pmatrix},\ C=\begin{pmatrix}0.1 & 0.2 & 0.03\\ 0.12 & 0.22 & 0.05\\ -0.17 & 0.05 & -0.21\end{pmatrix}.$$

We get $b_0=0.37$, $b_1=0.47$, $b_2=0.30$, and $c=0.35$.

For $\varrho=1$, $\theta=1$, $\gamma_1=0.4$, $\gamma_2=100$, and $T=1.05$, we get $\tilde{K}(\gamma_1,\varrho)\simeq 97<\gamma_2$.

Theorem 3 implies that the NFOTDSs is FTS w.r.t $\big(0.4,100,1,1.05\big)$.

5. Conclusions

In this paper, a new robust FTS for NFOTDSs was described. By suggesting an approach based on the fixed point theory, novel sufficient conditions for the robust FTS of such systems are obtained. Finally, two examples were described to show the validity and the useless of the suggested result.

Author Contributions: Formal analysis, A.B.M.; writing—original draft preparation, A.B.M.; Supervision, D.B.; Visualization, D.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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