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Finite-Time Synchronization of Coupled Memrisive Neural Network via Robust Control

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ABSTRACT In this paper, the synchronization of drive-response coupled memristive neural networks (CMNNs) and CMNN with multi-links is investigated. The memristors show the memory characteristics, low energy consumption, and nanometer scale so that CMNN can more truly simulate the working mechanism of brain neural networks. The classic treatment method is no longer being applied because of the parameter-dependent property in CMNN. The new approach is proposed that CMNN is transformed into a class of neural networks with interval parameters under the framework of Filippov solution. This method overcame the problem of mismatched parameters and be less conservative than those existing methods. Sufficient criteria are derived to guarantee the synchronization of the drive-response networks based on the drive-response concept and the Lyapunov function. Finally, the effectiveness of the proposed theories is validated with the numerical experiments.

INDEX TERMS Finite-time synchronization, couple memristive neural network, robust control, drive-response concept.

I. INTRODUCTION

Recently, as a result of the continuous development of brain science, the brain neural networks as a kind of complex networks play a decisive role in the field of artificial intelligence, brain science and neural disease [1]–[4]. What is the brain structure? How does the brain coordinate its work? There is an urgent issue to give a visual explanation. Therefore, some researchers have also started from the brain neural network to attempt to simulate the whole human brain. The main purpose is to study brain, understand brain and further to protect brain.

Based on the understanding of human brain, artificial neural network is firstly abstracted as a mathematical model from the biological neural network by means of mathematical, physical and information processing methods. With the development of artificial intelligence, researchers further actively explore the working mechanism of human brain and expect to simulate the learning and memory functions of human brain. The artificial neural network model are always developed to restore the real biological neural network as far as possible. The two most important parts of neural connections in neural networks are neurons and synapses, synapses are bridges among neurons. Due to synapses play a vital role in receiving signal plasticity, dynamic information memory in transmission and storage of receiving information among neurons. Therefore, the selection of elements for simulating synapses is very important when constructing artificial neural network models. The memristor is regarded as the equivalent electronic component of the synapse, which brings a new opportunity for the development of artificial neural network and makes the memristive neural network (MNN) be closer to the human brain in the way of information processing. The detail description of memristor can refer to reference [5].

Compared with the traditional artificial neural network, the MNN is more adaptive to the new model and new data, and will improve the computing speed and parallel

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processing capability. The application of the memory will better simulate the human brain and consciousness behavior in the robot application. At present, the research of memristor are mainly focused on the following two aspects: first is to carry out the research ideas of HP laboratory researchers, which is to study how to make electronic components with the characteristic of memristor using the most economical and substantial material; second is to follow the research idea of Professor Cai [6]–[8] and to construct mathematical model of the dynamic system by introducing the memristor, furthermore, the dynamic behavior and application of the MNN system are explored. Meanwhile, the system dynamics behavior with memristor plays a key role in practical applications and can also provide ideas for making memristor devices. At present, the research on the dynamic behavior of the MNN has just begun, especially some researchers have made some progress in the synchronous control of the MNN, where the stability of a class of memristor-based recurrent neural networks are researched in [9] and [10]. Wu etc. has given some new results in dynamic behaviors, exponential synchronization, anti-synchronization of a class of memristive recurrent neural networks [11]–[16]. Wen etc. has given another effective results in dynamics analysis, exponential stability analysis, dynamic behaviors of memristor-based recurrent networks [17]–[19]. After, a class of memristive neural networks with time-varying delays are further investigated in [20]–[23]. In order to further obtain the more abundant dynamics of the MNN, the researchers get some important conclusions in the stability analysis of the MNN and all kinds of synchronization control based on the Filippov solution, the set value mapping and the differential inclusion theory [24]–[26]. The research above is the exploration of a simple MNN model. In fact, the connection of neurons are complicated and have a large number of coupling phenomena. However, there are few studies on the MNN with coupling connection. Therefore, the study of CMNN is of theoretical and practical significance.

To achieve faster synchronization between the drive system and the respond system, many effective control theories have been introduced. Particularly, the finite time was introduced in 1961 [27], the studies about the finite-time stability and synchronization are of great significance in practical applications, which has much faster convergence time. In the secure communication, compared with asymptotic synchronization and exponential synchronization, the finite-time synchronization technique enables us to recover the transmitted signals in a setting time, which improves the efficiency and the confidentiality greatly.

In addition, the concept of network convergence has attracted wide attention in recent years, this idea of network convergence is a trend of future development, a gradual and complex process and its integration is very broad. Network convergence can not be understood as a simple synthesis of multiple networks, and can not be considered as a generalized substitution between networks, but, it is not only necessary to understand the advantages and disadvantages of each network with the idea of splitting the network, but also to consider the new network form with the macro pattern of network convergence. Especially the aspect of the brain neural network, researchers mainly focus on a single simple neural network, but the artificial neural network model which is integrated into complex and multiple neural networks is relatively less concerned. The MNN is the closest simulation tool with the biological brain. The dynamics of the CMNN may be directly related to the internal mechanism of the memristor as a neuron, which help us to reveal the information storage principle of the brain [28]. However, there are few related work, especially the stability and synchronous control of CMNN with multi-links. The multi-links complex network was introduced in detail by [29], which are more realistic than single link. The multi-links complex network are split into some subnetworks based on different time-delays, which are shown in human communication network, transportation network and relationship network etc. However, the form of multi-links are more practical significance in neural network because of nerve transmission delay and complex neuronal connections. In the past, we have given a series of explorations in general complex networks with multi-links [30]–[35]. But, there are few studies on the CMNN with multi-links. Therefore, we firstly introduce some new results of synchronization in CMNN, then give some extensions to CMNN with multi-links in the paper. The contributions of this paper are shown as follows: (1) Different from previous research methods for MNN or CMNN, we give a new method that the CMNN are transformed into a class of neural networks with unmatched uncertain parameters to be investigate. Furthermore, we overcome the difficulties of parameter mismatch to obtain some new results; (2) The main results are obtained to ensure finite-time synchronization of CMNN with time-varying delays by designing discontinuous controller, using robust control approach and finite time stability, which may be less conservative than the previous research methods of MNN [36]–[40]; (3) Additionally, results of this paper can be easily extended to CMNN with multi-links, we further give the effect of multiple delays on the dynamic performance of the whole network.

This paper is organized as follows. In Section 2, the corresponding preliminaries of this paper are given. In Section 3, some new results about the finite-time synchronization criteria of CMNN are obtain appropriate Lyapunov function and designing an appropriate controller, then we extend the above method to solve CMNN with multi-links and hope observe some interesting dynamic characteristics. In Section 4, numerical examples are given to demonstrate the effectiveness of proposed methods. Finally, conclusions and prospects are given in Section 5.

Notations: In this paper, if not explicitly stated, matrices are assumed to have compatible dimensions. R denotes the set of real numbers, R^n and $R^{m\times n}$ refer to, respectively, the n dimensional Euclidean space and the set of all $m \times n$ real matrices. $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$. The notation $P > 0$ means P is real symmetry positive definite and the superscript T

stands for the transpose for vector or matrices. In addition, $*$ refers to the ellipsis in symmetric matrices expressions and I denotes the identity matrix with the appropriate dimension.

II. PROBLEM FORMULATION AND PRELIMINARIES

Definition 1 [25]: Let X and Y be two sets, a map $F : X \to Y$ is called set-valued map, if any $x \in X$,there is always a corresponding set $F(x)$ is said to the value or image of F at x.

Definition 2 [25]: A set-valued map $F : X \to Y$ is said to be upper-semi-continuous at $x_0 \in x$, if for every neighborhood N_Y of $F(x_0) \subset Y$, there exists a neighborhood N_X of x_0 such that $F(N_X) = \bigcup_{x \in N_X F(x)} \subset N_Y$. If F is upper-semi-continuous for every $x \in X$, then the set-valued map F is upper-semi-continuous on the set X.

Definition 3 (Filippov Regularization [24]): For differential system, $\dot{x}(t) = f(t, x)$, where $f(t, x)$ is discontinuous in $x(t)$ and $x(t)$ is a solution of the differential system on $[t_0, t_1]$ in Filippov's sense, if $x(t)$ is absolutely continuous on any compact interval $[t_0, t_1]$, for almost all $t \in [t0, t1]$ such that

$$\dot{x} = K_F[f](t, x)$$

where

$$K_F[f](t, x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{co}[f(B(x, \delta) \setminus N), t]$$

where $\overline{co}[\cdot]$ is the convex closure hull of a set, $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$ is the ball of center x and radius δ, intersection is taken over all sets N of measure zero and over all $\delta > 0$, and $\mu(N)$is Lebesgue measure of set N.

Firstly, we give the mathematical model of memristive neural network without coupled connections as follows:

$$\dot{x}_m(t) = -c_m x_m(t) + \sum_{l=1}^{n} a_{ml}(x_m(t))\bar{g}_l(x_l(t)) + \sum_{l=1}^{n} b_{ml}(x_m(t))g_l(x_l(t - \tau_0(t))) + I_m(t), \quad (1)$$

where m = 1,2,...,n, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ and $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T \in R^n$ are, respectively, the voltage of capacitor and external input. $\bar{g}_l(x_l(t))$ and $g_l(x_l(t - \tau_0(t)))$ are, respectively, the bounded feedback functions without and with time-varying time delay between the memristor and $x(t)$; c_m is the mth neuron self-inhibitions, $A(x(t)) = (a_{ml}(x_m(t)))_{n\times n}$ and $B(x(t)) = (b_{ml}(x_m(t)))_{n\times n}$ represent the non-delayed and delayed memristive synaptic weights, respectively.

The parameters signification and performance of MNN are described as

$$c_m = \frac{1}{C_m}[\sum_{l=1}^{n}(W_{\bar{g}ml} + W_{gml}) \times sgn_{ml} + \frac{1}{R_m}],$$

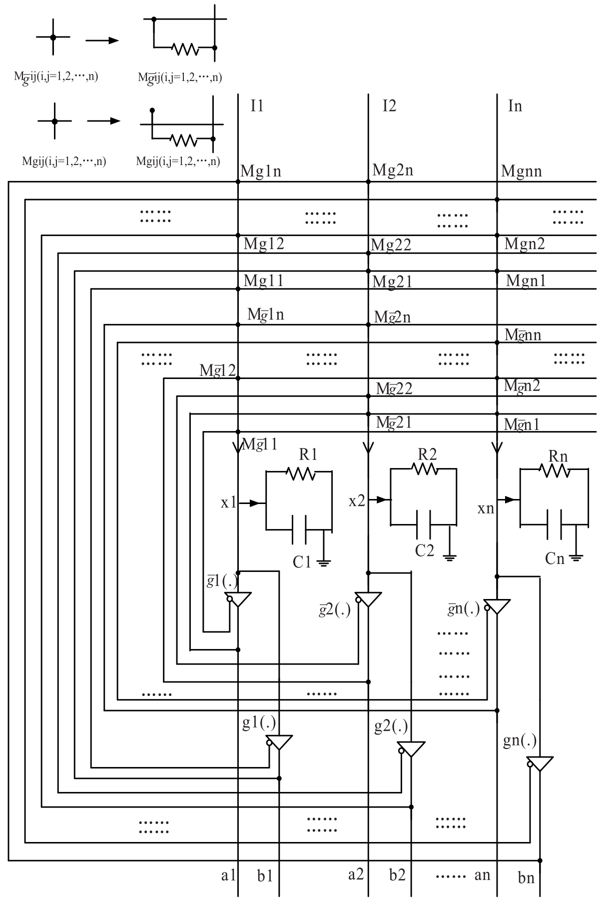

FIGURE 1. Circuit of MNN, where $x_i(\cdot)$ is the state of the ith subsystem, $\bar{f}_j(\cdot)$, $g_j(\cdot)$ are the amplifiers, $W_{\bar{g}ij}(\cdot)$ is the connection memristor between the amplifier $\bar{g}_j(\cdot)$ and state $x_i(\cdot)$ and $W_{gij}(\cdot)$ is the connection memristor between the amplifier $g_j(\cdot)$ and state $x_i(\cdot)$, R_i and C_i are the resistor and capacitor, I_i is the external input, a_i, b_i are the outputs.

$$a_{ml}(x_m(t)) = \frac{W_{\bar{g}ml}}{C_m} \times sgn_{ml},$$
$$b_{ml}(x_m(t)) = \frac{W_{gml}}{C_m} \times sgn_{ml},$$
$$a_{ml}(x_m(t)) = \begin{cases} \hat{a}_{ml}, & |x_m| \leq T_m, \\ \check{a}_{ml}, & |x_m| > T_m, \end{cases} \quad (2)$$
$$b_{ml}(x_m(t)) = \begin{cases} \hat{b}_{ml}, & |x_m| \leq T_m, \\ \check{b}_{ml}, & |x_m| > T_m, \end{cases} \quad (3)$$

Remark 1: For the model of MNN, compared with the electric circuits in Zhao *et al.* [40] in 2015 etc., the memductances of the memristors $W_{\bar{g}ij}$, W_{gij} and R_i, respectively take place of the resistors R_{ij}, F_{ij} and R_i of a general class of neural networks. MNN can be implemented by very large-scale integration (VLSI) circuits as shown in Figure 1. But in the coupled memristive neural networks, every MNN acts as a node connecting each other into the structure of coupled network.

Then, time-varying coupled memristive neural network (CMNN) as drive network contains N identical MNN, which is described by

$$\begin{aligned}\dot{x}_i(t) &= -Cx_i(t) + A(x_i(t))\bar{g}(x_i(t)) \\ &\quad + B(x_i(t))g(x_i(t-\tau_0(t))) \\ &\quad + I(t) + \sigma\sum_{j=1}^{N} w_{ij}^0\Gamma x_j(t) \\ &\quad + \sigma\sum_{j=1}^{N} w_{ij}^1\Gamma x_j(t-\tau_1(t)), \end{aligned} \tag{4}$$

where $\tau_1(t)$ and $\tau_2(t)$ is the time-varying delay satisfying $\dot{\tau}_0(t) \leq \mu_0 < 1$, $\dot{\tau}_1(t) \leq \mu_1 < 1$, $\mu = \max\{\mu_0, \mu_1\}$, where τ and μ are given constants. w_{ij}^0 and w_{ij}^1 are, respectively, non-delayed and delayed coupled matrices, which satisfy the following conditions:

H1: If there is a directed edge from j to i, then $w_{ij}^0 = 1$ and $w_{ij}^1 = 1$, otherwise $w_{ij}^0 = 0$ and $w_{ij}^1 = 0$;

H2: For $i = 1, 2, \ldots, N$, diffusive coupling conditions are satisfied as $w_{ii}^0 = -\sum_{j=1, j\neq i}^{N} w_{ij}^0$ and $w_{ii}^1 = -\sum_{j=1,j\neq i}^{N} w_{ij}^1$.

The corresponding response network can be described as follows:

$$\begin{aligned}\dot{y}_i(t) &= -Cy_i(t) + \sum_{j=1}^{n} A(y_i(t))\bar{g}(y_i(t)) \\ &\quad + \sum_{j=1}^{n} B(y_i(t))g(y_i(t-\tau_0(t))) \\ &\quad + I(t) + \sigma\sum_{j=1}^{N} w_{ij}^0\Gamma y_j(t) \\ &\quad + \sigma\sum_{j=1}^{N} w_{ij}^1\Gamma y_j(t-\tau_1(t)) \\ &\quad + u_i(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{5}$$

where $y(t) = (y_1^T(t), y_2^T(t), \ldots, y_N^T(t))^T$ is the response state scalar. $u_i(t)(i = 1, 2, \ldots, N)$ is a nonlinear controller to be designed and the remaining parameters are the same as drive system.

$$\begin{aligned} \hat{A} &= (\hat{a}_{ml})_{n\times n}, \quad \check{A} = (\check{a}_{ml})_{n\times n}. \\ \hat{B} &= (\hat{b}_{ml})_{n\times n}, \quad \check{B} = (\check{b}_{ml})_{n\times n}. \\ \tilde{A} &= (\tilde{a}_{ml})_{n\times n}, \quad \tilde{B} = (\tilde{B}_{ml})_{n\times n}. \end{aligned}$$

Denote

$$\begin{aligned} \bar{A} &= \max\{\hat{A}, \check{A}\}, \quad \underline{A} = \min\{\hat{A}, \check{A}\}. \\ \bar{B} &= \max\{\hat{B}, \check{B}\}, \quad \underline{B} = \min\{\hat{B}, \check{B}\}. \\ A &= \frac{1}{2}(\bar{A} + \underline{A}), \quad \tilde{A} = \frac{1}{2}(\bar{A} - \underline{A}). \\ B &= \frac{1}{2}(\bar{B} + \underline{B}), \quad \tilde{B} = \frac{1}{2}(\bar{B} - \underline{B}). \end{aligned}$$

Therefore, the Eq.(4) can be written as

$$\begin{aligned}\dot{x}(t) &\in -Cx(t) + (A + \overline{co}[-\tilde{A}, \tilde{A}])\bar{g}(x(t)) \\ &\quad + (B + \overline{co}(-\tilde{B}, \tilde{B}))g(x(t-\tau_0(t))), \\ &\quad + I(t) + \sigma W_0 x(t) + \sigma W_1 x(t-\tau_1(t)), \end{aligned} \tag{6}$$

where $x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t))^T$, $\bar{g}(x(t)) = (\bar{g}^T(x_1(t), \bar{g}^T(x_2(t), \ldots, \bar{g}^T(x_N(t))^T)$ and $g(x(t)) = (g^T(x_1(t), g^T(x_2(t), \ldots, g^T(x_N(t))^T)$. $W_0 = (w_{ij}^0)_{N\times N}$ and $W_1 = (w_{ij}^1)_{N\times N}$. Γ denotes inner couple matrix, which omits here due to it is as an identity matrix.

Furthermore, there exist the measurable functions

$$\triangle A_1(t) = \begin{cases} \tilde{A}, & A = \bar{A}, \\ -\tilde{A}, & A = \underline{A}, \end{cases} \tag{7}$$

$$\triangle B_1(t) = \begin{cases} \tilde{B}, & B = \bar{B}, \\ -\tilde{B}, & B = \underline{B}, \end{cases} \tag{8}$$

According to differential inclusions theory and set-valued mappings technique, we have

$$\begin{aligned}\dot{x}(t) &= -Cx(t) + (A + \triangle A_1(t))\bar{g}(x(t)) \\ &\quad + (B + \triangle B_1(t))g(x(t-\tau_0(t))) \\ &\quad + I(t) + \sigma W_0 x(t) + \sigma W_1 x(t-\tau_1(t)), \end{aligned} \tag{9}$$

where the initial conditions of CMNN (9) is $x(t) = \Psi(t)$, $t \in [-\tau, 0]$.

Remark 2: From CMNN (4) to CMNN (9), a novel technique is proposed to transform a CMNN system to a CNN with interval parameters, and the finite-time synchronization is obtained by investigating this CNN.

Remark 3: According to the conditions (2) and (3) of state-dependence, the variables $\triangle A_1$ and $\triangle B_1$ may not reach their maximum and minimum values at the same time. That is, when $\hat{A} > \check{A}$ or $\hat{A} < \check{A}$, it does not mean the corresponding $\hat{B} > \check{B}$ or $\hat{B} < \check{B}$ hold.

Similar to Eq.(4), the Eq.(5) is also rewritten as follows:

$$\begin{aligned}\dot{y}(t) &= -Cy(t) + (A + \triangle A_2(t))\bar{g}(y(t)) \\ &\quad + (B + \triangle B_2(t))g(y(t-\tau_0(t))) + I(t) \\ &\quad + \sigma W_0 y(t) + \sigma W_1 y(t-\tau_1(t)) \\ &\quad + u(t), \end{aligned} \tag{10}$$

where the initial conditions of Eq.(10) is $x(t) = \Phi(t)$, $t \in [-\tau, 0]$, $\triangle A_2 \in [-\tilde{A}, \tilde{A}]$ and $\triangle B_2 \in [-\tilde{B}, \tilde{B}]$.

Noting 1: Obviously, the variables $\triangle A_2$ and $\triangle B_2$ may not reach their maximum and minimum values at the same time.

Let $e(t) = y(t) - \alpha x(t)$ be the synchronization error, where $\alpha \subset R$ is a real scaling factor. Especially, the initial condition of the error system is $e(t) = \Phi(t) - \alpha\Psi(t)$.

Then we yield the error system from systems (9) and (10) as follows:

$$\begin{aligned}\dot{e}(t) &= -Ce(t) + (A + \triangle A_2(t))f(e(t)) \\ &\quad + (B + \triangle B_2(t))f(e(t-\tau_0(t))) + \Xi(t) \\ &\quad + \sigma W_0 e(t) + \sigma W_1 e(t-\tau_1(t)) + u(t), \end{aligned} \tag{11}$$

where $f(e(\cdot)) = \bar{g}(e(\cdot) + x(\cdot)) - \bar{g}(x(\cdot)) = g(e(\cdot) + x(\cdot)) - g(x(\cdot))$, $\Xi(t) = (\triangle A_2 - \triangle A_1(t))g(x(t)) + (\triangle B_2 - \triangle B_1(t))g(x(t-\tau_1(t)))$.

Through the above analysis, we can give the following Assumptions and Lemmas.

Remark 4: The parameters $\triangle A_i(t)$ and $\triangle B_i(t) (i = 1, 2)$ are time-varying but norm-bounded, which satisfy

$$\triangle A_1(t) = H_1F_1(t)M_1, \quad \triangle A_2(t) = H_1E_1(t)M_1,$$
$$\triangle B_1(t) = H_2F_2(t)M_2, \quad \triangle B_2(t) = H_2E_2(t)M_2.$$

where M_i and H_i $(i = 1, 2)$ are the known real constant matrices. In this paper, $M_1 = \bar{A} - \underline{A}$, $M_2 = \bar{B} - \underline{B}$ and $H_i = diag\{\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\}$, $F_i(t)$ and $E_i(t)$ are unknown real matrices with appropriate dimension and Lebesgue norm measurable elements and satisfying

$$F_i^T(t)F_i(t) \le I, \quad E_i^T(t)E_i(t) \le I.$$

Assumption 1: The activation function $g(x)$ is bounded and it satisfies a Lipschitz condition

$$|g_i(\xi_1) - g_i(\xi_2)| \le l_i|\xi_1 - \xi_2|, \quad i = 1, 2, \ldots n$$

for any $\xi_1, \xi_2 \in R$, where real constant $l_i > 0$ for any i and let $L = diag\{l_1, l_2, \ldots, l_n\}$.

Assumption 2: There exist constant Z_i such that $g_i(x) \leqslant Z_i$ for $\forall x \in R$, $i = 1, 2, \ldots, n$.

Assumption 3: For any two vectors $x, y \in R^n$ and a positive definite matrix $Q \in R^{n\times n}$, the following matrix inequality holds:

$$2x^Ty \le x^TQx + y^TQ^{-1}y.$$

Lemma 1 (See [42]): If X and Y are real matrices with appropriate dimensions, then there exists a number $\varepsilon > 0$, such that

$$X^TY + Y^TX \le \varepsilon X^TX + \frac{1}{\varepsilon}Y^TY.$$

Lemma 2 [43]: Assume that a continuous, positive-definite function $V(t)$ satisfies the following differential inequality:

$$\dot{V}(t) + \alpha V^{\eta}(t) \le 0, \quad \forall t \ge t_0, \ V(t_0) \ge 0,$$

where $\alpha > 0, 0 < \eta < 1$ are two constants. Then, for any given t_0, $V(t)$ satisfies the following differential inequality:

$$V^{1-\eta}(t) \le V^{1-\eta}(t_0) - \alpha(1-\eta)(t-t_0), \quad t_0 \le t \le t_1,$$

and

$$V(t) \equiv 0, \quad \forall t \ge t_1,$$

with t_1 given by

$$t_1 = t_0 + \frac{V^{1-\eta}(t_0)}{\alpha(1-\eta)}.$$

Lemma 3 [44]: Let $x_1, x_2, \ldots, x_n \in R^n$ are any vectors and $0 < q < 2$ is a real number, which satisfy as follows:

$$\|x_1\|^q + \|x_2\|^q + \cdots + \|x_n\|^q \ge (\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2)^{\frac{q}{2}}.$$

Lemma 4 [45] (See Wang et al., 2009): The linear matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0,$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ and $S(x)$ depend affinely on x, and it is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0.$$

III. MAIN RESULTS

We give the following controller $u(t)$

$$\begin{aligned} u(t) = &-Re(t) - \Lambda sign(e(t)) \\ &- k_1 sign(e(t))|e(t)|^{\beta} \\ &- k_1\frac{e(t)}{\|e(t)\|^2}\sum_{m=0}^{1}\left(\int_{t-\tau_m(t)}^{t} e^T(s)e(s)ds\right)^{\frac{\beta+1}{2}}, \end{aligned} \tag{12}$$

where $R = diag(r_1, r_2, \ldots, r_n)$ is the feedback gain matrix to be designed, $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, r_i and λ_i, $i = 1, 2, \ldots, n$ are positive constants, and

$$\lambda_i = \sum_{j=1}^{n}(|\hat{a}_{ij} - \check{a}_{ij}| + |\hat{b}_{ij} - \check{b}_{ij}|)Z_j. \tag{13}$$

$sign(e(t)) = (sign(e_1(t)), sign(e_2(t)), \ldots, sign(e_n(t)))^T$. $0 < \beta < 1$ and k_1 is a random constant. And sign(x) is the sign function which is defined as follows:

$$sign(x) = \begin{cases} -1, & if\ x < 0, \\ 0, & if\ x = 0, \\ 1, & if\ x > 0. \end{cases}$$

Remark 5 (See [41]): According to the boundedness of chaotic signals, the Assumption 2 can be guaranteed to achieve robust synchronization of CMNN.

Theorem 1: Under Assumptions 1-3, the drive network (4) and the response network (5) can realize finite-time projective synchronization with the controller (12), if there exist a positive matrix R and two positive constants θ_1 and θ_2 satisfying the following conditions:

$$\begin{bmatrix} \Omega_0 & L & A & H_1 & B & H_2 & W_1 \\ \star & 2\theta_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -\frac{1}{\theta_1}I & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\frac{1}{(\theta_1\|M_1\|^2)} & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{1}{\theta_2}I & 0 & 0 \\ \star & \star & \star & \star & \star & -\frac{1}{(\theta_2\|M_2\|^2)} & 0 \\ \star & \star & \star & \star & \star & \star & -\frac{1}{\sigma^2} \end{bmatrix} \le 0,$$

where

$$\Omega_0 = 2(\frac{I}{1-\mu} - R - C + \sigma W_0),$$

and

$$\frac{2}{\theta_2}L^2 - I \leq 0,$$

Then, synchronization is achieved in a finite time:

$$t_0 = \frac{V(0)^{1-(\beta+1)/2}}{\gamma(1-\frac{\beta+1}{2})},$$

and

$$V(0) = e^T(0)e(0) + \frac{1}{(1-\mu)}\sum_{m=0}^{1}(\int_{-\tau_m(0)}^{0} e^T(s)e(s)ds),$$

where $e(0), \hat{\upsilon}(0)$ are the initial conditions of e, υ, respectively.

Remark 6: Based on the research of the above CMNN, we extend the method to the general situation for the CMNN that CMNN with multi-links is introduced. Due to the connections between neurons in the neural network are complex and each of two neurons have more than one link. The study of CMNN with multi-links has more practical theoretical significance.

Next, CMNN with multi-links is described as follows:

$$\begin{aligned}\dot{x}_i(t) = &-Cx_i(t) + A(x_i(t))\bar{g}(x_i(t)) \\ &+ B(x_i(t))g(x_i(t-\tau_0(t))) + I(t) \\ &+ \sigma\sum_{j=1}^{N} w_{ij}^0 \Gamma x_j(t) \\ &+ \sigma\sum_{m=1}^{l}\sum_{j=1}^{N} w_{ij}^m \Gamma x_j(t-\tau_m(t)),\end{aligned} \tag{14}$$

where $w_{ij}^m, m = 1, \ldots, l$ is the lth sub-network's topological structure, which satisfy the conditions H1 and H2. Time-varying delay $\tau_m(t), m = 0, 1, \ldots l$ satisfy $\dot{\tau_m}(t) \leq \mu_m < 1, \mu = \max\{\mu_m, m = 0, 1, \ldots, l\}$. The reminding parameters are same to Eq.(4).

Similarly, Eq.(18) as drive network can be rewritten as

$$\begin{aligned}\dot{x}(t) = &-Cx(t) + (A + \triangle A_1(t))\bar{g}(x(t)) \\ &+ (B + \triangle B_1(t))g(x(t-\tau_0(t))) + I(t) \\ &+ \sigma W_0 x(t) + \sum_{m=1}^{l} \sigma W_1 x(t-\tau_m(t)),\end{aligned} \tag{15}$$

Then, the corresponding response network is given as

$$\begin{aligned}\dot{y}(t) = &-Cy(t) + (A + \triangle A_2(t))\bar{g}(y(t)) \\ &+(B + \triangle B_2(t))g(y(t-\tau_0(t))) + I(t) \\ &+\sigma W_0\, y(t) + \sum_{m=1}^{l} \sigma W_1 y(t-\tau_m(t)) \\ &+u(t),\end{aligned} \tag{16}$$

where

$$\begin{aligned}u(t) = &-Re(t) - k_1 sign(e(t))|e(t)|^\beta \\ &-\Lambda sign(e(t)) - k_1 \tfrac{e(t)}{\|e(t)\|^2} \\ &\times \sum_{m=0}^{l}(\textstyle\int_{t-\tau_m(t)}^{t} e^T(s)e(s)ds)^{\frac{\beta+1}{2}},\end{aligned} \tag{17}$$

The corresponding Theorem 2 is extended as follows, shown at the top of the next page.

Theorem 2: Under Assumptions 1-4, the drive network (15) and the response network (16) with the controller (17) can realize finite-time projective synchronization, if there exist a positive matrix R and two positive constants θ_1 and θ_2 satisfying the following conditions:

where

$$\Omega_0 = \frac{l+1}{1-\mu}I - 2(R - C + \sigma W_0),$$

and

$$\frac{2}{\theta_2}L^2 - I \leq 0,$$

Then, synchronization is achieved in a finite time:

$$t_1 = \frac{V(0)^{1-(\beta+1)/2}}{\gamma(1-\frac{\beta+1}{2})},$$

and

$$V(0) = e^T(0)e(0) + \frac{1}{(1-\mu)}\sum_{m=0}^{l}(\int_{-\tau_m(0)}^{0} e^T(s)e(s)ds),$$

where $e(0), \hat{\upsilon}(0)$ are the initial conditions of e, υ, respectively.

Proof: Similarly with the proof of Theorem 1. The detailed proofs are omitted here.

Remark 7: In previous research, differential inclusions theory and set-valued mappings technique have been recently introduced to deal with this CMNN system. But, we study the synchronization of CMNN without using the previous solution technique. A novel analytical technique is first proposed to transform the CMNN to a class of neural network with interval parameters, which is an extension of our paper [5]. In addition, we give a proper controller that a part of the uncertainties ($\Xi(t) = (\triangle A_2 - \triangle A_1(t))g(x(t)) + (\triangle B_2 - \triangle B_1(t))g(x(t-\tau_1(t)))$) is treated as a perturbation term to achieve robust control. In this paper, we use the boundedness of uncertain parameters to fill the gap for the previous researches, the results have less conservative than those previous analysis technique.

IV. NUMERICAL EXAMPLES

In this section, two numerical examples are given to demonstrate the effectiveness of our proposed scheme. Firstly,

$$\begin{bmatrix} \Omega_0 & L & A & H_1 & B & H_2 & W_1 & \cdots & W_l \\ \star & 2\theta_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & -\frac{1}{\theta_1}I & 0 & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & \star & -\frac{1}{(\theta_1\|M_1\|^2)} & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & \star & \star & -\frac{1}{\theta_2}I & 0 & 0 & \cdots & 0 \\ \star & \star & \star & \star & \star & -\frac{1}{(\theta_2\|M_2\|^2)} & 0 & \cdots & 0 \\ \star & \star & \star & \star & \star & \star & -\frac{1}{\sigma^2} & \cdots & 0 \\ \star & \star & \star & \star & \star & \star & \star & \cdots & 0 \\ \vdots & & & & \vdots & & & \ddots & \\ \star & \star & \star & \star & \star & \star & \star & \cdots & -\frac{1}{\sigma^2} \end{bmatrix} \leq 0,$$

we give the following memristive neural network without coupled connections:

$$\begin{aligned} \dot{x}_m(t) &= -c_m x_m(t) + \sum_{l=1}^{n} a_{ml}(x_m(t))\bar{g}_l(x_l(t)) \\ &\quad + \sum_{l=1}^{n} b_{ml}(x_m(t))g_l(x_l(t-\tau_0(t))) \\ &\quad + I_m(t), \quad m = 1, 2 \end{aligned} \tag{18}$$

where $n = 2$, $I_1 = I_2 = 0$, $\bar{g}_l(x) = g_l(x) = tanh(x)$, $l = 1, 2$, $\tau_0(t) = 0.2|cost|$ and $c_1 = 1$, $c_2 = 1.5$. The remaining parameters are given with (19)-(22)

$$a_{11}(x_1) = \begin{cases} 0.7, & |x_1(t)| \leq 1, \\ 0.3, & |x_1(t)| > 1, \end{cases}$$

$$a_{12}(x_1) = \begin{cases} 1.5, & |x_1(t)| \leq 1, \\ 0.5, & |x_1(t)| > 1, \end{cases} \tag{19}$$

$$a_{21}(x_2) = \begin{cases} -0.1, & |x_2(t)| \leq 1, \\ -0.3, & |x_2(t)| > 1, \end{cases}$$

$$a_{22}(x_2) = \begin{cases} 0.1, & |x_2(t)| \leq 1, \\ 0.9, & |x_2(t)| > 1. \end{cases} \tag{20}$$

$$b_{11}(x_1) = \begin{cases} -1.5, & |x_1(t)| \leq 1, \\ -1.3, & |x_1(t)| > 1, \end{cases}$$

$$b_{12}(x_1) = \begin{cases} -0.1, & |x_1(t)| \leq 1, \\ -0.05, & |x_1(t)| > 1, \end{cases} \tag{21}$$

$$b_{21}(x_2) = \begin{cases} -0.15, & |x_2(t)| \leq 1, \\ -0.2, & |x_2(t)| > 1, \end{cases}$$

$$b_{22}(x_2) = \begin{cases} -2.3, & |x_2(t)| \leq 1, \\ -2.5, & |x_2(t)| > 1. \end{cases} \tag{22}$$

Then, by the above Eqs.(19)-(22), one has

$$\begin{aligned} A &= \begin{pmatrix} 0.5 & 1 \\ -0.2 & 0.5 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0.2 & 0.5 \\ 0.1 & 0.4 \end{pmatrix}, \\ M_1 &= \begin{pmatrix} 0.4 & 1 \\ 0.2 & 0.8 \end{pmatrix}, \end{aligned} \tag{23}$$

$$B = \begin{pmatrix} -1.4 & -0.175 \\ -0.075 & -2.4 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0.1 & 0.025 \\ 0.025 & 0.1 \end{pmatrix}, \tag{24}$$

$$\begin{aligned} M_2 &= \begin{pmatrix} 0.2 & 0.05 \\ 0.05 & 0.2 \end{pmatrix}, \quad H_1 = H_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \\ L &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{25}$$

and $F_i(t) \in [-1, 1]$, $i = 1, 2$.

Example 1: Based on MNN system (18), we consider the following CMNN systems as drive system and response system with controller:

$$\begin{aligned} \dot{x}_i(t) &= -Cx_i(t) + A(x_i(t))\bar{g}(x_i(t)) \\ &\quad + B(x_i(t))g(x_i(t-\tau_0(t))) \\ &\quad + \sigma \sum_{j=1}^{3} w_{ij}^0 \Gamma x_j(t) \\ &\quad + \sigma \sum_{j=1}^{3} w_{ij}^1 \Gamma x_j(t-\tau_1(t)) \\ &\quad + I_i, \quad i = 1, 2, 3 \end{aligned} \tag{26}$$

$$\begin{aligned} \dot{y}_i(t) = &\quad -Cy_i(t) + A(y_i(t))\bar{g}(y_i(t)) \\ &\quad + B(y_i(t))g(y_i(t-\tau_0(t))) \\ &\quad + \sigma \sum_{j=1}^{3} w_{ij}^0 \Gamma y_j(t) \\ &\quad + \sigma \sum_{j=1}^{3} w_{ij}^1 \Gamma y_j(t-\tau_1(t)) + I_i + u_i(t). \end{aligned} \tag{27}$$

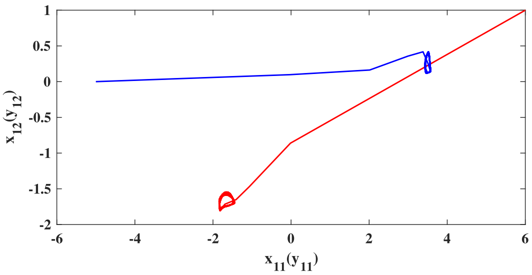

FIGURE 2. The phase curves of systems (26) and (27) in two dimensional cases.

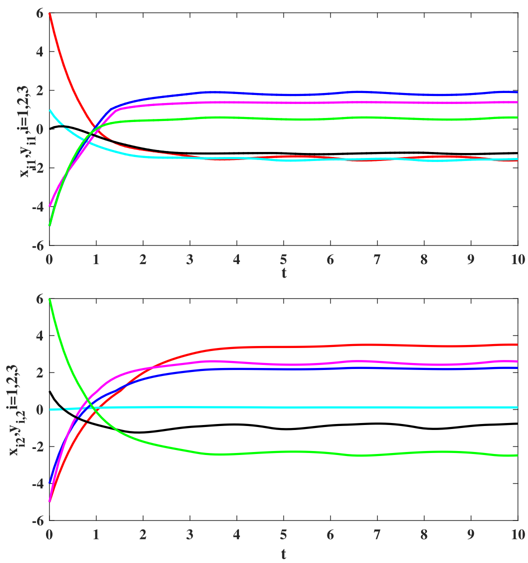

FIGURE 3. The two dimensional state trajectory of drive systems (26) and response system (27) without controller.

where $\tau_1(t) = 0.2 * |sint|$, which means that $\tau = 0.2, \mu = 0.2$ and $\sigma = 0.1$. Besides

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

We take the initial values of drive and response systems as $x_{i1}(0) = (cos(t) + 5, sin(t) - 5, sin(t))^T$, $x_{i2}(0) = (sin(t) - 5, cos(t) - 5, cos(t))^T$, $y_{i1}(0) = (cos(t), cos(t) - 5, sin(t) - 5)^T$ and $y_{i2}(0) = (sin(t), sin(t) - 5, cos(t) + 5)^T$. Let $r = 70$, according to the LMI toolbox, we obtain $\theta_1 = 98.1305$, $\theta_2 = 2.036$. Fig.2 show the phase curves of systems (26) and (27) in two dimensional cases, which indicates the boundedness of chaotic signals. Figs.3 and 4 show the two dimensional drive-response systems with three nodes and error state trajectory of CMNN without controller. Figs.5 and 6 show the two dimensional drive-response systems with three nodes and error state trajectory of CMNN with controller.

Example 2: Based on CMNN systems (26) and (27), we consider the following CMNN with multi-links as drive

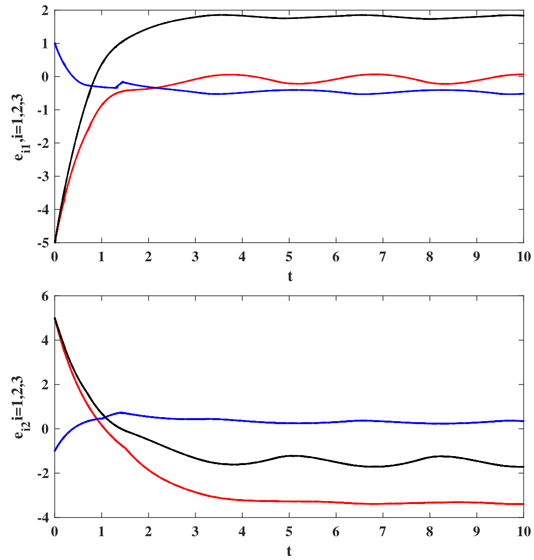

FIGURE 4. The two dimensional error state trajectory of CMNN system without controller.

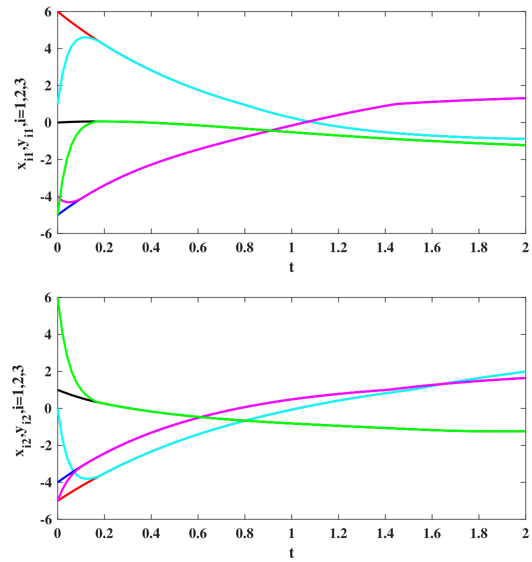

FIGURE 5. The two dimensional state trajectory of drive systems (26) and response system (27) with controller.

system and response system with controller:

$$\begin{aligned} \dot{x}_i(t) = & -Cx_i(t) + A(x_i(t))\bar{g}(x_i(t)) \\ & + B(x_i(t))g(x_i(t - \tau_0(t))) \\ & + \sigma \sum_{j=1}^{6} w_{ij}^0 \Gamma x_j(t) \\ & + \sigma \sum_{j=1}^{6} w_{ij}^1 \Gamma x_j(t - \tau_1(t)) \\ & + \sigma \sum_{j=1}^{6} w_{ij}^2 \Gamma x_j(t - \tau_2(t)) \end{aligned}$$

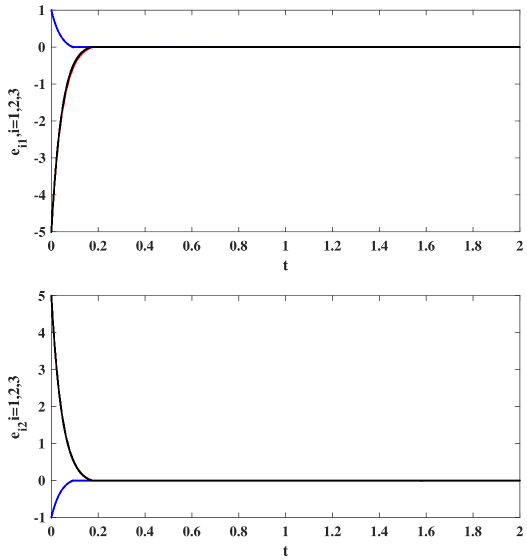

FIGURE 6. The two dimensional error state trajectory of CMNN system with controller.

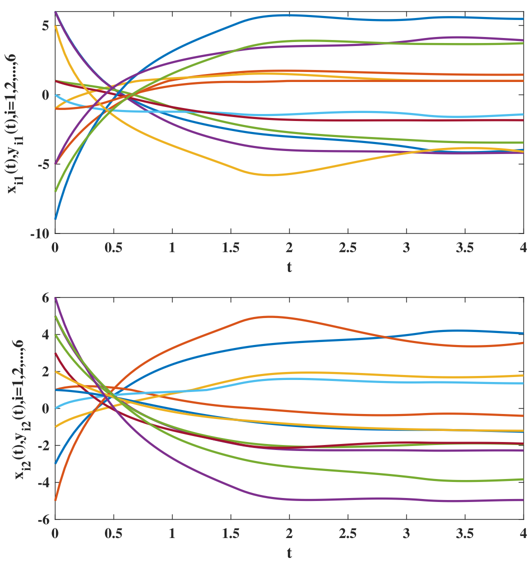

FIGURE 7. The two dimensional state trajectory of drive systems (28) and response system (29) without controller.

$$+I_i, \quad i=1,2,\ldots,6 \tag{28}$$

$$\begin{aligned}\dot{y}_i(t) = &-Cy_i(t)+A(y_i(t))\bar{g}(y_i(t)) \\ &+B(y_i(t))g(y_i(t-\tau_0(t))) \\ &+\sigma\sum_{j=1}^{6} w_{ij}^0\Gamma y_j(t) \\ &+\sigma\sum_{j=1}^{6} w_{ij}^1\Gamma y_j(t-\tau_1(t)) \\ &+\sigma\sum_{j=1}^{6} w_{ij}^2\Gamma y_j(t-\tau_2(t))+I_i+u_i(t). \end{aligned}\tag{29}$$

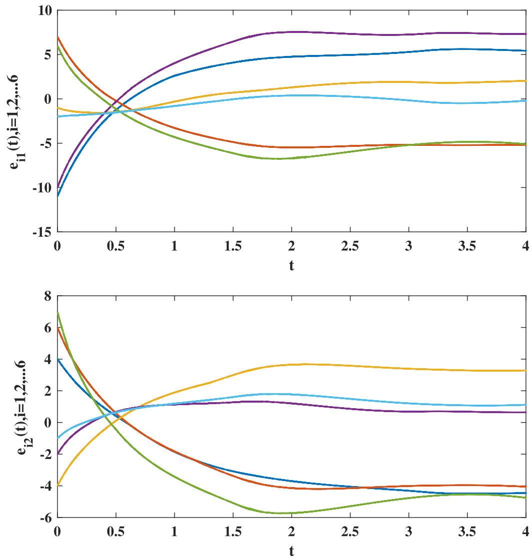

FIGURE 8. The two dimensional error state trajectory of CMNN with multi-links and without controller.

where $\tau_1(t)=0.2*|sint|$, $\tau_2(t)=0.15*|sint|$, which means that $\tau = 0.2$,$\mu = 0.2$. The topological structures of these three sub-networks (W_0,W_1 and W_2) in the network are as follows:

$$W_0 = \begin{bmatrix} -6 & 3 & 2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & -7 & 4 & 2 \\ 0 & 0 & 0 & 4 & -6 & 2 \\ 0 & 0 & 0 & 2 & 2 & -4 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -7 & 2 & 0 & 3 & 0 & 2 \\ 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 3 & 0 & 2 \\ 3 & 0 & 3 & -9 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 2 & 3 & 0 & -7 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} -3 & 2 & 0 & 0 & 0 & 1 \\ 2 & -5 & 3 & 0 & 0 & 0 \\ 0 & 3 & -7 & 4 & 0 & 0 \\ 0 & 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 0 & 5 & -11 & 6 \\ 1 & 0 & 0 & 0 & 5 & -6 \end{bmatrix}.$$

The initial values of drive and response systems as $x_{i1}(0) = (cos(t)+5, sin(t)-1, cos(t), cos(t), cos(t)-2, sin(t)-5)^T$, $x_{i2}(0) = (sin(t)-3, cos(t)-2, cos(t)+3, sin(t)+3, sin(t)-5, cos(t)+5)^T$, $y_{i1}(0) = (sin(t)-5, cos(t)+5, sin(t), sin(t)-9, sin(t)+5, cos(t)-8)^T$ and $y_{i2}(0) = (cos(t), sin(t)+5, sin(t), cos(t), cos(t)+1, sin(t)+5)^T$. In the same way, let $r = 90$, according to the LMI toolbox, we obtain $\theta_1 = 77.8076, \theta_2 = 6.9805$. Figs.7 and 8 show the two dimensional drive-response systems with three nodes and error state trajectory of CMNN

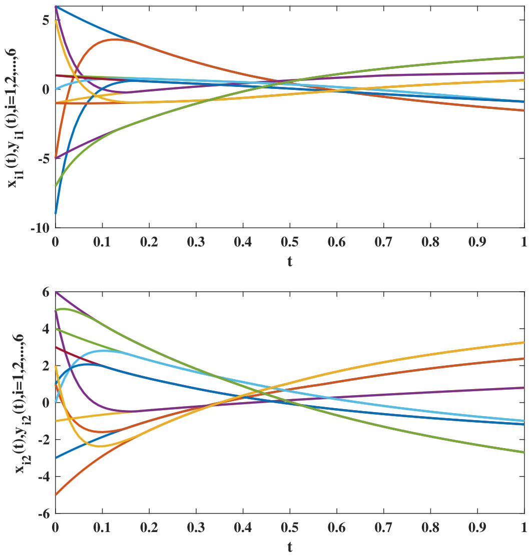

FIGURE 9. **The two dimensional state trajectory of drive systems (32) and response system (33) with controller.**

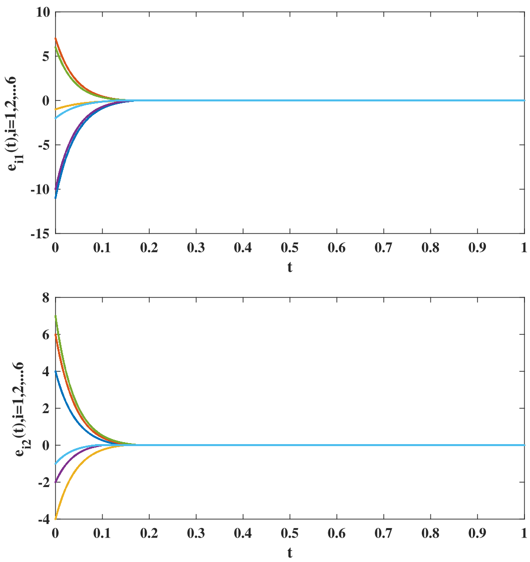

FIGURE 10. **The two dimensional error state trajectory of CMNN with multi-links and with controller.**

without controller. Figs.9 and 10 show the two dimensional drive-response systems with three nodes and error state trajectory of CMNN with controller.

V. CONCLUSION AND PROSPECT

In this paper, the finite-time synchronization of CMNN and CMNN with multi-links have been studied based on robust control technique. It is different from previous analysis technique, we give a new approach to study finite-time synchronization for CMNN and CMNN with multi-links based on the drive-response concept and the finite-time stability theory. The paper overcome some difficult of parameter unmatch and synchronization difficult of complex CMNN. The results filled the blank for CMNN and be less conservative than those previous analysis technique. Numerical simulations verify the effectiveness of our theoretical analysis.

Furthermore, we extend the CMNN model to simulate more specific form of brain neural network, especially some dynamical behavior of the associative memory CMNN and applications in network security communication.

APPENDIX

Proof of Theorem 1: We construct a Lyapunov function as follows

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = e^T(t)e(t),$$
$$V_2(t) = \frac{1}{1-\mu}\sum_{m=0}^{1}(\int_{t-\tau_m(t)}^{t} e^T(s)e(s)ds).$$

The derivative of $V_1(t)$ along with the trajectory of $e(t)$ is given as

$$\begin{aligned}\dot{V}_1(t) &= 2e^T(t)\dot{e}(t),\\ &= 2e^T(t)[-Ce(t) + (A + \triangle A_2(t))f(e(t))\\ &\quad + (B + \triangle B_2(t))f(e(t-\tau(t))) + \Xi(t)\\ &\quad + \sigma W_0 e(t) + \sigma W_1 e(t-\tau_1(t)) + u(t)],\\ &= -2e^T(t)Ce(t) + 2e^T(t)(A + \triangle A_2(t))f(e(t))\\ &\quad + 2e^T(t)(B + \triangle B_2(t))f(e(t-\tau(t)))\\ &\quad + 2e^T(t)\Xi(t) + 2\sigma e^T(t)W_0 e(t)\\ &\quad + 2\sigma e^T(t)W_1 e(t-\tau_1(t)) + 2e^T(t)u(t).\end{aligned}$$

According to Lemma 1 and Assumption 1, we get

$$\begin{aligned}&2e^T(t)(A + \triangle A_2(t))f(e(t))\\ &\le \theta_1\, e^T(t)AA^T e(t) + \frac{1}{\theta_1}f^T(e(t))f(e(t))\\ &\quad + \theta_1\, e^T(t)H_1E_1M_1M_1^TE_1^TH_1^T e(t)\\ &\quad + \frac{1}{\theta_1}f^T(e(t))f(e(t)),\\ &\le e^T(t)[\theta_1 AA^T + \frac{2}{\theta_1}L^2 + \theta_1\|M_1\|^2H_1H_1^T]e(t),\\ &2e^T(t)(B + \triangle B_2(t))f(e(t-\tau_0(t)))\\ &\le \theta_2\, e^T(t)BB^T e(t)\\ &\quad + \frac{1}{\theta_2}f^T(e(t-\tau_0(t)))f(e(t-\tau_0(t)))\\ &\quad + \theta_2\, e^T(t)H_2E_2M_2M_2^TE_2^TH_2^T e(t)\\ &\quad + \frac{1}{\theta_2}f^T(e(t-\tau_0(t)))f(e(t-\tau_0(t))),\\ &\le e^T(t)[\theta_2 BB^T + \theta_2\|M_2\|^2H_2H_2^T]e(t)\\ &\quad + \frac{2}{\theta_2}e^T(t-\tau_0(t))L^2e(t-\tau_0(t)).\end{aligned}$$

It is clear from Assumption 2 that

$$\begin{aligned}|\Xi(t)| \leq & |\triangle A_2(t) - \triangle A_1(t)||g(x(t))| \\ & + |\triangle B_2(t) - \triangle B_1(t)| \\ & \times |g(x(t-\tau(t))| \\ \leq & 2(\tilde{A}+\tilde{B})Z, \end{aligned} \quad (30)$$

Due to the controller $u(t)$,we can obtain

$$\begin{aligned}2e^T(t)u(t) = & 2e^T(t)[-Re(t) - \Lambda sign(e(t)) \\ & - k_1 sign(e(t))|e(t)|^{\beta} \\ & - k_1 \frac{e(t)}{\|e(t)\|^2}\sum_{m=0}^{1}(\int_{t-\tau_m(t)}^{t} e^T(s) \\ & \times e(s)ds)^{\frac{\beta+1}{2}}, \\ = & -2re^T(t)e(t) \\ & - 2k_1|e^T(t)e(t)|^{\frac{\beta+1}{2}} \\ & - 2e^T(t)\Lambda sign(e(t)) \\ & - 2k_1\sum_{m=0}^{1}(\int_{t-\tau_m(t)}^{t} e^T(s) \\ & \times e(s)ds)^{\frac{\beta+1}{2}}. \end{aligned} \quad (31)$$

From Eqs.(13),(30),(31), we get

$$\begin{aligned}& 2e^T(t)(\Xi(t) - \Lambda sign(e(t))) \\ & \leq 2\sum_{i=0}^{n}|e_i(t)|[2\sum_{j=0}^{n}(\tilde{a}_{ij}+\tilde{b}_{ij})Z_j - \lambda_i] \\ & = 0, \end{aligned} \quad (32)$$

Under Assumption 3, the following inequality is established

$$\begin{aligned}2\sigma e^T(t)W_1 e(t-\tau_1(t)) \leq & \sigma^2 e^T(t)W_1W_1^T e(t) \\ & + e^T(t-\tau_1(t))e(t-\tau_1(t)).\end{aligned}$$

Similarly, the derivative of $V_2(t)$, $V_3(t)$ are given as

$$\begin{aligned}\dot{V}_2(t) = & \frac{2}{1-\mu}e^T(t)e(t) \\ & - \frac{1-\dot{\tau}_0(t)}{1-\mu}e^T(t-\tau_0(t))e(t-\tau_0(t)) \\ & - \frac{1-\dot{\tau}_1(t)}{1-\mu}e^T(t-\tau_1(t))e(t-\tau_1(t)), \\ \leq & \frac{2}{1-\mu}e^T(t)e(t) - e^T(t-\tau_0(t))e(t-\tau_0(t)) \\ & - e^T(t-\tau_1(t))e(t-\tau_1(t)),\end{aligned}$$

Above all, $V(t)$ can be written as

$$\begin{aligned}\dot{V}(t) = & \sum_{i=1}^{2}\dot{V}_i(t), \\ \leq & e^T(t)\Omega_1 e(t) + e^T(t-\tau(t))\Omega_2 e(t-\tau(t)) \\ & - \gamma(|e^T(t)e(t)|^{\frac{\beta+1}{2}} \\ & + \sum_{m=0}^{1}(\frac{1}{1-\mu}\int_{t-\tau_m(t)}^{t} e^T(s)e(s)ds)^{\frac{\beta+1}{2}},\end{aligned}$$

where

$$\begin{aligned}\Omega_1 = & 2(\frac{1}{1-\mu} - r)I - 2C + 2\sigma W_0 + \sigma^2 W_1 W_1^T \\ & + \frac{2}{\theta_1}L^2 + \theta_1(AA^T + \|M_1\|^2 H_1 H_1^T) \\ & + \theta_2(BB^T + \|M_2\|^2 H_2 H_2^T), \\ \Omega_2 = & \frac{2}{\theta_2}L^2 - I,\end{aligned}$$

and $\gamma = min\{2k_1, 2k_1(1-\mu)^{\frac{\beta+1}{2}}\}$.

Based on the condition of Theorem 1 and Lemma 4, we get

$$\dot{V}(t) \leq -\gamma V^{\frac{\beta+1}{2}}(t). \quad (33)$$

Therefore, synchronization is achieved in a finite time:

$$t_0 = \frac{V(0)^{1-(\beta+1)/2}}{\gamma(1-\frac{\beta+1}{2})},$$

The proof of Theorem 1 is completed.

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