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OF SU3

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I hereby recommend that this dissertation prepared under my direction by Douglas Francis Holland

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ABSTRACT

The properties of SU3 finite transformations are investigated. These transformations on the defining three-dimensional complex space are parameterized in a form employing three factors, two of which are the Euler parameterization of an SU2 subgroup.

The irreducible representations of the factored parameterization are found explicitly. The formula for the volume element is derived, and the volume element is calculated for all parameterizations discussed. The orthogonality relation is verified for the factored parameterization.

Symmetries of the transformation matrices are discussed. A definition of triality results and a proof that it is additive modulus three follows. A generalization of G-Parity is presented which reduces correctly for pions.

Spherical harmonic basis states are derived as a specialization of the transformation matrix. Differential equations representing the infinitesimal generators are used to derive the infinitesimal generators of Biedenharn. These states are found to have symmetries that suggest their applicability as meson states. One parameter mass formulas for mesons and for baryons are derived. These results support the suggestion that the group parameters have some physical reality in the space of the eight-dimensional representation.

CHAPTER 1

INTRODUCTION

For some time now the group SU3 has been thought to carry the symmetry of the elementary particles. Since it contains SU2 as a subgroup, all the results concerning isospin can be incorporated into the scheme. The group has rank two which allows the introduction of another quantum number, the hypercharge or strangeness. Each irreducible representation (IR) is thought to correspond to a multiplet of particles; however the elusiveness of particles for the three-dimensional and six-dimensional representations leads some to believe the actual symmetry group might be SU3/Z3.

Summary of Results for Finite Transformations

Considerable work has been done concerning the infinitesimal generators of the group. As an alternative mathematical technique, we wish to investigate the global properties to see if the group parameters have any physical meaning.

A parameterization of all unitary groups was given by Murnaghan (1962, pp. 7-11). We derive the parameterization used by Nelson (1967) which is of special interest as it employs two factors which are SU2 transformations in the Euler form.

Chacon and Moshinsky (1966) derive the IR's in Murnaghan's parameterization by extensive use of Weyl reflections. Nelson, using his parameterization, restricted the representation matrices to a particular right-hand state, and thus derived a single column of the matrix which acts as a set of spherical harmonic basis states. Both Nelson and previously Beg and Ruegg (1965) employ differential operators representing the infinitesimal generators in their derivations. We derive the complete matrix for each IR; the SU2 factors are known and the third factor is evaluated by employing tensor basis states. By applying a finite transformation to these tensors the spherical harmonic basis states are derived.

Weyl (1946) and Murnaghan (1962) discuss the dependence of the volume element on the class parameters, for integration concerning the characters of the group. We evaluate the complete dependence of the volume element on all parameters for Nelson's, Murnaghan's and the more familiar

$$U(\alpha) = e^{i\alpha \cdot I}$$

parameterizations.

Symmetries of the transformations were investigated and one result is a natural definition of triality. Baird and Biedenharn (1964) and Hagan and Macfarlane (1964) prove triality is additive modulus three. We provide a proof that follows very simply from our definition.

Employing the spherical harmonic basis states, the infinitesimal generators derived by Biedenharn (1963) are easily obtained.

By requiring certain symmetries of the basis states, one-parameter mass formulas are obtained for mesons and baryons. These are specializations of the Gell-Mann-Okubo (in Okubo 1962) mass formula.

Review of SU2

To clarify the problem and procedure we first consider the group SU2. Transformations in SU2 can be parameterized using Euler angles.

$$D(\alpha,\beta,\gamma) = e^{-i\alpha J_3}\, e^{-i\beta J_2}\, e^{-i\gamma J_3} \qquad \begin{array}{l} 0 \le \alpha \le 4\pi \\ 0 \le \gamma \le 4\pi \\ 0 \le \beta \le \pi \end{array} \tag{1.1}$$

J_3 can be diagonalized and since J^2 commutes with all J_i, the basis states of the IR's can be labeled Ψ_{jm}

$$J^2\, \Psi_{jm} = j(j+1)\, \Psi_{jm} \qquad j \text{ specifies the IR} \tag{1.2}$$

$$J_3\, \Psi_{jm} = m\, \Psi_{jm} \qquad m \text{ specifies the states within an IR}$$

By considering the Lie algebra of the infinitesimal generators J_i, all their IR's can be derived.

Finally the representations

$$D^{j}_{m\,m'}(\alpha,\beta,\gamma) = \left\langle jm \right| e^{-i\alpha J_3}\, e^{-i\beta J_2}\, e^{-i\gamma J_3} \left| jm' \right\rangle \tag{1.3}$$

can in principle be determined by expanding the exponentials. However in practice this is difficult for the non-diagonal matrix $e^{-i\beta J_2}$.

An alternative method to derive the IR's of SU2 is to reduce the direct product $2 \otimes 2 \otimes 2 \ldots.$ One finds polynomial basis states or tensor states that transform irreducibly. The SU2 transformation $e^{-i\beta J_2}$ is applied to these states to derive the general form of $d^j_{m\,m'}(\beta)$.

The spherical harmonic basis states are found to be a specialization of the representations, to a particular right-hand state

$$\Psi_{jm}(\beta,\alpha) = \sqrt{\frac{2j+1}{4\pi}} \; D^{j\,*}_{m\,0}(\alpha,\beta) \,. \tag{1.4}$$

The volume element is

$$d\,R(\alpha,\beta,\gamma) = \sin\beta \, \frac{d\alpha}{4\pi} \, \frac{d\gamma}{4\pi} \, \frac{d\beta}{2} \tag{1.5}$$

and the orthogonality relations are derived by Clebsch-Gordan (C.G.) decomposition of the direct product.

$$\int D^{j_1\,*}_{m_1\,m_1'}(\alpha,\beta,\gamma)\; D^{j_2}_{m_2\,m_2'}(\alpha,\beta,\gamma)\; d\,R(\alpha,\beta,\gamma) \tag{1.6}$$

$$= (-1)^{m_1 - m_1'} \sum_{j\,m\,m'} C(j_1 j_2 j\,;-m_1\,m_2\,m)\,C(j_1 j_2 j\,;-m_1'\,m_2'\,m') \int D^{j}_{m\,m'}(\alpha,\beta,\gamma)\; d\,R(\alpha,\beta,\gamma)$$

$$= C(j_1 j_2 0\,;-m_1\,m_2\,0)\,C(j_1 j_2 0\,;-m_1'\,m_2'\,0) = \frac{\delta_{j_1 j_2}\;\delta_{m_1 m_2}\;\delta_{m_1' m_2'}}{2j_1+1}$$

In this paper we are guided by the strong analogy that exists between our parameterization and that for SU2. We follow closely the same procedure to derive the corresponding results for SU3.

We use the following set of infinitesimal generators.

$$I_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad I_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad I_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$I_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad I_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \qquad \operatorname{tr}(I_i I_j) = 2\delta_{ij}$$

The basis states are chosen such that I_3 and I_8 are diagonal in all representations and thus serve to partially label the states

$$Y \equiv \frac{1}{\sqrt{3}} I_8 \qquad Y\Psi\binom{Y}{M} = Y\Psi\binom{Y}{M} \tag{1.7}$$

$$M \equiv \frac{1}{2} I_3 \qquad M\Psi\binom{Y}{M} = M\Psi\binom{Y}{M}$$

The eigenvalues (Y, M) of the states within an IR are displayed in two-dimensional weight diagrams.

The addition of $I^2 = \frac{1}{4}(I_1{}^2 + I_2{}^2 + I_3{}^2)$, forms a complete set of commuting operators which serve to label the states

$$I^2 \Psi\binom{I\,Y}{M} = I(I+1)\Psi\binom{I\,Y}{M} . \tag{1.8}$$

We will derive a parameterization factored in a manner analogous to the SU2 Euler angles and give its explicit form.

$D^{\mu}_{\nu_1 \nu_2}(\alpha)$	where	μ	specifies the IR
		ν_1, ν_2	specify a state in an IR (I, Y,M)
		α	are the eight parameters

Again, in the explicit determination of the IR we encounter the matrix $e^{-i\nu I_4}$ (analogous to $e^{-i\beta J_2}$) which is determined by the use of tensor states.

Choosing $\nu_2 = 0$ (I = 0), we derive the spherical harmonic basis states for SU3. The volume element and orthogonality relations are derived by a method analogous to that used for SU2.

CHAPTER 2

PARAMETERIZATION OF GROUP TRANSFORMATIONS

Nelson (1967) uses the following parameterization:

$$U(\underline{\alpha}) = e^{-i\frac{\rho}{\sqrt{3}}I_8}\, e^{-i\frac{\alpha}{2}I_3}\, e^{-i\frac{\beta}{2}I_2}\, e^{-i\frac{\gamma}{2}I_3}\, e^{-i\nu I_4}\, e^{-i\frac{\alpha'}{2}I_3}\, e^{-i\frac{\beta'}{2}I_2}\, e^{-i\frac{\gamma'}{2}I_3} \quad . \tag{2.1}$$

The following is a derivation of this parameterization.

Transformations in SU3 map $Z \rightarrow Z'$ where Z and Z' are vectors in three-dimensional complex space such that the norm of Z' equals the norm of Z. We set $Z^\dagger Z = Z'^\dagger Z' = 1$.

Beg and Ruegg (1965) show that Z and Z' can be parameterized as follows:

$$Z = \begin{pmatrix} e^{i\phi_1}\cos\theta \\ e^{i\phi_2}\sin\theta\cos\varphi \\ e^{i\phi_3}\sin\theta\sin\varphi \end{pmatrix} \qquad \begin{matrix} 0 \le \phi_i \le 2\pi \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \varphi \le \frac{\pi}{2} \end{matrix} \tag{2.2}$$

(Z' with primed variables.) We now show that a solution for $(\alpha', \beta', \gamma')$ exists such that

$$T_2' Z = \begin{pmatrix} 0 \\ e^{i\phi_3}\cos\phi'' \\ e^{i\phi_3}\sin\phi'' \end{pmatrix} \tag{2.3}$$

where

$$T_2' \equiv e^{-i\frac{\alpha'}{2}I_3} e^{-i\frac{\beta'}{2}I_2} e^{-i\frac{\gamma'}{2}I_3} = \begin{pmatrix} e^{-\frac{i}{2}(\alpha'+\gamma')}\cos\frac{\beta'}{2} & -e^{-\frac{i}{2}(\alpha'-\gamma')}\sin\frac{\beta'}{2} & 0 \\ e^{\frac{i}{2}(\alpha'-\gamma')}\sin\frac{\beta'}{2} & e^{\frac{i}{2}(\alpha'+\gamma')}\cos\frac{\beta'}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

is in SU3.

The equations

$$e^{-\frac{i}{2}(\alpha'+\gamma'-2\phi_1)}\cos\frac{\beta'}{2}\cos\theta - e^{-\frac{i}{2}(\alpha'-\gamma'-2\phi_2)}\sin\frac{\beta'}{2}\sin\theta\cos\phi = 0 \tag{2.5a}$$

$$e^{\frac{i}{2}(\alpha'-\gamma'+2\phi_1)}\sin\frac{\beta'}{2}\cos\theta - e^{\frac{i}{2}(\alpha'+\gamma'+2\phi_2)}\cos\frac{\beta'}{2}\sin\theta\cos\phi = e^{i\phi_3}\cos\phi'' \tag{2.5b}$$

$$e^{i\phi_3}\sin\theta\sin\phi = e^{i\phi_3}\sin\phi'' \tag{2.5c}$$

are satisfied by

$$e^{-\frac{i}{2}\gamma'} = e^{-\frac{i}{2}(\phi_1-\phi_2)} \qquad 0 \le \alpha', \gamma' \le 4\pi$$

$$e^{\frac{i}{2}\alpha'} = e^{i(\phi_3 - \frac{\phi_1+\phi_2}{2})} \qquad 0 \le \beta' \le \pi \tag{2.6a}$$

$$\tan\frac{\beta'}{2} = \cot\theta\sec\phi \tag{2.6b}$$

and by defining

$$\cos\phi'' \equiv \sin\frac{\beta'}{2}\cos\theta + \cos\frac{\beta'}{2}\sin\theta\cos\phi \tag{2.6c}$$

$$0 \le \phi'' \le \frac{\pi}{2} .$$

We define

$$T_3 \equiv e^{-i\rho(I_3 + \frac{1}{\sqrt{3}} I_8)} \, e^{-i\nu I_7} \,. \qquad (2.7)$$

T_3 is in SU3.

Therefore if

$$e^{\frac{2}{3}i\rho} = e^{-i\phi_3} \qquad 0 \leq \rho \leq 3\pi \qquad (2.7a)$$

$$\nu = \frac{\pi}{2} - \phi'' \qquad 0 \leq \nu \leq \frac{\pi}{2} \qquad (2.7b)$$

then

$$T_3 \, T_2' \, Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,. \qquad (2.8)$$

Now consider T_2 defined as T_2' with unprimed variables, but the same ranges. T_2 is the most general special unitary matrix with (1) in the (3,3) position. Let U be a general matrix of SU3. From equation 2.8 and since the matrices are unitary, we have

$$T_3 \, T_2' \, U^{-1} = \begin{pmatrix} u'_{11} & u'_{12} & 0 \\ u'_{21} & u'_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv T_2^{-1} \,, \qquad (2.9a)$$

Therefore $U = T_2 T_3 T_2'$. (2.9b)

$e^{-i\rho I_3}$ can be absorbed into $e^{-i\frac{\gamma}{2} I_3}$ by a redefinition of γ .

Noting that

$$e^{-i\pi I_3} \, I_7 \, e^{i\pi I_3} = I_6 \qquad (2.10)$$

and another redefinition of γ and α' to absorb $e^{i\pi I_3}$ would allow the replacement of I_7 by I_6. Also an interchange of the role of the first and second components of Z would allow the replacement of I_6 or I_7 by I_4 or I_5. We choose I_4 to be consistent with Nelson's work (1967).

We therefore arrive at the final form.

$$D(\alpha) = e^{-i\frac{\rho}{\sqrt{3}}I_8}\, e^{-i\frac{\alpha}{2}I_3}\, e^{-i\frac{\beta}{2}I_2}\, e^{-i\frac{\gamma}{2}I_3}\, e^{-i\nu I_4}\, e^{-i\frac{\alpha'}{2}I_3}\, e^{-i\frac{\beta'}{2}I_2}\, e^{-i\frac{\gamma'}{2}I_3} \tag{2.11}$$

$0 \leq \alpha, \gamma \leq 4\pi$ $\qquad$ $0 \leq \beta \leq \pi$ $\qquad$ $0 \leq \nu \leq \frac{\pi}{2}$

$0 \leq \alpha', \gamma' \leq 4\pi$ $\qquad$ $0 \leq \beta' \leq \pi$ $\qquad$ $0 \leq \rho \leq 3\pi$

where these are the minimum ranges of the parameters. Other parameterizations are discussed in the appendix (Appendix A).

CHAPTER 3

EXPLICIT DETERMINATION OF THE TRANSFORMATION MATRICES

We now seek to generalize our result for the defining three-dimensional representation to all IR's. In what follows we shall use the integers (λ, μ) to denote an IR (Baird and Biedenharn, 1963), and (I, Y, M) to denote a state within the IR. The symbols (λ, μ) will be suppressed unless needed.

From the commutation rules, it follows that an SU2 subalgebra exists, composed of I_1, I_2, and I_3. Hence the basis states are chosen such that

$$I^2 \equiv I_1^2 + I_2^2 + I_3^2 \quad , \quad I_3 \quad \text{and} \quad I_8 \tag{3.1a}$$

are diagonal and

$$\left\langle \begin{smallmatrix} I\,Y \\ M \end{smallmatrix} \right| e^{-i\frac{\alpha}{2}I_3}\, e^{-i\frac{\beta}{2}I_2}\, e^{-i\frac{\gamma}{2}I_3} \left| \begin{smallmatrix} I'Y' \\ M' \end{smallmatrix} \right\rangle = D^{I}_{M\,M'}(\alpha,\beta,\gamma)\, \delta_{YY'}\, \delta_{II'} \tag{3.1b}$$

$$\left\langle \begin{smallmatrix} I\,Y \\ M \end{smallmatrix} \right| e^{-i\frac{\rho}{\sqrt{3}}I_8} \left| \begin{smallmatrix} I'Y' \\ M' \end{smallmatrix} \right\rangle = e^{-i\rho Y}\, \delta_{YY'}\, \delta_{II'}\, \delta_{MM'} \,. \tag{3.1c}$$

Now consider a weight diagram (Figure 1) which displays the basis states according to the (Y, M) values and consider

$$\left\langle \begin{smallmatrix} I\,Y \\ M \end{smallmatrix} \right| T_2\; T_3\; T_2' \left| \begin{smallmatrix} I'Y' \\ M' \end{smallmatrix} \right\rangle .$$

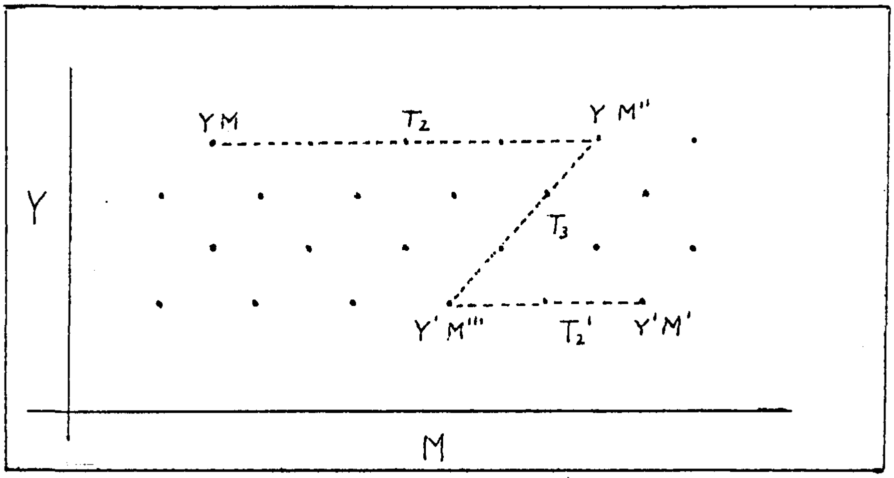

Fig. 1. Weight Diagram for Derivation of the Representations

T_2 and T_2' connect states in the same horizontal line, while T_3 connects states on the diagonal shown. This gives a relation between M''' and M''.

$$M'' \equiv M''' + \tfrac{1}{2}(Y - Y') \ . \tag{3.2}$$

Summarizing the above

$$\left\langle {}^{IY}_{M} \right| D(\alpha) \left| {}^{I'Y'}_{M'} \right\rangle = \sum_{M''} e^{-i\rho Y} D^{I}_{M\,M''}(\alpha,\beta,\gamma) \left\langle {}^{IY}_{M''} \right| e^{-i\nu I_4} \left| {}^{I'Y'}_{M'''} \right\rangle D^{I'}_{M'''\,M'}(\alpha',\beta',\gamma') \ . \tag{3.3}$$

One notices that the undetermined matrix $\left\langle {}^{IY}_{M''} \right| e^{-i\nu I_4} \left| {}^{I'Y'}_{M'''} \right\rangle$ plays a role analogous to $d^{j}_{m\,m'}(\beta)$ of SU2. In SU2, the $d^{j}_{m\,m'}(\beta)$ matrix is determined by applying $e^{-i\beta J_2}$ to a suitable polynomial basis state representation. A search for basis states in SU3 led to a paper by Mukunda and Pandit (1965). Their basis states Ψ^{IY}_{M} are given in terms of tensors $T^{m_1 \cdots m_\lambda}_{n_1 \cdots n_\mu}$ which transform under the three-dimensional defining representation and its complex conjugate.

$$T'^{m_1 \cdots m_\lambda}_{n_1 \cdots n_\mu} = \sum_{m' n'} A^{m_1}_{m_1'} \cdots A^{m_\lambda}_{m_\lambda'} A^{n_1'^c}_{n_1} \cdots A^{n_\mu'^c}_{n_\mu} \; T^{m_1' \cdots m_\lambda'}_{n_1' \cdots n_\mu'} \tag{3.4}$$

To have the Condon-Shortley phase convention hold for the T and V SU2 subgroups, a change in basis is made. We define new bases Ψ^c such that for the complex conjugate three-dimensional representation

$$\Psi_1^c \equiv -\Psi_1^* \qquad \Psi_2^c \equiv \Psi_2^* \qquad \Psi_3^c \equiv \Psi_3^* \tag{3.5a}$$

$$\text{or } \Psi^c = W\,\Psi^* \qquad W = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix} .$$

Correspondingly

$$D^c(\underline{\alpha}) \equiv W\, D^*(\underline{\alpha})\, W^{-1} \tag{3.5b}$$

or
$$e^{i\alpha \cdot I^c} = W\left[e^{i\alpha \cdot I}\right]^* W^{-1}$$

and
$$I_\kappa^c = W\,(-I_\kappa^*)\,W^{-1}$$

Following their notation $T^{j_1 m_1}_{j_2 m_2}$ has

$j_1 + m_1$ upper 1's, $j_1 - m_1$ upper 2's, and $\lambda - 2j_1$ upper 3's

$j_2 - m_2$ lower 1's, $j_2 + m_2$ lower 2's, and $\mu - 2j_2$ lower 3's.

We need the relation

$$\Psi_M^{IY} = N_3(I,Y) \sum_{m_1 m_2} C(j_1^{min}\, j_2^{min}\, I; m_1 m_2 M)\; N_1(j_1^{min} m_1\, j_2^{min} m_2)\; T^{j_1^{min} m_1}_{j_2^{min} m_2} \tag{3.6a}$$

and the inverse relation

$$T^{j_1 m_1}_{j_2 m_2} = \left[N_1(j_1 m_1 j_2 m_2)\right]^{-1} \sum_{I} C(j_1 j_2 I ; m_1 m_2 M) N_2(j_1 j_2 I)\left[N_3(I, Y)\right]^{-1} \Psi^{IY}_{M} \qquad (3.6b)$$

$$j_1^{min} = \tfrac{1}{2} I + \tfrac{1}{4} Y + \tfrac{1}{6}(\lambda - \mu) \qquad j_2^{min} = \tfrac{1}{2} I - \tfrac{1}{4} Y - \tfrac{1}{6}(\lambda - \mu) \qquad (3.6c)$$

$$N_1(j_1 m_1 j_2 m_2) = \left[\frac{\lambda!\ \mu!}{(j_1+m_1)!(j_1-m_1)!(\lambda-2j_1)!(j_2+m_2)!(j_2-m_2)!(\mu-2j_2)!}\right]^{\frac{1}{2}} \qquad (3.6d)$$

$$N_2(j_1 j_2 I) = \left[\frac{(2I+1)!(\lambda-j_1+j_2-I)!(\mu-j_2+j_1-I)!}{(j_1+j_2-I)!(j_1+j_2+I+1)!(\lambda-2j_1)!(\mu-2j_2)!}\right]^{\frac{1}{2}} \qquad (3.6e)$$

$$N_3(IY) = \left[\frac{(2I+1)!\ (\lambda+\mu+1)!}{\left(\frac{\lambda+2\mu}{3}+I+\frac{Y}{2}+1\right)!\left(\frac{2\lambda+\mu}{3}+I-\frac{Y}{2}+1\right)!}\right]^{\frac{1}{2}} \qquad (3.6f)$$

where (λ, μ) denote the IR.

For low-dimensional representations or the important special case where $I' = M' = 0$, the transformation of interest

$$e^{-i\nu I_y} = \begin{pmatrix} \cos\nu & 0 & -i\sin\nu \\ 0 & 1 & 0 \\ -i\sin\nu & 0 & \cos\nu \end{pmatrix} \qquad (3.7a)$$

can be applied directly to these tensors. Employing the standard phase convention,

$$A^{m\,c}_{n}(\nu) = \left[e^{-i\nu I_4^c}\right]^m_n = A^m_n(\nu) \tag{3.7b}$$

To illustrate the procedure, the results for the eight-dimensional IR is worked out. The numbering of the states is shown on the weight diagram (Figure 2).

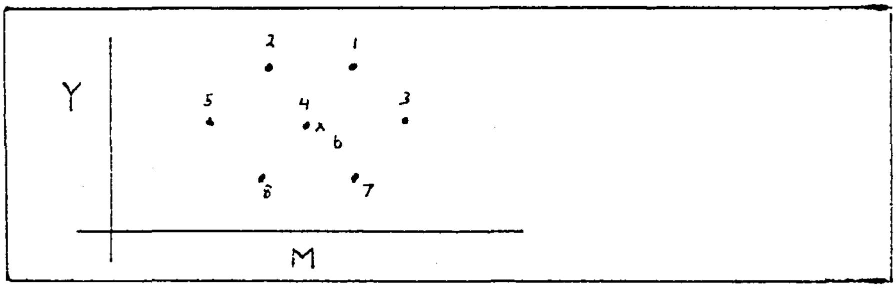

Fig. 2. Eight-dimensional Weight Diagram

Using this notation, the equations are

$$|1\rangle = T^1_3 \qquad |2\rangle = T^2_3 \qquad |3\rangle = T^1_2 \qquad |4\rangle = \tfrac{1}{\sqrt{2}}\left(T^2_2 + T^1_1\right) \tag{3.8a}$$

$$|5\rangle = T^2_1 \qquad |6\rangle = \sqrt{\tfrac{3}{2}}\,T^3_3 \qquad |7\rangle = T^3_2 \qquad |8\rangle = T^3_1 \;.$$

The inverse relation yields

$$T^1_1 = \tfrac{1}{\sqrt{6}}\,|6\rangle + \tfrac{1}{\sqrt{2}}\,|4\rangle \tag{3.8b}$$

$$T^2_2 = -\tfrac{1}{\sqrt{6}}\,|6\rangle + \tfrac{1}{\sqrt{2}}\,|4\rangle \,.$$

We now apply the transformation to the state $|1\rangle$.

$$e^{-i\nu I_4}|1\rangle = -i\sin\nu\cos\nu\, T_1^1 - \sin^2\nu\, T_1^3 + \cos^2\nu\, T_3^1 - i\sin\nu\cos\nu\, T_3^3 \tag{3.9}$$

$$= \cos^2\nu\,|1\rangle - \frac{i}{\sqrt{2}}\sin\nu\cos\nu\,|4\rangle - i\sqrt{\tfrac{3}{2}}\sin\nu\cos\nu\,|6\rangle - \sin^2\nu\,|8\rangle$$

The summary of transformations of all the states in the eight-dimensional IR follows.

$$
\langle i \,|\, e^{-i\nu I_4} \,|\, j \rangle =
\begin{array}{c c}
 & j \\
i & \begin{pmatrix}
\cos^2\nu & 0 & 0 & -\frac{i}{\sqrt{2}}\sin\nu\cos\nu & 0 & -i\sqrt{\frac{3}{2}}\sin\nu\cos\nu & 0 & -\sin^2\nu \\
0 & \cos\nu & 0 & 0 & -i\sin\nu & 0 & 0 & 0 \\
0 & 0 & \cos\nu & 0 & 0 & 0 & -i\sin\nu & 0 \\
-\frac{i}{\sqrt{2}}\sin\nu\cos\nu & 0 & 0 & \frac{1}{2}(1+\cos^2\nu) & 0 & -\frac{\sqrt{3}}{2}\sin^2\nu & 0 & -\frac{i}{\sqrt{2}}\sin\nu\cos\nu \\
0 & -i\sin\nu & 0 & 0 & \cos\nu & 0 & 0 & 0 \\
-i\sqrt{\frac{3}{2}}\sin\nu\cos\nu & 0 & 0 & -\frac{\sqrt{3}}{2}\sin^2\nu & 0 & -\frac{1}{2}\sin^2\nu+\cos^2\nu & 0 & -i\sqrt{\frac{3}{2}}\sin\nu\cos\nu \\
0 & 0 & -i\sin\nu & 0 & 0 & 0 & \cos\nu & 0 \\
-\sin^2\nu & 0 & 0 & -\frac{i}{\sqrt{2}}\sin\nu\cos\nu & 0 & -i\sqrt{\frac{3}{2}}\sin\nu\cos\nu & 0 & \cos^2\nu
\end{pmatrix}
\end{array}
\tag{3.10}
$$

A general result for all IR's would be very difficult with this method. An easier way presents itself by noting that

$$e^{i\frac{\pi}{2} I_6}\ I_4\ e^{-i\frac{\pi}{2} I_6} = I_2 \tag{3.11}$$

and of course

$$\left\langle {}^{I\,Y}_{M} \right| e^{-i\nu I_4} \left| {}^{I'Y'}_{M'} \right\rangle = d^{\,I}_{M\,M'}(2\nu)\ \delta_{YY'}\ \delta_{II'}\ . \tag{3.12}$$

Therefore we have only to determine matrices of the type

$$\left\langle {}^{I_\alpha Y_\alpha}_{M_\alpha} \right| e^{i\frac{\pi}{2} I_6} \left| {}^{I\,Y}_{M} \right\rangle$$

as

$$\left\langle {}^{I\,Y}_{M''} \right| e^{-i\nu I_4} \left| {}^{I'Y'}_{M'''} \right\rangle = \sum_{\alpha\,\beta} \left\langle {}^{I\,Y}_{M''} \right| e^{-i\frac{\pi}{2} I_6} \left| {}^{I_\beta Y_\beta}_{M_\beta} \right\rangle \delta_{I_\beta I_\alpha}\ d^{\,I_\beta}_{M_\beta M_\alpha}(2\nu)\ \delta_{Y_\beta Y_\alpha} \left\langle {}^{I_\alpha Y_\alpha}_{M_\alpha} \right| e^{i\frac{\pi}{2} I_6} \left| {}^{I'Y'}_{M'''} \right\rangle . \tag{3.13}$$

We first note that for the states obtained by the transformation $e^{i\frac{\pi}{2} I_6}$

$$\underline{M}\ e^{i\frac{\pi}{2} I_6} \left| {}^{I\,Y}_{M} \right\rangle = e^{i\frac{\pi}{2} I_6}\left(\frac{\underline{M}}{2} + \frac{3}{4}\underline{Y}\right) \left| {}^{I\,Y}_{M} \right\rangle \tag{3.14a}$$

$$\underline{Y}\ e^{i\frac{\pi}{2} I_6} \left| {}^{I\,Y}_{M} \right\rangle = e^{i\frac{\pi}{2} I_6}\left(-\frac{\underline{Y}}{2} + \underline{M}\right) \left| {}^{I\,Y}_{M} \right\rangle \tag{3.14b}$$

follow from the commutation rules. This implies that all states obtained from this transformation acting on a state, belong to the same point on the weight diagram.

Also $M_\alpha = \frac{M}{2} + \frac{3}{4}Y \;, \quad Y_\alpha = -\frac{Y}{2} + M$ (3.15)

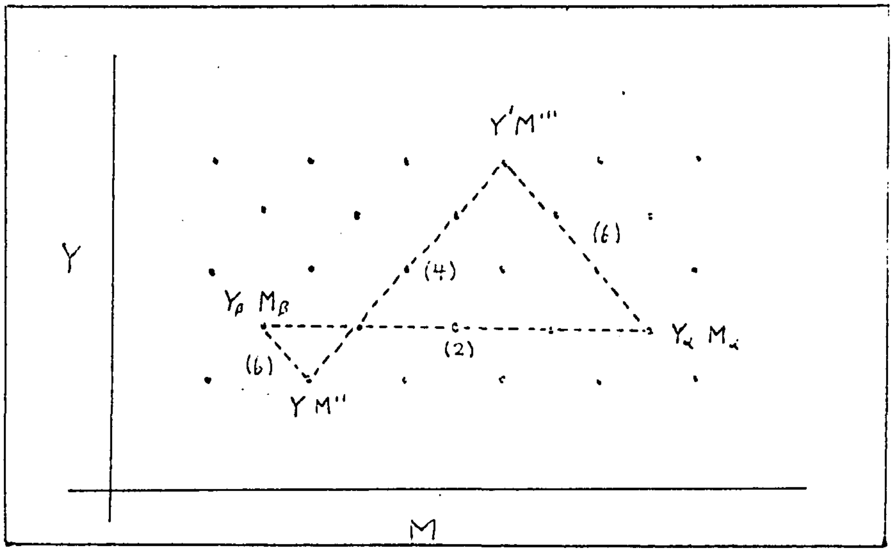

Fig. 3. Weight Diagram for Derivation of the $e^{-i\nu I_4}$ Factor

Transformations of this type map points into points along lines (6) on the weight diagram. Therefore from Figure 3 we see that the sum over (α) and (β) above can be replaced by a single sum over I_α .

$$\left\langle {}^{I\,Y}_{M''} \right| e^{-i\nu I_4} \left| {}^{I'Y'}_{M'''} \right\rangle = \sum_{I_\alpha} \left\langle {}^{I\,Y}_{M''} \right| e^{-i\frac{\pi}{2}I_6} \left| {}^{I_\beta Y_\beta}_{M_\beta} \right\rangle \delta_{I_\beta I_\alpha}\, d^{\,I_\alpha}_{\substack{(2\nu)\\ M_\beta M_\alpha}}\, \delta_{Y_\beta Y_\alpha} \left\langle {}^{I_\alpha Y_\alpha}_{M_\alpha} \right| e^{-i\frac{\pi}{2}I_6} \left| {}^{I'Y'}_{M'''} \right\rangle \tag{3.16}$$

The explicit form of this matrix can be found by applying the transformation to the tensor basis states. For the three-dimensional representation and its complex conjugate, the following come from the standard phase convention.

$$e^{i\frac{\pi}{2}I_6} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \qquad e^{i\frac{\pi}{2}I_6^c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \tag{3.17}$$

These affect the tensors $T^{j_1 m_1}_{j_2 m_2}$ in the following way. Each upper 2 and 3 is replaced by i3 and i2 respectively. Each lower 2 and 3 is replaced by -i3 and -i2 respectively.

$$T^{j_1 m_1}_{j_2 m_2} \longrightarrow (i)^{\lambda - j_1 - m_1} (-i)^{\mu - j_2 + m_2} \, T^{\frac{1}{2}(\lambda - 2j_1 + m_1),\, \frac{1}{2}(3j_1 + m_1 - \lambda)}_{\frac{1}{2}(\mu - j_2 - m_2),\, \frac{1}{2}(-3j_2 + m_2 + \mu)} \tag{3.18}$$

Using the previous relations connecting the states Ψ^{IY}_M and tensors, we finally obtain

$$\left\langle {}^{I_\alpha Y_\alpha}_{M_\alpha} \right| e^{i\frac{\pi}{2}I_6} \left| {}^{I\,Y}_{M} \right\rangle = \sum_{m_1 m_2} (i)^{\lambda - j_1^{min} - m_1} (-i)^{\mu - j_2^{min} + m_2} C\left(j_1^{min}\, j_2^{min}\, I;\, m_1 m_2 M\right) \times \tag{3.19}$$

$$C\left(\tfrac{1}{2}(\lambda - j_1^{min} + m_1),\, \tfrac{1}{2}(\mu - j_2^{min} - m_2),\, I_\alpha;\, \tfrac{1}{2}(3j_1^{min} + m_1 - \lambda),\, \tfrac{1}{2}(-3j_2^{min} + m_2 + \mu),\, M_\alpha\right) \times$$

$$N_2\left(\tfrac{1}{2}(\lambda - j_1^{min} + m_1),\, \tfrac{1}{2}(\mu - j_2^{min} - m_2)\right) \left[N_3(I_\alpha, Y_\alpha)\right]^{-1} .$$

This completes the explicit form of the transformation matrices.

It is interesting to note the connection between these finite group transformations and the Weyl reflections, which lie outside the group. In the interesting case of reflections and transformations across the vertical, the operations are:

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad e^{\pm i \frac{\pi}{2} I_1} = \begin{pmatrix} 0 & \pm i & 0 \\ \pm i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad e^{\pm i \frac{\pi}{2} I_2} = \begin{pmatrix} 0 & \pm 1 & 0 \\ \mp 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.20}$$

$$W^c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad e^{\pm i \frac{\pi}{2} I_1^c} = \begin{pmatrix} 0 & \pm i & 0 \\ \pm i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad e^{\pm i \frac{\pi}{2} I_2^c} = \begin{pmatrix} 0 & \mp 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A successive application of the group operation followed by the Weyl reflection leaves the tensors unchanged except for a phase factor. These product operators are therefore diagonal and may be of physical interest. Using the symbols $X_j^{\pm}$ to denote the operation $e^{\pm i \frac{\pi}{2} I_j}$ and applying these operators to the tensors we have

$$W X_1^{\pm} \Psi_M^{IY} = (\pm i)^{2I} (-1)^{\mu - I - \frac{Y}{2} - \frac{1}{3}(\lambda - \mu)} \Psi_M^{IY} \tag{3.21}$$

$$W X_2^{\pm} \Psi_M^{IY} = (-1)^{I \pm M} (-1)^{\mu - I - \frac{Y}{2} - \frac{1}{3}(\lambda - \mu)} \Psi_M^{IY} .$$

For the eight-dimensional representation $\lambda = \mu = 1$, and for the pion states $I = 1$. Therefore

$$W X_1^{\pm} \Psi_M^{10} = - \Psi_M^{10} \qquad \text{pions} \tag{3.22}$$

$$W X_2^{\pm} \Psi_M^{10} = -(-1)^M \Psi_M^{10} \qquad \text{pions} .$$

Since the Weyl reflection is related to charge conjugation and the group operations to isospin rotations, the product operations are related to G-Parity. In fact $WX_1^{\pm}$ may be a generalization of G-Parity as it reduces to the desired form for pions.

CHAPTER 4

SPHERICAL HARMONIC BASIS STATES

For the group SU2, the spherical harmonic basis states were obtained by specializing to the right-hand state $m' = 0$ in $D^{j}_{m\,m'}(\alpha,\beta,\gamma)$. In this way the dependence on γ was eliminated. Comparing this to SU3 we have

$$D^{j}_{m\,m'}(\alpha,\beta,\gamma) \longrightarrow \langle jm |\, e^{-i\alpha J_3}\, e^{-i\beta J_2}\, e^{-i\gamma J_3}\, | j\,0 \rangle \tag{4.1a}$$

$$\doteq \langle jm |\, e^{-i\alpha J_3}\, e^{-i\beta J_2}\, | j\,0 \rangle \qquad \text{For SU2.}$$

$$D^{\mu}_{\nu_1\,\nu_2}(\underline{\alpha}) \longrightarrow \left\langle {I\,Y \atop M} \right|\, e^{-i\frac{\rho}{\sqrt{3}} I_8}\, e^{-i\frac{\alpha}{2} I_3}\, e^{-i\frac{\beta}{2} I_2}\, e^{-i\frac{\gamma}{2} I_3}\, e^{-i\nu I_4}\, e^{-i\frac{\alpha'}{2} I_3}\, e^{-i\frac{\beta'}{2} I_2}\, e^{-i\frac{\gamma'}{2} I_3} \,|I=0\rangle \tag{4.1b}$$

$$= \left\langle {I\,Y \atop M} \right|\, e^{-i\frac{\rho}{\sqrt{3}} I_8}\, e^{-i\frac{\alpha}{2} I_3}\, e^{-i\frac{\beta}{2} I_2}\, e^{-i\frac{\gamma}{2} I_3}\, e^{-i\nu I_4}\, |I=0\rangle \qquad \text{For SU3.}$$

From the close analogy to SU2, we see that the choice of $I = 0$ for the left-hand state eliminates three variables.

We now derive the explicit form of this matrix. Using the weight diagram (Figure 4), starting with the right-hand state, we see this state is connected to states on the line marked 4.

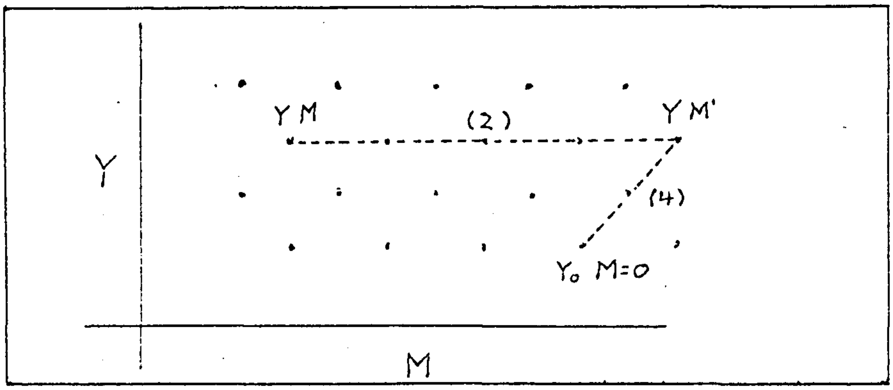

Fig. 4. Weight Diagram for Derivation of Spherical Harmonic States

The left-hand state is connected to states on the horizontal line and therefore

$$M' = \tfrac{1}{2}(Y - Y_0) \tag{4.2a}$$

$$\Psi^{IY}_{M} = e^{-i\rho Y} D^{I}_{M\,\frac{1}{2}(Y-Y_0)}(\alpha,\beta,\gamma) \left\langle \begin{smallmatrix} I & Y \\ \frac{1}{2}(Y-Y_0) & \end{smallmatrix} \right| e^{-i\nu I_4} \left| \begin{smallmatrix} 0 & Y_0 \\ 0 & \end{smallmatrix} \right\rangle \; . \tag{4.2b}$$

We can evaluate the last factor by applying this transformation $e^{-i\nu I_4}$ to the tensor basis states. From equation 3.6a we have

$$\left| \begin{smallmatrix} 0 & Y_0 \\ 0 & \end{smallmatrix} \right\rangle = \left[\frac{(\lambda+\mu+1)!}{\left(\frac{\lambda+2\mu}{3} + \frac{Y_0}{2} + 1\right)!\left(\frac{2\lambda+\mu}{3} - \frac{Y_0}{2} + 1\right)!} \right]^{\frac{1}{2}} T^{0\;0}_{0\;0} \; . \tag{4.3}$$

The effect of this transformation on $T^{0\;0}_{0\;0}$ follows from equation 3.4:

$$T^{0\;0}_{0\;0} \longrightarrow \sum_{t\;s} \binom{\lambda}{t}\binom{\mu}{s} (-i\sin\nu)^{t} (\cos\nu)^{\lambda-t} (-i\sin\nu)^{\mu-s} (\cos\nu)^{s}\; T^{\frac{t}{2}\;\;\frac{t}{2}}_{\frac{1}{2}(\mu-s),\,-\frac{1}{2}(\mu-s)} \; . \tag{4.4}$$

Each $T^{j_1 m_1}_{j_2 m_2}$ is a linear combination of basis states as shown by the inverse relations, equation 3.6b. We therefore take the projection of each tensor occurring in the sum onto the basis states Ψ^{IY}_{M}.

Also note that $M' = m_1 + m_2 = \frac{1}{2}(t + s - \mu)$ gives a relation that reduces the double sum to a single one.

In $T^{0\,0}_{0\,0}$ there are λ upper and μ lower 3's. Each upper 3 contributes a $-2/3$ to Y and each lower 3 a $+2/3$.

We now have

$$\left\langle {}^{I\,Y}_{M'} \right| e^{-i\nu I_4} \left| {}^{0\,Y_0}_{0} \right\rangle = \left[\frac{(\lambda+\mu+1)!}{(\lambda+1)!\,(\mu+1)!}\right]^{\frac{1}{2}} \left[N_3(I,Y)\right]^{-1} \sum \binom{\lambda}{t}\binom{\mu}{2M'+\mu-t} \times \tag{4.5}$$

$$(\sin\nu)^{-2M'+2t}\,(\cos\nu)^{2M'+\mu+\lambda-2t}\,(-i)^{-2M'+2t}\left[N_1\left(\tfrac{t}{2},\tfrac{t}{2},-M'+\tfrac{t}{2},M'-\tfrac{t}{2}\right)\right]^{-1} \times$$

$$C\left(\tfrac{t}{2},-M'+\tfrac{t}{2},I;\tfrac{t}{2},M'-\tfrac{t}{2},M'\right)\,N_2\left(\tfrac{t}{2},-M'+\tfrac{t}{2},I\right) \quad .$$

Using the explicit formula for $C(j_1 j_2 I_j m_1 m_2 m)$

$$C\left(\tfrac{t}{2},-M'+\tfrac{t}{2},I;\tfrac{t}{2},M'-\tfrac{t}{2},M'\right) = \left[\frac{(2I+1)\;t!\;(-2M'+t)!}{(t-M'-I)!\,(I-M'+t+1)!}\right]^{\frac{1}{2}} \tag{4.6}$$

and making the substitution $t = \ell + M' + I$ we have

$$\left\langle {}^{I\,Y}_{M'} \right| e^{-i\nu I_4} \left| {}^{0\,Y_0}_{0} \right\rangle = \left[\frac{(2I+1)(\lambda-M'-I)!\,(\mu+M'-I)!\,(\mu+M'+I+1)!\,(\lambda-M'+I+1)!}{(\lambda+1)\,(\mu+1)}\right]^{\frac{1}{2}} \times$$

$$\sum_{\ell} \frac{(-i)^{2(\ell+I)}\,(\sin\nu)^{2(\ell+I)}\,(\cos\nu)^{-2(\ell+I)+\mu+\lambda}}{\ell!\,(2I+\ell+1)!\,(\lambda-M'-I-\ell)!\,(M'+\mu-I-\ell)!}$$

$$= \frac{-(-i)^{2I}\,(2I+1)^{\frac{1}{2}}}{(\lambda+1)^{\frac{1}{2}}\,(\mu+1)^{\frac{1}{2}}}\,\csc\nu\; d^{\frac{1}{2}(\lambda+\mu+1)}_{\frac{1}{2}(\mu-\lambda+2M'+2I+1),\,\frac{1}{2}(\mu-\lambda+2M'-2I-1)}(2\nu) \quad . \tag{4.7}$$

The basis states are therefore

$$\Psi^{IY}_{M} = \frac{-(-1)^{2I}(2I+1)^{\frac{1}{2}}}{(\lambda+1)^{\frac{1}{2}}(\mu+1)^{\frac{1}{2}}} \csc\nu \; D^{I}_{M,\frac{Y}{2}+\frac{1}{3}(\lambda-\mu)}(\alpha,\beta,\gamma) \; d^{\frac{1}{2}(\lambda+\mu+1)}_{\frac{1}{6}(\mu-\lambda+3Y+6I+3),\frac{1}{6}(\mu-\lambda+3Y-6I-3)}(2\nu) \tag{4.8}$$

Nelson (1967) derived these functions by constructing differential operators representing the infinitesimal generators. With the following substitutions these results agree to within a phase factor.

$\lambda \longrightarrow m$ $\qquad \mu \longrightarrow n$ $\qquad Y \longrightarrow u$

$\rho \longrightarrow \beta$

$\beta \longrightarrow \alpha_2$ $\qquad \gamma \longrightarrow \gamma$

$\alpha \longrightarrow \alpha_3$

Perhaps mention should be made here concerning the ninth operator found by Nelson.

$$D^{(1)} = \frac{2}{i}\frac{\partial}{\partial Y} - \frac{1}{i}\frac{\partial}{\partial \rho} \qquad \text{in our notation} \tag{4.9a}$$

$$D^{(1)}\Psi^{IY}_{M} = -\frac{1}{3}(\lambda-\mu)\Psi^{IY}_{M} \tag{4.9b}$$

The triality of a representation is $\lambda-\mu$ mod. 3, so this operator is closely related to it. We shall have more to say about triality in Chapter 6.

CHAPTER 5

VOLUME ELEMENT

For finite groups if $F(A_i)$ is a function of the group elements A_i, then

$$\sum_{A_i \in G} F(A_i) = \sum_{A_i \in G} F(BA_i) \tag{5.1}$$

where B is any element of the group. This relation can be extended to infinite groups where

$A(\gamma(\sigma,\alpha)) = A(\sigma)\, A(\alpha)$. the Greek letters represent the group parameters

We want $\int_G F(\alpha)\, \rho(\alpha)\, d\alpha = \int_G F(\gamma(\sigma,\alpha))\, \rho(\alpha)\, d\alpha$. The group integral is then left-invariant.

The following is a brief summary of derivations by Smirnov (1961) and Murnaghan (1938). If

$$A(\gamma) = A(\sigma)\, A(\alpha)$$

then

$$\int_G F(\gamma)\, \rho(\gamma)\, d\gamma = \int_G F(\gamma(\sigma,\alpha)) \left|\frac{\partial \gamma_i}{\partial \alpha_j}\right| \rho(\gamma(\sigma,\alpha))\, d\alpha \ . \tag{5.2}$$

We therefore require that

$$\left|\frac{\partial \gamma_i}{\partial \alpha_j}\right| \rho(\gamma_{(\sigma,\alpha)}) = \rho(\alpha) \quad . \tag{5.3}$$

Therefore we must find a ρ such that this relation is satisfied.

Define $$S_{ik}(A(\sigma), A(\alpha)) \equiv \frac{\partial \gamma_i}{\partial \alpha_k} \quad . \tag{5.4}$$

Now let $$A(\gamma) = A(\beta)\, A(\delta) = A(\beta)\, A(\beta')\, A(\alpha) \quad . \tag{5.5}$$

From $$\frac{\partial \gamma_i}{\partial \alpha_k} = \frac{\partial \gamma_i}{\partial \delta_j} \frac{\partial \delta_j}{\partial \alpha_k} \tag{5.6}$$ using the summation convention

we have $$S_{ik}(A(\beta)A(\beta'), A(\alpha)) = S_{ij}(A(\beta), A(\delta))\, S_{jk}(A(\beta'), A(\alpha)) . \tag{5.7}$$

Let $A(\alpha) = I$ and $A(\beta') = A(\alpha)$;

Therefore $$S_{ik}(A(\beta)A(\alpha), I) = S_{ij}(A(\beta), A(\alpha))\, S_{jk}(A(\alpha), I) \tag{5.8}$$

Define $$\rho^{-1}(\alpha) = \left| S(A(\alpha), I)\right| \tag{5.9}$$

and taking determinants above, we arrive at the desired relation:

$$\rho(\alpha) = \left|\frac{\partial \gamma_i}{\partial \alpha_k}\right| \rho(\gamma) \tag{5.10}$$

Also in equation 5.8 setting $A(\beta) = A(\alpha^{-1})$ and noting that $S_{ij}(I, A(\alpha)) = \delta_{ij}$ implies $|S(I,I)| = 1$, we have

$$\rho(\alpha) = \left| S(A(\alpha^{-1}), A(\alpha))\right| = \left|\frac{\partial \gamma_i}{\partial \alpha_k}\right|_{\beta = \alpha^{-1}} . \tag{5.11}$$

Because of the complexity of relations involving the group parameters, this form is too difficult to evaluate. However from $A_{(\sigma)}A_{(\alpha)} = A(\gamma(\sigma,\alpha))$, taking derivatives with respect to α_i and setting $A_{(\sigma)} = A^{-1}(\alpha)$,

$$A_{(\sigma)} \frac{\partial A(\alpha)}{\partial \alpha_i} = \frac{\partial A(\gamma(\sigma,\alpha))}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial \alpha_i} \tag{5.12a}$$

becomes

$$A^{-1}_{(\alpha)} \frac{\partial A(\alpha)}{\partial \alpha_i} = \left.\frac{\partial \gamma_j}{\partial \alpha_i}\right|_{\sigma=\alpha^{-1}} I_j \equiv C_{ij}(\alpha)\, I_j \; . \tag{5.12b}$$

Since the I_j's are linearly independent over the real field, the expansion is unique and therefore

$$\rho(\alpha) = C\left|C_{ij}(\alpha)\right| \tag{5.13}$$

where C is an arbitrary constant to be determined later.

Up to this point we have discussed only left-invariance. However if the group is compact (the range of parameters is closed and bounded), the left-invariant integral is also right-invariant.

For SU3 the volume element is determined from

$$T^{-1}_{(\underline{\alpha})} \frac{\partial}{\partial \alpha_i} T(\underline{\alpha}) = C_{ij}(\underline{\alpha})\, I_j \qquad \rho(\underline{\alpha}) = C\left|C_{ij}(\underline{\alpha})\right| \; . \tag{5.14}$$

This procedure now requires the evaluation of an eight-by-eight determinant. Fortunately, this can be reduced to a four-by-four determinant in the following manner.

For the parameterization

$$T(\underline{\alpha}) = e^{-i\frac{\rho}{\sqrt{3}}I_8}\, e^{-i\frac{\alpha}{2}I_3}\, e^{-i\frac{\beta}{2}I_2}\, e^{-i\frac{\gamma}{2}I_3}\, e^{-i\nu I_4}\, e^{-i\frac{\alpha'}{2}I_3}\, e^{-i\frac{\beta'}{2}I_2}\, e^{-i\frac{\gamma'}{2}I_3} \tag{5.15}$$

we define $D(\rho,\alpha)$ such that $T(\underline{\alpha}) = D(\rho,\alpha)V$.

$$D(\rho,\alpha) \equiv e^{-i\frac{\rho}{\sqrt{3}}I_8}\, e^{-i\frac{\alpha}{2}I_3} \qquad \text{is diagonal.} \tag{5.16}$$

$$V = D^{-1}(\rho,\alpha)\; T(\underline{\alpha})$$

The volume element is unaffected by a unitary, similarity transformation. The proof is as follows:

$$V\, c_{ij}(\underline{\alpha})\, I_j V^{-1} = c_{ij}(\underline{\alpha})\, V I_j V^{-1} = c_{ij}(\underline{\alpha})\; a_{jk}\, I_k \tag{5.17}$$

V is unitary

Since VI_jV^{-1} is hermitian and the set $\{ I_k \}$ is a basis for three-by-three traceless hermitian matrices, therefore

$$V I_j V^{-1} = a_{jk}\, I_k \qquad \text{where } a_{jk} \text{ is real.} \tag{5.18}$$

Also since the I_k's are linearly independent and tr $I_iI_j = 2\,\delta_{ij}$ we have

$$2\,\delta_{ij} = tr\, I_i I_j = tr\; V I_i V^{-1} V I_j V^{-1} \tag{5.19}$$

$$= tr\;\; a_{i\ell}\; a_{jk}\, I_\ell\, I_k \qquad = 2\; a_{i\ell}\; a_{jk}\; \delta_{\ell k}$$

Therefore $\quad a_{i\ell}\; a_{j\ell} = \delta_{ij} \qquad$ and $\quad |a_{i\ell}| = \pm 1$

Therefore

$$\left| c_{ij}(\underline{\alpha})\, a_{jk} \right| = \pm \left| c_{ij}(\underline{\alpha}) \right| \tag{5.20}$$

so the volume element is unaffected except for a constant.

Now making use of this, we have

$$V\left[T^{-1}(\underline{\alpha}) \frac{\partial}{\partial \alpha_i} T(\underline{\alpha}) \right] V^{-1} = D^{-1}(\rho,\alpha) \frac{\partial}{\partial \alpha_i} D(\rho,\alpha) \quad \text{for } \alpha_i = \rho, \alpha \tag{5.21a}$$

$$= \frac{\partial V}{\partial \alpha_i} V^{-1} \qquad \text{for } \alpha_i \neq \rho, \alpha \,. \tag{5.21b}$$

We will order the determinant in the following way:

		order	1	2	3	4	5	6	7	8
Columns:	By	I_i	I_8	I_3	I_1	I_2	I_4	I_5	I_6	I_7
Rows:	By	I	ρ	α	β	γ	ν	α'	β'	γ'

Equation 5.21a yields: (A) $\frac{-i}{\sqrt{3}}$ in the (1,1) position with the remainder of the row all zeros.

(B) $-\frac{i}{2}$ in the (2,2) position with the remainder of the row all zeros.

By the rules of determinant operations, the remainder of the first and second columns can be set equal to zero. These elements can be set equal to one by moving the constant factor outside the determinant.

From equation 5.21b, we have

(A) $-\frac{i}{2}$ in the (3,4) position, the remainder of the row all zeros.

(B) $-\frac{i}{2}\sin\beta$ in the (4,3) position, and $-\frac{i}{2}\cos\beta$ in the (4,2) position, the remainder of the row is zeros.

Note: This follows from $e^{-i\frac{\beta}{2}I_2} I_3 = (\cos\beta\, I_3 + \sin\beta\, I_1)\, e^{-i\frac{\beta}{2}I_2}$

These two results clear their respective columns and give a constant factor with $\sin\beta$ thus reducing the eight-by-eight to a four-by-four determinant. The evaluation of this determinant is tedious but straightforward yielding

$$\rho(\alpha) = C \sin\beta \;\sin\beta' \;\sin(2\nu)\;\sin^2\nu \quad . \tag{5.22}$$

By explicit calculation, we find that the volume element is unaffected by the replacement of I_4 by I_5, I_6, or I_7 in the parameterization.

The volume element is arbitrary in regards to the coefficient. This constant is usually chosen such that $\int_R \rho(\alpha)\, d(\alpha) = 1$. We also wish the volume element corresponding to the SU2 parts of the parameterization to have the correct form for SU2. We therefore want

$$\int_0^{3\pi}\int_0^{4\pi}\int_0^{\pi}\int_0^{4\pi}\int_0^{\frac{\pi}{2}}\int_0^{4\pi}\int_0^{\pi}\int_0^{4\pi} C \sin\beta \;\sin\beta'\;\sin(2\nu)\;\sin^2\nu \frac{d\rho}{3\pi}\frac{d\alpha}{4\pi}\frac{d\beta}{2}\frac{d\gamma}{4\pi}(d2\nu)\frac{d\alpha'}{4\pi}\frac{d\beta'}{2}\frac{d\gamma'}{4\pi} \tag{5.23}$$

$$= 1 \quad ,$$

Therefore C = 1.

$$\rho(\alpha)\,d(\alpha) = \sin\beta\,\sin\beta'\,\sin(2\nu)\,\sin^2\nu\;\frac{d\rho}{3\pi}\,\frac{d\alpha}{4\pi}\,\frac{d\beta}{2}\,\frac{d\gamma}{4\pi}\,(d2\nu)\,\frac{d\alpha'}{4\pi}\,\frac{d\beta'}{2}\,\frac{d\gamma'}{4\pi} \tag{5.24}$$

The volume element for other parameterizations is given in Appendix B.

CHAPTER 6

SYMMETRIES AND THE ORTHOGONALITY RELATIONS

Consider the transformation

$$\left\langle {}^{I\,Y}_{M} \right| e^{-i\frac{\rho}{\sqrt{3}} I_8}\, e^{-i\frac{\alpha}{2} I_3} \left| {}^{I'\,Y'}_{M'} \right\rangle = e^{-i\rho Y}\, e^{-i\alpha M}\, \delta_{II'}\, \delta_{YY'}\, \delta_{MM'} \quad . \tag{6.1}$$

Table 1 lists this transformation evaluated at various values of (ρ, α).

Table 1. Finite Transformations Resulting in Scalar Matrices

		Dimension		
ρ	α	(3)	(6)	(8)
0	0	I	I	I
π	2π	$e^{i\frac{2}{3}\pi} I$	$e^{i\frac{4}{3}\pi} I$	I
2π	0	$e^{i\frac{4}{3}\pi} I$	$e^{i\frac{2}{3}\pi} I$	I

The three matrices $(I,\ e^{i\frac{2}{3}\pi} I,\ e^{i\frac{4}{3}\pi} I)$ are mapped onto the identity in the eight-dimensional representation. There is a three-to-one homomorphism from the group SU3 to the group represented by the eight-dimensional representation, SU3/Z3.

Also we see from the above table and equation 6.1 that

$$D^{\mu}_{\nu,\,\nu_1}(\rho+\pi,\ \alpha+2\pi,\ \alpha_1) = e^{i\frac{2}{3}\pi t}\, D^{\mu}_{\nu,\,\nu_1}(\rho,\ \alpha,\ \alpha_1) \tag{6.2}$$

where $e^{i\frac{2}{3}\pi t}\, I = e^{-i\pi(\overline{Y} + 2\overline{M})}$

where α_i represents all other parameters and (t) is an integer from the set $\{0, 1, 2\}$.

This provides a convenient way to define (t), the triality of a representation. It reduces to the standard definition, since when

$I=0 \quad M=0 \quad Y_0 = \frac{2}{3}(\mu-\lambda)$ and therefore $t = \lambda - \mu \mod 3$.

The representations can be classified according to the value of (t). In the decomposition of direct products we find

$$D^{\mu}_{\nu'\nu} = \sum_{\nu_1'\nu_1\nu_2'\nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \nu_1' & \nu_2' & \nu' \end{pmatrix} D^{\mu_1}_{\nu_1'\nu_1} D^{\mu_2}_{\nu_2'\nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \tag{6.3}$$

implies that

$$e^{i\frac{2}{3}\pi t} = e^{i\frac{2}{3}\pi t_1} e^{i\frac{2}{3}\pi t_2} \tag{6.4}$$

Therefore, for the triality we find $t = t_1 + t_2 \mod 3$. (6.5)

All representations of the group SU3/Z3 have triality zero.

The matrix $\left\langle {}^{I\,Y}_{M} \right| e^{-i\nu I_4} \left| {}^{I'Y'}_{M'} \right\rangle$ plays a role similar to $d^{j}_{MM'}(\beta)$, in SU2. From the commutation rules we have

$$e^{i\pi I_3} I_4 e^{-i\pi I_3} = -I_4 \tag{6.6a}$$

Therefore $e^{-i\nu I_4} = e^{i\pi I_3} e^{i\nu I_4} e^{-i\pi I_3}$ (6.6b)

or $\left\langle {}^{I\,Y}_{M} \right| e^{-i\nu I_4} \left| {}^{I'Y'}_{M'} \right\rangle = (-1)^{2(M+M')} \left\langle {}^{I\,Y}_{M} \right| e^{i\nu I_4} \left| {}^{I'Y'}_{M'} \right\rangle$ (6.6c)

The orthogonality relations are derived using a procedure similar to the one outlined in the introduction for SU2. A problem in phases arises in connection with the D* appearing in the orthogonality relation. The standard C.G. coefficients for SU3 are for $S\,D^*S^{-1} = D^c$ rather than D* where

$$S = e^{i\pi Q}$$

$$Q \equiv M + \frac{Y}{2} + \frac{t}{3} \quad . \tag{6.7}$$

t is triality (Carruthers 1966, pp. 46-47).

Using the C.G. coefficients to reduce the direct product, we have

$$D^{\mu_1 *}_{\nu_1' \nu_1}(\underline{\alpha})\; D^{\mu_2}_{\nu_1' \nu_2}(\underline{\alpha}) = e^{i\pi(M_1' + M_1 + \frac{Y_1'}{2} + \frac{Y_1}{2} + \frac{2}{3}t)} \sum_{\mu_Y \nu' \nu} \begin{pmatrix} \mu_1^* & \mu_2 & \mu_Y \\ -\nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1^* & \mu_2 & \mu_Y \\ -\nu_1' & \nu_2' & \nu' \end{pmatrix} D^{\mu_Y}_{\nu' \nu}(\underline{\alpha}) \tag{6.8}$$

$$-\nu_1 \equiv (I_1, -Y_1, -M_1)$$

We now show that

$$\int_R D^{\mu}_{\nu' \nu}(\underline{\alpha})\, \rho(\underline{\alpha})\, d(\underline{\alpha}) = \delta_{\mu 1}\, \delta_{\nu' 0}\, \delta_{\nu 0} \quad . \tag{6.9}$$

$\mu \equiv 1$ for the one-dimensional representation.

We have from equations 3.2 and 3.3 and using the result from SU2 which is

$$\int_R D^{I}_{M_1 M_2}(\alpha,\beta,\gamma)\, dR(\alpha,\beta,\gamma) = \delta_{I0}\, \delta_{M_1 0}\, \delta_{M_2 0} \tag{6.10}$$

together with equation 4.8

$$\int_R D^{\mu}_{\nu' \nu}(\underline{\alpha})\, \rho(\underline{\alpha})\, d(\underline{\alpha}) = \delta_{\lambda,\mu}\, \delta_{Y0}\, \delta_{I0}\, \delta_{M0}\, \delta_{I'0} \int_0^{\frac{\pi}{2}} \left\langle \begin{matrix} 0 & 0 \\ & 0 \end{matrix} \right| e^{-i\nu I_4} \left| \begin{matrix} 0 & 0 \\ & 0 \end{matrix} \right\rangle_{\lambda=\mu} \sin(2\nu)\, \sin^2\nu\, d(2\nu) \tag{6.11}$$

But from previous results on spherical harmonic basis states, equation 4.7, we have

$$\left\langle \begin{smallmatrix} 0\,0 \\ 0 \end{smallmatrix} \right| e^{-i\nu I_4} \left| \begin{smallmatrix} 0\,0 \\ 0 \end{smallmatrix} \right\rangle_{\lambda=\mu} = \frac{1}{(\lambda+1)\sin\nu} \; d^{\lambda+\frac{1}{2}}_{\frac{1}{2}\,-\frac{1}{2}}(2\nu) \tag{6.12}$$

and

$$\int_0^{\frac{\pi}{2}} d^{\lambda+\frac{1}{2}}_{\frac{1}{2}\,-\frac{1}{2}}(2\nu)\,\sin\nu\,\sin(2\nu)\,d(2\nu) = \int_0^{\frac{\pi}{2}} d^{\lambda+\frac{1}{2}}_{\frac{1}{2}\,\frac{1}{2}}(2\nu)\; d^{\frac{1}{2}}_{-\frac{1}{2}\,\frac{1}{2}}(2\nu)\,\sin(2\nu)\,d(2\nu) \tag{6.13}$$

$$= \delta_{\lambda 0} \quad .$$

Therefore
$$\int D^{\mu}_{\nu\,\nu}(\alpha)\,\rho(\alpha)\,d(\alpha) = \delta_{\mu,1} \quad . \tag{6.14}$$

We use this result in integrating equation 6.8.

$$\int D^{\mu_1 *}_{\nu_1'\,\nu_1}(\alpha)\; D^{\mu_2}_{\nu_2'\,\nu_2}(\alpha)\,\rho(\alpha)\,d(\alpha) = e^{i\pi(M_1' + M_1 + \frac{Y_1'}{2} + \frac{Y_1}{2} + \frac{2}{3}\tau_1)} \begin{pmatrix} \mu_1^* & \mu_2 & 1 \\ -\nu_1 & \nu_2 & 0 \end{pmatrix} \begin{pmatrix} \mu_1^* & \mu_2 & 1 \\ -\nu_1' & \nu_2' & 0 \end{pmatrix} \tag{6.15}$$

A result derived by deSwart (1963) for the SU3 C.G. coefficients generalized for non-zero triality reads

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} = \xi_2\, e^{-i\pi(M_1 + \frac{Y_1}{2} + \frac{\tau_1}{3})} \left(\frac{d}{d_2}\right)^{\frac{1}{2}} \begin{pmatrix} \mu_1 & \mu^* & \mu^*_{2\gamma'} \\ \nu_1 & -\nu_3 & -\nu_2 \end{pmatrix} \quad ; \tag{6.16}$$

also
$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} = \xi_1 \begin{pmatrix} \mu_2 & \mu_1 & \mu_\gamma \\ \nu_2 & \nu_1 & \nu \end{pmatrix} \tag{6.17}$$
where ξ_1, ξ_2 are ± 1.

$$\begin{pmatrix} \mu_1 & 1 & \mu_2 \\ \nu_1 & 0 & \nu_2 \end{pmatrix} = \delta_{\mu_1\mu_2}\;\delta_{\nu_1\nu_2}$$

Using these relations we obtain for the product of C.G. coefficients in equation 6.15,

$$\begin{pmatrix} \mu_1^* & \mu_2 & 1 \\ -\nu_1 & \nu_2 & 0 \end{pmatrix} \begin{pmatrix} \mu_1^* & \mu_1 & 1 \\ -\nu_1' & \nu_2' & 0 \end{pmatrix} = \frac{1}{d_1} \begin{pmatrix} \mu_2 & 1 & \mu_1 \\ \nu_2 & 0 & \nu_1 \end{pmatrix} \begin{pmatrix} \mu_2 & 1 & \mu_1 \\ \nu_2' & 0 & \nu_1' \end{pmatrix} e^{-i\pi(M_2' + M_2 + \frac{Y_2'}{2} + \frac{Y_2}{2} + \frac{2}{3}\tau_2)} \tag{6.18}$$

$$= \frac{1}{d_1} \delta_{\mu_1 \mu_2} \delta_{\nu_1 \nu_2} \delta_{\nu_1' \nu_2'} e^{-i\pi(M_2' + M_2 + \frac{Y_2'}{2} + \frac{Y_2}{2} + \frac{2}{3}\tau_2)} ,$$

Therefore the final result is

$$\int_R D^{\mu_1 *}_{\nu_1' \nu_1}(\underline{\alpha}) \, D^{\mu_2}_{\nu_2' \nu_2}(\underline{\alpha}) \, \rho(\underline{\alpha}) \, d(\underline{\alpha}) = \frac{\delta_{\mu_1 \mu_2} \delta_{\nu_1 \nu_2} \delta_{\nu_1' \nu_2'}}{d_1} . \tag{6.19}$$

CHAPTER 7

INFINITESIMAL GENERATORS

In the preceding work, the explicit form of the infinitesimal generators was not required. As noted in the introduction, Biedenharn has derived them using polynomial basis states. However, using group integration with the spherical harmonic basis states offers an alternative and straightforward method for their calculation. We will consider the following three-dimensional infinitesimal generators:

$V_\pm \equiv \frac{1}{2}\left(I_4 \mp i\, I_5\right)$ and we want to find $\left\langle {}^{I'Y'}_{M'} \right| V_\pm \left| {}^{IY}_{M} \right\rangle$

Nelson (1967) derives the differential operators $D(I_i)$ corresponding to the generators which act on the basis states as follows:

$$\Psi^{IY}_{M} \equiv \left\langle {}^{IY}_{M} \right| T_{(\alpha)} \left| I=0 \right\rangle \tag{7.1}$$

$$D(I_i) \left\langle {}^{IY}_{M} \right| T_{(\alpha)} \left| I=0 \right\rangle = - \left\langle {}^{IY}_{M} \right| I_i\, T_{(\alpha)} \left| I=0 \right\rangle$$

We now use the differential operators of Nelson to find the infinitesimal generators. We have

$$D(V_\pm)\Psi^{IY}_{M} = - \left\langle {}^{IY}_{M} \right| V_\pm T_{(\alpha)} \left| I=0 \right\rangle \tag{7.2}$$

$$= - \sum_{i} \left\langle {}^{IY}_{M} \right| V_\pm \left| i \right\rangle \left\langle i \right| T_{(\alpha)} \left| I=0 \right\rangle .$$

Taking an inner product with $\Psi^{I'Y'}_{M'}$ we have by integrating

$$\int \Psi^{I'Y'*}_{M'} D(V_\pm)\, \Psi^{IY}_{M}\, \rho(\alpha)\, d(\alpha) = -\frac{1}{d} \left\langle {}^{IY}_{M} \middle| V_\pm \middle| {}^{I'Y'}_{M'} \right\rangle \tag{7.3a}$$

where $d = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2)$ is the dimension. Therefore

$$\left\langle {}^{I'Y'}_{M'} \middle| V_\pm \middle| {}^{IY}_{M} \right\rangle = -\frac{d}{2} \int \Psi^{I'Y'*}_{M'} D(I_4 \pm i I_5)\, \Psi^{IY}_{M}\, \rho(\alpha)\, d(\alpha) \,. \tag{7.3b}$$

We wish to calculate the infinitesimal generator V+. Therefore using Nelson's differential equations

$$-\frac{d}{2} D(I_4 + i I_5) \Psi^{IY}_{M} = -\frac{d}{2} e^{i(\rho + \frac{\alpha}{2} + \frac{\gamma}{2})} \Big[-\frac{3}{2} \cos\frac{\beta}{2} \tan\nu \frac{\partial}{\partial\rho} + \sec\frac{\beta}{2} \cot\nu \frac{\partial}{\partial\alpha} \tag{7.4}$$

$$+ 2i \sin\frac{\beta}{2} \cot\nu \frac{\partial}{\partial\beta} + \left(\cos\frac{\beta}{2}\tan\nu + \sec\frac{\beta}{2}\cot\nu\right)\frac{\partial}{\partial\nu} - i\cos\frac{\beta}{2}\frac{\partial}{\partial\nu}\Big] \Psi^{IY}_{M} \,.$$

We now use the following relation from SU2 to eliminate $\frac{\partial}{\partial\beta}$.

$$-\frac{d}{d\beta} d^{\tau}_{m\,m'}(\beta) = \sqrt{(\tau+m)(\tau-m+1)}\; d^{\tau}_{m-1\,m'}(\beta) + \left(m\cot\beta - \frac{m'}{\sin\beta}\right) d^{\tau}_{m\,m'}(\beta) \tag{7.5}$$

and taking the other derivatives we have

$$-\frac{d}{2} D(I_4 + i I_5) \Psi^{IY}_{M} = \frac{-i d}{2} \Big\{ \Big[\left(Y + \tfrac{1}{3}(\mu-\lambda)\right) \tan\nu - \frac{\partial}{\partial\nu} - 2M\cot\nu \Big] D^{\frac{1}{2}}_{-\frac{1}{2}\,-\frac{1}{2}}(\alpha,\beta,\gamma)\, D^{I}_{M,\frac{Y}{2}+\frac{1}{3}(\lambda-\mu)}(\alpha,\beta,\gamma) \tag{7.6}$$

$$+ 2\sqrt{(I+M)(I-M+1)}\, \cot\nu\, D^{\frac{1}{2}}_{\frac{1}{2}\,-\frac{1}{2}}(\alpha,\beta,\gamma)\, D^{I}_{M-1,\frac{Y}{2}+\frac{1}{3}(\lambda-\mu)}(\alpha,\beta,\gamma) \Big\} \hat{\phi}^{IY}_{M}\, e^{-i\rho(Y-1)} \,,$$

where

$$\hat{\phi}^{IY}_{M} = \frac{(-i)^{2I}(2I+1)^{\frac{1}{2}}}{(\lambda+1)^{\frac{1}{2}}(\mu+1)^{\frac{1}{2}}} \csc\nu \; d^{\frac{1}{2}(\lambda+\mu+1)\,(2\nu)}_{\frac{1}{6}(\mu-\lambda+3Y+6I+3),\,\frac{1}{6}(\mu-\lambda+3Y-6I-3)} \; . \tag{7.7}$$

The orthogonality relations of SU2

$$\int D^{j_1 *}_{M_1 M_1'} \; D^{\frac{1}{2}}_{-\frac{1}{2}\,-\frac{1}{2}} \; D^{j_3}_{M_3 M_3'} \, dR = \frac{C(\frac{1}{2}\, j_3\, j_1\, ;\, -\frac{1}{2}\, M_3\, M_1)\, C(\frac{1}{2}\, j_3\, j_1\, ;\, -\frac{1}{2}\, M_3'\, M_1')}{2j_1+1} \tag{7.8}$$

$$|j_3 - \tfrac{1}{2}| \leq j_1 \leq j_3 + \tfrac{1}{2} \qquad M_1 = M_3 - \tfrac{1}{2} \quad . \quad M_1' = M_3' - \tfrac{1}{2}$$

together with integration over ρ show that only inner products with $\Psi^{I \pm \frac{1}{2}\; Y-1}_{M-\frac{1}{2}}$ give non-zero results.

First we consider $\Psi^{I+\frac{1}{2}\; Y-1}_{M-\frac{1}{2}}$.

$$-\frac{d}{2}\int \Psi^{I+\frac{1}{2}\,Y-1}_{M-\frac{1}{2}} \, D(I_4 + i I_5)\, \Psi^{IY}_{M}\, \rho(\underline{\alpha})\, d(\underline{\alpha}) = \frac{-id}{2}\int \hat{\phi}^{I+\frac{1}{2}\;Y-1}_{M-\frac{1}{2}} \times \tag{7.9}$$

$$\Big\{\Big[\big(Y+\tfrac{1}{3}(\mu-\lambda)\big)\tan\nu - 2M\cot\nu + 2(I+M)\cot\nu + \big(Y+\tfrac{1}{6}(\mu-\lambda)-I-\tfrac{1}{2}\big)(\cot\nu - \tan\nu)$$

$$- \frac{Y-\frac{1}{3}(\mu-\lambda)+I+\frac{1}{2}}{\sin\nu\,\cos\nu} + \cot\nu\Big]\hat{\phi}^{IY}_{M}$$

$$-2\sqrt{\left[\tfrac{1}{2}(\mu+\lambda+1)+\tfrac{Y}{2}+\tfrac{1}{6}(\mu-\lambda)-I-\tfrac{1}{2}\right]\left[\tfrac{1}{2}(\mu+\lambda+1)-\tfrac{Y}{2}-\tfrac{1}{6}(\mu-\lambda)+I+\tfrac{3}{2}\right]} \;\; \hat{\phi}^{I+\frac{1}{2}\;Y-1}_{M-\frac{1}{2}} \; \frac{(2I+1)^{\frac{1}{2}}}{(2I+2)^{\frac{1}{2}}}\Big\}$$

$$\frac{(I-M+1)^{\frac{1}{2}}\,(I-\frac{Y}{2}+\frac{1}{3}(\mu-\lambda)+1)^{\frac{1}{2}}}{(2I+1)(2I+2)} \quad \sin^2\nu \quad \sin(2\nu)\; d(2\nu)$$

Use has been made of the relation

$$-\frac{d}{d\nu}\, d^{t\,(2\nu)}_{m\, m'} = -2\sqrt{(t+m')(t-m'+1)}\;\; d^{t\,(2\nu)}_{m\, m'-1} + 2\left(m'\cot 2\nu - \frac{m}{\sin 2\nu}\right) d^{t\,(2\nu)}_{m\, m'} \; : \tag{7.10}$$

and the explicit form of the C.G. coefficients given in Gottfried (1966, p. 220) for $j_2 = \frac{1}{2}$.

The coefficient of the $\hat{\phi}^{IY}_{M}$ term is zero. Integration gives the result

$$\left[\frac{(I+M)\left[I+\frac{Y}{2}-\frac{1}{3}(\mu-\lambda)\right]\left[\frac{1}{3}(2\mu+\lambda)+\frac{Y}{2}+I+1\right]\left[\frac{1}{3}(2\lambda+\mu)-\frac{Y}{2}-I+1\right]}{(2I+1)(2I)}\right] \equiv a_+ \,. \qquad (7.11a)$$

The inner product with $\Psi^{I-\frac{1}{2}\;Y-1}_{M-\frac{1}{2}}$ is evaluated following the same procedure

$$\left[\frac{(I-M+1)\left[I-\frac{Y}{2}+\frac{1}{3}(\mu-\lambda)+1\right]\left[\frac{1}{3}(\lambda+2\mu)+\frac{Y}{2}-I\right]\left[\frac{1}{3}(2\lambda+\mu)-\frac{Y}{2}+I+2\right]}{2(I+1)(2I+1)}\right] \equiv a_- \qquad (7.11b)$$

therefore

$$V_+\left|{}^{IY}_{M}\right\rangle = a_+\left|\begin{matrix}I+\frac{1}{2}\\ Y-1\\ M-\frac{1}{2}\end{matrix}\right\rangle + a_-\left|\begin{matrix}I-\frac{1}{2}\\ Y-1\\ M-\frac{1}{2}\end{matrix}\right\rangle . \qquad (7.12)$$

CHAPTER 8

MASS FORMULA

Coordinate representations of the basis states are used to derive a mass formula. For SU2 the spherical harmonic basis states are related to the transformation matrices in the following way:

$$Y_{\ell m}(\beta,\alpha) = \sqrt{\frac{2\ell+1}{4\pi}}\; D^{\ell\,*}_{m\,0}(\alpha,\beta) \tag{8.1}$$

These functions can also be derived by construction of differential equations and their solutions. Following the close analogy to SU2, we derive basis states for mesons and baryons, and use these to find mass formulas.

Meson Mass Formula

For SU3 Beg and Ruegg (1965), using differential equations, constructed basis states. Nelson showed later that these states could be obtained from

$$D^{\mu}_{\nu\,0}(\underline{\alpha}) = \left\langle {I\,Y \atop M} \right| e^{-i\rho Y}\, e^{-i\frac{\alpha}{2} I_3}\, e^{-i\frac{\beta}{2} I_2}\, e^{-i\frac{\gamma}{2} I_3}\, e^{-i\nu I_4} \left| {0\,Y_0 \atop 0} \right\rangle \quad . \tag{8.2}$$

For SU2, restriction of m' to $2m' = 0$ mod.2 would have removed the unwanted half-integral states. In SU3, the restriction $3/2\ Y_0 = 0$ mod.3 would have eliminated the unwanted states of non-zero triality.

Also in SU2 a rotation of 2π about the Y-axis followed by a rotation about the Z-axis should correspond to the point ($\alpha = 0, \beta = 0$) in three-dimensional space.

For SU2

$$\langle jm | e^{-i\alpha J_3} e^{-i2\pi J_2} | jm' \rangle = \langle jm | e^{-i2\pi J_2} e^{-i\alpha J_3} | jm' \rangle \tag{8.3a}$$

$$= \eta \; D^{j}_{m0}(0,0) \qquad \text{if } m' = 0 \; .$$

For SU3

$$\left\langle {}^{IY}_{M} \right| e^{-i\frac{\alpha}{2}I_3} e^{-i\frac{\beta}{2}I_2} e^{-i\frac{\gamma}{2}I_3} e^{-i\pi I_4} \left| {}^{I'Y'}_{M'} \right\rangle = \left\langle {}^{IY}_{M} \right| e^{-i\pi I_4} e^{-i\frac{\alpha}{2}I_3} e^{-i\frac{\beta}{2}I_2} e^{-i\frac{\gamma}{2}I_3} \left| {}^{I'Y'}_{M'} \right\rangle \tag{8.3b}$$

$$= \eta \; D^{\mu}_{\nu 0}(\varrho) \qquad \text{if } I' = 0 \; .$$

This suggests that if I' is chosen to be zero, the five parameters remaining may have significance as coordinates in a space.

For meson states, we will require that conjugation transform a particle into its antiparticle or

$$\Psi^{IY*}_{M} = \frac{(2I+1)^{\frac{1}{2}}(i)^{2I}}{2} \csc\nu \; d^{\frac{3}{2}}_{\frac{Y}{2}+I+\frac{1}{2},\frac{Y}{2}-I-\frac{1}{2}}(2\nu) \; D^{I\,*}_{M\frac{Y}{2}}(\alpha,\beta,\gamma) \; e^{i\rho Y} \tag{8.4}$$

$$= (-1)^{M-\frac{Y}{2}+2I} \; \Psi^{I,-Y}_{-M} \; .$$

Therefore, these functions that arise quite naturally from analogy to SU2 appear to represent meson states. To find if these states have any physical meaning, we construct a mass formula in the usual manner. Assume that the mass operator transforms as the I = 0 component of the vector operator and a scalar. The expectation value of the mass correction is as follows:

$$\int \Psi_{M}^{IY*}(\alpha)\, \Psi_{c}^{o\,c}(\alpha)\, \Psi_{M}^{IY}(\alpha)\, \rho(\alpha)\, d(\alpha) \equiv \Delta m \quad . \tag{8.5}$$

Using equation 4.8

$$\begin{aligned} \Delta m &= \frac{2I+1}{4}\int_{R} D_{M\,\frac{Y}{2}}^{I\,*}(\alpha,\beta,\gamma)\; D_{M\,\frac{Y}{2}}^{I}(\alpha,\beta,\gamma)\, dR(\alpha,\beta,\gamma)\int_{0}^{\frac{\pi}{2}}\left[d^{\frac{3}{2}}_{\frac{Y}{2}+I+\frac{1}{2},\ \frac{Y}{2}-I-\frac{1}{2}}(2\nu)\right]^{2}\left(1+3\,d^{1}_{0\,0}(2\nu)\right)\sin 2\nu\, d\nu \\ &= \frac{1}{32}\left(1 + \frac{Y^{2}-(2I+1)^{2}}{5}\right) \quad , \end{aligned} \tag{8.6}$$

Therefore the mass formula for mesons is

$$m = m_{c} + a\left[Y^{2}-(2I+1)^{2}\right] \quad . \tag{8.7}$$

This result is independent of the parameterization and could as well have been determined from the C.G. of SU3. Equation 8.7 is the Okubo formula which agrees with the experimental determination of the mass if tradition is followed and the square of the masses is used.

Baryon Mass Formula

The spherical harmonic basis states were not suitable for baryons since the relation

$$\Psi_{M}^{IY\,*} = \eta\, \Psi_{-M}^{I,-Y} \qquad \text{(the } \eta \text{ is a phase factor)}$$

should not hold.

However with another choice of the right-hand state, this result can be avoided. Since the eight-dimensional representation is equivalent to its complex conjugate we have

$$\left\langle {}^{I\,Y}_{M} \right| T(\alpha) \left| {}^{I'Y'}_{M'} \right\rangle^{*} = \eta \left\langle {}^{I\,-Y}_{-M} \right| T(\alpha) \left| {}^{I'-Y'}_{-M'} \right\rangle \quad ; \tag{8.8}$$

we therefore seek a right-hand state where $Y \neq 0$ or $M \neq 0$.

Now if G-Parity was applied to these states in the following manner

$$G \left\langle {}^{I\,Y}_{M} \right| T(\alpha) \left| {}^{I'Y'}_{M'} \right\rangle = \left\langle {}^{I\,Y}_{M} \right| e^{-i\pi I_1} C^{-1} T(\alpha)\, C\, e^{i\pi I_1} \left| {}^{I'Y'}_{M'} \right\rangle , \tag{8.9}$$

and if the baryon states $|\Sigma\rangle$ are not to have definite G-Parity, then this requires $Y \neq 0$.

Therefore we assume the baryon basis states are

$$\phi^{I\,Y}_{M} = \left\langle {}^{I\,Y}_{M} \right| T(\alpha) \left| {}^{I=\frac{1}{2}}_{\substack{Y=1\\ M=\frac{1}{2}}} \right\rangle \tag{8.10}$$

The connection to the mass would be

$$\int \left\langle {}^{I\,Y}_{M} \right| T(\alpha) \left| {}^{I=\frac{1}{2}}_{\substack{Y=1\\ M=\frac{1}{2}}} \right\rangle^{*} \left\langle {}^{0\,0}_{0} \right| T(\alpha) \left| {}^{0\,0}_{0} \right\rangle \left\langle {}^{I\,Y}_{M} \right| T(\alpha) \left| {}^{I=\frac{1}{2}}_{\substack{Y=1\\ M=\frac{1}{2}}} \right\rangle \rho(\alpha)\, d(\alpha) \tag{8.11}$$

$$= \int \left\langle {}^{I\,Y}_{M} \right| T(\alpha) \left| {}^{I=\frac{1}{2}}_{\substack{Y=1\\ M=\frac{1}{2}}} \right\rangle^{*} \sum_{\gamma} \begin{pmatrix} 8 & 8 & 8_{\gamma} \\ 0 & {}^{I\,Y}_{M} & {}^{I\,Y}_{M} \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma} \\ 0 & \substack{I=\frac{1}{2}\\ Y=1\\ M=\frac{1}{2}} & \substack{I=\frac{1}{2}\\ Y=1\\ M=\frac{1}{2}} \end{pmatrix} \times$$

$$\left\langle {}^{I\,Y}_{M} \right| T(\alpha) \left| {}^{I=\frac{1}{2}}_{\substack{Y=1\\ M=\frac{1}{2}}} \right\rangle \rho(\alpha)\, d(\alpha)$$

$$= \frac{1}{8} \sum_{\gamma=1,2} \begin{pmatrix} 8 & 8 & 8_\gamma \\ 0 & \begin{smallmatrix} I\,Y \\ M \end{smallmatrix} & \begin{smallmatrix} I\,Y \\ M \end{smallmatrix} \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_\gamma \\ 0 & \begin{smallmatrix} I=\frac{1}{2} \\ Y=1 \\ M=\frac{1}{2} \end{smallmatrix} & \begin{smallmatrix} I=\frac{1}{2} \\ Y=1 \\ M=\frac{1}{2} \end{smallmatrix} \end{pmatrix} \equiv \sum_{\gamma=1,2} \frac{\Delta_\gamma}{8}$$

All SU2 C.G. coefficients are 1 here and the results depend only on the isoscalar factors (See deSwart 1963).

Table 2. Products of C.G. Coefficients Used in Baryon Mass Formula

	Δ_1	Δ_2
$Y = 1$	$\frac{1}{20}$	$\frac{1}{4}$
$Y = 0$, $I = 1$	$-\frac{1}{10}$	0
$Y = 0$, $I = 0$	$\frac{1}{10}$	0
$Y = -1$	$\frac{1}{20}$	$-\frac{1}{4}$

Therefore

$$\Delta_1 + \Delta_2 = \frac{Y}{4} + \frac{1}{10}\left[-I(I+1) + \frac{Y^2}{4} + 1\right] \tag{8.12}$$

and the mass correction is

$$\Delta m = b\left[4I(I+1) - 10Y - Y^2\right] . \tag{8.13}$$

With b = 18 Mev this result agrees to within 5% with the experimentally determined baryon masses.

APPENDIX A

OTHER PARAMETERIZATIONS

We now consider two parameterizations, other than the one used by Nelson, which are of interest.

$e^{i\alpha H}$ Parameterization

A matrix obeys its characteristic equation. Since the defining representation is three-dimensional we have for SU3

$$e^{i\alpha H} = \lambda(\alpha)\, I + \mu(\alpha)\, H + \nu(\alpha)\, H^2 \tag{A.1}$$

where $H = \sum_i \hat{\alpha}_i I_i$ and $\sum_i \hat{\alpha}_i^{\,2} = 1$ and α_i are real.

Murnaghan (1962, pp. 14-19) shows that all unitary matrices are similar to a diagonal form by unitary transformation. We will find it advantageous to use the diagonal form for derivations that do not involve the $\hat{\alpha}_i$'s.

$$H \equiv \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} = \begin{pmatrix} \frac{\hat{\alpha}_8}{\sqrt{3}} + \hat{\alpha}_3 & & \\ & \frac{\hat{\alpha}_8}{\sqrt{3}} - \hat{\alpha}_3 & \\ & & -\frac{2}{\sqrt{3}}\hat{\alpha}_8 \end{pmatrix} \tag{A.2}$$

$$x_1 + x_2 + x_3 = 0 \qquad \operatorname{tr} H^2 = 2$$

The characteristic equation is

$$(\chi_1 - \sigma)(\chi_2 - \sigma)(\chi_3 - \sigma) = 0 \quad . \tag{A.3}$$

Replacing σ by H and using the above results we have

$$H^3 = H + \gamma I \quad \text{where} \quad \gamma = |H| \quad . \tag{A.4}$$

Taking the trace of each side and making use of the anticommutation,

$$\{I_i, I_j\} = \frac{4}{3}\delta_{ij} I + 2\sum_k d_{ijk} I_k \tag{A.5}$$

where the d_{ijk}'s are listed by Carruthers (1966, p. 31).
We have

$$\gamma = \frac{1}{3} \operatorname{tr} H^3 = \frac{1}{3} \operatorname{tr} \sum_{ijk} \hat{\alpha}_i \hat{\alpha}_j \hat{\alpha}_k I_i I_j I_k \tag{A.6}$$

$$= \frac{1}{3} \operatorname{tr} \sum_{ijk} \hat{\alpha}_i \hat{\alpha}_j \hat{\alpha}_k \frac{1}{2}\{I_i, I_j\} I_k = \frac{2}{3} \sum_{ijk} d_{ijk} \hat{\alpha}_i \hat{\alpha}_j \hat{\alpha}_k \quad .$$

For the set $\{I, H, H^2\}$ to be linearly dependent we must have a solution for (a,b) such that $I + aH + bH^2 = 0$. By multiplying by I, H, and H^2 and taking the trace each time we find

$$3 + 2b = 0 \qquad 2a + 3\gamma b = 0 \qquad 3\gamma a = 1 \tag{A.7}$$

which requires $27\gamma^2 = 4.$ (A.8)

Since

$$\gamma = -\frac{2}{\sqrt{3}}\,\hat{\alpha}_8\left(\frac{\hat{\alpha}_8^2}{3} - \hat{\alpha}_3^2\right)$$

the solutions for $\hat{\alpha}_3$, $\hat{\alpha}_8$ such that equation A.8 holds, are listed in the table.

Table 3. Singular Transformations

$\hat{\underline{\alpha}}_i$	$\hat{\alpha}_3$	$\hat{\alpha}_8$
$\hat{\underline{\alpha}}_1$	$-\sqrt{\frac{3}{4}}$	$-\frac{1}{2}$
$\hat{\underline{\alpha}}_2$	$\sqrt{\frac{3}{4}}$	$-\frac{1}{2}$
$\hat{\underline{\alpha}}_3$	0	1

Note that these solutions correspond to, at least for one pair, $x_i = x_j$ and thus are the singular matrices of Weyl (1946).

We also note that

$$\begin{aligned} e^{i\alpha\hat{\underline{\alpha}}_i\cdot I} &= I & \alpha = 0 \\ &= e^{i\frac{2}{3}\pi} I & \alpha = \frac{2\pi}{\sqrt{3}} \\ &= e^{i\frac{4}{3}\pi} I & \alpha = \frac{4\pi}{\sqrt{3}} \\ &= I & \alpha = \frac{6\pi}{\sqrt{3}} \end{aligned} \tag{A.9}$$

for all $\hat{\underline{\alpha}}_i$ in the set $(\hat{\underline{\alpha}}_1, \hat{\underline{\alpha}}_2, \hat{\underline{\alpha}}_3)$. They generate the important abelian subalgebra Z3 of SU3. All matrices similar to these are therefore periodic in α with the period $\sqrt{3}\,\pi$.

This is not true in general as we now show. Consider equation A.1 in diagonal form:

$$\begin{pmatrix} e^{i\alpha x_1} \\ e^{i\alpha x_2} \\ e^{i\alpha x_3} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} \begin{pmatrix} \lambda(\alpha) \\ \mu(\alpha) \\ \nu(\alpha) \end{pmatrix} \equiv Q \begin{pmatrix} \lambda(\alpha) \\ \mu(\alpha) \\ \nu(\alpha) \end{pmatrix} , \tag{A.10}$$

Inverting this result, we have

$$\lambda(\alpha) = \frac{\gamma}{|Q|} \left[\frac{x_3 - x_2}{x_1} e^{i\alpha x_1} + \frac{x_1 - x_1}{x_2} e^{i\alpha x_2} + \frac{x_2 - x_1}{x_3} e^{i\alpha x_3} \right] \tag{A.11a}$$

$$\mu(\alpha) = \frac{1}{|Q|} \left[x_1 (x_3 - x_2) e^{i\alpha x_1} + x_2 (x_1 - x_3) e^{i\alpha x_2} + x_3 (x_2 - x_1) e^{i\alpha x_3} \right] \tag{A.11b}$$

$$\nu(\alpha) = \frac{1}{|Q|} \left[(x_3 - x_2) e^{i\alpha x_1} + (x_1 - x_3) e^{i\alpha x_2} + (x_2 - x_1) e^{i\alpha x_3} \right] . \tag{A.11c}$$

If the matrix was periodic with period α_I we have for any integer (n)

$$e^{i n \alpha_I H} = I = \lambda(n\alpha_I) I + \mu(n\alpha_I) H + \nu(n\alpha_I) H^2 . \tag{A.12}$$

If $\{I, H, H^2\}$ are linearly independent

$$\lambda(n\alpha_I) = 1 \qquad \mu(n\alpha_I) = \nu(n\alpha_I) = 0 . \tag{A.12a}$$

This requires $e^{in\alpha_I x_1} = e^{in\alpha_I x_2} = e^{in\alpha_I x_3} = 1$ which is only satisfied when

$$\chi_i = \frac{2\pi \ell_i}{n\alpha_I} \qquad \text{for } \ell_i \text{ integer.} \tag{A.12b}$$

This cannot be satisfied for all (n).

Murnaghan Parameterization

Murnaghan (1962, pp. 7-10) shows that the unitary three-dimensional matrices could be factored in the following form:

$$U(\alpha) = D(\delta_1\, \delta_2\, \phi_3)\, U_{23}(\phi_2, \sigma_3)\ U_{12}(\theta_1, \sigma_2)\ U_{13}(\phi_1, \sigma_1) \tag{A.13}$$

$$-\pi < \phi < \pi \qquad -\tfrac{\pi}{2} \le \sigma_i \le \tfrac{\pi}{2} \qquad -\tfrac{\pi}{2} \le \delta_1 \le \tfrac{\pi}{2} \qquad -\pi \le \delta_2 \le \pi$$

(A.13a)

$$U_{23}(\phi_2, \sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_2 & -\sin\phi_2 e^{-i\sigma_3} \\ 0 & \sin\phi_2 e^{i\sigma_3} & \cos\phi_2 \end{pmatrix} = e^{-i\frac{\sigma_3}{2}U_3}\, e^{-i\phi_2 I_7}\, e^{i\frac{\sigma_3}{2}U_3}$$

$$U_3 = -\frac{I_3}{2} + \frac{\sqrt{3}}{2} I_8$$

(A.13b)

$$U_{12}(\theta_1, \sigma_2) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 e^{-i\sigma_2} & 0 \\ \sin\theta_1 e^{i\sigma_2} & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{-i\frac{\sigma_2}{2}I_3}\, e^{-i\theta_1 I_2}\, e^{i\frac{\sigma_2}{2}I_3}$$

(A.13c)

$$U_{13}(\phi_1, \sigma_1) = \begin{pmatrix} \cos\phi_1 & 0 & -\sin\phi_1 e^{-i\sigma_1} \\ 0 & 1 & 0 \\ \sin\phi_1 e^{i\sigma_1} & 0 & \cos\phi_1 \end{pmatrix} = e^{-i\frac{\sigma_1}{2}V_3}\, e^{-i\phi_1 I_5}\, e^{i\frac{\sigma_1}{2}V_3}$$

$$V_3 = \frac{I_3}{2} + \frac{\sqrt{3}}{2} I_8$$

(A.13d)

$$D(\delta_1,\delta_2,\phi_3) = \begin{pmatrix} e^{i\delta_1} & & \\ & e^{i\delta_2} & \\ & & e^{i\phi_3} \end{pmatrix} \longrightarrow \underset{\text{For SU3}}{e^{i\alpha_1 I_3}\, e^{i\alpha_2 I_8}}$$

Therefore for $U_{12}(\theta_1, \sigma_2)$, we have

$$\left\langle \begin{smallmatrix} I\,Y \\ M \end{smallmatrix} \right| U_{12}(\theta_1,\sigma_2) \left| \begin{smallmatrix} I'Y' \\ M' \end{smallmatrix} \right\rangle = \delta_{I\,I'} \;\; \delta_{Y\,Y'} \; D^{I}_{M\,M'}(\sigma_2,\, 2\phi_1, -\sigma_2) \;\; . \tag{A.14}$$

Chacon and Moshinsky (1966) showed how to transform the other $U_{ij}(\phi,\sigma)$ into $U_{12}(\phi,\sigma)$ by Weyl reflections and in this way generalized this result to all IR's.

APPENDIX B

VOLUME ELEMENT IN OTHER PARAMETERIZATIONS

We now consider the volume element in two parameterizations, other than the one used by Nelson, which are of interest.

$e^{i\alpha H}$ Parameterization

The parameterization of interest is

$$U(\underline{\alpha}) = e^{i\alpha \sum_i \hat{\alpha}_i I_i} \qquad \alpha_i \equiv \alpha \hat{\alpha}_i \tag{B.1}$$

This problem can be greatly simplified by noting that the density function $\rho(\underline{\alpha})$ is a class function. Consider $H_D = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix}$. Then all matrices $U = V^{-1} e^{i\alpha H_D} V$, where V is a general SU3 transformation, form a class where (α, x_1, x_2, x_3) are the class variables. (Only two independent variables are required as there are two relations on the x's.)

We now show that $\rho(\underline{\alpha})$ is invariant under a similarity transformation and therefore depends only on the class variables.

$$U(\underline{\alpha}') = V^{-1} U(\underline{\alpha}) V \tag{B.2a}$$

Therefore

$$\sum_i \alpha_i' I_i = V^{-1} \sum_i \alpha_i I_i V \tag{B.2b}$$

The α 's are real and $\sum_i \alpha_i^2 = \sum \alpha_i'^2$.

Therefore

$$\alpha_i' = \sum_j a_{ij} \alpha_j \tag{B.2c}$$

where a_{ij} is a real orthogonal matrix.

Also

$$\frac{\partial \alpha_i'}{\partial \alpha_j} = a_{ij} \quad . \tag{B.2d}$$

Now

$$U^{-1}(\underline{\alpha}') \frac{\partial}{\partial \alpha_i'} U(\underline{\alpha}') = \sum_j C_{ij}(\underline{\alpha}') I_j$$

becomes

$$\sum_j U^{-1}(\underline{\alpha}) \frac{\partial U(\underline{\alpha})}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial \alpha_i'} = \sum_j C_{ij}(\underline{\alpha}') V I_j V^{-1} \quad . \tag{B.3}$$

Therefore

$$\sum_{jk} C_{jk}(\underline{\alpha}) a_{ji} I_k = \sum_{jk} C_{ij}(\underline{\alpha}') a_{kj} I_k \tag{B.4a}$$

and from the linear independence of the I_k's

$$\sum_j C_{jk}(\underline{\alpha}) a_{ji} = \sum_j C_{ij}(\underline{\alpha}') a_{kj} \tag{B.4b}$$

$$\rho(\underline{\alpha}) \equiv |C_{jk}(\underline{\alpha})| = |C_{ij}(\underline{\alpha}')| \equiv \rho(\underline{\alpha}') \quad . \tag{B.5}$$

In the volume element calculation, we will evaluate all derivatives at $U(\underline{\alpha})$ in diagonal form:

$$\underline{\alpha} = (0, 0, \alpha_3, 0, 0, 0, 0, \alpha_8) \qquad \frac{\partial \alpha}{\partial \alpha_i}\Big|_D = (0, 0, \hat{\alpha}_3, 0, 0, 0, 0, \hat{\alpha}_8) \quad ; \tag{B.6}$$

and for $i \neq 3,8$

$$\frac{\partial f(\alpha)}{\partial \alpha_i} = \frac{\partial f(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \alpha_i}$$

implies

$$\frac{\partial \gamma}{\partial \alpha_i}\Big|_D = \frac{\partial \lambda(\alpha)}{\partial \alpha_i}\Big|_D = \frac{\partial \mathcal{M}(\alpha)}{\partial \alpha_i}\Big|_D = \frac{\partial \mathcal{V}(\alpha)}{\partial \alpha_i}\Big|_D = 0 \quad . \tag{B.7}$$

From

$$e^{i\sum_k \alpha_k I_k} = \lambda(\alpha) I + \mathcal{M}(\alpha) \sum_i \hat{\alpha}_i I_i + \mathcal{V}(\alpha) \sum_{ij} \hat{\alpha}_i \hat{\alpha}_j I_i I_j \tag{B.8}$$

we have from the anticommutation relation equation A.5

$$e^{i\sum_k \alpha_k I_k} = \left[\lambda(\alpha) + \tfrac{2}{3}\mathcal{V}(\alpha)\right] I + \mathcal{M}(\alpha) \sum_i \frac{\alpha_i}{\alpha} I_i + \mathcal{V}(\alpha) \sum_{ijk} d_{ijk} \frac{\alpha_i \alpha_j}{\alpha^2} I_k \quad . \tag{B.9}$$

For $\ell \neq 3,8$

$$\frac{\partial}{\partial \alpha_\ell} e^{i\sum_k \alpha_k I_k}\Big|_D = \frac{\mathcal{M}(\alpha)}{\alpha} I_\ell + \frac{2\mathcal{V}(\alpha)}{\alpha^2} \sum_{jk} d_{\ell jk} \alpha_j I_k \quad . \tag{B.10}$$

Now using the explicit form of $\lambda(\alpha)$, $\mathcal{M}(\alpha)$, $\mathcal{V}(\alpha)$ and that

$$|Q| = (\chi_1 \chi_3 + 2\chi_2^2)(\chi_1 - \chi_3) \tag{B.11}$$

the indices may be permuted cyclically.

$$\frac{\partial}{\partial \alpha_\ell} e^{i\sum_k \alpha_k I_k}\Big|_D = \frac{1}{\alpha}\left(\mathcal{M}(\alpha) - \chi_3 \mathcal{V}(\alpha)\right) I_\ell = \frac{1}{\alpha(\chi_2 - \chi_1)}\left(e^{i\alpha\chi_2} - e^{i\alpha\chi_1}\right) I_\ell \qquad \ell = 1, 2 \tag{B.12a}$$

$$= \frac{1}{\alpha}\left(\mu(\alpha) - x_2\nu(\alpha)\right) I_\ell = \frac{1}{\alpha(x_1 - x_3)}\left(e^{i\alpha x_1} - e^{i\alpha x_3}\right) I_\ell \quad \ell = 4,5 \qquad \text{(B.12b)}$$

$$= \frac{1}{\alpha}\left(\mu(\alpha) - x_1\nu(\alpha)\right) I_\ell = \frac{1}{\alpha(x_2 - x_3)}\left(e^{i\alpha x_2} - e^{i\alpha x_3}\right) I_\ell \quad \ell = 6,7 \qquad \text{(B.12c)}$$

Multiplication on the left by $\begin{pmatrix} e^{-i\alpha x_1} & & \\ & e^{-i\alpha x_2} & \\ & & e^{-i\alpha x_3} \end{pmatrix}$ will give all rows except the third and eighth in the determinant.

These rows are easily determined by setting all other variables to zero and taking the derivative. Also use is made of $[I_3, I_8] = 0$.

$$e^{-i(\alpha_3 I_3 + \alpha_8 I_8)} \frac{\partial}{\partial \alpha_\ell} e^{i(\alpha_3 I_3 + \alpha_8 I_8)} = i \quad \text{for} \quad \ell = 3, 8 \qquad \text{(B.13)}$$

The determinant is the product of three two-by-two determinants and is easily evaluated. With the change of notation $y_i = \alpha x_i$

$$\rho(\underline{\alpha}) = C \frac{\sin^2\frac{1}{2}(y_1 - y_2)\,\sin^2\frac{1}{2}(y_2 - y_3)\,\sin^2\frac{1}{2}(y_3 - y_1)}{(y_1 - y_2)^2\,(y_2 - y_3)^2\,(y_3 - y_1)^2}$$

where C is a constant (B.14) to be determined by normalization.

The x_i's satisfy the characteristic equation

$$x_i^3 - x_i - \gamma = 0. \qquad \text{(B.15)}$$

Therefore

(B.16)

$$x_1 = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3} \qquad A = \sqrt[3]{\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \frac{1}{27}}}$$

$$x_2 = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}$$

$$x_3 = A + B \qquad B = \sqrt[3]{\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \frac{1}{27}}}$$

where γ is given by equation A.6.

Murnaghan Parameterization

The volume element for the Murnaghan parameterization was also calculated.

$$U(\underline{\alpha}) = D(\alpha_3, \alpha_8)\, U_{23}(\phi_2, \sigma_3)\, U_{12}(\theta_1, \sigma_1)\, U(\phi_1, \sigma_1) \qquad \text{(B. 17)}$$

To facilitate this evaluation we change the basis from $\{I_k\}$ to $\{I_3, I_8, E_{pq}\}$ where E_{pq} $(p \neq q)$ has a (1) in the pth row, qth column. Also define $U = D_{(\alpha_3, \alpha_8)}V$ and note that a similarity transformation of this type will not alter $\rho(\underline{\alpha})$.

$$V c_{ij}(\underline{\alpha}) I_j V^{-1} = c_{ij}(\underline{\alpha}) V I_j V^{-1} = c_{ij}(\underline{\alpha}) a_{jk} I_k \qquad \text{(B.18)}$$

where $\quad |a_{jk}| = 1 \quad$ implies $\quad |c_{ij}(\underline{\alpha})| = |c_{ij}(\underline{\alpha}) a_{jk}|$. (B.18a)

We therefore evaluate

$$V\left[U^{-1} \frac{\partial}{\partial \alpha_i} U\right] V^{-1} = D^{-1} \frac{\partial U}{\partial \alpha_i} V^{-1} \quad . \qquad \text{(B.19)}$$

$$D^{-1} \frac{\partial U}{\partial \alpha_3} V^{-1} = i I_3 \qquad \text{(B.19a)}$$

$$D^{-1} \frac{\partial U}{\partial \alpha_8} V^{-1} = i I_8 \qquad \text{(B.19b)}$$

$$D^{-1} \frac{\partial U}{\partial \phi_2} V^{-1} = - e^{-i\sigma_3} E_{23} + e^{i\sigma_3} E_{32} \qquad \text{(B.19c)}$$

$$D^{-1} \frac{\partial U}{\partial \sigma_3} V^{-1} = i \sin\phi_2 \cos\phi_2 \left(e^{-i\sigma_3} E_{23} + e^{i\sigma_3} E_{32} \right) \qquad \text{(B.19d)}$$

$$+ i \sin^2\phi_2 \left(\frac{I_3}{2} - \frac{\sqrt{3}}{2} I_8 \right)$$

With the above results the E_{23}, E_{32}, I_3, I_8 columns can be cleared of all but a single non-zero term to reduce the eight-by-eight determinant to a four-by-four with a factor $C \sin 2\phi_2$.

The evaluation of the remaining four-by-four determinant is straightforward and the result is as follows:

$$P(\alpha) = C \sin 2\phi_1 \sin 2\phi_2 \sin 2\theta_1 \cos^2 \theta_1 \qquad \text{C normalization constant} \qquad (B.20)$$

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