Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 461–475
S 0019-2082

FINITE TRANSITIVE PERMUTATION GROUPS AND BIPARTITE VERTEX-TRANSITIVE GRAPHS

CHERYL E. PRAEGER

Celebrating the 100th anniversary of the birth of Reinhold Baer

Abstract. We prove a structure theorem for a class of finite transitive permutation groups that arises in the study of finite bipartite vertex-transitive graphs. The class consists of all finite transitive permutation groups such that each non-trivial normal subgroup has at most two orbits, and at least one such subgroup is intransitive. The theorem is analogous to the O'Nan–Scott Theorem for finite primitive permutation groups, and this in turn is a refinement of the Baer Structure Theorem for finite primitive groups. An application is given for arc-transitive graphs.

1. Introduction

By a permutation group on a set Ω we mean a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$ of all permutations of Ω. A transitive permutation group G on a set Ω is *primitive* if, for $\alpha \in \Omega$, the stabiliser G_α is a maximal subgroup of G. The term primitive can also be used for abstract groups, namely an abstract group is said to be *primitive* if it is isomorphic to a primitive permutation group on some set, or equivalently, G is primitive if it has a maximal subgroup H such that the core $\mathrm{Core}_G(H)$ of H in G is trivial, where $\mathrm{Core}_G(H) := \bigcap_{g\in G} H^g$. In 1957 Reinhold Baer [1] identified three types of (abstract) finite primitive groups according to the structure of the socle, where the *socle* $\mathrm{soc}(G)$ of a finite group G is the product of its minimal normal subgroups. This result of Baer has been called, for example by Förster in [5], the Baer Structure Theorem for primitive groups. It shows that, for a finite primitive group G, exactly one of the following holds.

(I) $\mathrm{soc}(G)$ is an abelian minimal normal subgroup of G;
(II) $\mathrm{soc}(G)$ is a non-abelian minimal normal subgroup of G;

Received July 23, 2002.
2000 *Mathematics Subject Classification.* 20B25, 20B05, 05C25.

(III) $\mathrm{soc}(G)$ is a direct product of two (isomorphic) non-abelian minimal normal subgroups of G.

Twenty years after Baer's paper [1] appeared, his result was refined independently by L. L. Scott and M. E. O'Nan into what is now called the O'Nan–Scott Theorem; see [13]. The framework it provides has proved to be the most useful modern method for identifying the possible structures of finite primitive permutation groups and is now used routinely for their analysis. For many families of point-transitive combinatorial objects, it is possible to describe the objects in terms of the sub–family of point–primitive objects, and in these cases the O'Nan–Scott Theorem has proved a powerful analytical tool. An especially successful example of this is the study of finite distance–transitive graphs, where this approach has led to an almost complete classification of the finite primitive distance–transitive graphs; see [6], [14].

However in some applications, for example to 2-arc-transitive graphs, it is not possible to relate a typical object in the family with a point-primitive object and so the theory of primitive groups cannot be used. Fortunately some of these families can be studied using quasiprimitive permutation groups. A permutation group $G \leq \mathrm{Sym}(\Omega)$ is *quasiprimitive* if each non-trivial normal subgroup N is transitive, or equivalently if $G = NG_\alpha$ for each such N (where $\alpha \in \Omega$). If G is primitive then G_α is maximal in G and G_α contains no non-trivial normal subgroup N since $\mathrm{Core}_G(G_\alpha) = 1$, and therefore $G = NG_\alpha$ for each such N. Thus each primitive group is quasiprimitive, but the converse is not true since each transitive permutation representation of a non-abelian simple group is quasiprimitive. Finite quasiprimitive permutation groups have been described by a theorem similar to the O'Nan–Scott Theorem in [11], and played a central role in elucidating the structure of finite non-bipartite 2-arc-transitive graphs. They are well-suited for studying families of combinatorial objects that are closed under a quotient operation; see [12]. Further discussion of the family of 2-arc transitive graphs will be given in Section 4.

In studies of vertex– and edge–transitive graphs, the bipartite graphs have always been far more difficult to handle than the non-bipartite ones. A graph Γ is *bipartite* if its vertex set Ω can be partitioned into two parts, say Δ, Δ', in such a way that every edge of Γ joins a vertex of Δ to a vertex of Δ'. If G is a vertex-transitive group of automorphisms of a connected bipartite graph Γ then G preserves the bipartition $\{\Delta, \Delta'\}$ and the set of elements of G that fix Δ and Δ' setwise forms a normal subgroup G^+ of G of index 2; moreover, if Γ is connected then the bipartition $\{\Delta, \Delta'\}$ is uniquely determined by Γ. This means in particular that, provided Γ has more than two vertices, then G is not quasiprimitive on vertices. In Section 4 we discuss results from [8] that show in particular that each finite bipartite 2-arc transitive graph Γ is a normal cover of a bipartite graph admitting a group that is both 2-arc transitive and bi-quasiprimitive on vertices.

By a *bi-quasiprimitive* permutation group G on Ω we mean a subgroup $G \leq \mathrm{Sym}(\Omega)$ that is not quasiprimitive and has the property that each non-trivial normal subgroup has at most two orbits. The main result of this paper, Theorem 1.1, is a structure theorem for finite bi-quasiprimitive groups similar to the Baer Structure Theorem for primitive groups. A study of bi-quasiprimitive 2-arc transitive automorphism groups was begun in [10]. We believe that the analysis in this paper of the finite bi-quasiprimitive groups, as permutation groups, will lead to a better understanding of bipartite 2-arc transitive graphs and also of other families of bipartite graphs. In particular we believe that it will shed light on the Weiss Conjecture for locally-primitive graphs (see Section 4).

For a finite bi-quasiprimitive permutation group G on Ω, there is at least one non-trivial intransitive normal subgroup N (since G is not quasiprimitive) and N must therefore have two orbits, say Δ, Δ'. Each element of G either fixes these two orbits or interchanges them. Thus the elements of G that fix Δ, Δ' setwise form a subgroup G^+ of index 2, and G^+ induces a transitive permutation group H on Δ. By the embedding theorem for permutation groups, G is conjugate in $\mathrm{Sym}(\Omega)$ to a subgroup of the wreath product $H \wr S_2 = (H \times H) \cdot S_2$. The set Ω may be identified with $\Delta \times \{1, 2\}$ such that, for (y_1, y_2) in the base group $H \times H$, and $(12) \in S_2$,

$$(\delta, i)^{(y_1,y_2)} = (\delta^{y_i}, i) \quad \text{and} \quad (\delta, i)^{(12)} = (\delta, i^{(12)})$$

for all $(\delta, i) \in \Omega$. With this identification of Ω the parts of the bipartition are $\{(\delta, i) \mid \delta \in \Delta\}$ for $i = 1, 2$. Theorem 1.1 identifies various distinct possibilities for $\mathrm{soc}(G)$.

The statement uses the following notation. Let M be a group. For each $\varphi \in \mathrm{Aut}(M)$, $\mathrm{Diag}_\varphi(M \times M)$ denotes the full diagonal subgroup $\{(x, x^\varphi) \mid x \in M\}$ of $M \times M$ (and we write the identity automorphism as 1). There is a natural embedding of M into $\mathrm{Sym}(M)$ with elements of M acting by right multiplication. This subgroup, which we identify with M, is *regular* in the sense that only the identity element fixes a point. The normaliser of this subgroup M in $\mathrm{Sym}(M)$ is called the *holomorph* $\mathrm{Hol}(M)$ of M and is the semidirect product $M \cdot \mathrm{Aut}(M)$ with elements of $\mathrm{Aut}(M)$ acting naturally. The centraliser $C(M)$ of M in $\mathrm{Sym}(M)$ consists of the elements of M acting by left multiplication. It is a subgroup isomorphic to M, $C(M) \cap M = Z(M)$, and $\mathrm{Aut}(M) \cap (C(M)M) = \mathrm{Inn}(M)$ consists of the inner automorphisms $\iota_y : x \mapsto y^{-1}xy$ induced by the elements $y \in M$.

Theorem 1.1. *Let G be a bi-quasiprimitive permutation group on a finite set Ω, let G^+ be a subgroup of G of index 2 with two orbits Δ, Δ' in Ω, and let H be the permutation group induced by G^+ on Δ. Then, replacing G by a conjugate in $\mathrm{Sym}(\Omega)$ if necessary, $G \leq H \wr S_2$, $G \setminus G^+$ contains an element $g = (x, 1)(12)$, for some $x \in H$, and one of the following holds.*

(a) *H is quasiprimitive and one of:*
  (i) $G = \langle G^+, g\rangle$ *and* $G^+ = \mathrm{Diag}_\varphi(H \times H)$, *where* $\varphi \in \mathrm{Aut}(H)$, $\varphi^2 = \iota_x$. *Moreover g centralises* G^+ *if and only if* $|\Omega| = 4$ *and* $G = Z_4$ *or* $Z_2 \times Z_2$.
  (ii) $\mathrm{soc}(G) = \mathrm{soc}(H) \times \mathrm{soc}(H)$.
  (iii) $\mathrm{soc}(H) = C(M) \times M \leq H \leq \mathrm{Hol}(M)$, *where* $M, C(M)$ *are isomorphic non-abelian, regular minimal normal subgroups of* H, *and* $\mathrm{soc}(G) = (C(M) \times C(M)) \times \mathrm{Diag}_\varphi(M \times M)$ *where* $\varphi \in \mathrm{Aut}(M)$ *and* φ^2 *is the restriction of* ι_x *to* M.

(b) *H is not quasiprimitive, but has a unique transitive minimal normal subgroup M, M is non-abelian and* $\mathrm{soc}(G) = M \times M$.

(c) *H is not quasiprimitive,* $G^+ = \mathrm{Diag}_\varphi(H \times H)$, *where* $\varphi \in \mathrm{Aut}(H)$, $\varphi^2 = \iota_x$; *there exists an intransitive minimal normal subgroup R of H such that* $R^\varphi \neq R$, $M := R \times R^\varphi$ *is a transitive normal subgroup of H, and* $N := \mathrm{Diag}_\varphi(M \times M)$ *is a minimal normal subgroup of G; and one of:*
  (i) $\mathrm{soc}(G) = N$.
  (ii) $\mathrm{soc}(G) = N \times \bar{N}$, *where* $N, \bar{N}$ *are isomorphic non-abelian minimal normal subgroups of G, and* $\bar{N} = \mathrm{Diag}_\varphi(\bar{M} \times \bar{M})$; $M, \bar{M}$ *are isomorphic regular normal subgroups of H,* $\mathrm{soc}(H) = M \times \bar{M} \leq H \leq \mathrm{Hol}(M)$, *and* $\bar{M} = \bar{R} \times \bar{R}^\varphi$ *for an intransitive minimal normal subgroup* $\bar{R} \cong R$.

Moreover there are examples in each of the cases.

REMARKS 1.1.

(1) The subgroup G^+ is the unique subgroup of G of index 2 except for the case where $|\Omega| = 4$ and $G = Z_2 \times Z_2$ in part (a)(i).
(2) In part (a)(iii) the group H is a primitive subgroup of $\mathrm{Hol}(M)$, while in part (c)(ii), the subgroup H of $\mathrm{Hol}(M)$ has four minimal normal subgroups.
(3) Groups H occurring in part (b) are called innately transitive permutation groups, and have been studied in [2].

We give in Section 2 examples of finite bi-quasiprimitive groups for each of the parts of Theorem 1.1, and in Section 3 we prove that each finite bi-quasiprimitive group satisfies the conditions of exactly one of the parts, thus completing the proof of Theorem 1.1. In Section 4 we explain how each finite arc-transitive bipartite graph is a multi-cover of a graph admitting a group of automorphisms that is both transitive on arcs and bi-quasiprimitive on vertices. Here, by an arc we mean an ordered pair of vertices that are joined by an edge. One family of arc-transitive, vertex bi-quasiprimitive graphs is the family comprising the *complete bipartite graphs* $K_{n,n}$ for which each part of the bipartition has size n and each vertex of one part is joined to each vertex of the other part. The full automorphism group $S_n \wr S_2$ of $K_{n,n}$ is bi-quasiprimitive of type (a)(ii), and for certain values of n there are arc-transitive, bi-quasiprimitive subgroups of $S_n \wr S_2$ of types (a)(iii) or (b). For graphs other than $K_{n,n}$, all arc-transitive, vertex bi-quasiprimitive groups must be of the other types.

THEOREM 1.2. *Let Γ be a finite connected bipartite graph and suppose that $G \leq \mathrm{Aut}(\Gamma)$ is arc-transitive and vertex bi-quasiprimitive. Then either $\Gamma = K_{n,n}$ for some n, or G satisfies part* (a)(i) *or* (c) *of Theorem* 1.1.

Proof. Suppose that $\Gamma \neq K_{n,n}$ and let $\alpha \in \Delta$. Then the set of vertices adjacent to α is a proper subset of Δ' and it follows that G_α is not transitive on Δ'. Thus G is of type (a)(i) or (c) of Theorem 1.1. □

The paper [10] contains an investigation along the lines of Theorem 1.2 in the special case where G is in addition 2-arc transitive, and in particular the first result Theorem 2.1 of [10] follows immediately from Theorem 1.2. We discuss the results of [10] in more detail in Section 4. In [7] a larger class of graphs, the locally-quasiprimitive graphs, were investigated also from this point of view. In that paper the need for a better understanding of bi-quasiprimitive groups was highlighted, and Theorem 1.1 essentially solves Problem 6.2 of that paper. The extra structural details for bi-quasiprimitive groups given in Theorem 1.1 should increase our understanding of bi-quasiprimitive 2-arc transitive graphs and other bipartite edge-transitive graphs.

2. Examples of bi-quasiprimitive groups

In this section we give several constructions of finite bi-quasiprimitive permutation groups to demonstrate that each of the cases in Theorem 1.1 can arise. We label the examples according to the case in the main theorem. We shall construct bi-quasiprimitive subgroups G of $H \wr S_2$, where H is a transitive permutation group on Δ and $H \wr S_2 = (H \times H) \cdot S_2$ acts as in Section 1 on $\Omega = \Delta \times \{1, 2\}$. For each group G, the subgroup $G^+ := G \cap B$ will project onto each of the direct factors H of B. Note that in order to show that G is bi-quasiprimitive it is sufficient to prove that every minimal normal subgroup of G has at most two orbits in Ω.

EXAMPLE (a)(i). Let H be a quasiprimitive permutation group on a set Δ, and let $\varphi \in \mathrm{Aut}(H)$ such that $\varphi \neq 1$ and φ^2 is an inner automorphism ι_x of H, for some $x \in H$. Then the subgroup $G = \langle \mathrm{Diag}_\varphi(H \times H), (x, 1)(12) \rangle$ of $H \wr S_2$ is bi-quasiprimitive as in part (i).

Proof. Let $G^+ = \mathrm{Diag}_\varphi(H \times H)$. Then G^+ is normal in G of index 2, and each minimal normal subgroup of G contained in G^+ is of the form $N = \mathrm{Diag}_\varphi(M \times M)$ for some normal subgroup M of H. Since H is quasiprimitive,

M is transitive on Δ and so N has two orbits in Ω. If N were a minimal normal subgroup of G and $N \not\leq G^+$, then $N \cap G^+ = 1$ by minimality, and so $N \cong Z_2$ and $G = G^+ \times N$. Now N is generated by $(y, y^\varphi)g = (yx, y^\varphi)(12)$ for some $y \in H$. A straightforward computation shows that such an element centralises G^+ if and only if $\varphi = 1$, which is not the case. Thus every minimal normal subgroup of G has two orbits. □

If in Example (a)(i) we were to take $\varphi = 1$ then $x \in Z(H)$ and $(x, 1)(12)$ would centralise G^+. The only possibility for H that leads to a bi-quasiprimitive group G in this case is $H = S_2$ on $\Delta = \{1, 2\}$, giving $G = Z_4$ if $x \neq 1$ or $G = Z_2 \times Z_2$ if $x = 1$ (see Lemma 3.1).

Example (a)(ii). Let H be a quasiprimitive permutation group on a set Δ, and take $G = H \wr S_2$. Then it is easy to check that G is bi-quasiprimitive as in part (a)(ii).

Example (a)(iii). Let $H = \mathrm{Hol}(M) = M \cdot \mathrm{Aut}(M)$ acting on $\Delta = M$, where $M = T^k$ for some non-abelian simple group T and $k \geq 1$. Let $\varphi \in \mathrm{Aut}(M)$ and set $x := \varphi^2 \in \mathrm{Aut}(M) < H$ and $C(M) := C_H(M)$. Let $G^+ = (C(M) \times C(M))\,\mathrm{Diag}_\varphi(H \times H)$ (where we take conjugation by $\varphi \in \mathrm{Aut}(M) < H$ as the corresponding inner automorphism of H), $g = (x, 1)(12)$, and $G = \langle G^+, g\rangle$. Then G is bi-quasiprimitive as in part (a)(iii).

Proof. The element g normalises $\mathrm{Diag}_\varphi(H \times H)$ and hence normalises G^+. Also $g^2 = (x, x) = (x, x^\varphi) \in \mathrm{Diag}_\varphi(H \times H)$, so $|G : G^+| = 2$. All the conditions of part (a)(iii) hold for G. We just have to show it is bi-quasiprimitive. If N were a minimal normal subgroup of G not contained in G^+, then as in the proof of Example (a)(i), $G = G^+ \times N \cong G^+ \times Z_2$, but no element of $G \backslash G^+$ centralises G^+. Thus if N is a minimal normal subgroup of G then $N \leq G^+$, and hence $N \leq G \cap (\mathrm{soc}(H) \times \mathrm{soc}(H)) = (C(M) \times C(M)) \times \mathrm{Diag}_\varphi(M \times M)$. It follows that N is $C(M) \times C(M)$ or $\mathrm{Diag}_\varphi(M \times M)$, each of which has two orbits in Ω. □

It is rather more difficult to provide examples for part (b). We give one small example to show that this case does arise.

Example (b). Let $H = S \times A$ where $S = S_5$ and $A = \langle a\rangle \cong Z_2$, acting by right multiplication on the set Δ of 20 right cosets of $K := \langle (123), (12), (45)a\rangle$ (where here the permutations refer to elements of S). Let $M = A_5$, let $b \in S \setminus M$ be an element of order 2, and let g be the element (12) in the top group S_2 of $H \wr S_2$. Then the subgroup $G = \langle M \times M, (a, b), g\rangle$ is bi-quasiprimitive as in part (b).

Proof. Since $M \cap K = \langle (123)\rangle$, M is transitive on Δ. Thus H has a unique transitive minimal normal subgroup M that is non-abelian, and also an intransitive normal subgroup A, so H is not quasiprimitive. Now $G^+ := G \cap (H \times H) = \langle M \times M, (a,b), (b,a) \rangle$. The projection of G^+ onto each direct factor H is $\langle M, a, b \rangle = H$, and $G/(M \times M) \cong D_8$. It is easy to check that $M \times M$ has trivial centraliser in G. Now $M \times M$ is a minimal normal subgroup of G and has two orbits in Ω. Since $M \times M$ has trivial centraliser it follows that $\mathrm{soc}(G) = M \times M$, so G is bi-quasiprimitive as in (b). □

It is clear how to generalise this example to any instance of an almost simple permutation group S with transitive socle of index 2 such that a point stabiliser S_δ fixes 2 points, or equivalently $|N_S(S_\delta) : S_\delta| = 2$. For the remaining cases of Theorem 1.1 we give again a reasonably general construction.

Example (c)(i) and (ii). For $i = 1, 2$, let K_i be a quasiprimitive permutation group on Δ_i such that $Z(K_i) = 1$ and let R_i be a minimal normal subgroup. Suppose that there exists an isomorphism $\sigma : K_1 \to K_2$ such that $R_1^\sigma = R_2$. Note that the actions on the Δ_i need not be permutationally isomorphic; we only assume that the groups K_i are isomorphic as abstract groups. Let $H = K_1 \times K_2$ acting naturally on $\Delta = \Delta_1 \times \Delta_2$ by $(\delta_1, \delta_2)^{(a_1,a_2)} = (\delta_1^{a_1}, \delta_2^{a_2})$ for $\delta_i \in \Delta_i, a_i \in K_i$. Let $\varphi \in \mathrm{Aut}(H)$ be the map $\varphi : (a, b^\sigma) \mapsto (b, a^\sigma)$ for $a, b \in K_1$. Then the subgroup $G = \langle \mathrm{Diag}_\varphi(H \times H), (12) \rangle$ of $H \wr S_2$ is bi-quasiprimitive.

Proof. Since $\varphi^2 = 1$, it is easily checked that the element $g := (12)$ of the top group of $H \wr S_2$ normalises, but does not centralise, $G^+ := \mathrm{Diag}_\varphi(H \times H)$. In fact, G^+ has trivial centraliser in G. Thus each minimal normal subgroup N of G is contained in G^+ and hence $N = \mathrm{Diag}_\varphi(M \times M)$ for some normal subgroup M of H. Now $\mathrm{Diag}_\varphi(M \times M)^g = \mathrm{Diag}_\varphi(M^\varphi \times M^\varphi)$, and since $N^g = N$ we require $M^\varphi = M$. Also, if M_0 is a normal subgroup of H such that $M_0^\varphi = M_0$ then $\mathrm{Diag}_\varphi(M_0 \times M_0)$ is normal in G. Thus the minimal normal subgroups of G are those subgroups $\mathrm{Diag}_\varphi(M \times M)$ where M is normal in H, $M^\varphi = M$, and M is minimal with respect to these properties. Let M be such a subgroup, and let R be a minimal normal subgroup of H contained in M. If $R \neq R^\varphi$ then $R^\varphi \leq M$ and $R \times R^\varphi$ is a normal subgroup of H invariant under φ, so $M = R \times R^\varphi$. In the case where $R \leq K_i$ for some i, we have $R^\varphi \neq R$, so $M = R \times R^\varphi$ and M is transitive on Δ, and hence $\mathrm{Diag}_\varphi(M \times M)$ has two orbits in Ω. For example, the subgroup $R = R_1$ gives rise to a minimal normal subgroup of this type. Suppose then that R is not contained in K_1 or K_2. By the minimality of R, $R \cap K_1 = R \cap K_2 = 1$. However, in order for a subgroup of this type to be normal in $H = K_1 \times K_2$ it must be contained in $Z(K_1) \times Z(K_2)$, and this is impossible since $Z(K_i) = 1$. Thus G is bi-quasiprimitive and is of type (c). If K_i has a unique minimal normal subgroup then (c)(i) holds, while if this is not the case then K_i has two isomorphic non-abelian regular minimal normal subgroups and (c)(ii) holds. □

3. Proof of Theorem 1.1

Suppose that G, G^+, H are as in Theorem 1.1, and that $G \leq H \wr S_2 = (H \times H) \cdot S_2$ acting naturally on $\Omega = \Delta \times \{1, 2\}$. We shall write the base group $B = H \times H$ as $B = H_1 \times H_2$ when we wish to distinguish the two direct factors. Note that $G^+ = B \cap G$ and by the definition of H, G^+ projects onto, say H_1. As G interchanges H_1 and H_2 while normalising G^+, it follows that G^+ projects also onto H_2, that is to say, G^+ is a subdirect subgroup of B. Let $g \in G \setminus G^+$. Then $g = (x, y)(12)$ for some $x, y \in H$, and since G^+ projects onto H_2, multiplying g by an element of G^+ if necessary, we may assume that $y = 1$, so $g = (x, 1)(12)$. First we consider the exceptional case in which some element of $G \setminus G^+$ centralises G^+. Note that, since $G^+ \neq 1$, the cardinality of Ω is at least 4.

Lemma 3.1. *Some element of $G \setminus G^+$ centralises G^+ if and only if $|\Omega| = 4$ and $G = Z_4$ or $Z_2 \times Z_2$. In particular Theorem* 1.1 (a)(i) *holds in this case.*

Proof. If $|\Omega| = 4$ and $G = Z_4$ or $Z_2 \times Z_2$, then g centralises G^+. Also $H = S_2$ is quasiprimitive and Theorem 1.1 (a)(i) holds. Suppose conversely that $g = (x, y)(12) \in G \setminus G^+$ and g centralises G^+. Then for each $(a, b) \in G^+$ we have

$$(a, b) = (a, b)^g = (a, b)^{(x,y)(12)} = (b^y, a^x).$$

Thus $a = b^y, b = a^x$, and since G^+ is a subdirect subgroup of B it follows that $G^+ = \mathrm{Diag}_{\iota_x}(H \times H)$ and $xy, yx \in Z(H)$. Now $g^2 = (xy, yx) \in G^+ \cap (Z(H) \times Z(H))$. Suppose first that $g^2 = 1$. Then $N = \langle g \rangle \cong Z_2$ is a normal subgroup of G and hence has at most two orbits in Ω. It follows that $|\Omega| = 4$ and $G = N \times G^+ = Z_2 \times Z_2$. Suppose now that $g^2 \neq 1$. Then $G^+ \cap (Z(H) \times Z(H))$ is a nontrivial normal subgroup of G and hence has two orbits in Ω. In particular $Z(H)$ is transitive on Δ, and as transitive abelian permutation groups are regular it follows that $Z(H)$ is regular on Δ and we may identify Δ with $Z(H)$ in such a way that, for $\delta = 1 \in \Delta$, $H = Z(H)H_\delta$ with H_δ acting by conjugation. Since H_δ centralises $Z(H)$ we have that $H_\delta = 1$ and $H = Z(H)$. It follows that G is abelian and hence regular on Ω. Let p be a prime divisor of $|G^+|$ and M a subgroup of G^+ of order p. Since G is abelian M is normal in G and hence has two orbits in Ω. It follows that $G^+ = M$. If p were odd then we would have $G = Z_{2p}$, and G would have a normal subgroup of order 2 having $p > 2$ orbits, which is a contradiction. Hence $p = 2$, $|\Omega| = 4$, and since $g^2 \neq 1$ we have $G = Z_4$. In both of these cases with $|\Omega| = 4$, $H = S_2$ is quasiprimitive and Theorem 1.1 (a)(i) holds. □

From now on we shall assume that no element of $G \setminus G^+$ centralises G^+, and in particular, $g = (x, 1)(12)$ does not centralise G^+. Our next step is to study the minimal normal subgroups of G. Let π_i denote the projection map $\pi_i : B \to H_i$ and note that $H_i = \pi_i(G^+)$ for $i = 1, 2$.

LEMMA 3.2. *Let N be a minimal normal subgroup of G, and let $N_i = N \cap H_i$ for $i = 1, 2$.*

(a) *Then $N \subseteq G^+$, and either N is a minimal normal subgroup of G^+, or $N = M \times M^g$ where both M and M^g are minimal normal subgroups of G^+. In particular $\mathrm{soc}(G) \subseteq \mathrm{soc}(H) \times \mathrm{soc}(H)$. Moreover, for any minimal normal subgroup M of G^+ that is not normal in G, $M \times M^g$ is a minimal normal subgroup of G.*

(b) *If $N_1 \neq 1$, then N_1 is a transitive minimal normal subgroup of H_1 and $N = N_1 \times N_2$. Moreover, for any minimal normal subgroup M of H_1 that is contained in G^+, M is transitive on Δ and $M \times M^g$ is a minimal normal subgroup of G.*

Proof. Suppose that $N \not\leq G^+$. Then $N \cap G^+ = 1$ by the minimality of N, and hence N centralises G^+, contrary to our assumption. Thus $N \leq G^+$. Suppose that N is not a minimal normal subgroup of G^+ and let M be a minimal normal subgroup of G^+ contained in N. Then $M^g \neq M$ since otherwise M would be normalised by $\langle G^+, g\rangle = G$ contradicting the minimality of N. Also M^g is a minimal normal subgroup of G^+ since g normalises G^+, and hence $M \cap M^g = 1$ and $M \times M^g$ is normal in G^+. Now $M^g \subset N$ since N is normal in G, and so $M \times M^g \leq N$. Also $(M^g)^g = M$ since $g^2 \in G^+$, and hence $M \times M^g$ is normalised by $\langle G^+, g\rangle = G$. Now the minimality of N implies that $N = M \times M^g$. It follows that $\mathrm{soc}(G) \subseteq \mathrm{soc}(H) \times \mathrm{soc}(H)$.

If M is a minimal normal subgroup of G^+ that is not normal in G, then the argument of the previous paragraph shows that $M \times M^g$ is a normal subgroup of G. If M is a direct product of isomorphic non-abelian simple groups then G^+ is transitive on the simple direct factors of both M and M^g and hence G is transitive on the simple direct factors of $M \times M^g$ implying its minimality. If M is elementary abelian then $M \times M^g$ is also elementary abelian, is irreducible as a G-module, and so is minimal normal in G. Thus part (a) is proved.

Now suppose that $N_1 = N \cap H_1 \neq 1$. Then N_1 is a normal subgroup of G^+ properly contained in N. Let M be a minimal normal subgroup of G^+ contained in N_1. By the first paragraph of the proof, $N = M \times M^g$. Now $M^g \leq N_2$ and it follows that $M = N_1$ and $M^g = N_2$. Since G is bi-quasiprimitive, it follows that the N_i are transitive on Δ.

Finally suppose that M is a minimal normal subgroup of H_1 that is contained in G^+. Then $M^g \leq H_2 \cap G$ and $M \times M^g$ is a normal subgroup of G contained in G^+. The fact that it is a minimal normal subgroup of G follows as in the second paragraph of the proof. Since G is bi-quasiprimitive, it follows that M is transitive on Δ. □

As preparation for treating the cases where $G^+ \cap H_1 \neq 1$, we make the following observations. Note that, for a transitive permutation group N on

Δ, $|C_{\mathrm{Sym}(\Delta)}(N)|$ is equal to the number of fixed points of N_δ, where $\delta \in \Delta$; and $C_{\mathrm{Sym}(\Delta)}(N)$ is *semiregular*, that is, the only element fixing a point is the identity.

LEMMA 3.3. *Suppose that a permutation group H on Δ has transitive normal subgroups M and N such that $M \cap N = 1$. Then both are regular, $Z(N) = Z(M) = 1$, and we may identify Δ with N so that $H \leq \mathrm{Hol}(N)$ and $M = C_{\mathrm{Sym}(N)}(N) \cong N$. Further if N is a minimal normal subgroup then H is primitive and* $\mathrm{soc}(H) = N \times M$.

Proof. Since N, M are normal and $N \cap M = 1$, we have that $M \subseteq C_{\mathrm{Sym}(\Delta)}(N)$ and since M is transitive, $|M| \geq |\Delta|$. It follows that $M = C_{\mathrm{Sym}(\Delta)}(N)$ and both M and N are regular. Thus we may identify Δ with N so that $H \leq \mathrm{Hol}(N)$. Also $C_{\mathrm{Sym}(N)}(N) \cong N$ and since $N \cap C_{\mathrm{Sym}(N)}(N) = Z(N)$ it follows that $Z(N) = Z(M) = 1$. Finally if N is a minimal normal subgroup then, since $M = C_{\mathrm{Sym}(\Delta)}(N)$, we have that $\mathrm{soc}(H) = N \times M$. In particular H has two minimal normal subgroups N and M, and each is transitive, so H is quasiprimitive. By [11], H is primitive. $\square$

LEMMA 3.4. *Suppose that M is a normal subgroup of H such that G^+ has a subgroup* $\mathrm{Diag}_\varphi(M \times M)$*, for some $\varphi \in \mathrm{Aut}(M)$, that is left invariant by $g = (x, 1)(12)$. Then the automorphism of M induced by conjugation by x is φ^2.*

Proof. Each element of $\mathrm{Diag}_\varphi(M \times M)$ is of the form (y, y^φ) where $y \in M$, and we have $(y, y^\varphi)^g = (y^\varphi, y^x)$. Hence $y^x = (y^\varphi)^\varphi$ for all $y \in M$, that is the automorphism of M induced by conjugation by x is equal to φ^2. $\square$

Now we deal with the case where $G^+ \cap H_1 \neq 1$.

LEMMA 3.5. *Suppose that $G^+ \cap H_1 \neq 1$. Then one of Theorem* 1.1 (a)(ii), (a)(iii), (b) *holds, and in case* (a)(iii)*, H is primitive.*

Proof. Let M_1 be a minimal normal subgroup of H_1 contained in $N_1 := G^+ \cap H_1$ and let $M_2 = M_1^g$, $N_2 = G^+ \cap H_2$, so $M_2 \leq N_2$. By Lemma 3.2 (b), M_1 is transitive on Δ and $M_1 \times M_2$ is a minimal normal subgroup of G. If N_1 contained a second minimal normal subgroup L_1 of H_1 distinct from M_1 then L_1 would also be transitive and $L_1 \times L_1^g$ would be a second minimal normal subgroup of G. In this case, by Lemma 3.3, H is primitive and $\mathrm{soc}(H) = M_1 \times L_1$. Also by Lemma 3.2 (a), $\mathrm{soc}(G) = (M_1 \times M_2) \times (L_1 \times L_1^g) = \mathrm{soc}(H) \times \mathrm{soc}(H)$, and Theorem 1.1 (a)(ii) holds.

Thus we may assume that $\mathrm{soc}(G) \cap N_1 = M_1$. If $\mathrm{soc}(H_1) = M_1$ then again Theorem 1.1 (a)(ii) holds, so we may assume that $\mathrm{soc}(H_i) = M_i \times L_i$, where $L_i \neq 1$ and g interchanges L_1 and L_2. Recall that M_1 is transitive. If M_1 were abelian then it would be regular and self-centralising in $\mathrm{Sym}(\Delta)$.

However since $L_1 \leq C_{\mathrm{Sym}(\Delta)}(M_1)$, it follows that M_1 is non-abelian. Suppose that G has a minimal normal subgroup $K \neq M_1 \times M_2$. By Lemma 3.2, $K \times M_1 \times M_2 \leq \mathrm{soc}(H_1) \times \mathrm{soc}(H_2)$ and hence $K \leq L_1 \times L_2$. Since G is bi-quasiprimitive, K has two orbits in Ω and so L_1 is transitive on Δ. By Lemma 3.3, H_1 is primitive, M_1, L_1 are isomorphic non-abelian minimal normal subgroups and are regular on Δ, $M_1 = C_{\mathrm{Sym}(\Delta)}(L_1)$, and we may identify Δ with L_1 so that $H_1 \leq \mathrm{Hol}(L_1)$. Hence $\pi(K) = L_i$ for each i, and so, identifying $L_1 = L_2 = L$, $K = \mathrm{Diag}_\varphi(L \times L)$ for some $\varphi \in \mathrm{Aut}(L)$. By Lemma 3.4, φ^2 is the restriction of ι_x to L. Clearly $K \times (M_1 \times M_2) = G^+ \cap \mathrm{soc}(B)$, and hence $\mathrm{soc}(G) = K \times (M_1 \times M_2)$, and Theorem 1.1 (a)(iii) holds.

We may now assume that $\mathrm{soc}(G) = M_1 \times M_2$. If L_1 is intransitive, then Theorem 1.1 (b) holds. So suppose finally that L_1 is transitive. Arguing as in the previous paragraph, H_1 is primitive, M_1, L_1 are isomorphic non-abelian minimal normal subgroups and are regular on Δ, $M_1 = C_{\mathrm{Sym}(\Delta)}(L_1)$, and we may identify Δ with L_1 so that $H_1 \leq \mathrm{Hol}(L_1) = L_1 \cdot \mathrm{Aut}(L_1)$. Since $\mathrm{soc}(G) = M_1 \times M_2 = \mathrm{soc}(B) \cap G$, we have $G^+/\mathrm{soc}(G) \cong G^+\mathrm{soc}(B)/\mathrm{soc}(B)$, which is isomorphic to a subgroup of $\mathrm{Out}(L_1)^2$. On the other hand, since $\pi_i(G^+) = H_i$, it follows that $G^+/\mathrm{soc}(G)$ has a chief factor isomorphic to L_1. Since $L_1 = T^k$ for some non-abelian simple group T and $k \geq 1$, we have $\mathrm{Out}(L_1) = \mathrm{Out}(T) \wr S_k$, and no subgroup of $\mathrm{Out}(L_1)^2$ has a chief factor isomorphic to T^k. Thus this final case does not arise. □

LEMMA 3.6. *If* $G^+ \cap H_1 = 1$*, then one of Theorem* 1.1 (a)(i), (c)(i), (c)(ii) *holds.*

Proof. Identifying $H_1 = H_2 = H$, we have $G^+ = \mathrm{Diag}_\varphi(H \times H)$ for some $\varphi \in \mathrm{Aut}(H)$, and by Lemma 3.4, $\varphi^2 = \iota_x$. If H is quasiprimitive then Theorem 1.1 (a)(i) holds. So suppose that H is not quasiprimitive, and let R be an intransitive minimal normal subgroup of H. Then $S = \{(y, y^\varphi) \mid y \in R\}$ is the corresponding minimal normal subgroup of G^+. Since R is intransitive on Δ, S has more than two orbits in Ω, and hence S is not normal in G. By Lemma 3.2, $S^g \cap S = 1$ and $N := S \times S^g$ is a minimal normal subgroup of G. For $(y, y^\varphi) \in S$, $(y, y^\varphi)^g = (y^\varphi, y^x) = (y^\varphi, (y^\varphi)^\varphi)$ and hence S^g is the minimal normal subgroup of G^+ corresponding to R^φ, whence $R^\varphi \neq R$. Moreover, N consists of all elements $(z, z^\varphi)(y, y^\varphi)^g = (zy^\varphi, (zy^\varphi)^\varphi)$, for $y, z \in R$. Thus N is the normal subgroup of G^+ corresponding to $M := R \times R^\varphi$, that is, $N = \mathrm{Diag}_\varphi(M \times M)$. Since N has two orbits in Ω it follows that M is transitive on Δ. If $\mathrm{soc}(G) = N$ then Theorem 1.1 (c)(i) holds.

Suppose then that $\mathrm{soc}(G) \neq N$ and let $\bar{N}$ be a minimal normal subgroup of G, $\bar{N} \neq N$. By Lemma 3.2, $\bar{N} \leq G^+$ and $\bar{N}$ has two orbits in Ω. Thus $\bar{M} := \pi_1(\bar{N})$ and M are two transitive normal subgroups of $H_i = H$ and $\bar{M} \cap M = 1$. By Lemma 3.3, $\bar{M}$ and M are isomorphic, non-abelian, and regular on Δ, $\mathrm{soc}(H) = M \times \bar{M} \leq H \leq \mathrm{Hol}(M)$, and since H is not primitive, $\bar{M}$ is not a minimal normal subgroup of H. By Lemma 3.2, $\mathrm{soc}(G) = N \times \bar{N}$, and also $\bar{N} = L \times L^g$ where L is a minimal normal subgroup of G^+. By the definition of $\bar{M}$, $\bar{N} = \mathrm{Diag}_\varphi(\bar{M} \times \bar{M}) \cong \bar{M}$ and so $\bar{N}$ is non-abelian. As in the previous paragraph, since $\bar{N} = L \times L^g$, we see that $\bar{M} = \bar{R} \times \bar{R}^\varphi$ for an intransitive minimal normal subgroup $\bar{R}$ of H, and since $\bar{M} \cong M$ we have $\bar{R} \cong R$ and $\bar{N} \cong N$. Thus Theorem 1.1 (c)(ii) holds. □

The proof of Theorem 1.1 follows from Lemmas 3.1, 3.5, 3.6 and the examples given in Section 2.

4. Quotients and covers of graphs

A graph $\Gamma = (\Omega, E)$ consists of a vertex set Ω and a subset E of unordered pairs from Ω that we call edges. As in Section 1, an arc is an ordered pair (α, β) such that $\{\alpha, \beta\} \in E$. More generally for $s \geq 1$, we define an s-arc in Γ as a vertex sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ such that each (α_i, α_{i+1}) is an arc and $\alpha_{i-1} \neq \alpha_{i+1}$. We say that Γ is (G, s)-arc transitive if G acts as a group of automorphisms of Γ and is transitive on s-arcs. If G is both vertex-transitive and s-arc transitive on a graph Γ, then G is also t-arc transitive for all $t \leq s$.

In [8], [9], quotient graphs of vertex-transitive graphs were studied in an attempt to find sufficient conditions under which a quotient inherits various properties such as s-arc transitivity. Quotient graphs are defined for each partition $\mathcal{P}$ of the vertex set of a graph Γ: the *quotient graph* $\Gamma_\mathcal{P} = (\mathcal{P}, E_\mathcal{P})$ where $\{A, B\} \in E_\mathcal{P}$ if and only if there exist at least one $\alpha \in A$ and at least one $\beta \in B$ such that $\{\alpha, \beta\}$ is an edge of Γ. Clearly if Γ is connected then also $\Gamma_\mathcal{P}$ is connected. If $G \leq \mathrm{Aut}(\Gamma)$, then G induces a subgroup $G^\mathcal{P}$ of $\Gamma_\mathcal{P}$ provided $\mathcal{P}$ is G-invariant, that is, for all $A \in \mathcal{P}$ and $g \in G$, the image $A^g = \{\alpha^g | \alpha \in A\}$ is also in $\mathcal{P}$. In this case, the action of $G^\mathcal{P}$ on $\Gamma_\mathcal{P}$ inherits from G any of the properties: vertex-transitivity, edge-transitivity, arc-transitivity. Thus if Γ is connected and G-arc-transitive, then by choosing a maximal G-invariant vertex-partition $\mathcal{P}$, we obtain a quotient $\Gamma_\mathcal{P}$ on which $G^\mathcal{P}$ is vertex-primitive as well as arc-transitive. Moreover $\Gamma_\mathcal{P}$ is a *k-multicover* of Γ for some k, that is, if (A, B) is an arc of $\Gamma_\mathcal{P}$ then each vertex of A is adjacent to exactly k vertices of B. The theory of finite primitive permutation groups can often be used effectively to study connected G-arc-transitive graphs Γ through their primitive quotients $G^\mathcal{P}$.

However the property of s-arc transitivity, for $s \geq 2$, is not in general inherited by a quotient $\Gamma_\mathcal{P}$ modulo a G-invariant partition $\mathcal{P}$, and in fact such quotients are often far from s-arc transitive. On the other hand, in [8, Section 1] it was found that s-arc transitivity is inherited by normal quotients and that each finite (G, s)-arc transitive graph Γ, where $s \geq 2$, is a *cover* (that is a 1-multicover) of a (G, s)-arc transitive quotient $\Gamma_\mathcal{P}$ such that G is also quasiprimitive or bi-quasiprimitive on the vertices of $\Gamma_\mathcal{P}$. Thus quasiprimitive and bi-quasiprimitive s-arc transitive graphs play a central role in elucidating the structure of finite s-arc transitive graphs. By a *normal quotient* of a G-arc transitive graph Γ we mean a quotient $\Gamma_{\mathcal{P}}$ such that $\mathcal{P}$ is the set of orbits on vertices of a normal subgroup of G. It was this discovery that inspired the development of the theory of finite quasiprimitive groups, and in particular the quasiprimitive O'Nan-Scott Theorem in [11].

It was shown in [11] that only a few of the possible types of quasiprimitive groups can act 2-arc transitively, and by Theorem 1.2, if Γ is not a complete bipartite graph, then a biquasiprimitive 2-arc transitive subgroup of automorphisms must be of type (a)(i) or (c) of Theorem 1.1. In [10, Theorem 2.3] it was shown that, for a bi-quasiprimitive group G of type (a)(i) acting 2-arc transitively, the quasiprimitive action of G^+ on each part of the bipartition can only be of one of the types possible for a quasiprimitive 2-arc transitive action. The possibility of bi-quasiprimitive groups G of type (c) acting 2-arc transitively is rather problematic. Some information about this case is provided in [9, Theorem 2.1B] and [10, Section 2]. One motivation for proving Theorem 1.1 was to gain a better understanding of this case, and indeed we are able to do so in Proposition 4.1.

A likely further application of this theory will be towards a proof of a conjecture made by Richard Weiss [15] in 1978 about finite locally primitive graphs and a similar conjecture (see [7]) for finite locally-quasiprimitive graphs. A graph Γ is said to be *G-locally-primitive*, or *G-locally-quasiprimitive*, if $G \leq \mathrm{Aut}(\Gamma)$ and, for each vertex α, the stabiliser G_α induces a primitive or quasiprimitive action respectively on $\Gamma(\alpha)$. Each vertex-transitive 2-arc transitive graph is locally-primitive, and each vertex-transitive locally-primitive graph is locally-quasiprimitive. The conjectures are that there exist functions f, f' such that, for a finite vertex-transitive locally-primitive or locally-quasiprimitive graph of valency v, the number of automorphisms fixing a given vertex is at most $f(v)$ or $f'(v)$ respectively. In [4] it was shown that the conjecture for locally-primitive graphs is true for non-bipartite graphs if and only if it holds for such graphs with an almost simple automorphism group. We believe that Theorem 1.1 and Proposition 4.1 will help in attacking these conjectures for bipartite graphs. The statement of Proposition 4.1 uses the notation of Theorem 1.1. The assertion in part (b) that Γ is not $(G, 2)$-arc transitive relies on the classification of finite almost simple 2-transitive groups and hence on the finite simple group classification.

PROPOSITION 4.1. *Let* $\Gamma = (\Omega, E)$ *be a finite connected bipartite graph and suppose that* $G \leq \mathrm{Aut}(\Gamma)$ *is locally-quasiprimitive and vertex bi-quasiprimitive and that Theorem* 1.1 (c) *holds. Let* $\alpha \in \Omega$ *and let* $\Gamma(\alpha)$ *denote the set of vertices adjacent to* α.

(a) *If* G *is of type* (c)(i), *then* R, R^φ *are semi-regular on* Δ *and* $\mathrm{soc}(G)_\alpha$ *is isomorphic to a (possibly trivial) diagonal subgroup of* $M = R \times R^\varphi$.

(b) *If G is of type* (c)(ii)*, then Γ is not $(G,2)$-arc transitive. Here $H \leq \mathrm{Hol}(M) = M \cdot \mathrm{Aut}(M)$ and up to isomorphism we may identify Ω with $M \times \{1,2\}$ such that, for $(y,i) \in \Omega$, $m \in M$, and $\sigma \in \mathrm{Aut}(M)$,*

$$(y,i)^m = \begin{cases} (ym,1) & \textit{if } i=1 \\ (ym^{\varphi},2) & \textit{if } i=2 \end{cases} \quad \textit{and} \quad (y,i)^{\sigma} = \begin{cases} (y^{\sigma},1) & \textit{if } i=1 \\ (y^{\sigma^{\varphi}},2) & \textit{if } i=2. \end{cases}$$

Moreover, for $\alpha = (1,2)$, $\mathrm{soc}(G)_\alpha = X \times Y$ where $X := \mathrm{Diag}_\varphi(R \times R) \cong R$ is transitive on $\Gamma(\alpha)$, $Y := \mathrm{Diag}_\varphi(R^\varphi \times R^\varphi) \cong R$ fixes $\Gamma(\alpha)$ pointwise, and for $\gamma = (y,1) \in \Gamma(\alpha)$, $X_\gamma = \mathrm{Diag}_\varphi(C_R(y) \times C_R(y))$.

Proof. Suppose that G is of type (c)(i), so $\mathrm{soc}(G) = \mathrm{Diag}_\varphi(M \times M)$ and $M = R \times R^\varphi$. If M is regular on Δ then the assertions of (a) hold, so we may assume that M is not regular. Then G has no non-trivial semiregular normal subgroups, and the assertions in (a) follow from [10, Theorem 2.1B].

Suppose that G is of type (c)(ii). Then $\mathrm{soc}(G) = N \times \bar{N}$, where N, $\bar{N}$ are isomorphic non-abelian minimal normal subgroups of G, and each is semiregular on the vertex set Ω with orbits Δ, Δ'. Moreover, using the notation of Theorem 1.1, $G^+ = \mathrm{Diag}_\varphi(H \times H)$, $N = \mathrm{Diag}_\varphi(M \times M) \cong M$, $H \leq \mathrm{Hol}(M) = M \cdot \mathrm{Aut}(M)$, and we may identify the vertex set Ω of Γ with $M \times \{1,2\}$ with the action specified in (b). For $\alpha = (1,2), \beta = (1,1) \in \Omega$, we have $\mathrm{soc}(G)_\alpha = \mathrm{soc}(G)_\beta = \mathrm{Diag}_\varphi(\mathrm{Inn}(M) \times \mathrm{Inn}(M)) \cong M$. Thus $\mathrm{soc}(G)_\alpha$ acts on $M \times \{1\}$ essentially with the natural action of $\mathrm{Inn}(M)$. Also $\mathrm{soc}(G)_\alpha$ is a direct product of two isomorphic minimal normal subgroups of G_α, namely $X := \mathrm{Diag}_\varphi(R \times R) \cong R$ and $Y := \mathrm{Diag}_\varphi(R^\varphi \times R^\varphi) \cong R$. Since Γ is connected it follows that $\mathrm{soc}(G)_\alpha$ acts non-trivially on the set $\Gamma(\alpha)$ of vertices adjacent to α, and since the group $G_\alpha^{\Gamma(\alpha)}$ induced on $\Gamma(\alpha)$ is quasiprimitive, it follows that each of its minimal normal subgroups in $(\mathrm{soc}(G)_\alpha)^{\Gamma(\alpha)}$ is transitive. Let $\gamma = (y,1) \in \Gamma(\alpha) \subseteq M \times \{1\}$ with $y = y_1 y_2 \in R \times R^\varphi$. Suppose that X acts nontrivially on $\Gamma(\alpha)$. Then X does not fix γ, so $y_1 \neq 1$ and for some $a \in R$ we have $y_1^a \neq y_1$ and $\gamma' := (y_1^a y_2, 1) \in \Gamma(\alpha)$. However no element of Y can map γ to γ' and hence Y cannot act transitively on $\Gamma(\alpha)$. Thus Y must fix $\Gamma(\alpha)$ pointwise, and this implies that $\Gamma(\alpha) \subseteq R \times \{1\}$. Moreover $X_\gamma = \mathrm{Diag}_\varphi(C \times C)$ where $C = C_R(y_1)$. Suppose now that Y is non-trivial on $\Gamma(\alpha)$. Then an analogous argument shows that $\Gamma(\alpha) \subseteq R^\phi \times \{1\}$, X fixes $\Gamma(\alpha)$ pointwise, and $Y_\gamma = \mathrm{Diag}_\varphi(C \times C)$ where $C = C_{R^\varphi}(y_2)$. Now the map $\bar{\varphi} : (y,i) \mapsto (y^\varphi, i)$, for $(y,i) \in \Omega$, is a bijection, fixes α, normalises G^+, conjugates $g = (x,1)(12)$ to $g^{\bar{\varphi}} = (x^\varphi,1)(12)$, and we have $gg^\varphi = (x, x^\varphi) \in G^+$. Thus $\bar{\varphi} \in N_{\mathrm{Sym}(\Omega)}(G)$ and $\bar{\varphi}$ induces an isomorphism from Γ to a graph for which X is non-trivial on the vertices adjacent to α.

Finally suppose that Γ is $(G,2)$-arc transitive. Then $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive and has socle isomorphic to R. It follows from a result of Burnside (see [16, Theorem 11.7]) that R is simple. However we have just shown that a point stabiliser in this action of R is isomorphic to $C_R(y_1)$ for some $y_1 \neq 1$, and there is no finite almost simple 2-transitive group with this property; see [3]. □

References

[1] R. Baer, *Classes of finite groups and their properties*, Illinois J. Math. **1** (1957), 115–187.

[2] J. Bamberg and C. E. Praeger, *Finite permutation groups with a transitive minimal normal subgroup*, Proc. London Math. Soc. (3), to appear.

[3] P. J. Cameron, *Finite permutation groups and finite simple groups*, Bull. London Math. Soc. **13** (1981), 1–22.

[4] M. D. Conder, C. H. Li, and C. E. Praeger, *On the Weiss conjecture for finite locally primitive graphs*, Proc. Edinburgh Math. Soc. (2) **43** (2000), 129–138.

[5] P. Förster, *Projektive Klassen endlicher Gruppen 1*, Math. Z. **186** (1984), 149–178.

[6] A. A. Ivanov, *Distance-transitive graphs and their classification*, Investigations in algebraic theory of combinatorial objects, Kluwer Acad. Publ., Dordrecht, 1994, pp. 283–378.

[7] C. H. Li, C. E. Praeger, A. Venkatesh, and S. Zhou, *Finite locally-quasiprimitive graphs*, Discrete Math. **246** (2002), 197–218.

[8] C. E. Praeger, *Imprimitive symmetric graphs*, Ars Combin. **19** (1985), 149–163.

[9] ———, *On automorphism groups of imprimitive symmetric graphs*, Ars Combin. **23** (1987), 207–224.

[10] ———, *On a reduction theorem for finite, bipartite* 2*-arc-transitive graphs*, Australas. J. Combin. **7** (1993), 21–36.

[11] ———, *An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to* 2*-arc transitive graphs*, J. London Math. Soc. (2) **47** (1993), 227–239.

[12] ———, *Permutation groups and normal subgroups*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 67–76.

[13] L. L. Scott, *Representations in characteristic p*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Amer. Math. Soc., Providence, R.I., 1980, pp. 319–331.

[14] J. van Bon and A. M. Cohen, *Prospective classification of distance-transitive graphs*, Combinatorics '88, Vol. 1 (Ravello, 1988), Mediterranean, Rende, 1991, pp. 25–38.

[15] R. Weiss, *s-transitive graphs*, Algebraic methods in graph theory, I,II (Szeged, 1978), Colloq. Math. Soc. János Bolyai, vol. 25, North Holland, Amsterdam, 1978, pp. 827–847.

[16] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964.

Department of Mathematics & Statistics, University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009, Australia

E-mail address: praeger@maths.uwa.edu.au