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FINITE TYPE RULED SURFACES IN LORENTZ-MINKOWSKI SPACE

Dong-Soo Kim, Young Ho Kim and Dae Won Yoon

Abstract. In this article, we study ruled surfaces in a Lorentz-Minkowski space, which has finite type immersion. We give a complete classification of ruled surfaces of finite type immersion into a Lorentz-Minkowski space with arbitrary codimension.

1. INTRODUCTION

In late 1970's B.-Y. Chen ([3, 4]) introduced the notion of finite type immersion into a Euclidean space. A lot of works have been done in this field of study since then. He also extended the notion of finite type immersion of submanifolds into a pseudo-Euclidean space in 1980's. It can be defined formally in the following: A pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbb{E}^m_s$ with signature $(s, m-s)$ is said to be of *finite type* if its position vector field x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \cdots, x_k$ non-constant maps such that $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, k$. If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are different, then M is said to be of k- *type*. Such a notion can be developed to a smooth map such as the Gauss map of finite type, that is, the Gauss map G of the submanifold of a Euclidean or a pseudo-Euclidean space is said to be of *finite type* if G can be expressed as a finite sum of the ambient manifold valued eigenfunctions of Δ ([1, 2, 6, 9, 10, 11, 12]).

Ruled surfaces in Euclidean space of finite type were studied by B.-Y. Chen et al. ([5]). On the other hand, F. Dillen et al. ([7]) classified ruled surfaces of finite type in 3-dimensional Lorentz-Minkowski space as an open portion of minimal, circular or hyperbolic cylinders and isoparametric surfaces with null rulings. Recently, the

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authors ([9, 10]) completely classified the family of ruled surfaces of a Lorentz-Minkowski space with finite type Gauss map. Therefore, we may raise a natural question: What kind of ruled surfaces have finite type immersion into a Lorentz-Minkowski space?

In this article, we study ruled surfaces in a Lorentz-Minkowski m-space $\mathbb{L}^m$, and we give the complete classification theorem of such ruled surfaces.

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless otherwise mentioned.

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(s, m-s)$ with the metric $ds^2 = -dx_1^2 - \cdots - dx_s^2 + dx_{s+1}^2 + \cdots + dx_m^2$, where $(x_1, x_2, \cdots, x_m)$ denotes the standard coordinate system in $\mathbb{E}^m_s$. In particular, for $m \geq 2$, $\mathbb{E}^m_1$ is called a *Lorentz-Minkowski m - space*. For simplicity, we denote $\mathbb{E}^m_1$ by $\mathbb{L}^m$ from now on.

Let $x : M \longrightarrow \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian submanifold M into $\mathbb{E}^m_s$. From now on, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, that is, the induced metric on the submanifold is non-degenerate.

For the components g_{ij} of the induced pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ on M from that of $\mathbb{E}^m_s$ we denote by (g^{ij}) (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). Then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x^j}\right). \tag{2.1}$$

Now, we define a ruled surface M in $\mathbb{L}^m$. Let I and J be open intervals containing 0 in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve of J into $\mathbb{L}^m$ and $\beta = \beta(s)$ a vector field along α with $\alpha'(s) \wedge \beta(s) \neq 0$ for every $s \in J$. Then, a ruled surface M is defined by the parametrization given as follows:

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I.$$

For such a ruled surface, α and β are called the *base curve* and the *director vector field*, respectively. The director vector field β is a curve along the base curve α. In particular, if β is constant, the ruled surface over a base curve α is said to be a *cylinder*. In this case, a base curve can be chosen as a curve in $(m-1)$-dimensional affine space $\mathbb{E}^{m-1}_s$ $(s = 0, 1)$ orthogonal to the constant vector β, where the index s is determined according to β. Thus, a cylinder means the right cylinder in $\mathbb{L}^m$. Or, else it is called *non-cylindrical*.

In case that the base curve α and the director vector field β are non-null, the base curve α can be chosen to be orthogonal to the director vector field β and β can be normalized satisfying $\langle\beta(s),\beta(s)\rangle = \varepsilon(=\pm1)$ for all $s \in J$, i.e., β can be regarded as a spherical curve lying in a pseudo-Riemannian sphere $S_1^{m-1} = \{p \in \mathbb{L}^m | \langle p,p\rangle = 1\}$ or a hyperbolic space $H^{m-1} = \{p \in \mathbb{L}^m | p_m > 0, \langle p,p\rangle = -1\}$, where $p = (p_1, p_2, \cdots, p_m)$. In this case, according to the character of vector fields α' and β, we have ruled surfaces of five different kinds as follows: If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. If the vector field β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or null, respectively. When the vector field β is time-like, β' is space-like because of the causal character. In this case, M is said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, the director vector field is always space-like. According as its derivative β' is non-null or null, it is also said to be of type M_-^1 or M_-^2, respectively.

A curve in $\mathbb{E}_s^m$ is called *null* if its tangent vector field is null along it. If the base curve α is a null curve and the director vector field β along α is a null vector field, then the ruled surface M is called a *null scroll*.

Remark. Let M be a ruled surface in $\mathbb{L}^m$ defined by a null base curve α and a non-null director vector field β. In this case, passing to a curve defined by $\tilde{\alpha} = \alpha(s) + f(s)\beta(s)$ as a base curve for a certain function f, M can be determined by a non-null base curve $\tilde{\alpha}$ and a non-null director vector field β, i.e., M is reduced to one of $M_\pm^1, M_\pm^2$ or M_+^3-type. A ruled surface M with a non-null base curve α and a null director vector field β is turned out to be a null scroll by taking a null base curve $\tilde{\alpha} = \alpha(s) + f(s)\beta(s)$ for a suitable function f.

The authors found a class of flat ruled surfaces of type M_+^2 with finite type Gauss map in [9, 11], which is the only class of non-cylindrical ruled surfaces with finite type Gauss map in $\mathbb{L}^m$. We call such flat ruled surfaces as ruled surfaces of type *FNC*-M_+^2.

If the base curve and the director vector field regarded as a curve are of finite type in $\mathbb{L}^m$, the ruled surfaces are said to be of *bi-finite type*. In particular, a null scroll with Cartan frame in $\mathbb{L}^3$ is said to be a *B-scroll* ([1, 8]). The authors ([9, 10]) defined an *extended* B*-scroll* and a *generalized* B*-scroll* in a Lorentz-Minkowski m-space $\mathbb{L}^m$.

A generalized B-scroll in $\mathbb{L}^m$ is defined as follows: Let M be a null scroll generated by a null curve $\alpha = \alpha(s)$ in $\mathbb{L}^m$ ($m \geq 4$) and $\beta = \beta(s)$ a null vector field along α, which is up to congruences parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s), \qquad s \in J, \quad t \in I$$

such that $\langle\alpha',\alpha'\rangle = 0$, $\langle\beta,\beta\rangle = 0$, where I and J are some open intervals. Without loss of generality, we may assume $\langle\alpha',\beta\rangle=-1$ by passing β to another null vector field $\tilde{\beta}=-\frac{1}{\langle\alpha',\beta\rangle}\beta$.

Let $\alpha=\alpha(s)$ be a null curve in $\mathbb{L}^m$ and let $A(s), B(s), C_1(s), \cdots, C_{m-2}(s)$ be a null frame along α satisfying

$$\langle A,A\rangle=\langle B,B\rangle=\langle A,C_i\rangle=\langle B,C_i\rangle=0,$$

$$\langle A,B\rangle=-1, \qquad \langle C_i,C_j\rangle=\delta_{ij}, \qquad \alpha'(s)=A(s)$$

for $1\le i,j\le m-2$. Let $X(s)$ be the matrix $(A(s) \quad B(s) \quad C_1(s) \quad \cdots \quad C_{m-2}(s))$ consisting of column vectors of $A(s), B(s), C_1(s), \cdots, C_{m-2}(s)$ with respect to the standard coordinate system in $\mathbb{L}^m$.

A system of ordinary differential equations

$$X'(s)=X(s)M(s)$$

with

$$M(s)=\begin{pmatrix} 0 & 0 & -a & 0 & 0 & \cdots & 0 \\ 0 & 0 & -k_1(s) & -k_2(s) & -k_3(s) & \cdots & -k_{m-2}(s) \\ -k_1(s) & -a & 0 & -w_2(s) & -w_3(s) & \cdots & -w_{m-2}(s) \\ -k_2(s) & 0 & w_2(s) & 0 & 0 & \cdots & 0 \\ -k_3(s) & 0 & w_3(s) & 0 & \cdots & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -k_{m-2}(s) & 0 & w_{m-2}(s) & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

has a unique solution with respect to a given initial condition $X(0)=(A(0) \quad B(0) \quad C_1(0)\cdots C_{m-2}(0))$. Making use of a solution vector field B, we can define a null scroll $x(s,t)=\alpha(s)+tB(s)$ that is called a *generalized B-scroll*. (For the details, see [10]).

2. Ruled Surfaces with Non-null Base Curves

In this section, we study finite type ruled surfaces in $\mathbb{L}^m$ with non-null base curve and non-null director vector field.

Theorem 3.1. *If a cylinder with space-like or time-like base curve in $\mathbb{L}^m$ is of finite type, then the base curve is of finite type.*

Proof. Let M be a cylinder in $\mathbb{L}^m$ with a space-like or a time-like smooth $(m-1)-$curve $\alpha=\alpha(s)$ and a space-like or time-like unit constant vector field

$\beta = \beta(s)$ along α orthogonal to α, where s is the arc length of α. Then, M is parametrized by

$$x = x(s,t) = \alpha(s) + t\beta, \qquad s \in J, \quad t \in I$$

such that $\langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1), \langle \alpha', \beta \rangle = 0, \langle \beta, \beta \rangle = \varepsilon_2 (= \pm 1)$.

In this case, the cylinder M is only of type M^1_+, M^3_+ or M^1_-.

We divide it by two cases.

Case 1. Let M be a cylinder of type M^1_+ or M^1_-, that is, $\varepsilon_2 = 1$. Then, by (2.1) the Laplacian Δ of M is given in terms of s and t by

$$\Delta = -\varepsilon_1 \frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2}.$$

We denote by Δ' the Laplacian of α, that is $\Delta' = -\varepsilon_1 \frac{\partial^2}{\partial s^2}$. In this case, by the straightforward computation, the surface M is of finite type if and only if each component of $\alpha(s)$ can be written as a finite sum of eigenfunctions of Δ, i.e.,

$$\alpha(s) = \Gamma_0 + \sum_{i=1}^{k} \Gamma_i(s,t) \tag{3.1}$$

where $\Delta\Gamma_i = \lambda_i \Gamma_i$. We may assume that all the λ_i are mutually different. If we apply $\prod_{i=2}^{k}(\Delta - \lambda_i)$ to (3.1), we obtain that Γ_1 does not depend on t. Similarly, we find that none of the Γ_i depends on t. Moreover, we see

$$\Delta'\Gamma_i(s) = -\varepsilon_1 \frac{\partial^2}{\partial s^2}\Gamma_i(s) = -\varepsilon_1 \frac{\partial^2}{\partial s^2}\Gamma_i(s) - \frac{\partial^2}{\partial t^2}\Gamma_i(s) = \Delta\Gamma_i(s) = \lambda_i \Gamma_i(s)$$

for all i. Hence, every component of α can be written as a finite sum of eigenfunctions of Δ'. This means that α is of finite type. Thus, M is of finite type if and only if α is of finite type.

Case 2. Let M be a cylinder of type M^3_+, that is, $\varepsilon_1 = 1, \varepsilon_2 = -1$. In this case the Laplacian Δ of M is given by

$$\Delta = -\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}.$$

Similarly to Case 1, we can obtain the same result.

Consequently, by combining the Case 1 and Case 2, the proof is completed. ■

Let M be a non-cylindrical ruled surface of one of three types M^1_+, M^3_+ or M^1_- according to the character of the base curve α and the director vector field β. In dealing with this case, we suppose that s is the arc-length of the director vector field β. Since it is viewed as a spherical curve, the director vector field β can be normalized with unit speed.

Thus, the ruled surface M can be parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s), \qquad s \in J, \quad t \in I$$

such that $\langle \alpha', \beta \rangle = 0, \langle \beta, \beta \rangle = \varepsilon_2 (= \pm 1)$ and $\langle \beta', \beta' \rangle = \varepsilon_3 (= \pm 1)$. Since M is non-degenerate, the tangent vector field $x_s = \frac{\partial x}{\partial s}$ cannot be null. For later use we define smooth functions q, u and v as follows:

$$q = \|x_s\|^2 = \varepsilon_4 \langle x_s, x_s \rangle, \quad u = \langle \alpha', \beta' \rangle, \quad v = \langle \alpha', \alpha' \rangle,$$

where $\varepsilon_4 = \text{sign}\langle x_s, x_s \rangle$. In turn, q can be given as

$$q = \varepsilon_4(\varepsilon_3 t^2 + 2ut + v). \tag{3.2}$$

It is easy to show that the Laplacian Δ of M can be expressed as by using (2.1)

$$\Delta = -\varepsilon_4 \left(\frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} \right) - \varepsilon_2 \left(\frac{\partial^2}{\partial t^2} + \frac{1}{2q}\frac{\partial q}{\partial t}\frac{\partial}{\partial t} \right).$$

Let P be a polynomial in t with functions in s as coefficients with degree $\deg(P) = d$. By a straightforward computation, for each positive integer m we have

$$\frac{\partial}{\partial s}\left(\frac{P(t)}{q^m}\right) = \frac{1}{q^{m+1}}\tilde{P}_1(t), \frac{\partial^2}{\partial s^2}\left(\frac{P(t)}{q^m}\right) = \frac{1}{q^{m+2}}\tilde{P}_2(t), \frac{\partial}{\partial t}\left(\frac{P(t)}{q^m}\right)$$
$$= \frac{1}{q^{m+1}}\tilde{P}_3(t), \frac{\partial^2}{\partial t^2}\left(\frac{P(t)}{q^m}\right) = \frac{1}{q^{m+2}}\tilde{P}_4(t),$$

where $\tilde{P}_i(t)$ $(i = 1,2,3,4)$ are some polynomials in t with functions in s as coefficients with $\deg \tilde{P}_1(t) \le d+2$, $\deg \tilde{P}_2(t) \le d+4$, $\deg \tilde{P}_3(t) \le d+1$, and $\deg \tilde{P}_4(t) \le d+2$.

Thus,

$$\Delta\left(\frac{P(t)}{q^m}\right) = \frac{1}{q^{m+3}}\{-\varepsilon_4 \tilde{P}_2(t) + \frac{1}{2}\varepsilon_4 \frac{\partial q}{\partial s}\tilde{P}_1(t) - \varepsilon_2 q \tilde{P}_4(t) - \frac{1}{2}\varepsilon_2 \frac{\partial q}{\partial t} q \tilde{P}_3(t)\} = \frac{1}{q^{m+3}}\tilde{P}(t),$$

where $\tilde{P}$ is a polynomial in t with functions in s as coefficients with $\deg \tilde{P}(t) \le d+4$.

Therefore, we have the following

Lemma 3.2. *If P is a polynomial in t with functions in s as coefficients and $deg(P) = d$, then $\Delta(\frac{P(t)}{q^m}) = \frac{1}{q^{m+3}}\tilde{P}(t)$ where $\tilde{P}(t)$ is a polynomial in t with functions in s as coefficients and $deg(\tilde{P}) \le d+4$.*

By a straightforward computation, we obtain

$$\Delta x = \frac{P_1(t)}{q^2}$$

where $P_1(t)$ is a polynomial in t with functions in s such that $\deg(P_1) = d$ $(0 \leq d \leq 3)$. Suppose that M is of k-type. Then, there exist numbers $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that

$$\Delta^{k+1} x + \lambda_1 \Delta^k x + \cdots + \lambda_k \Delta x = 0. \tag{3.3}$$

Let $r \geq 1$ be an integer. By applying Lemma 3.2, we have

$$\Delta^r x = \frac{P_r(t)}{q^{3r-1}},$$

where $P_r(t)$ is a vector whose components are polynomials in t with functions in s as coefficients and $\deg(P_r) \leq d + 4(r-1)$. Substituting this into (3.3) yields

$$\frac{P_{k+1}}{q^{3k+2}} + \lambda_1 \frac{P_k}{q^{3k-1}} + \cdots + \lambda_k \frac{P_1}{q^2} = 0.$$

Suppose $\lambda_k \neq 0$, that is, M is of non-null k-type. Multiplying q^{3k+2} to the last equation, we get

$$P_{k+1} + \lambda_1 q^3 P_k + \cdots + \lambda_k q^{3k} P_1 = 0.$$

The degree of $q^{3k} P_1$ is $6k + d$ and those of the rest terms are less than or equal to $6k + d - 2$. Thus, the sum in (3.3) can never be zero unless $\Delta x = 0$. Therefore, M is minimal.

In the case of $\lambda_k = 0$, that is, M is of null k-type. Then, $\lambda_{k-1} \neq 0$. By applying the same argument as above, we see that $\Delta^2 x = 0$, in other words, M is of bi-harmonic. Quite similarly to the case of bi-harmonic submanifolds of Euclidean space of finite type, it is easily proved that a bi-harmonic submanifold of pseudo-Euclidean space of finite type is minimal (cf. [3]). Hence, we have

Proposition 3.3. *A non-cylindrical ruled surface of the type M_+^1, M_+^3 or M_-^1 in $\mathbb{L}^m$ is of finite type if and only if it is minimal.*

We now consider a non-cylindrical ruled surface M of type M_+^2 or M_-^2 with finite type immersion into $\mathbb{L}^m$. The parametrization for M may be given by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \beta, \beta \rangle = 1, \quad \langle \alpha', \beta \rangle = 0, \quad \langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1)$ and β' is null. Similarly to proof of Proposition 3.3, we also define functions q and u by

$$q = ||x_s||^2 = \varepsilon_4 \langle x_s, x_s \rangle, \qquad u = \langle \alpha', \beta' \rangle, \qquad \varepsilon_4 = \text{sign}\langle x_s, x_s \rangle,$$

which give

$$q = \varepsilon_4(2ut + \varepsilon_1),$$

where the parameter t runs in such a way that $q > 0$. By the straightforward computation, we easily have the Laplacian Δ of M in the form of

$$\Delta = -\varepsilon_4(-\frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} + \frac{1}{q}\frac{\partial^2}{\partial s^2}) - (\frac{1}{2q}\frac{\partial q}{\partial t}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}). \tag{3.4}$$

Quite similarly to prove Lemma 3.2, we can obtain the following lemma.

Lemma 3.4. *If P is a polynomial in t with functions in s as coefficients and $deg(P) = d$, then $\Delta(\frac{P(t)}{q^m}) = \frac{1}{q^{m+3}}\tilde{P}(t)$ where $\tilde{P}(t)$ is a polynomial in t with functions in s as coefficients and $deg(\tilde{P}) \le d + 2$.*

A straightforward computation yields

$$\Delta x = \frac{1}{q^2}\left\{(u'\beta' - 2u\beta'')t^2 + (u'\alpha' - 2u\alpha'' - \varepsilon_1\beta'' - 2u^2\beta)t - \varepsilon_1\alpha'' - \varepsilon_1 u\beta\right\}. \tag{3.5}$$

Let $P_1(t) = (u'\beta' - 2u\beta'')t^2 + (u'\alpha' - 2u\alpha'' - \varepsilon_1\beta'' - 2u^2\beta)t - \varepsilon_1\alpha'' - \varepsilon_1 u\beta$. Then, $\deg(P_1) = d$ with $0 \le d \le 2$. By using Lemma 3.4, we have

$$\Delta^r x = \frac{P_r(t)}{q^{3r-1}}, \qquad (r \ge 1)$$

where $P_r(t)$ is a vector whose components are polynomial in t with functions in s as coefficients and $\deg(P_r) \le 2(r-1) + d$.

Suppose that M is not minimal. Then, there exists a non-empty open subset $U = \{p \in M | \Delta x \neq 0\}$. If $u(p) \neq 0$ at $p \in U$, then $u \neq 0$ at every point in a neighborhood V of p contained in U. Suppose there exists a non-empty open subset $W \subset V$ such that $u'\beta' - 2u\beta'' \neq 0$ or $u'\alpha' - 2u\alpha'' - \varepsilon_1\beta'' - 2u^2\beta \neq 0$ on W. Then, P_1 and q are polynomials in t with functions in s as coefficients such that $\deg(P_1) \ge 1$ and $\deg(q) = 1$ on W. Using the similar argument adapted to a non-cylindrical ruled surface of type $M^1_\pm$ or M^3_+ and Lemma 3.4, (3.3) cannot be achieved unless $\Delta x = 0$ on W, which is a contradiction. Therefore, W is empty and thus we have

$$u'\beta' - 2u\beta'' = 0, \tag{3.6}$$

$$u'\alpha' - 2u\alpha'' - \varepsilon_1\beta'' - 2u^2\beta = 0 \tag{3.7}$$

on V. From (3.6), $\beta'' = \frac{u'}{2u}\beta'$ on V. Taking the scalar product with α' in (3.7) and using $\beta'' = \frac{u'}{2u}\beta'$, we obtain $u' = 0$ and thus $\beta'' = 0$ on V. Putting these into

(3.7), we see that $\alpha'' + u\beta = 0$ that makes $P(t) = 0$ and hence $\Delta x = 0$ on V, a contradiction. Therefore, the subset V is empty and the function u is identically zero on U.

Next, on the subset $M - U$, we have from (3.5)

$$u'\beta' - 2u\beta'' = 0, \tag{3.8}$$

$$u'\alpha' - 2u\alpha'' - \varepsilon_1\beta'' - 2u^2\beta = 0, \tag{3.9}$$

$$\alpha'' + u\beta = 0 \tag{3.10}$$

on $M - U$. Substituting (3.10) into (3.9), we get $\beta'' = \varepsilon_1 u'\alpha'$. Together this with (3.8), $u'(\beta' - 2\varepsilon_1 u\alpha') = 0$ on $M - U$. Suppose that $S = \mathrm{int}(M - U)$ is not empty, where $\mathrm{int}(M - U)$ denotes the interior of $M - U$. If $u'(p) \neq 0$ at some point $p \in S$, $u' \neq 0$ at every point of an open neighborhood $S_1 \subset S$. Then, $\beta' = 2u\varepsilon_1\alpha'$ on S_1. Taking the scalar product with β' yields $u = 0$ on S_1, which is a contradiction. Thus, the function u is constant on each component of S. Since u is zero on U and continuous on M, u is identically zero on M. In this case, $\alpha'' = 0$ and $\beta'' = 0$ on $M - U$. If $\mathrm{int}(M - U) = \emptyset$, there is nothing to say that $u \equiv 0$ on M by continuity.

Since there is no time-like vector orthogonal to a null vector in a Lorentz-Minkowski space, the base curve α must be space-like and $\varepsilon_1 = 1$, i.e., M cannot be of M^2_--type and the only possibility is that M is of M^2_+-type. Therefore, we have $\alpha'' = \beta'' = 0$ on $M - U$ and $\Delta x = -\alpha'' - \beta'' t$ on U. Thus, we have

Proposition 3.5. *Let M be a non-cylindrical ruled surface of the type M^2_+ or M^2_- in $\mathbb{L}^m$ is of finite type. Then, it is either minimal or bi-finite of type M^2_+.*

Remark. In Proposition 3.5, if we regard the director vector field β as a curve in $\mathbb{L}^m$, it is a null curve. Therefore, the meaning of finiteness of curve β is formally defined.

Corollary 3.6. *The only finite type non-cylindrical ruled surfaces of M^2_--type are minimal.*

Putting together Proposition 3.3 and Proposition 3.5, we get

Theorem 3.7. *A non-cylindrical ruled surface M in a Lorentz-Minkowski space $\mathbb{L}^m$ is of finite type if and only if it is either minimal or bi-finite of type M^2_+.*

Example 3.8. Let α be a space-like curve of the form $\alpha = \alpha(s) = (\sin s, \sin s, 0, s)$ and β a vector field along α such that $\beta = \beta(s) = (\cos s, \cos s, 1, 0)$ in $\mathbb{L}^4$. Consider a ruled surface M parametrized by $x(s,t) = \alpha(s) + t\beta(s)$ on $s \in I$ and $t \in J$ for some open intervals I and J. Then, M is a non-cylindrical finite type ruled surface of type M^2_+.

4. Null Scrolls of Finite Type and Classification of Ruled Surfaces in $\mathbb{L}^m$

First of all, we consider a null scroll M in $\mathbb{L}^m$ parametrized by

$$x(s,t) = \alpha(s) + t\beta(s)$$

where α is a null curve in $\mathbb{L}^m$ and β a null vector field along α satisfying $\langle \alpha'(s), \alpha'(s)\rangle = \langle \beta(s), \beta(s)\rangle = 0$, $\langle \alpha'(s), \beta(s)\rangle = -1$ for some open intervals I and J. Furthermore, without loss of generality we may choose α as a null geodesic of M. We then have $\langle \alpha'(s), \beta'(s)\rangle = 0$ for all s. Therefore, we have the natural frame $\{x_s, x_t\}$ given by

$$x_s = \alpha' + t\beta', \quad x_t = \beta.$$

Using the induced Lorentz metric on M and formula (2.1), we have the Laplacian operator Δ on M given by

$$\Delta = 2\frac{\partial^2}{\partial s \partial t} + \frac{\partial q}{\partial t}\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2},$$

where $q = \langle x_s, x_s\rangle$. From which, using the Beltrami-equation, we have the mean curvature vector field H given by

$$H(s,t) = -\beta'(s) - tv(s)\beta(s), \tag{4.1}$$

where $v = \langle \beta', \beta'\rangle$. We need the following lemma for later use.

Lemma 4.1 ([9]). *Let $V(s)$ be a smooth $l(\geq 2)$-dimensional non-degenerate distribution of index 1 in a Lorentz-Minkowski $m(\geq 3)$-space $\mathbb{L}^m$ along a curve $\alpha = \alpha(s)$. Then, we can choose orthonormal vector fields $C_1(s), \cdots, C_{m-l}(s)$ along α which generate the orthogonal complement $V^{\perp}(s)$ satisfying $C_i'(s) \in V(s)$ for $1 \leq i \leq m-l$.*

Suppose that M is of finite type. Then, there exist constants λ_1, λ_2, $\cdots$, λ_k such that

$$\Delta^{k+1}x + \lambda_1\Delta^k x + \cdots + \lambda_{k-1}\Delta^2 x + \lambda_k \Delta x = 0$$

for some positive integer k. By a direct computation, the last equation holds if and only if

$$2^k(v^k\beta)' + 2^{k-1}\lambda_1(v^{k-1}\beta)' + \cdots + 2\lambda_{k-1}(v\beta)' + \lambda_k\beta' = 0, \tag{4.2}$$

$$2^k v^{k+1} + 2^{k-1}\lambda_1 v^k + \cdots + 2\lambda_{k-1}v^2 + \lambda_k v = 0, \tag{4.3}$$

where $v = \langle \beta', \beta'\rangle$.

Suppose v is identically zero. From the causal character of the vector field β, β' must be space-like. Thus (4.1) implies that M is minimal. We now consider an open subset $\mathcal{U} = \{p \in M | v(p) \neq 0\}$ of M. Suppose $\mathcal{U} \neq \emptyset$. We then have from (4.3)

$$\lambda_k + 2\lambda_{k-1}v + \cdots + 2^{k-1}\lambda_1 v^{k-1} + 2^k v^k = 0$$

on $\mathcal{U}$. v is a solution of an algebraic equation $\lambda_k + 2\lambda_{k-1}z + \cdots + 2^{k-1}\lambda_1 z^{k-1} + 2^k z^k = 0$ with real coefficients and thus v is a real constant. By the causal character of β, v is a positive constant on each component of $\mathcal{U}$. Since v vanishes on $M - \mathcal{U}$, by continuity, $\mathcal{U} = M$. In this case, we obtain $\Delta H = 2vH$. Therefore, we have the following:

Proposition 4.2. *Let M be a null scroll of finite type in a Lorentz-Minkowski m-space $\mathbb{L}^m$. Then, it is of either 1-type or null 2-type.*

Considering a result in [10], we obtain

Corollary 4.3. *Let M be a null scroll in a Lorentz-Minkowski m-space $\mathbb{L}^m$. Then, M is of finite type if and only if M has finite type Gauss map.*

Let v be a nonzero constant. By the causal character of β, v is a positive number a^2 ($a > 0$). Let $A(s) = \alpha'(s), \quad B(s) = \beta(s), \quad C_1(s) = -\frac{1}{a}\beta'(s)$. Let $V(s)$ be the vector space spanned by $A(s), B(s), C_1(s)$ along α. According to Lemma 4.1, we have orthonormal vector fields $C_2(s), \cdots, C_{m-2}(s)$ generating the orthogonal complement $V^{\perp}(s)$ satisfying $C_j'(s) \in V(s)$ for $j = 2, \cdots, m-2$. Let $k_j(s) = \langle C_j'(s), A(s)\rangle = -\langle A'(s), C_j(s)\rangle$ ($j = 1, \cdots, m-2$) and $w_j(s) = \langle C_1'(s), C_j(s)\rangle$ ($j = 2, \cdots, m-2$). Then, we have

$$C_1'(s) = -aA(s) - k_1(s)B(s) + \sum_{j=2}^{m-2} w_j(s)C_j(s),$$
$$A'(s) = -\sum_{j=1}^{m-2} k_j(s)C_j(s), \quad C_i'(s) = -k_i(s)B(s) - w_i(s)C_1(s)$$

for $i = 2, \cdots, m-2$. It defines the so-called generalized B-scroll given by $x(s,t) = \alpha(s) + tB(s)$. Therefore, we have

Theorem 4.4. *Let M be a null scroll of finite type in a Lorentz-Minkowski m-space $\mathbb{L}^m$. Then, M is minimal or an open part of a generalized B-scroll.*

Combining Theorem 3.1 and Theorem 3.7 in the previous section with Theorem 4.4, we obtain the following classification theorem for ruled surfaces in a Lorentz-Minkowski space $\mathbb{L}^m$.

Theorem 4.5 (Classification). *Let M be a ruled surface in a Lorentz-Minkowski m-space $\mathbb{L}^m$ of finite type. Then, M is an open portion of a cylinder over finite type base curve, a non-cylindrical minimal ruled surface, a bi-finite M_+^2-type ruled surface, a minimal null scroll or a generalized B-scroll.*

By considering this theorem and the last section of [9], we have

Corollary 4.6 ([7]). *A ruled surface M in $\mathbb{L}^3$ is of finite type if and only if M is one of the following:*

(1) *M is minimal,*

(2) *M is a part of a circular cylinder,*

(3) *M is a part of a hyperbolic cylinder,*

(4) *M is an isoparametric surface with null rulings.*

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Dong-Soo Kim
Department of Mathematics,
Chonnam National University,
Kwangju 500-757, Korea
E-mail: dosokim@chonnam.ac.kr

Young Ho Kim
Department of Mathematics,
Kyungpook National University,
Taegu 702-701, Korea
E-mail: yhkim@knu.ac.kr

Dae Won Yoon
Department of Mathematics,
College of Education and RINS,
Gyeongsang National University,
Chinju 660-701, Korea
E-mail: dwyoon@gsnu.ac.kr