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FINITE TYPE SUBMANIFOLDS IN PSEUDO-EUCLIDEAN SPACES AND APPLICATIONS

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§ 1. Introduction.

Let E_s^m be the m-dimensional pseudo-Euclidean space with (flat) pseudo-Riemannian metric of signature $(s, m-s)$. And let M be a compact space-like submanifold of E_s^m. By using the induced Riemannian structure on M, we can define two well-defined numbers p and q associated with the submanifold M in E_s^m. Here p is a positive integer and q is either $+\infty$ or an integer $\geqq p$. The pair $[p, q]$ is called the *order of the submanifold* M (cf. [1]). The submanifold M is said to be of *finite type* if q is finite. Otherwise, M is said to be of *infinite type*. The submanifold M is of finite type if and only if there is a non-trivial polynomial P such $P(\Delta)H=0$; where Δ is the Laplacian Δ on M and H the mean curvature vector of M in E_s^m.

In this paper, we will give some general results for finite type submanifolds in the pseudo-Euclidean space E_s^m. By applying these results, we will prove the following. (1) There exist no compact space-like hypersurfaces with constant mean curvature and constant scalar curvature in the anti-de Sitter space-time; (2) Every compact hypersurface with constant mean curvature and constant scalar curvature in a hyperbolic space is a small hypersphere; and (3) If M is a compact space-like hypersurface of the de Sitter space-time, then M has non-zero constant mean curvature and constant scalar curvature when and only when M is mass-symmetric and of 2-type in the Lorentz-Minkowski world.

For the general knowledge on Finite-Type Submanifolds in Euclidean spaces, see [1, 2]. And for the general knowledge on Relativity, see for instance [3, 4].

§ 2. Preliminaries.

Let E_s^m be the m-dimensional pseudo-Euclidean space with metric tensor given by

$$g_0=-\sum_{i=1}^{s} dx_i^2+\sum_{j=s+1}^{m} dx_j^2, \tag{2.1}$$

where $(x_1, \cdots, x_m)$ is a rectangular coordinate system of E_s^m. (E_s^m, g_0) is a flat pseudo-Riemannian manifold of signature $(s, m-s)$.

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Let c be a point in E_s^m and $r>0$. We put

$$S_s^m(c, r)=\{x\in E_s^{m+1} \mid \langle x-c, x-c\rangle=r^2\}, \tag{2.2}$$

$$H_s^m(c, r)=\{x\in E_{s+1}^{m+1} \mid \langle x-c, x-c\rangle=-r^2\}, \tag{2.3}$$

where $\langle\, ,\rangle$ denotes the indefinite inner product on the pseudo-Euclidean space. It is known that $S_s^m(c, r)$ and $H_s^m(c, r)$ are complete pseudo-Riemannian manifolds of signature $(s, m-s)$ and respective constant sectional curvatures r^{-2} and $-r^{-2}$. $S_s^m(c, r)$ is simply-connected for $s<m-1$; $S_{m-1}^m(c, r)$ is connected and has infinite cyclic fundamental group; and $S_m^m(c, r)$ has two simply-connected components. $S_s^m(c, r)$ and $H_s^m(c, r)$ are called the *pseudo-Riemannian sphere* and the *pseudo-hyperbolic space*, respectively (cf. [6, p. 67].) The point c is called the *center* of $S_s^m(c, r)$ and of $H_s^m(c, r)$. In the following, $S_s^m(0, 1)$ and $H_s^m(0, 1)$ are simply denoted by S_s^m and H_s^m, respectively. S_1^m is called the *de Sitter space-time* (=*de Sitter world*) and H_1^m the *anti-de Sitter space-time* (=*anti-de Sitter world*). Both S_1^m and H_1^m are pseudo-Riemannian manifold of signature $(1, m-1)$. The hyperbolic space H^m is defined by

$$H^m=\{x\in E_1^{m+1} \mid \langle x, x\rangle=-1 \text{ and } t>0\}, \tag{2.4}$$

where $t=x_1$ is the first coordinate in E_1^m. H^m is a complete, simply-connected Riemannian manifold of constant sectional curvature -1. E_1^m is called the *Lorentz-Minkowski space-time* (=*Lorentz-Minkowski world*.)

Let $\tilde{M}$ be a pseudo-Riemannian manifold with pseudo-Riemannian metric $\tilde{g}$. Denote by $\langle\, ,\rangle$ the associated non-degenerate inner product and by $\tilde{\nabla}$ the metric connection on $\tilde{M}$. A tangent vector X to $\tilde{M}$ is said to be *space-like* (respectively, *time-like* or *light-like*) if $\langle X, X\rangle>0$ or $X=0$ (respectively, if $\langle X, X\rangle<0$ or $\langle X, X\rangle=0$ and $X\neq 0$).

Let M be a submanifold of $\tilde{M}$. If the pseudo-Riemannian metric tensor $\tilde{g}$ of $\tilde{M}$ induces a pseudo-Riemannian metric (respectively, Riemannian metric) on M, then M is called a *psuedo-Riemannian* (respectively, *space-like*) *submanifold* of $\tilde{M}$.

If M is a pseudo-Riemannian (or space-like) submanifold of $\tilde{M}$, each tangent space is, $T_x(M)$ by definition, a nondegenerate subspace of $T_x(\tilde{M})$. Hence, we have the direct sum decomposition:

$$T_x(\tilde{M})=T_x(M)\oplus T_x^{\perp}(M), \tag{2.5}$$

where the normal space $T_x^{\perp}(M)$ is also nondegenerate.

Let ∇ denote the induced metric connection on M. Then, for any vector fields X, Y tangent to M, we have the following Gauss formula:

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X, Y), \tag{2.6}$$

where h is the second fundamental form of M in $\tilde{M}$.

Denote by R and $\tilde{R}$ the curvature tensors of M and $\tilde{M}$, respectively. The

Gauss equation is given by

$$\langle R(X, Y)Z, W\rangle=\langle\tilde{R}(X, Y)Z, W\rangle + \langle h(X, W), h(Y, Z)\rangle-\langle h(X, Z), h(Y, W)\rangle . \tag{2.7}$$

Denote by D the linear connection induced on the normal bundle $T^{\perp}(M)$. For each vector field ξ normal to M, the Weingarten formula is given by

$$\tilde{\nabla}_X\xi=-A_\xi X+D_X\xi , \tag{2.8}$$

where A_ξ is the Weingarten map with respect to ξ. A_ξ is a self-adjoint endomorphism of the tangent bundle $T(M)$ which can be diagonalized when M is space-like. It is well-known that h and A are related by

$$\langle h(X, Y), \xi\rangle=\langle A_\xi X, Y\rangle . \tag{2.9}$$

For the second fundamental form h, we define the covariant differentiation of h by

$$(\bar{\nabla}_X h)(Y, Z)=D_X h(Y, Z)-h(\nabla_X Y, Z)-h(Y, \nabla_X Z) . \tag{2.10}$$

The Codazzi equation is given by

$$(\tilde{R}(X, Y)Z)^{\perp}=(\bar{\nabla}_X h)(Y, Z)-(\bar{\nabla}_Y h)(X, Z) . \tag{2.11}$$

A normal vector field ξ is said to be *parallel* is $D_X\xi=0$ for any vector X tangent to M.

Let M be a pseudo-Riemannian submanifold with signature $(t, n-t)$ in M. Let $E_1, \cdots, E_n$ be an orthonormal local basis on M such that $E_1, \cdots, E_t$ are time-like and $E_{t+1}, \cdots, E_n$ are space-like. If F is an endomorphism of TM such that $F=(F_{ij})$ with respect to the basis $E_1, \cdots, E_n$, then the *trace* of F is defined by

$$\operatorname{tr} F=\sum_{i=1}^{n}\varepsilon_i F_{ii} ,$$

where $\varepsilon_1=\cdots=\varepsilon_t=-1$, $\varepsilon_{t+1}=\cdots=\varepsilon_n=1$ and $F_{ij}=\langle F(E_i), E_j\rangle$. From these, we find

$$\operatorname{tr} h=\sum_{i=1}^{n}\varepsilon_i h(E_i, E_i) . \tag{2.12}$$

The *mean curvature vector* H of M in $\tilde{M}$ is defined by $H=1/n \operatorname{tr} h$, where $n=\dim M$. A pseudo-Riemannian submanifold M of $\tilde{M}$ is said to be *minimal* if the mean curvature vector H vanishes identically. A pseudo-Riemannian submanifold M in $\tilde{M}$ is said to be *pseudo-umbilical* if $\langle H, H\rangle\neq 0$ and $A_H=\mu I$ for some function μ on M, where I is the identity transformation.

We need the following.

LEMMA 1. *Let M be a pseudo-Riemannian (in particular, space-like) submanifold of the pseudo-Riemannian sphere $S_s^m(c, r)$ (respectively, the pseudo-hyperbolic* *space $H_{s-1}^m(c, r)$) in E_s^{m+1}. Then the mean curvature vectors H and H' of M in E_s^{m+1} and in $S_s^m(c, r)$ (respectively, in $H_{s-1}^m(c, r)$) satisfy*

$$H=H'-\frac{1}{r^2}(x-c) \quad \left(\text{respectively, } H=H'+\frac{1}{r^2}(x-c)\right). \tag{2.13}$$

Proof. Let $\varepsilon=1$ or -1 according to M being a submanifold of $S_s^m(c, r)$ or of $H_{s-1}^m(c, r)$. Let x denote the position vector of M in E_s^{m+1}. Then we have

$$\langle x-c, x-c\rangle=\varepsilon r^2. \tag{2.14}$$

For each vector X tangent to $S_s^m(c, r)$ or to $H_{s-1}^m(c, r)$ we have $\tilde{\nabla}_X x=X$. Thus, $x-c$ is normal to $S_s^m(c, r)$ and to $H_{s-1}^m(c, r)$. Moreover, from $\tilde{\nabla}_X x=X$, we have

$$A_{x-c}=\tilde{A}_{x-c}=-I, \tag{2.15}$$

where A and $\tilde{A}$ denote the Weingarten maps of M and $S_s^m(c, r)$ (or $H_{s-1}^m(c, r)$) in E_s^{m+1}. Let h' and h be the second fundamental forms of M in $S_s^m(c, r)$ (or in $H_{s-1}^m(c, r)$) and of $S_s^m(c, r)$ (or $H_{s-1}^m(c, r)$) in E_s^{m+1}, respectively. Then we have

$$h(X, Y)=h'(X, Y)+\tilde{h}(X, Y). \tag{2.16}$$

Since $h(X, Y)=\varepsilon\langle A_{x-c}X, Y\rangle(x-c)/r^2$, (2.12), (2.15) and (2.16) give (2.13).

(Q. E. D.)

§ 3. Some General Results.

Throughout the remaining part of this paper, we assume that M is a connected, n-dimensional, pseudo-Riemannian submanifold of E_s^{m+1}.

First, we give the following.

LEMMA 2. *A submanifold M of the pseudo-Euclidean space E_s^{m+1} is a pseudo-umbilical submanifold with parallel mean curvature vector if and only if M is either a minimal submanifold of a pseudo-Riemannian sphere $S_s^m(c, r)$ or a minimal submanifold of a psedo-hyperbolic space $H_{s-1}^m(c, r)$ for some $c\in E_s^{m+1}$ and $r>0$.*

Proof. Assume that M is a pseudo-umbilical submanifold of E_s^{m+1} with parallel mean curvature vector. Then, $\langle H, H\rangle\neq 0$ and, by definition, for any X tangent to M, we have $X\langle H, H\rangle=2\langle\tilde{\nabla}_X H, H\rangle=0$. Thus, $\langle H, H\rangle$ is a non-zero constant. We put

$$\langle H, H\rangle=\frac{\varepsilon}{r^2}, \qquad \varepsilon=1 \text{ or } -1. \tag{3.1}$$

Let $A_H=\mu I$. Then from (2.9) and (2.12), we find

$$\varepsilon=\mu r^2. \tag{3.2}$$

We put

(3.3) $$y=x+\varepsilon r^2 H\,,$$

where x is the position vector of M in E_s^{m+1}. Then, for any vector X tangent to M, we have

(3.4) $$\begin{aligned}\tilde{\nabla}_X y &= \tilde{\nabla}_X x+\varepsilon r^2 \tilde{\nabla}_X H \\ &= X-\varepsilon r^2 A_H X=0\,.\end{aligned}$$

Thus y is a constant vector, say c, in E_s^{m+1}. Hence, we find $\langle x-c,\, x-c\rangle = r^4\langle H, H\rangle=\varepsilon r^2$. This shows that M lies either in the pseudo-Riemannian sphere $S_s^m(c, r)$ or in the pseudo-hyperbolic space $H_{s-1}^m(c, r)$. Since (3.3) gives

(3.5) $$H=-\frac{\varepsilon}{r^2}(x-c)\,,$$

Lemma 1 implies that $H'=0$, i.e., M is a minimal submanifold of $S_s^m(c, r)$ or of $H_{s-1}^m(c, r)$.

Conversely, if M is a minimal submanifold of $S_s^m(c, r)$ or of $H_{s-1}^m(c, r)$, then Lemma 1 shows that $H=-\varepsilon(x-c)/r^2$. This implies that

$$\tilde{\nabla}_X H=-\frac{\varepsilon}{r^2}\tilde{\nabla}_X x=-\frac{\varepsilon}{r^2}X\,.$$

Thus, $A_H=\varepsilon I/r^2$ and $DH=0$. Moreover, by applying (2.13) and (3.5) we also find $\langle H, H\rangle=\varepsilon/r^2\neq 0$. Consequently, M is a pseudo-umbilical submanifold with parallel mean curvature vector. (Q. E. D.)

Remark 1. If E_s^{m+1} is the Euclidean space E^{m+1}, Lemma 2 is due to Yano and Chen [7]. For pseudo-Euclidean case the "only if" part of Lemma 2 was given in [5].

LEMMA 3. *If M is a submanifold of S_s^m (or H_{s-1}^m) in E_s^{m+1}, then the mean curvature vectors H' and H in S_s^m (or in H_{s-1}^m) and in E_s^{m+1} satisfy*

(3.6) $$DH=D'H', \qquad A_H=A_{H'}+\varepsilon I\,.$$

Proof. Obvious from (2.8), (2.14) and Lemma 1.

In the following, by a *hyperplane section* N of S_s^m (or of H_{s-1}^m) we mean the intersection of S_s^m (or of H_{s-1}^m) and a hyperplane L of E_s^{m+1}.

By applying Lemma 2, we have the following.

PROPOSITION 1. *Let M be a submanifold of the pseudo-Riemannian sphere S_s^m (respectively, of the pseudo-hyperbolic space H_{s-1}^m.) If M is a pseudo-umbilical submanifold with parallel mean curvature vector, then M is a minimal submanifold of a hyperplane section of S_s^m (respectively, of H_{s-1}^m).*

Proof. Under the hypothesis, the mean curvature vector H' of M in S_s^m (or in H_{s-1}^m) satisfies

(3.7) $$\langle H', H'\rangle\neq 0\,, \qquad A'_{H'}=\mu I \quad\text{and}\quad D'H'=0\,,$$

where A' and D' denote the Weingarten map and the normal connection of M in S_s^m (or in H_{s-1}^m). Let A be the Weingarten map of M in E_s^{m+1}. Then, by (3.7) and Lemma 3, we know that M has parallel mean curvature vector H in E_s^{m+1}, too. Since H is parallel, $\langle H, H\rangle$ is constant.

If $\langle H, H\rangle=0$, then $H=H'-\varepsilon x$ is a light-like vector. Thus, we have $\langle H', H'\rangle=-\varepsilon$. Because $A'_{H'}=\mu I$, we find $\mu=\langle H', H'\rangle=-\varepsilon$. Thus, we get

$$A_{H'}=A'_{H'}=-\varepsilon I. \tag{3.8}$$

Applying (3.7), (3.8) and Lemma 3, we find

$$\tilde{\nabla}_X H=\tilde{\nabla}_X H'-\varepsilon\tilde{\nabla}_X x=0. \tag{3.9}$$

This shows that H is a constant vector in E_s^{m+1}. Let $H=-\varepsilon c$. Then we have

$$\langle x-c, x-c\rangle=-\varepsilon. \tag{3.10}$$

Since $\langle x, x\rangle=\varepsilon$, we obtain $\langle x, c\rangle=\varepsilon-1/2\langle c, c\rangle$. This shows that M lies in the hyperplane section N given by

$$N=\{x\in E_s^{m+1} \mid \langle x, x\rangle=\varepsilon \quad\text{and}\quad \langle x, c\rangle=\varepsilon-1/2\langle c, c\rangle\}$$

for some constant c. Since $H=-\varepsilon c$ is normal to the hyperplane $\{x\in E_s^{m+1} \mid \langle x, c\rangle =\varepsilon-1/2\langle c, c\rangle\}$, the mean curvature vector of M in N vanishes. Thus, M is a minimal submanifold of N.

If $\langle H, H\rangle\neq 0$, then M is pseudo-umbilical in E_s^{m+1} by Lemma 2. Thus, Lemma 2 implies that M is a minimal submanifold of a $S_s^m(c, r)$ or of a $H_{s-1}^m(c, r)$ for some $c\in E_s^{m+1}$ and $r>0$. Thus, we have

$$\langle x-c, x-c\rangle=\bar{\varepsilon}r^2, \tag{3.11}$$

where $\bar{\varepsilon}=1$ or -1 according to M is a minimal submanifold of $S_s^m(c, r)$ or of $H_{s-1}^m(c, r)$. Since $\langle x, x\rangle=\varepsilon$, (3.11) gives $2\langle x, c\rangle=\varepsilon-\bar{\varepsilon}r^2+\langle c, c\rangle$. Thus, M lies in the hyperplane section given by

$$N=\{x\in E_s^{m+1} \mid \langle x, x\rangle=\varepsilon \quad\text{and}\quad \langle x-c, x-c\rangle=\bar{\varepsilon}r^2\}.$$

Since M is minimal in $\{x\in E_s^{m+1} \mid \langle x-c, x-c\rangle=\bar{\varepsilon}r^2\}$, M is minimal in N, too.

(Q. E. D.)

Remark 2. If M is a minimal submanifold of a hyperplane section N of S_s^m (or of H_{s-1}^m), then either N is totally geodesic in S_s^m (or in H_{s-1}^m) or N is a pseudo-umbilical submanifold with parallel mean curvature vector.

Let M be a pseudo-Riemannian submanifold with orthonormal local basis $E_1, \cdots, E_n$. For any real function f on M, the Laplacian Δf of f is defined by

$$\Delta f=-\sum_{i=1}^{n}\varepsilon_i\{E_iE_if-\nabla_{E_i}E_if\}.$$

We mention the following lemma for later use.

LEMMA 4. *Let M be an n-dimensional submanifold of E_s^{m+1}. Then we have*

$$\Delta x = -nH. \tag{3.12}$$

Proof. Let a be any fixed vector in E_s^{m+1} and p a point in M. Let $E_1, \cdots, E_n$ be an orthonormal local basis about p such that $(\nabla_{E_i}E_j)(p)=0$; $i, j = 1, \cdots, n$. Then we have

$$\begin{aligned}(\Delta\langle x, a\rangle)_p &= -\sum_{i=1}^{n} \varepsilon_i (E_i)_p \langle E_i, a\rangle \\ &= -\sum \varepsilon_i \langle \tilde{\nabla}_{E_i} E_i, a\rangle(p) \\ &= -\sum \varepsilon_i \langle h(E_i, E_i), a\rangle_p \\ &= -n\langle H, a\rangle_p .\end{aligned}$$

Since both Δx and H are independent of the choice of the local basis, we have $\langle \Delta x, a\rangle = -\langle nH, a\rangle$. Because the inner product $\langle\ ,\ \rangle$ is nondegenerate, this implies equation (3.12). (Q. E. D.)

Combining Lemmas 1 and 4 we have the following.

PROPOSITION 2. *There exist no compact space-like minimal submanifolds in any pseudo-hyperbolic space H_s^m.*

Proof. If M is a compact space-like minimal submanifold of H_s^m, then Lemma 1 gives $H = x$. Thus, Lemma 4 implies $\Delta x = -nx$. This shows that $-n$ is an eigenvalue of M. Since M is a compact Riemannian manifold, eigenvalues of Δ on M are non-negative. (Q. E. D.)

Remark 3. In contrast to Proposition 2, there exist compact minimal space-like submanifolds in pseudo-Riemannian spheres.

LEMMA 5. *If M is compact space-like submanifold of E_s^{m+1}, then we have*

$$\int_M H\, dV = 0, \tag{3.13}$$

$$\int_M \langle H, x\rangle dV + \int_M dV = 0. \tag{3.14}$$

Proof. Since M is compact, Lemma 4 and Hopf's Lemma implies (3.13).

By using (3.12) we have $\Delta\langle x, x\rangle = -2n(1 + \langle H, x\rangle)$. Thus, we also have (3.14). (Q. E. D.)

In the following, a compact submanifold M of $S_s^m(c, r)$ (or of $H_{s-1}^m(c, r)$) in E_s^{m+1} is called *mass-symmetric* if the center of mass of M in E_s^{m+1} is just the center c of $S_s^m(c, r)$ (or of $H_{s-1}^m(c, r)$). Lemmas 1 and 4 imply the following.

LEMMA 6. *If M is a compact, space-like, minimal submanifold of the pseudo-Riemannian sphere S_s^m, then M is mass-symmetric in E_s^{m+1}.*

Proof. Under the hypothesis, we have $\Delta x=-nH=nx$. Thus $\int_M x\,dV=0$.
(Q. E. D.)

Remark 4. From the proof of Lemma 6, we see that if M is an n-dimensional, compact, space-like, minimal submanifold of S_s^m, then the first non-zero eigenvalue λ_1 of Δ on M satisfies $\lambda_1 \leqq n$.

§4. Finite-type submanifolds in E_s^{m+1}.

Let M be a compact, space-like submanifold of E_s^{m+1}. Then M with the induced metric is a Riemannian manifold. Thus, the Laplacian Δ of M is an elliptic differential operator and it has infinite sequence of eigenvalues:

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty\,.$$

Let $V_k=\{f\in C^\infty(M)\mid \Delta f=\lambda_k f\}$ be the eigenspace of Δ with eigenvalue λ_k. Then each V_k is finite-dimensional. If we define an inner product on $C^\infty(M)$ by $(f, g)=\int_M fg\,dV$, then the decomposition $\sum_{k=0}^{\infty} V_k$ is orthogonal and dense in $C^\infty(M)$ (in L^2-sense). V_0 is 1-dimensional and it consists of constant functions.

For each $f\in C^\infty(M)$, let f_t be the projection of f onto V_t. Then we have the following decomposition:

$$f=\sum_{t=0}^{\infty} f_t \qquad (\text{in } L^2\text{-sense})\,. \tag{4.2}$$

If f is a non-constant function on M, there is a positive integer $p\geqq 1$ such that $f_p\neq 0$ and $f=f_0+\sum_{t\geqq p} f_t$. If there are infinite f_t's which are nonzero, we put $q=\infty$. Otherwise, there is an integer $q\geqq p$ such that $f_q\neq 0$ and $f=f_0+\sum_{t=p}^{q} f_t$. Thus, in both cases, we have

$$f-f_0=\sum_{t=p}^{q} f_t\,, \tag{4.3}$$

where q is either ∞ or an integer $\geqq p$.

For the compact, space-like submanifold M in E_s^{m+1}, we put

$$x=(x_1, \cdots, x_{m+1})\,, \tag{4.4}$$

where x_A is the A-th rectangular coordinate function of M in E_s^{m+1}. For each x_A, we have

$$x_A-(x_A)_0=\sum_{t=p_A}^{q_A} (x_A)_t\,, \qquad A=1, \cdots, m+1\,. \tag{4.5}$$

We put

$$p=\inf\{p_A\}\,, \qquad q=\sup\{q_A\}\,, \tag{4.6}$$

where A ranges among all A such that $x_A\neq(x_A)_0$. It is clear that p is an integer $\geqq 1$ and q is either ∞ or an integer $\geqq p$. The pair of these invariants $[p, q]$ is called the *order of* M in E_s^{m+1} (cf. [1]). The submanifold M in E_s^{m+1} is said to be *of finite type* if q is finite. Otherwise, M is of infinite type. The submanifold M in E_s^{m+1} is said to be *of k-type* if there exist exactly k nonzero x_t's in the following decomposition:

$$x = x_0 + \sum_{t=p}^{q} x_t, \qquad \Delta x_t = \lambda_t x_t. \tag{4.7}$$

We need the following.

PROPOSITION 3. *Let M be a compact, space-like submanifold of E_s^{m+1}. Then M is of finite type if and only if there exists a non-trivial polynomial P such that $P(\Delta)H=0$ (or equivalently, $P(\Delta)(x-x_0)=0$).*

Proof. Consider the decomposition (4.7). If M is of finite type, q is finite. Thus, we have from Lemma 4,

$$-n\Delta^i H = \sum_{t=p}^{q} \lambda_t^{i+1} x_t, \qquad i=0, 1, 2, \cdots. \tag{4.8}$$

Let $c_1 = -\sum_{t=p}^{q} \lambda_t$, $c_2 = \sum_{t<s} \lambda_t \lambda_s$, $\cdots$, $c_{q-p+1} = (-1)^{q-p+1} \lambda_p \cdots \lambda_q$. Then we find

$$\Delta^k H + c_1 \Delta^{k-1} H + \cdots + c_k H = 0, \qquad k = q-p+1. \tag{4.9}$$

Conversely, if H satisfies (4.9) for some $k \geqq 0$, then $k \geqq 1$ by Lemma 4. Thus, by (4.7) and (4.9), we find

$$\sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_k) x_t = 0. \tag{4.10}$$

Let

$$x_t = ((x_t)_1, \cdots, (x_t)_{m+1}).$$

Then (4.10) gives

$$\sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_k)(x_t)_A = 0, \qquad A = 1, \cdots, m+1. \tag{4.11}$$

If $x_s \neq 0$, then

$$(x_s)_A \neq 0$$

for some A. Thus, we find

$$0 = \sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_k)((x_t)_A, (x_s)_A) = 0, \tag{4.12}$$

where $(f, g) = \int_M fg \, dV$. Since

$$\lambda_t((x_t)_A, (x_s)_A) = (\Delta(x_t)_A, (x_s)_A)$$
$$= ((x_t)_A, \Delta(x_s)_B) = \lambda_s((x_t)_A, (x_s)_A),$$

we find

$$((x_t)_A, (x_s)_A)=0$$

if $t\neq s$. Therefore, (4.12) implies

$$\lambda_t^k+c_1\lambda_t^{k-1}+\cdots+c_k=0 \qquad \text{whenever} \quad x_t\neq 0. \tag{4.13}$$

Since equation (4.13) has at most k real solutions, at most k of the x_t's in (4.7) are nonzero. Thus, M is of finite-type. Similar argument applies to $P(\Delta)(x-x_0)=0$. (Q. E. D)

From the proof of Proposition 3, we also have the following.

PROPOSITION 4. *Let M be a compact, space-like submanifold of E_s^{m+1}. If M is of k-type, then there exists a polynomial P of degree k such that $P(\Delta)H=0$ (or $P(\Delta)(x-x_0)=0$.) Conversely, if $P(\Delta)H=0$ (or $P(\Delta)(x-x_0)=0$) for a polynomial P of degree $d\geqq 1$, then M is of k-type for some k; $d\geqq k\geqq 1$.*

Now, we give the following Lemmas for later use.

LEMMA 7. *Let M be a compact, space-like submanifold of E_s^{m+1}. If M is not of 1-type and if H satisfies $\Delta^2H+b\Delta H+cH=0$ for some constant b and c, then $b=-(\lambda_p+\lambda_q)<0$ and $c=\lambda_p\lambda_q>0$ where $[p, q]$ is the order of M in E_s^{m+1}.*

Proof. Under the hypothesis, Proposition 4 shows that M is of 2-type. Thus, we have $x=x_0+x_p+x_q$. Therefore, we find

$$(\lambda_p^3+b\lambda_p^2+c\lambda_p)x_p+(\lambda_q^3+b\lambda_q^2+c\lambda_q)x_q=0.$$

Thus, we have $\lambda_p^2+b\lambda_p+c=0$, $\lambda_q^2+b\lambda_q+c=0$. Since λ_p and λ_q are positive real roots of $t^2+bt+c=0$, we have $b=-(\lambda_p+\lambda_q)<0$ and $c=-\lambda_p\lambda_q>0$. (Q. E. D)

LEMMA 8. *Let M be a compact, space-like submanifold of E_s^{m+1}. Then x_0 is the center of mass in E_s^{m+1}.*

Proof. Since $\int_M x_t\, dV=1/\lambda_t\int_M \Delta x_t\, dV=0$, (4.9) gives to $x_0=\int_M x\, dV/\int_M dV$. (Q. E. D)

LEMMA 9. *Let M be a space-like submanifold of E_s^{m+1}. Then we have*

$$\Delta H=\Delta^D H+\sum_{i=1}^{n}\{(\nabla_{E_i}A_H)E_i+A_{D_{E_i}H}E_i+h(E_i, A_HE_i)\}, \tag{4.15}$$

where Δ^D is the Laplacian associated with D and $E_1, \cdots, E_n$ an orthonormal local basis of M.

Proof. Let a be any vector in E_s^{m+1} and X, Y any vector fields tangent to M. We have

$$YX\langle H, a\rangle=\langle D_YD_XH, a\rangle-\langle\nabla_Y(A_HX), a\rangle -\langle A_{D_XH}Y, a\rangle-\langle h(Y, A_HX), a\rangle. \tag{4.16}$$

Since $\langle\ ,\ \rangle$ is non-degenerate, this gives (4.15) where

$$\Delta^D H = -\sum_{i=1}^{n} \{D_{E_i} D_{E_i} H - D_{\nabla_{E_i} E_i} H\}. \qquad \text{(Q.E.D)}$$

§5. 1-type Submanifolds in E_s^{m+1}.

In this section, we give the following *characterization theorem* for 1-type submanifolds.

THEOREM 1. *Let M be a compact, space-like submanifold of E_s^{m+1}. Then M is of 1-type in E_s^{m+1} if and only if M is a minimal submanifold in a pseudo-Riemannian sphere $S_s^m(c, r)$ for some $c \in E_s^{m+1}$ and $r>0$.*

Proof. If M is of 1-type in E_s^{m+1}, we have $\Delta x = \lambda_p(x-x_0)$. Thus, by Lemma 4, we find

$$nH = \lambda_p(x_0 - x). \tag{5.1}$$

This shows that $n\tilde{\nabla}_X H = -\lambda_p X$. Thus, we have

$$A_H = \frac{\lambda_p}{n} I \quad \text{and} \quad DH = 0. \tag{5.2}$$

Moreover, by (5.1), we also have

$$X\langle x-x_0, x-x_0\rangle = 2\langle X, x-x_0\rangle = 0$$

for any X tangent to M. Thus, $\langle x-x_0, x-x_0\rangle$ is constant.

If $\langle x-x_0, x-x_0\rangle = 0$, then M lies in the light cone $C=\{x \in E_s^{m+1} \mid \langle x-x_0, x-x_0\rangle = 0\}$ with vertex at x_0. Since M lies in C and x_0 is the center of mass of M in E_s^{m+1}, M must lies in both parts of the light cone C. This is impossible since M is assumed to be a space-like submanifold. Thus, $\langle H, H\rangle \neq 0$. Consequently, M is pseudo-umbilical in E_s^{m+1} with parallel mean curvature vector. Thus, by Lemma 2, M is either a minimal submanifold of a pseudo-Riemannian sphere or a minimal submanifold of a pseudo-hyperbolic space. Since M is compact, Proposition 2 shows that the second case cannot occurs.

Conversely, if M is a minimal submanifold of a pseudo-Riemannian sphere, then Lemmas 1 and 4 imply that $\Delta H + \lambda H = 0$ for some constant λ. Thus, by Proposition 4, M is of 1-type in E_s^{m+1}. (Q.E.D.)

In the following, a compact hypersurface N of the pseudo-hyperbolic space H_{s-1}^m is called a *small hypersphere* of H_{s-1}^m if N is the intersection of H_{s-1}^m with a pseudo-Riemannian sphere $S_s^m(c, r)$ in E_s^{m+1}. A small hypersphere of H_{s-1}^m is a totally umbilical submanifold in both H_{s-1}^m and E_s^{m+1}. Moreover, a small hypersphere of H_{s-1}^m is a totally umbilical hypersurface of a linear hyperplane L in E_s^{m+1} with c as its hyperplane normal.

By applying Theorem 1, we may obtain the following.

COROLLARY 1. *Let M be a compact, space-like submanifold of a pseudo-*

hyperbolic space $H_{s-1}{}^{m}$ *in* $E_s{}^{m+1}$. *Then* M *is of* 1-*type in* $E_s{}^{m+1}$ *if and only if* M *is a minimal submanifold of a small hypersphere of* $H_{s-1}{}^{m}$.

In particular, if $E_s{}^{m-1}$ is the Lorentz-Minkowski space-time, Corollary 1 reduces to the following.

COROLLARY 2. *A compact submanifold* M *of the hyperbolic space* $H^m \subset E_1{}^{m+1}$ *is of* 1-*type in the Lorentz-Minkowski world if and only if* M *is a minimal submanifold of a small hypersphere of* H^m.

§ 6. Hypersurfaces in Anti-de Sitter World.

It is known that there exist abundant examples of compact hypersurfaces with constant mean curvature and constant scalar curvature in a Riemannian sphere S^m. In this section, by applying the theory of finite-type submanifolds, we prove the following non-existence theorem in the anti-de Sitter world.

THEOREM 2. *There exist no compact space-like hypersurfaces with constant mean curvature and constant scalar curvature in the anti-de Sitter space-time* $H_1{}^{n+1}$.

Proof. We regard the anti-de Sitter space-time $H_1{}^{n+1}$ as a hypersurface of $E_2{}^{n+2}$ defined by

$$H_1{}^{n+1}=\{x\in E_2{}^{n+2} \mid \langle x, x\rangle=-1\} \tag{6.1}$$

Let M be a compact space-like hypersurface of $H_1{}^{n+1}$. Then the position vector x of M in $E_2{}^{n+2}$ is a time-like unit normal vector. From Lemma 1, we have

$$H=H'+x\,, \tag{6.2}$$

where H' is the mean curvature vector of M in $H_1{}^{n+1}$. Since M is space-like, H' is either zero or time-like. We put

$$\langle H', H'\rangle=-\alpha^2. \tag{6.3}$$

Then $H'=\alpha\xi$ for a unit time-like vector field ξ normal to M and tangent to $H_1{}^{n+1}$. Moreover, we have $\operatorname{tr} A_\xi=-n\alpha$. By applying (2.9), we find

$$\begin{aligned}\sum_i \langle h(E_i, A_H E_i), x\rangle&=\sum_i \langle A_H E_i, A_x E_i\rangle=-\sum_i \langle A_H E_i, E_i\rangle\\&=-n\langle H, H\rangle=n+n\alpha^2.\end{aligned} \tag{6.4}$$

Similarly, applying (2.9) and (5.2) and Lemma 2, we also have

$$\begin{aligned}\sum_i \langle h(E_i, A_H E_i), \xi\rangle&=\alpha\|A_\xi\|^2-\operatorname{tr} A_\xi\\&=\alpha\|A_\xi\|^2+n\alpha\,.\end{aligned} \tag{6.5}$$

Consequently, (6.4) and (6.5) give

(6.6)
$$\sum_i h(E_i, A_H E_i) = -(\|A_\xi\|^2 + n)H' - (n + n\alpha^2)x .$$

Since A_ξ is self-adjoint and M is space-like, we may diagonalize A_ξ. Let $E_1, \cdots, E_n$ be an orthonormal local basis of M such that

(6.7)
$$A_\xi E_i = \mu_i E_i , \qquad i = 1, \cdots, n .$$

If we put

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \omega_j^k(E_i) E_k ,$$

then (6.7) implies

(6.8)
$$(\nabla_{E_i} A_H) E_j = (E_i \alpha)\mu_j E_j + \alpha(E_i \mu_j) E_j + \sum_k \alpha(\mu_j - \mu_k)\omega_j^k(E_i) E_k .$$

Therefore, by the Codazzi equation, we obtain

(6.9)
$$\alpha\{(E_i\mu_j)E_j - (E_j\mu_i)E_i + \sum_k (\mu_j - \mu_k)\omega_j^k(E_i)E_k - \sum_k (\mu_i - \mu_k)\omega_i^k(E_j)E_k\} = 0 ,$$

where we have used the fact $D\xi = 0$. If $j \neq i$, (6.9) gives

(6.10)
$$\alpha(E_i \mu_j) = \alpha(\mu_i - \mu_j)\omega_i^j(E_j) , \qquad j \neq i .$$

Since $\omega_i^j = -\omega_j^i$, (6.8) and (6.10) imply

(6.11)
$$\sum_i (\nabla_{E_i} A_H) E_i = \sum_i [(E_i \alpha)\mu_i E_i + \alpha(E_i \mu_i) E_i] + \alpha \sum_{j \neq i} (E_j \mu_i) E_j .$$

Since $\sum_i \mu_i = \operatorname{tr} A_\xi = -n\alpha$, (5.11) gives

(6.12)
$$\sum_i (\nabla_{E_i} A_H) E_i = \sum_i \Big[A_{D'_{E_i} H'} E_i - \frac{n}{2}(E_i \alpha^2) E_i \Big] .$$

Consequently, by applying Lemma 2, Lemma 8, (6.6), (6.12), we find

(6.13)
$$\Delta H = \Delta^{D'} H' + 2 \operatorname{tr} A_{D' H'} + \frac{n}{2} \operatorname{grad}\langle H', H' \rangle - (\|A_\xi\|^2 + n)H' - (n + n\alpha^2)x .$$

In particular, if M has constant mean curvature α in H_1^{n+1}, then $D'H' = 0$. Thus (6.13) reduces to

(6.14)
$$\Delta H = -(\|A_\xi\|^2 + n)H' - (n + n\alpha^2)x .$$

On the other hand, since the second fundamental form h of M in E_2^{n+2} is given by

(6.15)
$$h(X, Y) = -\langle A_\xi X, Y \rangle \xi + \langle X, Y \rangle x ,$$

equation (2.7) of Gauss gives

$$\tau=\|A_\xi\|^2-n^2\alpha^2-n(n-1)\,, \tag{6.16}$$

where τ is the scalar curvature of M. Combining (6.14) with Lemma 1, we find

$$\Delta H+(\|A_\xi\|^2+n)H+(n\alpha^2-\|A_\xi\|^2)x=0\,. \tag{6.17}$$

If the scalar curvature τ of M is also constant, then (6.16) shows that

$$b=\|A_\xi\|^2+n \quad \text{and} \quad c=n\alpha^2-\|A_\xi\|^2 \tag{6.18}$$

are constant. Therefore, Lemmas 4 and 7 imply that M is of 1-type. Therefore, $x=x_0+x_p$ for some integer $p\geqq 1$. Combining this with Lemma 4, we obtain

$$nH=\lambda_p(x_0-x)\,. \tag{6.19}$$

On the other hand, (6.17) and Lemma 5 give

$$(n\alpha^2-\|A_\xi\|^2)\int_M x\, dV=0\,. \tag{6.20}$$

If $n\alpha^2=\|A_\xi\|^2$, then $\Delta H=-(n+\|A_\xi\|^2)H$ which is impossible. Therefore, we find $x_0=0$. Thus, (6.19) gives $nH=-\lambda_p x$. Since x and ξ are orthonormal, Lemma 1 implies that this case is also impossible. (Q.E.D)

A space-like hypersurface in H_1^{n+1} (or in H^{n+1}) is said to be isoparametric if the Weingarten map A_ξ has constant eigenvalues. Since a space-like isoparametric hypersurface in H_1^{n+1} (or in H^{n+1}) has constant mean curvature and constant scalar curvature, Theorem 2 implies immediately the following.

COROLLARY 3. *There exist no compact space-like isoparametric hypersurfaces in the anti-de Sitter space-time.*

§7. Hypersurfaces in Hyperbolic Space.

In this section, we apply our previous results to give the following classification theorem in hyperbolic space.

THEOREM 3. *The only compact hypersurfaces with constant mean curvature and constant scalar curvature in the hyperbolic space H^{n+1} are small hyperspheres of H^{n+1}.*

Proof. We regard the hyperbolic space H^{n+1} as a space-like hypersurface of the Lorentz-Minkowski world E_1^{n+2} defined by

$$H^{n+1}=\{x\in E_1^{n+2} \mid \langle x, x\rangle=-1 \quad \text{and} \quad t>0\}\,. \tag{7.1}$$

Let M be a compact hypersurface of H^{n+1} with mean curvature vector H'. Let ξ be a unit vector field in the direction of H'. We put $H'=\alpha\xi$. Then, by an argument similar to that given in section 6, we may obtain

(7.2) $$\Delta H=\Delta^{D'}H'+2\,\mathrm{tr}\,A_{D'H'}-\frac{n}{2}\mathrm{grad}\langle H',\, H'\rangle +(\|A_\xi\|^2-n)H'+(n\alpha^2-n)x\,.$$

If M has constant mean curvature, (7.2) reduces to

(7.3) $$\Delta H=(\|A_\xi\|^2-n)H'+(n\alpha^2-n)x\,.$$

On the other hand, we have

(7.4) $$H=H'+x\,,$$

(7.5) $$\tau=n^2\alpha^2-\|A_\xi\|^2-n(n-1)\,.$$

Thus, we obtain

(7.6) $$\Delta H+bH+cx=0\,,$$

where $b=n-\|A_\xi\|^2$ and $c=\|A_\xi\|^2-n\alpha^2$. If M has constant scalar curvature τ, too, then (7.5) shows that b and c are constants. Thus, by applying Lemma 5, we obtain

(7.7) $$cx_0\int_M dV=c\int_M x\, dV=0\,.$$

Since M lies in H^{n+1}, $t>0$. Therefore the center of mass cannot be the origin of E_1^{n+2}. Thus, $c=\|A_\xi\|^2-n\alpha^2=0$. This implies that M is totally umbilical in H^{n+1}. Therefore, M is a small hypersphere of H^{n+1} (cf. [1, p. 129].) (Q.E.D.)

Remark 4. It is well-known that a small hypersphere of H^{n+1} has constant mean curvature and constant scalar curvature.

COROLLARY 4. *The only compact isoparametric hypersurfaces of the hyperbolic space H^{n+1} are small hyperspheres.*

§ 8. Hypersurfaces in de Sitter World.

In this section, we study 2-type hypersurfaces in the de Sitter world.

THEOREM 4. *Let M be a compact space-like hypersurface in the de Sitter space-time $S_1^{n+1}\subset E_1^{n+2}$. Then M has nonzero constant mean curvature and constant scalar curvature in S_1^{n+1} if and only if M is mass-symmetric and of 2-type in the Lorentz-Minkowski world E_1^{n+2}.*

Proof. We recall that the de Sitter space-time S_1^{n+1} is a hypersurface of E_1^{n+2} defined by

(8.1) $$S_1^{n+1}=\{x\in E_1^{n+2}\mid \langle x,\, x\rangle=1\}\,.$$

Let H' be the mean curvature vector of the compact, space-like hypersurface M in S_1^{n+1}. Then either $H'=0$ or H' is time-like. Let ξ be the time-like unit vector in S_1^{n+1} normal to M. We put $H'=\alpha\xi$. Then we have

$$H=H'-x, \qquad \operatorname{tr} A_\xi=-n\alpha. \tag{8.2}$$

By applying an argument similar to that given in section 6, we may obtain

$$\Delta H=\Delta^{D'} H'+2\operatorname{tr} A_{D'H'}+\frac{n}{2}\operatorname{grad}\langle H', H'\rangle +(n-\|A_\xi\|^2)H'-(n-n\alpha^2)x. \tag{8.3}$$

If M is mass-symmetric and of 2-type in E_1^{n+2}, then there exist two constants b and c such that (Proposition 3)

$$\Delta H+bH+cx=0. \tag{8.4}$$

Combining (8.2), (8.3) and (8.4) we find

$$\Delta^{D'} H'+2\operatorname{tr} A_{D'H'}+\frac{n}{2}\operatorname{grad}\langle H', H'\rangle =(\|A_\xi\|^2-n-b)H'+(b-c+n-n\alpha^2)x. \tag{8.5}$$

Since x is normal to S_1^{n+1} and other terms in (8.5) are tangent to S_1^{n+1}, we obtain $n\alpha^2=b-c+n$. Thus, M has constant mean curvature α in the de Sitter world. Therefore, (7.5) gives

$$\|A_\xi\|^2=n+b \tag{8.6}$$

which is constant. From equation (2.7) of Gauss, we see that the scalar curvature τ of M satisfies

$$\tau=\|A_\xi\|^2-n^2\alpha^2+n(n-1). \tag{8.7}$$

Therefore, M has constant scalar curvature.

Conversely, if M has constant mean curvature in S_1^{n+1} and constant scalar curvature, then (8.3) implies

$$\Delta H+bH+cx=0, \tag{8.8}$$

where $b=\|A_\xi\|^2-n$ and $c=\|A_\xi\|^2-n\alpha^2$ are constant. Therefore, by applying Lemma 5 and (8.8), we see that the center of mass of M in E_1^{n+2} is the origin. Thus, $M\subset S_1^{n+1}$ is mass-symmetric in E_1^{n+2}. Moreover, (8.8) and Proposition 3 show that M is either of 1-type or of 2-type. If M is of 1-type, Lemma 4 gives $nH=-\lambda_p x$. Thus, Lemma 1 implies that M is minimal in the de Sitter world S_1^{n+1}. This is a contradiction. (Q. E. D.)

COROLLARY 5. *Let M be a compact, space-like hypersurface in the de Sitter space-time S_1^{n+1}. If M has nonzero constant mean curvature and constant scalar* *curvature, then*

(8.9) $$\alpha^2=\left(1-\frac{\lambda_p}{n}\right)\left(1-\frac{\lambda_q}{n}\right),$$

(8.10) $$\tau=(n-1)(\lambda_p+\lambda_q)-\lambda_p\lambda_q>0,$$

(8.11) $$\|A_\xi\|^2=n-(\lambda_p+\lambda_q).$$

This corollary follows from Lemma 7 and the proof of Theorem 4. From Lemma 6 and Theorem 4 we also have the following.

COROLLARY 6. *Let M be a compact, space-like, isoparametric hypersurface of the de Sitter space-time $S_1^{n+1}\subset E_1^{n+2}$. Then M is mass-symmetric. Moreover, if M is not minimal in S_1^{n+1}, then M is of 2-type in E_1^{n+2}.*

Remark 5. We can prove that if M is a compact, space-like, 2-type hypersurface of S_1^{n+1}, then M is always mass-symmetric in S_1^{n+1}.

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