Advances in Pure Mathematics, 2014, 4, 209-212
Published Online May 2014 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2014.45027

Finite Type Transcendental Entire Functions Whose Buried Points Set Contains Unbounded Positive Real Interval

Feng Guo

Department of Mathematics, China University Mining & Technology, Beijing, China
Email: sdsxguofeng@163.com

Received 4 March 2014; revised 4 April 2014; accepted 9 April 2014

Abstract

Let $f_\mu(z) = z \cdot e^{p(z)+\mu}$ with $p(z)$ being real coefficient polynomial and it's leading coefficient be positive, $\mu \in \mathbb{R}^+$, when $p(z)$ and μ satisfy two certain conditions, buried point set of $f_\mu(z)$ contains unbounded positive real interval.

Keywords

Transcendental Entire Functions, Julia Set, Buried Points Set, Real Interval

1. Introduction and Main Result

Let $f(z)$ be an entire function on the complex plane $\mathbb{C}$. Define the iterated sequence $\{f^n\}$ of f as

$$f^0 = z, \quad f^{n+1}(z) = f \circ f^n(z) \quad (n = 0,1,2,\cdots)$$

$\mathbb{C}$ can be divided into two sets:

$$F(f) = \{z \mid \{f^n\} \text{ is normal at } z\}, \quad J(f) = \mathbb{C} \setminus F(f)$$

$F(f)$ is called Fatou set, which is open and contains at most countably many components. $J(f)$ is called Julia set, and it's closed and perfect. The fundamental theory of complex dynamical system can refer to [1]-[4].

For an entire function f, let $\mathcal{S} = \{f \text{ is entire function} \mid \operatorname{sing}(f^{-1}) \text{ is a finite set}\}$, with $\operatorname{sing}(f^{-1})$ be the set of singular value. If f is not a smooth covering map over any neighborhood of α, then α is a singular

How to cite this paper: Guo, F. (2014) Finite Type Transcendental Entire Functions Whose Buried Points Set Contains Unbounded Positive Real Interval. *Advances in Pure Mathematics*, **4**, 209-212.
http://dx.doi.org/10.4236/apm.2014.45027

value. If $f \in \mathcal{S}$, we call f is finite type entire function. The basic properties of the type entire function can refer to [5].

I.N. Baker has first structured the transcendental entire function whose Julia set is $\mathbb{C}$ (See [6]):

Theorem A:

For a certain real value μ, $f_\mu = ze^{z+\mu}$ *has the whole complex plane for its Julia set.*

Notice the set $\{\mu \in \mathbb{R}^+ : J(f_\mu(z)) = \mathbb{C}, f_\mu = ze^{z+\mu}\}$. Baker's result show that the set is nonempty; Jang, C.M. proved that this set contains infinitely many elements in [7]; Qiao, J. proved that the set is unbounded in [8]. What's more, Qiao has researched the buried sets in [9], which contains unbounded positive real interval:

Theorem B:

If $\mu \in [0, +\infty)$ *for* $f_\mu = ze^{z+\mu}$ *and* $J(f_\mu(z)) \neq \mathbb{C}$, *then* $\mathbb{R}^+$ *belongs to the set of buried points.*

Here we study the function $f_\mu(z) = z \cdot e^{p(z)+\mu}$ with $p(z)$ is real coefficient polynomial and it's leading coefficient is positive, expend the function in Theorem B:

Theorem 1 *Let* $f_\mu(z) = z \cdot e^{p(z)+\mu}$, $p(z)$ *is real coefficient polynomial and it's leading coefficient is positive, the zeros of* $zp'(z)+1=0$ *are* $\{a_k | k = 1, \cdots, d\}$ *which are real, unbounded positive real interval* $L = (c, +\infty) \cap \mathbb{R}^+$ *with* $c = \max_{1 \le k \le d}\{a_k\}$. $J(f_\mu(z)) \neq \mathbb{C}$ *and* μ *satisfy*:

(1) $p(z) + \mu > 0 \ (z \in \mathbb{R}^+ \cup \{0\})$;

(2) $x_\mu p'(x_\mu) \in (-\infty, -2) \cup (0, +\infty)$ with x_μ is real zeros of $p(z) + \mu = 0$.

Then L belongs to the set of buried points set.

Remark: Qiao has given the example that satisfy the condition of Theorem B in [9], then the example show that the function satisfy conditions in Theorem 1 is nonempty.

2. Proof of Theorem 1

Lemma 1 *Let* $f(z)$ *be an entire funcion of finite type. Then each Fatou component is eventually periodic, and* $F(f)$ *has only finitely many periodic compinents. They are attractive domains, superattractive domains, parabolic domains or Siegel discs.*

Lemma 2 *Let* $f(z)$ *be a transcendental entire function,* D *be a component of* $F(f)$. *If* D *is an attractive, a superattractive or a parabolic periodic domain, then the cycle of D contains at least one singularity of* f^{-1}; *if D is a Siegel disc, then the forward orbits of the singularities of* f^{-1} *are dense on* ∂D.

Lemma 3 *Let* $f(z)$ *be transcendental entire function of finite type. Then* $f^n(z) \not\to \infty \ (n \to \infty)$ *for* $z \in F(f)$.

Proof of Theorem 1: The singularities of f_μ^{-1} are 0 and $f_\mu(a_k) \in \mathbb{R} \ (1 \le k \le d)$, then $\{f_\mu^n(a_k)\}_{n=1}^{\infty} \in \mathbb{R} \ (1 \le k \le d)$ if $J(f_\mu(z)) \neq \mathbb{C}$, so from Lemma 2 $F(f_\mu)$ have no Siegel disc and from Lemma 1,the periodic component of $F(f_\mu)$ only be attractive, superattractive or parabolic. $f_\mu(0) = 0$ and from $p(z) + \mu > 0 \ (z \in \mathbb{R}^+)$ we can have $f'_\mu(0) = e^{p(0)+\mu} > 1$, then 0 is a repelling fixed point, from Lemma 1 and 2, $F(f_\mu)$ has at most d cycles of periodic components $\{D_j^k\}_{j=1}^{p_k - 1} (1 \le k \le d, p_k \in \mathbb{N})$:

$$\forall 1 \le k \le d, \ \ f_\mu(D_j^k) = D_{j+1}^k \ \ (j = 0, 1, 2, \cdots, p_k - 2), \ \ f_\mu(D_{p_k - 1}^k) = D_0^k$$

such that $f_\mu(a_k) \in \bigcup_{j=0}^{p_k - 1} D_j^k$ and there exist $\{x_k \in \mathbb{R}(1 \le k \le d)\}$ such that

$$* \ \ f_\mu^{np_k + 1}(a_k) \to x_k \ \ (n \to \infty)$$

Here we first proof $L = (c, \infty) \cap \mathbb{R}^+$ belong to $J(f_\mu)$. If there exist $t \in (c, \infty) \cap \mathbb{R}^+$ and $t \in F(f_\mu)$, then t is contained in a component of $F(f_\mu)$ and from above we have that there exist $m \in \mathbb{N}$, $a \in \{a_k | k = 1, \cdots, d\}$, a cycle of component $\{D_j\}_{j=1}^{p}$ with $p \in \{p_k | k = 1, \cdots, d\}$ and $x_0 \in \{x_k | k = 1, \cdots, d\}$ such that $f_\mu^m(t)$ and $f_\mu(a)$ are in the same domain D_j, from $*$ we have that $f_\mu^{m+np}(t) \to x_0 \ (n \to \infty)$. However,

$p(z)+\mu>0\left(z\in\mathbb{R}^+\right)$ means $f_\mu(z)>z$ when $z\in\mathbb{R}^+$, $f'_\mu(z)>0$ when $z\in(c,\infty)\cap\mathbb{R}^+$, then $\left\{f_\mu^n(z)\right\}_{n=1}^{\infty}\in L$ is montone increasing sequence, by the relation

$$f_\mu^{n+1}(z)=f_\mu^n(z)\exp\left(p\left(f_\mu^n(z)\right)+\mu\right)$$

we have $f_\mu^n(z)\to+\infty\left(z\in L\cap F\left(f_\mu\right);n\to\infty\right)$ which give a contradiction to Lemma 3.

Then we will proof that $L=(c,\infty)\cap\mathbb{R}^+$ belongs to the set of buried points. If there exist a point $a_0\in(c,\infty)\cap\mathbb{R}^+$ and a_0 is on the boundary of a component of $F\left(f_\mu\right)$, from the discussion above, we know there exist some $N\in\mathbb{N}$, a cycle of component $\left\{D_j\right\}_{j=1}^{p}\in\left\{D_j^k\right\}_{j=1}^{p_k-1}$ $(1\le k\le d)$ with $p\in\left\{p_k\,\middle|\,k=1,\cdots,d\right\}$ such that when $n>N$,

$$f_\mu^n(a_0)\in\bigcup_{j=0}^{p-1}\partial D_j\ \ (n=1,2,3,\cdots)$$

and there exist $x_0\in\left\{x_k\in\mathbb{R}(1\le k\le d)\right\}$ and some $a\in\left\{a_k\,\middle|\,k=1,\cdots,d\right\}$, $a\in\left\{D_j\right\}_{j=1}^{p}$ such that $f_\mu^{np+1}(a)\to x_0\ (n\to\infty)$.

Let $a_n=f_\mu^n(a_0)$ $(n=1,2,3,\cdots)$, it's easy to have that $a_n\in\mathbb{R}^+$ and $a_{n+1}>a_n$ $(n=1,2,\cdots)$, $a_n\to\infty\ (n\to\infty)$. Without the loss of generality, we can let $n>N$, $a_{np+j}\in\partial D$ $(j=0,1,2,\cdots,p-1)$.

Here we prove a_{np+j} $(n=1,2,3,\cdots)$ are all in the same connected component of ∂D_j. If not, there exist $a_{kp+j},a_{sp+j}\ (k<s)$ and two different component of ∂D_j called α_1 and α_2 such that

$$a_{kp+j}\in\alpha_1,\ a_{sp+j}\in\alpha_2$$

We can make curve ω, such that α_1 and α_2 belong to different components of $\mathbb{C}\setminus\omega$, then α_1 and α_2 belong to different components of $J\left(f_\mu\right)$, that gives a contradiction to $\left[a_{kp+j},a_{sp+j}\right]\subset J\left(f_\mu\right)$.

Let $\delta_n\subset\partial D_j$ be an bounded continuum containing a_{np+j} and $a_{(n+1)p+j}$, we will prove that $\left[a_{np+j},a_{(n+1)p+j}\right]\subset\delta_n$. If not, δ_n and $\left[a_{np+j},a_{(n+1)p+j}\right]$ can form a bounded domain and $J\left(f_\mu\right)$ have no interior point, therefore $F\left(f_\mu\right)$ have to have a bounded domain, notice that $\left\{a_n\right\}_{n=1}^{\infty}$, let $\lim_{n\to\infty}a_n=A$, due to $a_{n+1}=a_n\mathrm{e}^{p(a_n)+\mu}$, then $A=A\mathrm{e}^{p(A)+\mu}$, notice that $a_n,A\in\mathbb{R}^+$ and $p(z)+\mu>0$ when $z\in\mathbb{R}^+$ therefore $A=0,\infty$, from $a_n\in\mathbb{R}^+$ we have $A=+\infty$. That means D_j is a unbounded component, however any component of $F\left(f_\mu\right)$ have to turn into cycle $\left\{D_j\right\}_{j=0}^{p-1}$ from Lemma 1, therefore the components of $F\left(f_\mu\right)$ are all unbounded, it's contradiction. What's more, we can have that

$$\bigcup_{n=N}^{\infty}\left[a_{np+j},a_{(n+1)p+j}\right]\subset\partial D_j\quad(j=1,2,3,\cdots,p-1)$$

$$\left[a_{Np+j},\infty\right]\subset\partial D_j\quad(j=1,2,3,\cdots,p-1)$$

that means $\left[a_{N_0},\infty\right]$ is the common boundary of $D_0,D_1,\cdots,D_{p-1}$ with $N_0=Np+p-1$. The common boundary is at most of two domains, therefore $p\le2$. Here we divide two cases to discuss:

Case 1: $p=1$. $f_\mu(a)\in\bigcup_{j=0}^{p-1}D_j$ is $f_\mu(a)\in D_0$. Considering

$$f_\mu^{n+1}(a)=f_\mu^n(a)\mathrm{e}^{p\left(f_\mu^n(a)\right)+\mu}$$

and $*$ we have

$$x_0=\lim_{n\to\infty}f_\mu^{n+1}(a)=0\ \text{ or } x_0 \text{ are the zeros of } p(z)+\mu=2k\pi i\ \ (k=0,\pm1,\pm2,\cdots)$$

Notice $a\in\mathbb{R}$, then $x_0\in\mathbb{R}$, we only need to consider 0 and the real zeros of $p(z)+\mu=0$, from Lemma 2

x_0 is an attractive, superattractive, or rational indifferent fixed point, but from conditions (1) and (2) in Theorem 1, 0 and x_0 are repelling fixed point, it's a contradiction.

Case 2: $p = 2$. Without the loss of generality, we take D_0 be the component above $\left[a_{N_0}, +\infty\right)$ and D_1 be the component under $\left[a_{N_0}, +\infty\right)$, for $r \in \left[a_{N_0}, +\infty\right)$ with r is large enough, take a sequence

$\{z_n\}_{n=1}^{\infty} \in D_0 \cap \{z \mid \operatorname{Im} z > 0\}$ such that $z_n \to r \ (n \to \infty)$.

Let $z_n = r_n e^{i\theta_n}$ with $r_n > 0, \theta_n \in \left(0, \frac{\pi}{2}\right)$, then

$$f_\mu(z_n) = r_n e^{i\theta_n} e^{p(r_n i\theta_n) + \mu}$$

We can suppose that $p(z) = c_n z^d + \cdots + c_1 z + c_0$, then we have

$$\operatorname{Im} p\left(r_n e^{i\theta_n}\right) = c_n r_n^d \sin(\theta_n d) + \cdots + c_1 r_n \sin(\theta_n) > \left(c_n r_n^d + \cdots + c_1 r_n\right) \sin\theta_n = p(r_n) \sin\theta_n$$

Notice that $p(z) + \mu > 0 \ \left(z \in \mathbb{R}^+ \cup \{0\}\right)$ and we can easily deduce that $\operatorname{Im} p\left(r_n e^{i\theta_n}\right) > p(r_n) \sin\theta_n > 0$ and $f_\mu(z_n)$ belong to the above half plane when n and r is large enough, but it contradicts that $f_\mu(z_n) \in D_1$. The proof is complete.

Acknowledgements

Author thanks professor Jianyong Qiao and Yuhua Li for their discussions and suggestions.

References

[1] Eremenko, A.E. and Lyubich, M.Yu (.1990) The Dynamics of Analytic Transformations. *Leningrad Mathematical Journal*, **1**, 563.

[2] Beardon, A.F. (1991) Iteration of Rational Functions. Springer, Berlin. http://dx.doi.org/10.1007/978-1-4612-4422-6

[3] Milnor, J. (2006) Dynamics in One Complex Variable. 3rd Edition, Princeton University Press, Princeton and Oxford.

[4] Qiao, J. (2010) Complex Dynamics on Renormalization Transformations. Science Press, Beijing. (in Chinese)

[5] Morosawa, S., Nishimura, Y., Taniguchi, M. and Ueda, T. (2000) Holomorphic Dynamics, Cambridge University Press.

[6] Baker, I.N. (1970) Limit Functions and Sets of Non-Normality in Iteration Theory. *Annales Academiae Scientiarum Fennicae. Series A 1, Mathematica*, **467**, 1-11.

[7] Jang, C.M. (1992) Julia Set of the Function z exp(z + μ). *Tohoku Mathematical Journal*, **44**, 271-277.

[8] Qiao, J. (1994) The Set of the Mapping $z \to$ exp(z + μ). *Chinese Science Bulletin*, **39**, 529.

[9] Qiao, J. (1995) The Buried Points on the Julia Sets of Rational and Entire Functions. *Science in China Series A*, **38**, 1409-1419.