

## FINITE VOLUME AND FUNDAMENTAL GROUP ON MANIFOLDS OF NEGATIVE CURVATURE

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### 1. Introduction

Let  $V$  be a complete Riemannian manifold of dimension  $n$  and sectional curvature  $K \leq 0$ . Then  $V$  is a  $K(\pi, 1)$ -manifold with  $\pi = \pi_1(V)$  [8, p. 103] and hence determined up to homotopy by the fundamental group. In particular, the homology  $H_*(V)$  of  $V$  is isomorphic to the group homology  $H_*(\pi_1(V))$  (see [1]). Therefore  $V$  is compact if and only if  $H_n(\pi_1(V), \mathbf{Z}_2) = \mathbf{Z}_2$ . Hence the compactness of  $V$  can be read off from  $\pi_1(V)$ .

We give a similar characterization for the condition of finite volume:

**Theorem.** *Let  $V$  be a complete Riemannian manifold of dimension  $n \geq 3$  with curvature  $-b^2 \leq K \leq -a^2 < 0$ . Then the volume of  $V$  is finite if and only if:*

(1)  $\pi_1(V)$  contains only finitely many conjugation classes of maximal almost nilpotent subgroups of rank  $n - 1$ .

(2) If  $\Delta$  is the amalgamated product of  $\pi_1(V)$  with itself on these subgroups, then  $H_n(\Delta, \mathbf{Z}_2) = \mathbf{Z}_2$ .

For a full definition of  $\Delta$  we refer to §4.

For  $n = 2$ , the statement is wrong: Let  $V$  be a noncompact surface with constant negative curvature and finite volume. It is known that  $V$  has an end  $E$  diffeomorphic to  $S^1 \times (0, \infty)$  with a warped product metric  $f^2 ds^2 + dt^2$ . The curvature is given by  $-f''/f$  and the volume of  $E$  by  $2\pi \int_0^\infty f dt$ . Using a suitable function  $\bar{f}$  we can deform  $E$  to an expanding end, such that the new end has bounded negative curvature but infinite volume.

The first part of our proof (§3) leads to a description of the ends of finite volume in terms of the fundamental group. This part is based on the investigations of Heintze [6], Gromov [5] and Eberlein [3]. A topological argument then finishes the proof (§4).

This paper is a condensed version of parts of my thesis [10] written under the guidance of Professor Wolfgang Meyer at Münster. I am also deeply

grateful to Mikhael Gromov who proposed the result and pointed out essential ideas for the proof.

## 2. Notation and basic results

(Compare [3], [4].) Let  $X$  be a Hadamard manifold, i.e., a complete simply connected Riemannian manifold with curvature  $K \leq 0$ , let  $d(\cdot, \cdot)$  be the distance function on  $X$  and let  $\bar{X} = X \cup X(\infty)$  be the Eberlein-O'Neill compactification. For  $x \in X$  and  $z \in X(\infty)$  let  $HS(x, z)$  be the horosphere at  $z$  which contains  $x$  and  $HB(x, z)$  the corresponding (open) horoball. For an isometry  $\gamma$  of  $X$  we define the convex displacement function  $d_\gamma: x \rightarrow d(x, \gamma x)$ .  $\gamma$  is called elliptic (hyperbolic, parabolic), if  $d_\gamma$  has zero minimum (positive minimum, no minimum). An isometry  $\gamma$  can be extended to a homeomorphism of  $\bar{X}$ . If  $X$  has curvature  $K \leq -a^2 < 0$ , a nonelliptic isometry  $\gamma$  can be characterized by the fixed points  $\text{Fix}(\gamma)$  on  $X(\infty)$ : a hyperbolic isometry fixes exactly two points of  $X(\infty)$  and translates the unique geodesic joining these points. A parabolic isometry  $\gamma$  has exactly one fixed point  $z \in X(\infty)$  and leaves the horospheres  $HS(x, z)$  invariant.

For a complete manifold  $V$  of negative curvature let  $X$  be the Riemannian universal covering,  $\pi: X \rightarrow V$  the projection. Then  $V = X/\Gamma$ , where  $\Gamma$  is a freely acting, discrete group of isometries on  $X$ ,  $\Gamma \cong \pi_1(V)$ . We define the  $\Gamma$ -invariant function  $d_\Gamma: X \rightarrow (0, \infty)$  by  $d_\Gamma(x) := \min_{\gamma \in \Gamma - \text{id}} d_\gamma(x)$ . Then  $d_\Gamma(x) = 2 \text{Inj Rad}(\pi(x))$ , where  $\text{Inj Rad}$  is the injectivity radius.  $\text{Inj Rad}(p) \geq \varepsilon$  and  $K \leq 0$  imply that the volume of the distance ball  $B_\varepsilon(p)$  is larger than the volume of the  $\varepsilon$ -ball in euclidean space. Therefore  $\text{vol}(V) < \infty$  implies that the set  $\{\text{Inj Rad} \geq \varepsilon\}$  is compact for all  $\varepsilon > 0$ .

An end of  $V$  is a function  $E$  that assigns to each compact subset  $K$  of  $V$  a connected component  $E(K)$  of  $V - K$  with the condition that  $E(K) \supset E(K')$  if  $K \subset K'$ . An open set  $U \subset V$  is a neighborhood of an end  $E$  if  $E(K) \subset U$  for some compact subset  $K$ . An end  $E$  has finite volume if there is a neighborhood  $U$  of  $E$  with  $\text{vol}(U) < \infty$ .

For the proof of our theorem, we can assume (by scaling the metric) that  $V$  satisfies the curvature condition  $-1 \leq K \leq -a^2$ , where  $a$  is positive. This enables us to use the Margulis lemma in the following form.

**Margulis Lemma.** *There is a number  $\mu = \mu(n) > 0$ , depending only on  $n$ , with the following property: let  $X$  be an  $n$ -dimensional Hadamard manifold with curvature  $-1 \leq K \leq 0$ , let  $\Gamma$  be a discrete group of isometries on  $X$ ,  $x \in X$ , and let  $\Gamma_\mu(x)$  be the subgroup of  $\Gamma$  generated by the elements  $\gamma \in \Gamma$  with  $d_\gamma(x) \leq \mu$ . Then  $\Gamma_\mu(x)$  is almost nilpotent, that is,  $\Gamma_\mu(x)$  contains a nilpotent subgroup of finite index.*

For a proof see [11, p. 5.51], [2, p. 27], [5], [10].

**Lemma 1.** *Let  $X$  be a Hadamard manifold with curvature  $K \leq -a^2$  and let  $\Gamma$  be a freely acting, discrete and almost nilpotent group of isometries on  $X$ . Then  $\text{Fix}(\gamma_1) = \text{Fix}(\gamma_2)$  for all  $\gamma_1, \gamma_2 \in \Gamma - \text{id}$ . Hence the elements of  $\Gamma - \text{id}$  are either all parabolic with a common fixed point  $z \in X(\infty)$ , or all hyperbolic with common axis  $c$ . In the second case  $\Gamma$  is infinite cyclic.*

For a proof see [3, Lemma 3.1b].

### 3. Ends of finite volume

The main result of this section is the following description of the ends of finite volume.

**Proposition.** *Let  $V = X/\Gamma$  satisfy  $-1 \leq K \leq -a^2, 0 < r \leq \mu$ .*

(1) *If  $E$  is an end of finite volume, then there is a unique connected component  $U_r(E)$  of  $\{\text{Inj Rad} < r/2\}$  such that  $U_r(E)$  is a neighborhood of  $E$ . The volume of  $U_r(E)$  is finite. For two different ends  $E$  and  $E^*$  of finite volume, the neighborhoods  $U_r(E)$  and  $U_r(E^*)$  are disjoint.*

(2) *If  $n = \dim V \geq 3$ , then the ends of finite volume correspond one-to-one to the conjugation classes of the maximal almost nilpotent subgroups of rank  $n - 1$  in  $\Gamma$ .*

(3) *The ends of finite volume have disjoint neighborhoods  $U$  diffeomorphic to  $B \times (0, \infty)$ , where  $B$  is a compact codimension 1 submanifold of  $V$ .*

Before we will prove this result, we need some preparations. Our manifold  $V$  was represented as  $V = X/\Gamma$ . Now we look for a similar description for subsets  $U \subset V$  as  $U = W/\Gamma_W$ , where  $W \subset X$  is precisely invariant, i.e. for any  $\gamma \in \Gamma$  either  $\gamma W = W$  or  $\gamma W \cap W = \emptyset$ , and  $\Gamma_W$  is the subgroup  $\{\gamma \in \Gamma | \gamma W = W\}$ .

**Lemma 2.** *Let  $\Gamma$  be a discrete group of isometries acting on a Hadamard manifold  $X$ . Let  $r > 0$  and let  $W \subset X$  be a connected component of  $\{d_\Gamma < r\}$ . Then:*

(1)  *$W$  is precisely invariant.*

(2) *If  $\gamma \in \Gamma, x \in W$  and  $d_\gamma(x) < r$ , then  $\gamma \in \Gamma_W$ .*

*Proof.* (1) Because  $d_\Gamma$  is  $\Gamma$ -invariant,  $\gamma W$  is also a connected component of  $\{d_\Gamma < r\}$  for all  $\gamma \in \Gamma$ . Thus  $\gamma W \cap W \neq \emptyset$  implies  $\gamma W = W$ .

(2)  $d_\gamma(x) = d_\gamma(\gamma x) < r$ . The convexity of  $d_\gamma$  now implies  $d_\gamma < r$  hence  $d_\Gamma < r$  on the geodesic from  $x$  to  $\gamma x$ . Thus both  $x$  and  $\gamma x$  are in  $W$ . By (1),  $\gamma \in \Gamma_W$ . q.e.d.

Let  $U$  be a component (i.e., a connected component) of  $\{\text{Inj Rad} < r/2\}$  and  $W$  be a component of  $\pi^{-1}(U) \subset X$ . Then  $W$  is a component of  $\{d_\Gamma < r\}$

and, by Lemma 2,  $U = W/\Gamma_W$ . With regard to the Margulis Lemma we will study components  $U$  of  $\{\text{Inj Rad} < r/2\}$  and the corresponding components  $W$  of  $\{d_\Gamma < r\}$ , where  $r$  is smaller than the constant  $\mu$  of the Margulis Lemma.

**Lemma 3.** *Let  $V$  be complete,  $-1 \leq K \leq -a^2$ ,  $0 < r \leq \mu$ . Let  $U \subset V$  be a component of  $\{\text{Inj Rad} < r/2\}$  in  $V$ ,  $W$  a component of  $\pi^{-1}(U)$  in  $X$  and  $\Gamma_W = \{\gamma \in \Gamma \mid \gamma W = W\}$ .*

(1) *Either there is a unique geodesic  $c$  in  $X$ , such that  $\Gamma_W$  is the infinite cyclic group  $\Gamma_W = \Gamma_c := \{\gamma \in \Gamma \mid \gamma \text{ has axis } c\}$  or  $\Gamma_W$  is a group of parabolic isometries and there is a unique  $z \in X(\infty)$  with  $\Gamma_W = \Gamma_z := \{\gamma \in \Gamma \mid \gamma(z) = z\}$ .  $W$  is bounded in the first and unbounded in the second case.*

(2)  $W = \{d_{\Gamma_W} < r\}$ .

(3) *If  $W_1$  and  $W_2$  are distinct components of  $\{d_\Gamma < r\}$ , then  $\Gamma_{W_1}$  and  $\Gamma_{W_2}$  intersect only in the identity.*

*Proof.* (1) Using Lemma 1 it is easy to prove (see [3, Lemma 3.1c]): if  $x, y \in W$ ,  $d_\alpha(x), d_\beta(y) < r$  for nontrivial  $\alpha, \beta \in \Gamma$ , then  $\text{Fix}(\alpha) = \text{Fix}(\beta)$ . Thus for  $A := \{\gamma \in \Gamma - \text{id} \mid \text{there exists } x \in W \text{ with } d_\gamma(x) < r\}$ , the classification of isometries yields: either all  $\alpha \in A$  are hyperbolic with a unique common axis  $c$ , or all  $\alpha \in A$  are parabolic with a unique common fixed point  $z$ . If  $\gamma \in \Gamma_W - \text{id}$ ,  $x \in W$ , then  $\gamma x \in W$  and there is an  $\alpha \in A$  with  $r > d_\alpha(\gamma x) = d_{\gamma^{-1}\alpha\gamma}(x)$ . Hence  $\gamma^{-1}\alpha\gamma \in A$ .

If  $\alpha \in A$  is hyperbolic with axis  $c$ , then  $\gamma^{-1}c$  is the axis of  $\gamma^{-1}\alpha\gamma \in A$  and hence  $\gamma^{-1}c = c$ . Therefore  $\gamma$  leaves  $c$  invariant and  $\gamma$  is hyperbolic with axis  $c$ .

If  $\alpha \in A$  is parabolic with fixed point  $z \in X(\infty)$ , the same argument shows that  $\gamma z = z$ .  $\gamma$  is also parabolic by [4, Proposition 6.8].

Hence we have proved that the elements of  $\Gamma_W$  are either all hyperbolic with axis  $c$  ( $\Gamma_W \subset \Gamma_c$ ) or all parabolic with fixed point  $z$  ( $\Gamma_W \subset \Gamma_z$ ). In the first case  $c$  is contained in  $W$  and hence  $\Gamma_c \subset \Gamma_W$ . The discreteness of  $\Gamma$  then implies that  $\Gamma_c$  is infinite cyclic. In the second case let  $g: [0, \infty) \rightarrow X$  be a geodesic ray with  $g(0) \in W$  and  $g(\infty) = z$ . Because  $K \leq -a^2 < 0$ ,  $d_\gamma(g(t)) \rightarrow 0$  for all  $\gamma \in \Gamma_z$  as  $t$  goes to  $\infty$ . Hence  $g$  is contained in  $W$  and, by Lemma 2(2),  $\Gamma_z \subset \Gamma_W$ .

If  $U$  is bounded, then  $\text{Inj Rad}$  assumes a minimum in  $p \in U$ . Let  $x \in W$  with  $\pi(x) = p$  and  $d_\Gamma(x) = d_\gamma(x)$  for some  $\gamma \in \Gamma_W$ . If  $\gamma$  is parabolic, then there is a nearby  $y$  with  $d_\gamma(y) < d_\gamma(x)$ , hence  $\text{Inj Rad}(\pi(y)) < \text{Inj Rad}(\pi(x))$ , a contradiction.

On the other hand let  $\Gamma_W$  be an infinite cyclic group of isometries with common axis  $c$ . Then the curvature assumption implies that  $d_{\Gamma_W}(y) > r$  for all  $y \in X$  with  $d(y, c) > R$  for a suitable  $R$ . Therefore  $d(q, \pi(c)) < R$  for all  $q \in U$  and  $U$  is bounded.

(2) By Lemma 2(2),  $W \subset \{d_{\Gamma_W} < r\}$ . Now it is easy to see that for

a geodesic  $c$  or a point  $z \in X(\infty)$ , the sets  $\{d_{\Gamma_c} < r\}$  and  $\{d_{\Gamma_z} < r\}$  are connected. Therefore  $W = \{d_{\Gamma_w} < r\}$ .

(3) Let  $\gamma \in \Gamma_{W_1} \cap \Gamma_{W_2}$  be a nontrivial element. If  $\gamma$  is hyperbolic with axis  $c$ , then  $\Gamma_{W_1} = \Gamma_c = \Gamma_{W_2}$  and if  $\gamma$  is parabolic with fixed point  $z$ , then  $\Gamma_{W_1} = \Gamma_z = \Gamma_{W_2}$ . By (2),  $\Gamma_{W_1} = \Gamma_{W_2}$  implies  $W_1 = W_2$ .

**Lemma 4.** *Let  $V = X/\Gamma$  satisfy  $-1 \leq K \leq -a^2$ ,  $0 < r \leq \mu$ . Let  $U \subset V$  be an unbounded component of  $\{\text{Inj Rad} < r/2\}$ , and let  $W$  be a component of  $\pi^{-1}(U)$  with  $\Gamma_w$  as above. Then the volume of  $U$  is finite if and only if  $\Gamma_w$  is an almost nilpotent group of rank  $n - 1$ .*

**Remark.** The rank of an almost nilpotent group is the rank of a nilpotent subgroup of finite index. For the definition of rank and other facts about nilpotent groups compare Chapter II of [9].

*Proof.* We divide the proof into three steps:

(a) If  $\text{vol}(U) < \infty$ , then  $\Gamma_z$  is almost nilpotent and operates with compact quotient on the horospheres  $HS(x, z)$ :

The proof of Lemma 3.1g of [3] shows that  $\Gamma_z$  operates with compact quotient on the horospheres and therefore  $\Gamma_z$  is finitely generated. Let  $\gamma_1, \dots, \gamma_m$  be a system of generators.  $K \leq -a^2$  implies that there is a point  $g(t_0)$  with  $d_{\gamma_i}(g(t_0)) \leq r$ . By the Margulis Lemma,  $\Gamma_z$  is almost nilpotent with nilpotent subgroup  $N$  of finite index. Then  $N$  also operates with compact quotient on the horospheres.

(b) rank  $N = n - 1$ :  $N$  is nilpotent, finitely generated and without torsion. By a theorem of Malcev  $N$  is isomorphic to a lattice in a simply connected nilpotent Lie group  $A$  with  $\dim A = \text{rank } N =: m$  [9, Theorem II.2.18]. Because every lattice in a nilpotent Lie group has a compact quotient and  $A$  is homeomorphic to  $\mathbf{R}^m$ ,  $N$  operates with compact quotient on  $\mathbf{R}^m$ . Because  $N$  operates also on a horosphere, hence on  $\mathbf{R}^{n-1}$  with compact quotient, we conclude  $m = n - 1$  by comparing the homology groups of these  $K(\pi, 1)$ -manifolds.

(c) If  $\Gamma_z$  contains a nilpotent subgroup  $N$  of finite index and rank  $n - 1$ , then  $N$  and hence  $\Gamma_z$  operate with compact quotient on the horospheres  $HS(x, z)$  by inversion of the arguments of b. Because  $d_{\Gamma_z}(g(t)) \rightarrow \infty$  as  $t \rightarrow -\infty$ , we conclude easily that there is a horoball  $HB(x_0, z)$  with  $W \subset HB(x_0, z)$ , and thus  $\text{vol}(U) \leq \text{vol}(HB(x_0, z)/\Gamma_z)$ . We prove that the latter is finite:  $HB(x_0, z)/\Gamma_z$  is diffeomorphic to  $B \times (0, \infty)$ , where the projection on  $(0, \infty)$  is a riemannian submersion and  $B_t = B \times \{t\}$  is the quotient of a horosphere. Because of the curvature condition, we control the stable Jacobi-fields (see [7]). This implies  $\text{vol}(B_t) \leq ke^{-at}$  with a constant  $k$ . Hence

$$\text{vol}(HB(x_0, z)/\Gamma_z) \leq \int_0^\infty ke^{-at} dt < \infty.$$

**Lemma 5.** *Let  $V = X/\Gamma$  satisfy  $-1 \leq K \leq -a^2$ ,  $0 < r_1 \leq r_2 \leq \mu$ . Let  $U_i$  be components of  $\{\text{Inj Rad} < r_i/2\}$  with  $U_1 \subset U_2$  and let  $W_i$  be components of  $\pi^{-1}(U_i)$  with  $W_1 \subset W_2$ . Then:*

- (1)  $\Gamma_{W_1} = \Gamma_{W_2}$ .
- (2)  $U_1$  is the only component of  $\{\text{Inj Rad} < r_1/2\}$  which is contained in  $U_2$ .

*Proof.* (1)  $W_1 \subset W_2$  immediately implies  $\Gamma_{W_1} \subset \Gamma_{W_2}$ . Using Lemma 3(1) we conclude that either  $\Gamma_{W_1} = \Gamma_c = \Gamma_{W_2}$  or  $\Gamma_{W_1} = \Gamma_z = \Gamma_{W_2}$  for a geodesic  $c$  or a point  $z \in X(\infty)$ .

(2) is a consequence of (1) and Lemma 3(3).

Now we are able to prove our proposition.

*Proof.* (1) Because  $E$  has finite volume, there is a compact set  $K \subset V$  with  $\text{vol}(E(K)) < \infty$  and  $\text{Inj Rad}|_{E(K)} < r/2$ . Let  $U_r(E)$  be the component of  $\{\text{Inj Rad} < r/2\}$  which contains  $E(K)$ . If  $U'$  is another component of  $\{\text{Inj Rad} < r/2\}$  which is a neighborhood of  $E$ , then  $U' \cap U_r(E) \neq \emptyset$  and hence  $U' = U_r(E)$ .

We now prove that  $\text{vol}(U_r(E)) < \infty$ . Let  $K$  be as above. Then there is an  $r'$  with  $0 < r' < r$  and  $\text{Inj Rad}|_K > r'/2$ . By construction  $U_{r'}(E) \subset E(K) \subset U_r(E)$  and hence  $\text{vol}(U_{r'}(E)) < \infty$ . Let  $W_{r'} \subset W_r$  be components of  $\pi^{-1}(U_{r'}(E))$  and  $\pi^{-1}(U_r(E))$ . By Lemma 5,  $\Gamma_{W_{r'}} = \Gamma_{W_r}$  and, by Lemma 4, the finiteness of the volume of  $U_{r'}(E)$  implies  $\text{vol}(U_r(E)) < \infty$ .

If  $E, E^*$  are different ends of finite volume, there is a compact set  $K \subset V$  with  $E(K) \neq E^*(K)$  and hence  $E(K)$  and  $E^*(K)$  are disjoint. As above there is an  $r', 0 < r' < r$ , with  $U_{r'}(E) \subset E(K)$  and  $U_{r'}(E^*) \subset E^*(K)$ . By Lemma 5(2),  $U_r(E)$  and  $U_r(E^*)$  are distinct, hence disjoint.

(2) For an end  $E$  of finite volume let  $U_r(E), W_r$  be as in (1). By Lemma 4,  $\Gamma_{W_r}$  is almost nilpotent of rank  $n - 1$  and  $\Gamma_{W_r} = \Gamma_z$  for some  $z \in X(\infty)$ .  $\Gamma_z$  is maximal almost nilpotent: if  $\Gamma' \supset \Gamma_z$  is almost nilpotent, then, by Lemma 1, all  $\gamma \in \Gamma'$  have a common fixed point in  $X(\infty)$  and hence  $\Gamma' \subset \Gamma_z$ .

If  $W'_r = \gamma W_r$  is another component of  $\pi^{-1}(U_r(E))$ , then  $\Gamma_{W'_r} = \gamma \Gamma_{W_r} \gamma^{-1}$ . Thus we assign to every end of finite volume a conjugation class of the maximal almost nilpotent subgroups of rank  $n - 1$ . We prove that this map is bijective:

(a) Different ends  $E$  and  $E^*$  have disjoint  $U_r(E)$  and  $U_r(E^*)$ . If  $W_r$  and  $W_r^*$  are components of  $\pi^{-1}(U_r(E))$  and  $\pi^{-1}(U_r(E^*))$ , then there is no  $\gamma \in \Gamma$  with  $\gamma W_r = W_r^*$ . Therefore  $\Gamma_{W_r}$  and  $\Gamma_{W_r^*}$  define different conjugation classes by Lemma 3(3).

(b) On the other hand let  $\Delta \subset \Gamma$  be a maximal almost nilpotent subgroup of rank  $n - 1 \geq 2$ . Then  $\Delta$  is not infinite cyclic and hence, by Lemma 1,  $\Delta$  is a group of parabolic isometries with a common fixed point  $z \in X(\infty)$ . Thus

$\Delta \subset \Gamma_z$ . By the arguments of Lemma 4,  $\Delta$  operates with compact quotient on the horospheres  $HS(x, z)$  and  $\text{vol}(HB(x, z)) < \infty$ . Then  $\Gamma_z$  also operates with compact quotient on the horospheres and the argument of Lemma 4(a) proves that  $\Gamma_z$  is almost nilpotent. Hence  $\Delta = \Gamma_z$  by maximality. Part (c) of that lemma shows that for suitable  $x \in X$  the volume of  $HB(x, z)/\Gamma_z$  is arbitrarily small, and hence also the injectivity radius on  $\pi(HB(x, z))$  is small. For  $0 < r \leq \mu$  let  $U_r$  be the component of  $\{\text{Inj Rad} < r/2\}$  which contains  $\pi(HB(x, z))$  for suitable  $x$ . Let  $W_r$  be the component of  $\pi^{-1}(U_r)$  containing  $HB(x, r)$ . Then  $\Gamma_{W_r} = \Gamma_z$  and, by Lemma 4,  $\text{vol}(U_r) < \infty$ . By definition  $U_{r'} \subset U_r$  for  $0 < r' \leq r \leq \mu$ , and therefore one checks that the following function  $E$  defines an end of finite volume:

For compact  $K \subset V$  let  $E(K)$  be the component of  $V - K$  which contains  $U_r$ , where  $r$  is chosen such that  $\text{Inj Rad}_{|K} > r/2$ . By construction the conjugation class assigned by  $E$  is the class of  $\Delta$ .

(3) The proof of (2) shows that an end  $E$  of finite volume has a neighborhood of the form  $E(B) = HB(x, z)/\Gamma_z$  which is diffeomorphic to  $B \times (0, \infty)$  with  $B = HS(x, z)/\Gamma_z$ . These neighborhoods are contained in  $U_r(E)$ , hence different ends have disjoint neighborhoods.

**Remark.** Part (1) implies the theorem, due to Heintze [6, p. 33], that a complete manifold  $V$  with  $\text{vol}(V) < \infty$  and  $-1 \leq K \leq -a^2$  has only finitely many ends: the ends have disjoint neighborhoods  $U_r(E)$ . In  $U_r(E)$  we will find an injectively imbedded  $r/4$ -ball, thus  $\text{vol}(U_r(E))$  is larger than a constant depending on  $r$  and  $n$ .

#### 4. Finite volume and fundamental group

Let  $V$  be a complete Riemannian manifold of dimension  $n \geq 3$ , which satisfies  $-1 \leq K \leq -a^2$ . Using the result of Heintze remarked above, we see that the volume of  $V$  is finite if and only if  $V$  has only finitely many ends and every end has finite volume. This is equivalent to the conditions:

- (1)  $V$  has only finitely many ends of finite volume, and
- (2)  $V$  has no further ends.

According to the proposition, condition (1) is equivalent to the finiteness of the conjugation classes of the maximal almost nilpotent subgroups of rank  $n - 1$  in  $\pi_1(V)$ .

We will prove that (2) also is equivalent to a condition on the fundamental group. Therefore let us assume that  $V$  has finitely many ends  $E_0, \dots, E_k$  of finite volume. By our proposition the ends  $E_i$  have disjoint neighborhoods diffeomorphic to  $B_i \times (0, \infty)$ . We identify  $B_i \times (0, \infty)$  with subsets of  $V$ . Then

$M := V - \bigcup_{i=0}^k (B_i \times (0, \infty))$  is a manifold with  $k + 1$  boundary components  $B_0, \dots, B_k$ . It is easily checked that  $V$  has no further ends if and only if  $M$  is compact. Now we define a manifold  $W$  without boundary by glueing two copies  $M^1, M^2$  of  $M$  canonically along their common boundary. Clearly  $M$  is compact if and only if  $W$  is compact. Therefore condition (2) is equivalent to:

(2\*)  $W$  is compact.

To prove that (2\*) is a condition on  $\pi_1(V)$ , we show:

(a) The fundamental group of  $W$  can be computed purely algebraically from  $\pi_1(V)$ .

(b)  $W$  is a  $K(\pi, 1)$ -manifold, hence  $W$  is compact if and only if  $H_n(\pi_1(W), \mathbf{Z}_2) = \mathbf{Z}_2$ .

*Proof of (a).* By the theorem of Zaidenman ([12], compare Steenrod's reviews, Part I, Amer. Math. Soc., 1968, p. 52) we can compute the fundamental group of  $W$  in the following way: we choose points  $p_i \in B_i$ , and by arcs from  $p_i$  to  $p_0$  we define imbeddings  $\phi_i^j: \pi_1(B_i, p_i) \rightarrow \pi_1(M^j, p_0)$ . Let  $F_k$  be the free group with  $k$  generators  $\gamma_1, \dots, \gamma_k$ . Then  $\pi_1(W)$  is isomorphic to the quotient of the free product  $\pi_1(M^1, p_0) * \pi_1(M^2, p_0) * F_k$  divided by the normal subgroup generated by the elements  $\phi_0^1(\alpha_0)\phi_0^2(\alpha_0)^{-1}$ ,  $\phi_i^1(\alpha_i)\gamma_i\phi_i^2(\alpha_i)^{-1}\gamma_i^{-1}$ ,  $1 \leq i \leq k$ , where  $\alpha_i \in \pi_1(B_i, p_i)$ . This computation is purely algebraic, because by the construction of our proposition  $\phi_i^j(\pi_1(B_i, p_i))$  is a maximal system of pairwise nonconjugate maximal almost nilpotent subgroups of rank  $n - 1$ :  $\pi_1(W)$  is an amalgamated product with itself on the maximal almost nilpotent subgroups of rank  $n - 1$ .

*Proof of (b).* To prove that  $W$  is a  $K(\pi, 1)$ -manifold, we note:

(i)  $B_i \subset M$  is, as a quotient of a horosphere, a  $K(\pi, 1)$ -manifold.

(ii) By construction, the inclusion  $B_i \subset M$  induces an injection  $\pi_1(B_i) \rightarrow \pi_1(M)$ .

(iii) It is easy to see that the inclusions  $M^1, M^2 \subset W$  induce injections  $\pi_1(M^j) \rightarrow \pi_1(W)$ .

Now  $W$  is a  $K(\pi, 1)$ -manifold by the following lemma, which is an easy consequence of Whitehead's theorem [1, p. 49].

**Lemma 6.** *Let  $W$  be a CW-complex which is the union of two connected subcomplexes  $M^1$  and  $M^2$  whose intersection consists of  $k + 1$  components  $B_0, \dots, B_k$ . Let  $M^1, M^2, B_0, \dots, B_k$  be  $K(\pi, 1)$ -spaces and the maps  $\pi_1(B_i) \rightarrow \pi_1(W)$ ,  $\pi_1(M^j) \rightarrow \pi_1(W)$ , induced by the inclusions, be injective. Then  $W$  is a  $K(\pi, 1)$ -manifold.*



## References

- [1] K. S. Brown, *Cohomology of groups*, Graduate Texts in Math., No. 87, Springer, Berlin, 1982.
- [2] P. Buser & H. Karcher, *Gromov's almost flat manifolds*, Astérisque 81, Paris, 1981.
- [3] P. Eberlein, *Lattices in spaces of nonpositive curvature*, Ann. of Math. **111** (1980) 435–476.
- [4] P. Eberlein & B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973) 45–109.
- [5] M. Gromov, *Manifolds of negative curvature*, J. Differential Geometry **13** (1978) 223–230.
- [6] E. Heintze, *Mannigfaltigkeiten negativer Krümmung*, Habilitationsschrift, Universität Bonn, 1976.
- [7] E. Heintze & H. C. Im Hof, *Geometry of horospheres*, J. Differential Geometry **12** (1977) 481–491.
- [8] J. Milnor, *Morse theory*, Annals of Math. Studies No. 51, Princeton University Press, Princeton, 1963.
- [9] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, Berlin, 1972.
- [10] V. Schroeder, *Über die Fundamentalgruppe von Räumen nichtpositiver Krümmung und endlichem Volumen*, Münster, 1984.
- [11] W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton, 1978.
- [12] I. A. Zaidenman, *On the fundamental group of the sum of two connected polyhedrons with unconnected intersection*, Moskov. Gos. Univ. Uch. Zap. **163** (1952) Mat. 6, 69–71. (Russian)

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