

FINITE VOLUME METHODS FOR ELLIPTIC PDE'S: A NEW APPROACH

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Abstract. We consider a new formulation for finite volume element methods, which is satisfied by known finite volume methods and it can be used to introduce new ones. This framework results by approximating the test function in the formulation of finite element method. We analyze piecewise linear conforming or nonconforming approximations on nonuniform triangulations and prove optimal order H^1 -norm and L^2 -norm error estimates.

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1. INTRODUCTION

Finite volume method is a discretization technique widely used in the approximation of conservation laws, in computational fluid dynamics, in nonlinear problems in two or more dimensions, and in convection–diffusion problems see *e.g.* [6, 11, 15, 16, 21, 24]. It can be viewed as a generalization of the finite difference method in triangular meshes. Apart from an approximation of the solution at discrete points, like in finite differences, we can seek an approximation in a finite element space. This version of the finite volume method is often called finite volume element method. It is possible to consider finite volume methods as an approximation of the finite element method through an appropriate quadrature rule, in 1–D, *cf.*, *e.g.*, Roos *et al.* [26, Chap. I, 2.3, p. 77], or in 2–D in quadrilateral meshes, *cf.*, *e.g.*, Weiser and Wheeler [29].

We are interested in investigating a possible connection between the finite element and finite volume element methods using piecewise linear functions in triangular meshes, for second order elliptic boundary value problems. This will increase the understanding of these methods, will help us in the analysis of finite volumes and motivate us in developing new finite volume methods. Here we present a unified framework that describes known finite volume element methods and it provides a connection with the finite element method. In addition based on the proposed framework we construct new finite volume element methods having as control volumes the support of the basis functions of the corresponding finite element spaces.

We consider an elliptic boundary value problem of the form: Given $f \in L^2(\Omega)$, find a function $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$Lu = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

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with Ω a bounded, convex, polygonal domain in \mathbb{R}^2 , $Lv \equiv -\operatorname{div}(A\nabla v)$, $A = (a_{ij})_{i,j=1}^2$ a given symmetric matrix function with real-value entries $a_{ij} \in W^{1,\infty}(\Omega)$, $1 \leq i, j \leq 2$. We assume that the matrix A is uniformly positive definite, *i.e.*, there exists a positive constant α_0 such that

$$\xi^T A(x)\xi \geq \alpha_0 \xi^T \xi, \quad \forall \xi \in \mathbb{R}^2, \quad \forall x \in \bar{\Omega}. \quad (1.2)$$

Problem (1.1) can be equivalently written in the form

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad (1.3)$$

where $a(v, w) = \sum_{K \in T_h} \int_K A\nabla v \cdot \nabla w \, dx$, (\cdot, \cdot) the usual inner product in $L^2(\Omega)$, T_h a given triangulation of Ω and h the maximum diameter of the triangles of T_h . Given a finite dimensional space V_h , $V_h \subset H_0^1(\Omega)$ (conforming elements), or $V_h \not\subset H_0^1(\Omega)$ (nonconforming elements), the finite element approximate solution $u_h \in V_h$ satisfies (1.3) with test functions in V_h , *i.e.*

$$a(u_h, \chi) = (f, \chi), \quad \forall \chi \in V_h. \quad (1.4)$$

Here we will restrict our attention to piecewise linear functions in T_h , thus V_h will be either the standard finite element space of piecewise linear functions or the Crouzeix–Raviart finite element space, *cf.* [14].

In contrast to the finite element method which is based on a global variational form, finite volume methods rely on the local property of (1.1),

$$-\int_{\partial\mathcal{O}} (A\nabla u) \cdot n \, ds = \int_{\mathcal{O}} f \, dx, \quad \forall \mathcal{O} \subset \Omega. \quad (1.5)$$

In the finite volume method we seek an approximation $u_h \in V_h$ of u which satisfies (1.5) on a finite collection T_h' of polygonal domains $b \subset \Omega$, called control volumes;

$$-\int_{\partial b} (A\nabla u_h) \cdot n \, ds = \int_b f \, dx, \quad \forall b \in T_h'. \quad (1.6)$$

In order to construct an approximate solution of (1.1) the number of control volumes must be equal to the dimension of V_h . For these methods H^1 - and L^2 -norm error estimates are known, *cf.*, *e.g.* [2, 5, 19] and [12]. In addition in [19] there is a particularly interesting counterexample of a finite volume method which does not converge with optimal rate in the L^2 -norm. However, in [12] and [15], optimal order L^2 -norm convergence are proved if the control volumes are constructed using the barycenter of the triangles.

Finite volume schemes using nonconforming piecewise linear approximations were considered by the author in [7] and for Stokes problem in [11] and [20].

New formulation

It is known that in the case of Poisson's equation the corresponding bilinear forms of the finite element and finite volume method when using either conforming or nonconforming piecewise linear approximations, are identical, *cf.*, *e.g.* [2, Lem. 3] and [7, Lem. 3.1 and 3.2]. In Section 3 we will show that these "classic" finite volume methods can be written in the form

$$a(u_h, \chi) = (f, Q_1\chi), \quad \forall \chi \in V_h, \quad (1.7)$$

where $Q_1\chi$ is defined as an appropriate linear combination of point values of χ , *cf.* (3.2) and (3.6).

However, in the general case of the operator A with variable coefficients, (1.7) is no longer valid and matters are more complicated. Here we propose the following formulation

$$a_h(u_h, \chi) \equiv \sum_{K \in T_h} \{(A \nabla u_h \cdot n, Q_2^K \chi)_{\partial K} + (Lu_h, Q_1 \chi)_K\} = (f, Q_1 \chi), \quad (1.8)$$

with Q_1 as in (1.7) and Q_2^K similarly defined to Q_1 . Then in Section 3 we show that the finite volume methods of [2, 5], (conforming) and of [7] (nonconforming) can be written in this form, if $Q_2^K \equiv Q_1$. The formulation (1.8) covers also the finite element method since if $Q_1 = Q_2^K \equiv 1$ we have

$$\sum_{K \in T_h} \{(A \nabla u_h \cdot n, \chi)_{\partial K} + (Lu_h, \chi)_K\} = a(u_h, \chi), \quad \forall \chi \in V_h,$$

with $(\cdot, \cdot)_{\partial K}$ and $(\cdot, \cdot)_K$ the usual inner products in $L^2(\partial K)$ and $L^2(K)$, respectively. In addition (1.8) is a generalization of (1.7). Indeed, in the case that A is a 2×2 constant matrix, in view the required assumption $\int Q_2^K \chi = \int \chi$ over the sides of the triangles, *cf.* Section 6, and that $L\chi = 0$, for $\chi \in V_h$, we get

$$a_h(\chi, \psi) = \sum_K (A \nabla \chi \cdot n, \psi)_{\partial K} = a(\chi, \psi), \quad \forall \chi, \psi \in V_h.$$

The fact that the finite volume methods can be written in the general form of (1.7) and (1.8) clarifies the connection with the finite element method. Thus using this formulation we can significantly simplify known proofs and prove new results in L^2 , without any assumption on the mesh other than regularity. Similar approaches were also developed in [12, 15] and [23]. Note that according to our knowledge there is not another unified formulation that covers the analysis of conforming and nonconforming finite volume element methods.

New finite volume methods

The discrete method (1.8) leads to the introduction of new finite volume schemes, with the characteristic property of having as control volumes the support domains of the basis functions of the finite element space, (in this case the control volumes overlap), *cf.* Section 4. These methods are clearly simpler to implement since the dual mesh is constructed using elements of the primal mesh. They admit a local form which in the case of constant coefficients with conforming piecewise linear approximations reads

$$\int_{\partial b} \nabla u_h \cdot n \, ds = \frac{2}{3} \int_b f \, dx, \quad (1.9)$$

i.e., the discrete local conservation over the control volumes (1.6) is no longer satisfied, (a rather expected fact since the control volumes overlap). We show error estimates of optimal order in H^1 - and L^2 -norm. An implicit conclusion of our results is that the finite volume scheme on these control volumes

$$\int_{\partial b} \nabla u_h \cdot n \, ds = \int_b f \, dx,$$

diverges. A rather surprising fact is the factor $\frac{2}{3}$ in (1.9) leads to a convergent method with optimal order. Similar results hold for the nonconforming case.

New convergence results

Even though H^1 -norm error estimates are optimal in order and in regularity, the L^2 -norm error estimates require additional regularity assumptions for the exact solution than the finite element method, *i.e.*, $u \in W^{3,p}(\Omega)$, *cf.* [12] and [15]. In Section 5 we show that the solution u_h of (1.8) satisfies optimal order H^1 - and L^2 -norm error estimates. The L^2 -norm estimate holds if $f \in W^{1,p}$, $1 < p \leq 2$, $A \in W^{2,\infty}$ and $\int Q_1$ defines a

quadrature rule for area integrals exact for piecewise linear functions. Thus, we derive L^2 -norm error estimates with $u \in H^2$ and additional assumptions posed only on the data f . Similar error estimates were derived by Hackbusch, in [18], in the case of a diagonal matrix function A . Note that the error analysis presented here works for both conforming and nonconforming finite element spaces.

Extensions

- *New methods.* In principle the proposed formulation (1.8) is not restricted to the finite dimensional spaces V_h of piecewise linear functions. Thus using different finite dimensional spaces V_h and operators Q_1 and Q_2^K , than the ones proposed here, we can construct new methods, possibly better than the ones considered here.
- *High order methods–Overlapping.* The analysis of finite volume methods with nonoverlapping control volumes, using piecewise polynomials of degree greater than one is involved and restricted, to our knowledge, to quadratic polynomials, *cf.* [22]. One reason is that the construction of the dual mesh becomes more complex as the degree of the polynomial approximation increases. However, the use of the support of the basis functions as control volumes seems to overcome this problem. The general approach, suggested here, has been used by Plexousakis and Zouraris in [25] for the construction and analysis of conservative high order finite volume methods.
- *New results.* Based on the formulation (1.8) we can derive error estimates in various problems related to (1.1). Thus, for the Stokes problem, the *a posteriori* error estimators for the H^1 - and L^2 -norm are similar to the ones used in finite element method, *e.g.* for nonconforming piecewise linear approximations, *cf.* [10]. Also, known error estimates for the parabolic problem can be improved, regarding the required regularity of the exact solution, *cf.* [8]. In addition note that in the case of a nonconvex polygonal domain Ω the exact solution u of (1.1) will not be $W^{3,p}(\Omega)$, $p > 1$, *cf.*, *e.g.* [17, Chap. 5]. In [9], we derive H^1 - and L^2 -norm error estimates for (1.1), where Ω is a nonconvex polygonal domain, with minimum assumptions on the exact solution u .

An outline of the paper is the following. In Section 2 we introduce notation and in Section 3 we show that the classic finite volume methods can be written in the formulation (1.8). In Section 4 we describe new methods with overlapping control volumes whose construction is motivated by (1.8). Then in Sections 5 and 6, under sufficient assumptions for Q_1 and Q_2^K , we analyze the discrete scheme (1.8) and prove optimal order error estimates.

2. PRELIMINARIES

Consider a bounded, convex polygonal domain Ω and a triangulation T_h of Ω . Let h_K be the diameter of the triangle $K \in T_h$ and $h = \max_{K \in T_h} h_K$. We assume that the family $\{T_h\}_{0 < h < 1}$ of triangulations is nondegenerate, *i.e.* there exists a positive constant σ , independent of h , such that, for every $K \in T_h$, $\rho_K \geq \sigma h_K$, with ρ_K denoting the diameter of the largest circle contained in $K \in T_h$, see, *e.g.* [3, p. 106]. This assumption is equivalent to the minimal angle condition.

Given a domain $K \subset \mathbb{R}^2$, denote by $L^2(K)$ the space of square integrable real functions over K , $(\cdot, \cdot)_K$ the inner product in $L^2(K)$, $|\cdot|_{s,K}$ and $\|\cdot\|_{s,K}$ the seminorm and norm, respectively, of the Sobolev space $H^s(K)$, $|\cdot|_{W^{s,p}(K)}$ and $\|\cdot\|_{W^{s,p}(K)}$ the seminorm and norm, respectively, of the Sobolev space $W^{s,p}(K)$, $s \in \mathbb{N}$, $p \geq 1$. Then, we introduce the norm $\|\cdot\|_{s,h}$, $\|v\|_{s,h} = \{\sum_K \|v\|_{s,K}^2\}^{1/2}$, and the seminorm $|\cdot|_{s,h}$, $|v|_{s,h} = \{\sum_K |v|_{s,K}^2\}^{1/2}$, $s \in \mathbb{N}$. In addition, if $K = \Omega$ we suppress the index K , and if $s = 0$ we suppress the index s . Also with $|\cdot|$ we denote either the area of a region in \mathbb{R}^2 or the length of a line segment, and with $\|\cdot\|_{\mathbb{R}^2}$ the Euclidean norm on \mathbb{R}^2 and also the subordinate matrix norm.

In the sequel we introduce notation. Given a triangle $K \in T_h$, let $Z_h(K)$ and $E_h(K)$ be the set of the vertices and sides, respectively, of K , $Z_h = \cup_{K \in T_h} Z_h(K)$ and $E_h = \cup_{K \in T_h} E_h(K)$. Next, let Z_h^{in} and E_h^{in} be the set of the interior vertices and sides, respectively, of T_h and denote by m_e the middle point of a side $e \in E_h$.

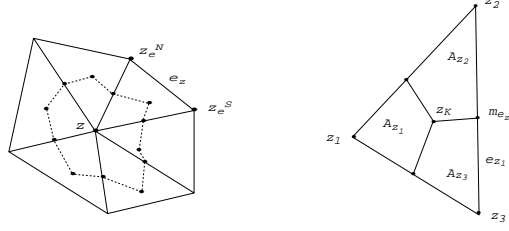


FIGURE 1. *Left:* A sample region V_z . With dotted lines the corresponding box b_z . *Right:* A triangle K partitioned into three subregions A_z .

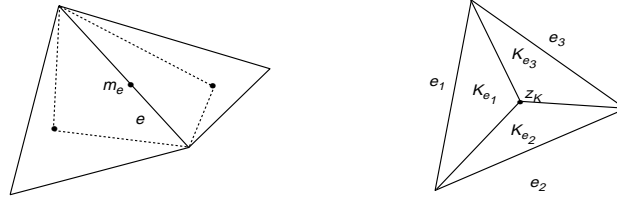


FIGURE 2. *Left:* A sample region V_e . With dotted lines the corresponding box b_e . *Right:* A triangle K partitioned into three subtriangles K_e .

Next we will consider the finite collections of control volumes \mathcal{V}_c , \mathcal{B}_c , \mathcal{V}_n and \mathcal{B}_n . With each vertex $z \in Z_h^{\text{in}}$ we associate a region $V_z \subset \Omega$, consisting of those triangles $K \in T_h$ that have z as a vertex and denote $\mathcal{V}_c = \{V_z : z \in Z_h^{\text{in}}\}$.

We construct \mathcal{B}_c in the following way: For each triangle $K \in T_h$ consider an interior point z_K . Connect z_K with line segments to the edge midpoints of K . This partitions K into three subregions A_z , $z \in Z_h(K)$, see Figure 1. Let $\mathcal{A}(K) = \{A_z : z \in Z_h(K)\}$. With each vertex $z \in Z_h^{\text{in}}$ we associate a box $b_z \subset V_z$, which consists of the union of the subregions A_z in V_z , which have z as a corner, see Figure 1, and let $\mathcal{B}_c = \{b_z : z \in Z_h^{\text{in}}\}$.

Also, with any $e \in E_h^{\text{in}}$ we associate a region V_e , consisting of the two triangles of T_h with common side e , and let $\mathcal{V}_n = \{V_e : e \in E_h^{\text{in}}\}$, see Figure 2.

Finally consider an interior point z_K of $K \in T_h$ and connect it with line segments to the vertices of K . Thus we partition K to three subtriangles K_e , $e \in E_h(K)$. Let $\tilde{T}_h(K)$ be the set $\{K_e : e \in E_h(K)\}$. Now, with each side $e \in E_h^{\text{in}}$ we associate a box $b_e \subset V_e$ consisting of the two subtriangles K_e , in V_e , which have e as a common side and let $\mathcal{B}_n = \{b_e : e \in E_h^{\text{in}}\}$.

Consider now, the finite dimensional space $X_h^0 = \{\chi \in C(\Omega) : \chi|_K \text{ is linear for all } K \in T_h \text{ and } \chi|_{\partial\Omega} = 0\}$ and the finite dimensional space S_h^0 which consists of the piecewise linear functions in $K \in T_h$ that are continuous at the middle points of a side $e \in E_h$, m_e , and $\chi(m_e) = 0$ if $e \in E_h \setminus E_h^{\text{in}}$. Note that S_h^0 is not a subspace of H_0^1 . In general we will denote X_h^0 and S_h^0 with V_h .

We have that X_h^0 and S_h^0 satisfy the following approximation property

$$\inf_{\chi \in V_h} \|v - \chi\|_{j,h} \leq C \left(\sum_{K \in T_h} h_K^{2(2-j)} \|v\|_{2,K}^2 \right)^{1/2}, \quad \forall v \in H^2 \cap H_0^1, \quad j = 0, 1, \quad (2.1)$$

cf., e.g. [13, Th. 15.3].

3. APPLICATIONS I: CLASSIC FINITE VOLUME METHODS

In this section we consider two “classic” finite volume schemes for (1.1), using as an approximation space conforming or nonconforming piecewise linear functions on a given triangulation T_h . Then for appropriate

choices of operator Q_1 given by (3.2) and (3.6), respectively, we can rewrite them in the form (1.8). Therefore, the error estimates proved for (1.8) in Section 5, *cf.* Theorems 5.1 and 5.2, will also hold for these methods.

Method I: We consider the finite volume approximations in X_h^0 of (1.1), with control volumes in \mathcal{B}_c , *i.e.* Find $u_h \in X_h^0$ such that

$$-\int_{\partial b_z} (A\nabla u_h) \cdot n \, ds = \int_{b_z} f \, dx, \quad \forall z \in Z_h^{\text{in}}, \quad (3.1)$$

which can be equivalently rewritten

$$\bar{a}_{CN}(u_h, \chi) = \sum_{z \in Z_h^{\text{in}}} \chi(z) \int_{b_z} f \, dx, \quad \forall \chi \in X_h^0,$$

with $\bar{a}_{CN}(\cdot, \cdot) : H_h^2 \times H_h^2 \rightarrow \mathbb{R}$ defined by $\bar{a}_{CN}(v, w) = -\sum_{z \in Z_h^{\text{in}}} w(z) \int_{\partial b_z} (A\nabla v) \cdot n \, ds$ and $H_h^2 = H^2 + X_h^0$.

Next, we will show that this discrete scheme can be written in the form (1.8). Let us consider the linear operator $Q_1|_K : \mathbb{P}_1(K) \rightarrow L^2(K)$ defined by,

$$Q_1\chi|_K = \sum_{z \in Z_h(K)} \chi(z) q_{A_z}, \quad (3.2)$$

with q_S the characteristic function of S and $A_z \in \mathcal{A}(K)$ the regions in which we partition $K \in T_h$, see Figure 1. Then, for every $f \in L^2$ and $\chi \in X_h^0$, it is easily seen that

$$(f, Q_1\chi) = \sum_K \sum_{z \in Z_h(K)} \chi(z) \int_{A_z} f \, dx = \sum_{z \in Z_h^{\text{in}}} \chi(z) \int_{b_z} f \, dx. \quad (3.3)$$

In the case of Poisson's equation it is known, *cf.* [2, Lem. 3], that

$$a(\chi, \psi) = \bar{a}_{CN}(\chi, \psi), \quad \forall \chi, \psi \in X_h^0. \quad (3.4)$$

Thus, in view of (3.3) the finite volume method (3.1) can be written equivalently in the form (1.7). In the general case now, by integration by parts we obtain

$$\int_{A_z} L\chi \, dx + \int_{\partial A_z \cap \partial K} A\nabla\chi \cdot n \, ds = - \int_{\partial A_z \cap \partial b_z} A\nabla\chi \cdot n \, ds, \quad \forall \chi \in X_h^0,$$

with $z \in Z_h$ and $K \in T_h$. Multiplying then with $\psi(z)$, $\psi \in X_h^0$, and summing subsequently over the triangles that have z as a common vertex and the vertices $z \in Z_h^{\text{in}}$, we get $a_h(\chi, \psi) = \bar{a}_{CN}(\chi, \psi)$. Hence, the solution u_h of (3.1) satisfies (1.8).

Method II: In [7] we consider finite volume approximations for (1.1) in S_h^0 , with control volumes in \mathcal{B}_n , *i.e.*, find $u_h \in S_h^0$ such that

$$-\int_{\partial b_e} (A\nabla u_h) \cdot n \, ds = \int_{b_e} f \, dx, \quad \forall e \in E_h^{\text{in}}, \quad (3.5)$$

which can be equivalently rewritten

$$\bar{a}_{NN}(u_h, \chi) = \sum_{e \in E_h^{\text{in}}} \chi(m_e) \int_{b_e} f \, dx, \quad \forall \chi \in S_h^0,$$

with $\bar{a}_{NN}(\cdot, \cdot) : \tilde{H}_h^2 \times \tilde{H}_h^2 \rightarrow \mathbb{R}$ defined by $\bar{a}_{NN}(v, w) = -\sum_{e \in E_h^{\text{in}}} w(m_e) \int_{\partial b_e} (A\nabla v) \cdot n \, ds$ and $\tilde{H}_h^2 = H^2 + S_h^0$.

Similarly, as in Method I, we can show that this discrete scheme can be written in the form (1.8), where the linear operator $Q_1|_K : \mathbb{P}_1(K) \rightarrow L^2(K)$ is defined by

$$Q_1\chi|_K = \sum_{e \in E_h(K)} \chi(m_e) q_{K_e}, \quad (3.6)$$

with $K_e \in \tilde{T}_h(K)$ are the regions we partition $K \in T_h$ see Figure 2 and q_S the characteristic function of a set $S \subset \mathbb{R}^2$.

Remark 3.1. If in the construction of the control volumes \mathcal{B}_c and \mathcal{B}_c we choose z_K to be the barycenter of $K \in T_h$, we have

$$\int_K Q_1\chi \, dx = \int_K \chi \, dx, \quad \forall \chi \in \mathbb{P}_1(K),$$

with Q_1 defined by either (3.2) or (3.6). Therefore, $\int Q_1$ defines a quadrature rule for area integrals which is exact for linear functions. \square

4. APPLICATIONS II: OVERLAPPING CONTROL VOLUMES

In this section we consider two new finite volume schemes for (1.1) with control volumes in \mathcal{V}_c and \mathcal{V}_n , using as an approximation space conforming and nonconforming piecewise linear functions, respectively, on a given triangulation T_h . For appropriate choices of operators Q_1 and Q_2^K we can rewrite them in the form (1.8). In contrast to Methods I and II, here the solution u of (1.1) and its approximation u_h do not satisfy the same local conservation property, *cf.*, *e.g.*, (4.4) and (4.5). However, the construction of the dual mesh is simpler. This approach helps in developing new high order finite volume methods, *cf.* [25].

In Method III we seek $u_h \in X_h^0$ such that

$$-\int_{\partial V_z} (A\nabla u_h) \cdot n \, ds + \frac{1}{2} \sum_{e \in E_h^*(V_z)} \int_e [A\nabla u_h] \cdot n_e \, ds = \int_{V_z} f \, dx, \quad \forall z \in Z_h^{\text{in}}, \quad (4.1)$$

and in Method IV we seek $u_h \in S_h^0$ such that

$$-\int_{\partial V_e} (A\nabla u_h) \cdot n \, ds + 2 \int_e [A\nabla u_h] \cdot n_e \, ds = \int_{V_e} f \, dx, \quad \forall e \in E_h^{\text{in}}, \quad (4.2)$$

where

$$[A\nabla \chi] \cdot n_e = (A\nabla \chi_{K^{+e}}) \cdot n_e - (A\nabla \chi_{K^{-e}}) \cdot n_e, \quad (4.3)$$

with K^{+e} , K^{-e} denoting two triangles having $e \in E_h^{\text{in}}$ as a common edge and n_e the normal vector of $e \in E_h(K^{+e})$, *cf.* Figure 3, and $E_h^*(V_z) = \{e \in E_h : z \in e\}$, *cf.* Figure 3.

Method III: We consider finite volume approximations in X_h^0 with control volumes in \mathcal{V}_c . As a first step we will construct the method for Poisson's equation and then for the general equation (1.1).

The balance equation for Poisson's equation over the control volume $V_z \in \mathcal{V}_c$ is

$$-\int_{\partial V_z} \nabla u \cdot n \, ds = \int_{V_z} f \, dx. \quad (4.4)$$

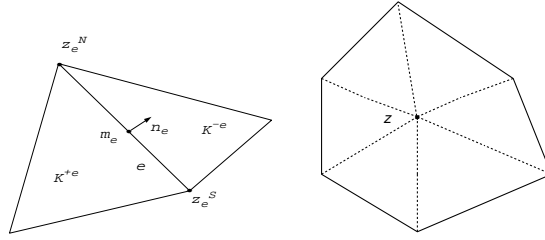


FIGURE 3. *Left:* The triangles K^{+e} and K^{-e} with common side e . *Right:* A box V_z ; with dotted lines the sides of the set $E_h^*(V_z)$.

The appropriate finite volume method in this case is: Find $u_h \in X_h^0$ such that

$$-\int_{\partial V_z} \nabla u_h \cdot n \, ds = \frac{2}{3} \int_{V_z} f \, dx, \quad \forall z \in Z_h^{\text{in}}, \quad (4.5)$$

which can be equivalently rewritten

$$\bar{a}_{CO}(u_h, \chi) = \frac{2}{3} \sum_{z \in Z_h^{\text{in}}} \chi(z) \int_{V_z} f \, dx,$$

where $\bar{a}_{CO}(\cdot, \cdot) : H_h^2 \times H_h^2 \rightarrow \mathbb{R}$ is defined by $\bar{a}_{CO}(v, w) = -\sum_{z \in Z_h^{\text{in}}} w(z) \int_{\partial V_z} \nabla v \cdot n \, ds$.

For this method the corresponding relation to (3.4) does not hold; however we can prove the following lemma.

Lemma 4.1. *Let $\chi, \psi \in X_h^0$. Then*

$$\bar{a}_{CO}(\chi, \psi) = 2a(\chi, \psi).$$

Proof. Let $\{\omega_z\}_{z \in Z_h}$ be the usual basis of X_h^0 , where ω_z is one on vertex z and zero in every other. Then, in order to get the desired relation it is sufficient to show it for $\psi = \omega_z$. For $z \in K$ and $\chi \in X_h^0$, we have

$$(\nabla \chi, \nabla \omega_z)_K = \int_{\partial K} \omega_z \nabla \chi \cdot n \, ds = \sum_{e \in E_h(K)} \nabla \chi \cdot n \int_e \omega_z \, ds. \quad (4.6)$$

Since ω_z is a linear polynomial on every side of K , we get

$$\int_e \omega_z \, ds = \frac{|e|}{2} (\omega_z(z_e^S) + \omega_z(z_e^N)), \quad \forall e \in E_h(K), \quad (4.7)$$

where by $|e|$ we denote the length of e . Therefore,

$$\int_{e_z} \omega_z \, ds = 0 \quad \text{and} \quad \int_\ell \omega_z \, ds = \frac{|e|}{2}, \quad \text{for } \ell \in E_h(K) \setminus \{e_z\}, \quad (4.8)$$

with e_z the side of K opposite to z , see Figure 1. Combining now (4.6)–(4.8), we easily obtain

$$\int_K \nabla \chi \cdot \nabla \omega_z \, dx = \frac{1}{2} \sum_{e \in E_h(K) \setminus \{e_z\}} \int_e \nabla \chi \cdot n \, ds = -\frac{1}{2} \int_{e_z} \nabla \chi \cdot n \, ds.$$

□

Next, let us consider the linear operator $Q_1|_K : \mathbb{P}_1(K) \rightarrow L^2(K)$ defined by

$$Q_1\chi|_K = \frac{1}{3} \sum_{z \in Z_h(K)} \chi(z), \quad (4.9)$$

with $K \in T_h$ and $\chi \in X_h^0$. We can easily see for every $f \in L^2$ and $\chi \in X_h^0$

$$(f, Q_1\chi) = \frac{1}{3} \sum_K \sum_{z \in Z_h(K)} \chi(z) \int_K f \, dx = \frac{1}{3} \sum_{z \in Z_h^{\text{in}}} \chi(z) \int_{V_z} f \, dx. \quad (4.10)$$

Thus, (4.5) can be written equivalently in the form (1.7).

In the general case of equation (1.1) the obvious generalization of (4.5) does not seem to be the correct one. In order to derive the “correct” method with control volumes in \mathcal{V}_c we will consider appropriate operators Q_1 and Q_2^K in the discrete method (1.8). Let Q_1 be defined the same way as above and Q_2^K be a linear operator $Q_2^K|_e : \mathbb{P}_1(K) \rightarrow L^2(e)$, be defined by

$$Q_2^K\chi|_e = \frac{1}{2}(\chi(z_e^N) + \chi(z_e^S)), \quad (4.11)$$

with $e \in E_h(K)$, and z_e^N and z_e^S the two vertices of the e , see Figure 3. Then, the bilinear form a_h can be written in the form

$$a_h(\chi, \psi) = \frac{1}{2} \sum_K \sum_{e \in E_h(K)} (A\nabla\chi \cdot n, \psi(z_e^N) + \psi(z_e^S))_e + \frac{1}{3} \sum_{z \in Z_h^{\text{in}}} \psi(z) \sum_{K \in T_h(V_z)} \int_K L\chi \, dx,$$

with $T_h(V_z)$ denoting the triangles forming V_z .

Let us consider now a triangle $K \in T_h$ and a vertex $z \in Z_h(K) \cap Z_h^{\text{in}}$; by integration by parts we obtain

$$\int_K L\chi \, dx + \frac{3}{2} \int_{\partial K \setminus \partial V_z} A\nabla\chi \cdot n \, ds = - \int_{\partial K \cap \partial V_z} A\nabla\chi \cdot n \, ds + \frac{1}{2} \int_{\partial K \setminus \partial V_z} A\nabla\chi \cdot n \, ds.$$

Multiplying then with $\psi(z)$, $\psi \in X_h^0$, and summing subsequently over the triangles that have z as a common vertex and the vertices $z \in Z_h^{\text{in}}$, we get

$$3a_h(\chi, \psi) = \bar{a}_{CO}(\chi, \psi) + \frac{1}{2} \sum_{z \in Z_h^{\text{in}}} \psi(z) \sum_{e \in E_h^*(V_z)} \int_e [A\nabla\chi] \cdot n_e \, ds,$$

with $[A\nabla\chi] \cdot n_e$ and n_e as in (4.3), $E_h^*(V_z) = \{e \in E_h : z \in e\}$, cf. Figure 3 and $\bar{a}_{CO}(\cdot, \cdot) : H_h^2 \times H_h^2 \rightarrow \mathbb{R}$ defined by $\bar{a}_{CO}(v, w) = - \sum_{z \in Z_h^{\text{in}}} w(z) \int_{\partial V_z} (A\nabla v) \cdot n \, ds$. Thus the relation above and (4.10) suggest that the method (1.8) can be written equivalently in the form (4.1).

Method IV: This time we consider finite volume approximations in S_h^0 with control volumes in \mathcal{V}_n . As we did in Method III, we develop this method first for Poisson’s equation. The balance equation over the control volume $V_e \in \mathcal{V}_n$ is

$$- \int_{\partial V_e} \nabla u \cdot n \, ds = \int_{V_e} f \, dx.$$

The appropriate finite volume method in this case is: Find $u_h \in S_h^0$ such that

$$- \int_{\partial V_e} \nabla u_h \cdot n \, ds = \frac{1}{3} \int_{V_e} f \, dx, \quad \forall e \in E_h^{\text{in}}, \quad (4.12)$$

which can be equivalently rewritten

$$\bar{a}_{NO}(u_h, \chi) = \frac{1}{3} \sum_{e \in E_h^{\text{in}}} \chi(m_e) \int_{V_e} f \, dx, \quad \forall \chi \in S_h^0,$$

with $\bar{a}_{NO}(\cdot, \cdot) : \tilde{H}_h^2 \times \tilde{H}_h^2 \rightarrow \mathbb{R}$ defined by $\bar{a}_{NO}(v, w) = - \sum_{e \in E_h^{\text{in}}} w(m_e) \int_{\partial V_e} \nabla v \cdot n \, ds$.

Using Green's formula and the fact that $\chi|_K \in \mathbb{P}_1$, for every $\chi \in S_h^0$, we can prove

$$\bar{a}_{NN}(\chi, \psi) = \bar{a}_{NO}(\chi, \psi), \quad \forall \chi, \psi \in S_h^0.$$

Hence, in view of the fact that $\bar{a}_{NN}(\chi, \psi) = \bar{a}_{NO}(\chi, \psi)$, for every $\chi, \psi \in S_h^0$, cf. [7, Lem. 3.1 and 3.2], we have $\bar{a}_{NO}(\chi, \psi) = a(\chi, \psi)$.

Next, let us consider the linear operator $Q_1|_K : \mathbb{P}_1(K) \rightarrow L^2(K)$ defined by

$$Q_1 \chi|_K = \frac{1}{3} \sum_{e \in E_h(K)} \chi(m_e), \quad (4.13)$$

with $K \in T_h$. We can easily see for every $f \in L^2$ and $\chi \in S_h^0$

$$(f, Q_1 \chi) = \frac{1}{3} \sum_K \int_K f \sum_{e \in E_h(K)} \chi(m_e) \, dx = \frac{1}{3} \sum_{e \in E_h^{\text{in}}} \chi(m_e) \int_{V_e} f \, dx. \quad (4.14)$$

Thus, (4.12) can be written equivalently in the form (1.7).

In the general case of equation (1.1) the obvious generalization of (4.12) does not seem to be the correct one. In order to derive the ‘‘correct’’ method with control volumes in \mathcal{V}_n we will consider appropriate operators Q_1 and Q_2^K in the discrete method (1.8). Let Q_1 be defined as above and the linear operator $Q_2^K|_e : \mathbb{P}_1(K) \rightarrow L^2(e)$, be defined by

$$Q_2^K \chi|_e = \chi(m_e), \quad (4.15)$$

with $e \in E_h(K)$. Then the bilinear form a_h can be written in the form

$$a_h(\chi, \psi) = \sum_K \sum_{e \in E_h(K)} (A \nabla \chi \cdot n, \psi(m_e))_e + \frac{1}{3} \sum_{e \in E_h^{\text{in}}} \psi(z) \sum_{K \in T_h(V_e)} \int_K L \chi \, dx,$$

with $T_h(V_e)$ denoting the triangles with common edge e . Let us consider now a triangle $K \in T_h$ and an edge $e \in E_h(K) \cap E_h^{\text{in}}$; by integration by parts we obtain

$$\int_K L \chi \, dx + 3 \int_e A \nabla \chi \cdot n \, ds = - \int_{\partial K \cap \partial V_e} A \nabla \chi \cdot n \, ds + 2 \int_e A \nabla \chi \cdot n \, ds.$$

Multiplying then with $\psi(m_e)$, $\psi \in S_h^0$, and summing subsequently over the triangles that have e as a common edge and the vertices $e \in E_h^{\text{in}}$, we get

$$3a_h(\chi, \psi) = \bar{a}_{NO}(\chi, \psi) + 2 \sum_{E \in E_h^{\text{in}}} \psi(m_e) \int_e [A \nabla \chi] \cdot n_e \, ds, \quad \forall \chi, \psi \in S_h^0,$$

with $[A \nabla \chi] \cdot n_e$ and n_e as in (4.3) and $\bar{a}_{NO}(\cdot, \cdot) : \tilde{H}_h^2 \times \tilde{H}_h^2 \rightarrow \mathbb{R}$ defined by $\bar{a}_{NO}(v, w) = - \sum_{e \in E_h^{\text{in}}} w(m_e) \int_{\partial V_e} (A \nabla v) \cdot n \, ds$. Thus, the relation above and (4.14) suggest that the method (1.8) can be written equivalently in the form (4.2).

Remark 4.1. In contrast to Methods I and II the discrete schemes considered here do not satisfy the same local balance equation as the continuous problem. An issue to examine is the coefficients $2/3$ and $1/3$ on the right-hand side of (4.5) and (4.12), respectively. Motivated by the discussion in [4], we can see that the scheme (4.5) satisfies a discrete analog of the conservation property (1.5) in a dilation of the control volume V_z with ratio $\lambda = 2/3$, *i.e.*, let us assume that f is constant, and let $V_{z,2/3} = \{z + (2/3)(x - z) : x \in V_z\}$. Then if u_h satisfies (4.5),

$$-\int_{\partial V_{z,2/3}} \nabla u_h \cdot n \, ds = -\frac{2}{3} \int_{\partial V_z} \nabla u_h \cdot n \, ds = \left(\frac{2}{3}\right)^2 \int_{V_z} f \, dx = \int_{V_{z,2/3}} f \, dx;$$

cf. Schmidt [27], for a finite volume method with overlapping control volumes that preserves local conservation. Similarly, we can see that the scheme (4.12) satisfies a discrete version of the conservation property (1.5) in a dilation of the control volume V_e with ratio $\lambda = 1/3$. \square

Remark 4.2. It is obvious that for Q_1 defined by either (4.9) or (4.13), $\int Q_1$ defines a quadrature rule for area integrals exact for linear functions, *i.e.*

$$\int_K Q_1 \chi \, dx = \int_K \chi \, dx, \quad \forall \chi \in \mathbb{P}_1(K).$$

5. A GENERAL DISCRETE PROBLEM

In this section we will analyze the discrete method (1.8), and derive error estimates in H^1 - and L^2 -norm. Then, in view of the previous two sections, these error estimates will also hold for Methods I–IV.

We consider the problem: Find $u_h \in V_h$ such that

$$a_h(u_h, \chi) = (f, Q_1 \chi), \quad \forall \chi \in V_h, \quad (5.1)$$

with $a_h(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}$ defined by $a_h(\chi, \psi) = \sum_K \{(A \nabla \chi \cdot n, Q_2^K \psi)_{\partial K} + (L \chi, Q_1 \psi)_K\}$.

Essential part in our analysis have the errors ε_h and ε_a

$$\begin{aligned} \varepsilon_h(f, \psi) &= (f, \psi) - (f, Q_1 \psi), \quad \forall f \in L^2, \psi \in V_h, \\ \varepsilon_a(\chi, \psi) &= a(\chi, \psi) - a_h(\chi, \psi), \quad \forall \chi, \psi \in V_h. \end{aligned}$$

In Section 6 we will give appropriate assumptions for Q_1 and Q_2^K and the proof of Lemmas 5.1 and 5.2.

Lemma 5.1. *There exists a constant $C = C(\sigma)$, such that*

$$|\varepsilon_h(f, \chi)| \leq C \sum_K h_K \|f\|_K |\chi|_{1,K}, \quad \forall f \in L^2, \forall \chi \in V_h, \quad (5.2)$$

$$|\varepsilon_h(f, \chi)| \leq C \sum_K h_K^2 |f|_{W^{1,p}(K)} |\chi|_{W^{1,q}(K)}, \quad \forall f \in W^{1,p}, \forall \chi \in V_h, \quad (5.3)$$

with $\int_K Q_1 \chi = \int_K \chi$, for $\chi \in \mathbb{P}_1(K)$, $K \in T_h$, and $\frac{1}{p} + \frac{1}{q} = 1$ in (5.3).

Lemma 5.2. *There exists a positive constant $C = C(\sigma, A)$ such that*

$$|\varepsilon_a(\psi, \chi)| \leq C \sum_K h_K |\psi|_{1,K} |\chi|_{1,K}, \quad \forall \chi, \psi \in V_h, \quad (5.4)$$

$$|\varepsilon_a(u_h, \chi)| \leq Ch(|u_h - u|_{1,h} + h|u|_2) |\chi|_{1,h} + Ch|u|_2 |\chi - \psi|_{1,h}, \quad (5.5)$$

with $\chi \in V_h$, $\psi \in X_h^0$ and $A \in W^{2,\infty}$ in (5.5).

Further, we have that the Poincaré–Friedrichs inequality holds in V_h , *i.e.*

$$\|\chi\| \leq C\|\chi\|_{1,h}, \quad \forall \chi \in V_h. \quad (5.6)$$

Remark 5.1. Clearly, (5.6) is satisfied when V_h is X_h^0 . In the case that V_h is S_h^0 , in view of [28, Prop. 4.13], we have that (5.6) also holds. \square

In view of Lemma 5.2 we can easily show that for h sufficiently small, method (5.1) satisfies

$$a_h(\chi, \chi) \geq \alpha_1 |\chi|_{1,h}^2.$$

Then, due to (5.6), we have that the solution u_h of (5.1) is uniquely defined. In the sequel we will show an optimal order H^1 –norm error estimation.

Theorem 5.1. *Let u be the solution of (1.1) and u_h the solution of (5.1). Then, there exists a constant C , independent of h , such that*

$$\|u - u_h\|_{1,h} \leq Ch\|u\|_2. \quad (5.7)$$

Proof. Due to the triangle inequality we have,

$$\|u - u_h\|_{1,h} \leq \|u - \chi\|_{1,h} + \|u_h - \chi\|_{1,h}, \quad \forall \chi \in V_h. \quad (5.8)$$

Thus, in view of the approximation properties (2.1) of V_h , it suffices to estimate the last part of (5.8). By the definition of $a(\cdot, \cdot)$ and (1.2) we have,

$$\|u_h - \chi\|_{1,h}^2 \leq C|a(u_h - \chi, u_h - \chi)|, \quad \forall \chi \in V_h. \quad (5.9)$$

Also, for every $\chi, \psi \in V_h$

$$|a(u_h - \chi, \psi)| = |a(u_h - u, \psi) + a(u - \chi, \psi)| \leq |a(u_h - u, \psi)| + C\|u - \chi\|_{1,h} \|\psi\|_{1,h}.$$

Therefore, choosing $\psi = u_h - \chi$ in the relation above and combining (5.9) we get

$$\|u_h - \chi\|_{1,h}^2 \leq C|a(u_h - u, u_h - \chi)| + C\|u - \chi\|_{1,h} \|u_h - \chi\|_{1,h}, \quad \forall \chi \in V_h. \quad (5.10)$$

We can easily see that for every $\psi \in V_h$

$$a(u - u_h, \psi) = a(u, \psi) - a_h(u_h, \psi) - \varepsilon_a(u_h, \psi) = \varepsilon_h(f, \psi) - \varepsilon_a(u_h, \psi). \quad (5.11)$$

Thus, choosing $\psi = u_h - \chi$ and combining Lemmas 5.1 and 5.2 and (5.10), for h sufficiently small, we obtain

$$\|u_h - \chi\|_{1,h} \leq C\|u - \chi\|_{1,h} + Ch\|u\|_2, \quad \forall \chi \in V_h.$$

\square

In the sequel we will estimate the difference $u - u_h$ in the L^2 –norm.

Theorem 5.2. *Let u be the solution of (1.1) and u_h the solution of (5.1). Assume that $f \in W^{1,p}$, $1 < p \leq 2$, $A \in W^{2,\infty}$ and*

$$\int_K \chi \, dx = \int_K Q_1 \chi \, dx, \quad \forall K \in T_h, \quad \forall \chi \in V_h.$$

Then, there exists a constant C , independent of h , such that

$$\|u - u_h\| \leq Ch^2 (\|u\|_2 + \|f\|_{W^{1,p}}).$$

Proof. We consider the following auxiliary problem: Seek $\varphi \in H^2$ such that

$$L\varphi = u - u_h \quad \text{in } \Omega \quad \text{and} \quad \varphi = 0 \quad \text{on } \partial\Omega. \quad (5.12)$$

It is well known, *cf.*, *e.g.* [17, Chap. 4], that

$$\|\varphi\|_2 \leq C\|u - u_h\|. \quad (5.13)$$

Using (5.12) and Green's formula, we easily obtain

$$\begin{aligned} \|u - u_h\|^2 &= (u - u_h, L\varphi) = a(u - u_h, \varphi) - \sum_K (A\nabla\varphi \cdot n, u - u_h)_{\partial K} \\ &= a(u - u_h, \varphi - \chi) + a(u - u_h, \chi) - \sum_K (A\nabla\varphi \cdot n, u - u_h)_{\partial K} \\ &= I + II + III. \end{aligned} \quad (5.14)$$

The first term can obviously be estimated in the following way

$$|I| \leq C\|u - u_h\|_{1,h} \|\varphi - \chi\|_{1,h}, \quad \forall \chi \in V_h. \quad (5.15)$$

In the sequel we will estimate *II* and *III*. In view of (5.11), we have

$$II = \varepsilon_h(f, \chi) - \varepsilon_a(u_h, \chi), \quad \forall \chi \in V_h.$$

Thus, in view of Lemmas 5.1 and 5.2, we have

$$\begin{aligned} |II| &\leq Ch^2 \sum_K |f|_{W^{1,p}(K)} |\chi|_{W^{1,q}(K)} + Ch(|u_h - u|_{1,h} + h|u|_2) |\chi|_{1,h} \\ &\quad + Ch|u|_2 |\chi - \psi|_{1,h}, \quad \forall \chi \in V_h, \quad \forall \psi \in X_h^0, \end{aligned} \quad (5.16)$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Next, let I_h^c be the Lagrange interpolant $I_h^c : H^2 \cap H_0^1 \rightarrow X_h^0$. It is well known that

$$\|I_h^c v - v\|_1 \leq Ch\|v\|_2, \quad \forall v \in H^2 \cap H_0^1.$$

Choosing now $\psi = I_h^c \varphi$ in (5.16) and using the estimation above we obtain

$$\begin{aligned} |II| &\leq Ch^2 \sum_K |f|_{W^{1,p}(K)} |\chi|_{W^{1,q}(K)} + Ch(|u_h - u|_{1,h} + h|u|_2) |\chi|_{1,h} \\ &\quad + Ch|u|_2 (|\chi - \varphi|_{1,h} + h\|\varphi\|_2), \quad \forall \chi \in V_h. \end{aligned} \quad (5.17)$$

Let us now introduce the operator \mathcal{M}_ℓ^μ as the orthogonal projection from $L^2(\ell)$ to \mathbb{P}_μ , defined by

$$\int_\ell w \mathcal{M}_\ell^\mu g \, ds = \int_\ell w g \, ds, \quad \forall w \in \mathbb{P}_\mu, \quad (5.18)$$

with ℓ a line segment of \mathbb{R}^2 and \mathbb{P}_μ the set of polynomials of two variables of degree at most $\mu \in \mathbb{N}$. Then according to [14, Lem. 3], for $K \in T_h$, we have

$$\left| \int_e \varphi (g - \mathcal{M}_e^0 g) \, ds \right| \leq Ch_K |\varphi|_{1,K} |g|_{1,K}, \quad \forall e \in E_h(K), \quad \forall \varphi, g \in H^1(K). \quad (5.19)$$

Also, using the fact that $u, \varphi \in H^2$, and for any $e \in E_h$,

$$\begin{aligned} \int_e (\chi|_{K_1} - \chi|_{K_2}) ds &= 0, \quad \forall \chi \in V_h, \text{ if } e = K_1 \cap K_2, \quad K_1, K_2 \in T_h, \\ \int_e \chi ds &= 0, \quad \forall \chi \in V_h, \text{ if } e \in E_h \setminus E_h^{\text{in}}, \end{aligned}$$

we obtain, *cf.*, *e.g.* [14],

$$\sum_K (A\nabla\varphi \cdot n, u - u_h)_{\partial K} = \sum_K \sum_{e \in E_h(K)} (A\nabla\varphi \cdot n - \mathcal{M}_e^0((A\nabla\varphi) \cdot n), u - u_h)_e.$$

Hence, in view of (5.19), we get

$$|III| = \left| \sum_K (A\nabla\varphi \cdot n, u - u_h)_{\partial K} \right| \leq Ch|\varphi|_2 \|u - u_h\|_{1,h}. \quad (5.20)$$

Next, let I_h be the Lagrange interpolant $I_h : H^2 \cap H_0^1 \rightarrow V_h$. It is known that

$$\begin{aligned} \|I_h v - v\|_{1,h} &\leq Ch\|v\|_2, \quad \forall v \in H^2 \cap H_0^1, \\ \|I_h v\|_{W^{1,q}(K)} &\leq C\|v\|_{2,K}, \quad \forall v \in H^2 \cap H_0^1, \quad 2 \leq q < \infty, \quad K \in T_h, \end{aligned} \quad (5.21)$$

cf., *e.g.* [13, Th. 16.1]. Thus, choosing $\chi = I_h\varphi$ in (5.15) and (5.17) and combining (5.7), (5.13), (5.14), (5.20) and (5.21) we obtain the desired estimate. \square

6. AUXILIARY RESULTS

In this section we will give appropriate assumptions for operators Q_1 and Q_2^K and prove Lemmas 5.1 and 5.2 of the previous section.

We consider a linear operator $Q_1, Q_1|_K : \mathbb{P}_1(K) \rightarrow L^2(K)$, with $K \in T_h$, satisfying $\int_K Q_1 c = \int_K c$, for all constants c and $K \in T_h$ and also

$$\|\chi - Q_1\chi\|_{L^q(K)} \leq h_K|\chi|_{W^{1,q}(K)}, \quad \forall \chi \in V_h, \quad 1 \leq q < \infty. \quad (6.1)$$

We can easily see that the operators Q_1 considered in (3.2), (3.6), (4.9) and (4.13), satisfy these assumptions. In addition, in view of Remarks 3.1 and 4.2, these operators may satisfy:

$$\int_K \chi dx = \int_K Q_1\chi dx, \quad \forall K \in T_h, \quad \forall \chi \in V_h. \quad (6.2)$$

Also, we consider a linear operator $Q_2^K, Q_2^K|_e : \mathbb{P}_1(K) \rightarrow L^2(e)$, with $K \in T_h$ and e a side of K , satisfying the following properties:

$$Q_2^K c|_e = c, \quad \text{for all constants } c, \quad e \text{ a side of } K, \quad (6.3)$$

$$\int_e \chi ds = \int_e Q_2^K \chi ds, \quad \forall \chi \in \mathbb{P}_1(K), \quad e \text{ a side of } K, \quad (6.4)$$

$$\|Q_2^K \chi\|_{L^\infty(e)} \leq \|\chi\|_{L^\infty(e)}, \quad \forall \chi \in V_h + X_h^0; \quad (6.5)$$

$$Q_2^{K_1} \chi|_e = Q_2^{K_2} \chi|_e, \quad \forall \chi \in V_h + X_h^0, \quad e = K_1 \cap K_2, \quad K_1, K_2 \in T_h, \quad (6.6)$$

$$Q_2^K \chi|_e = 0, \quad \forall \chi \in V_h + X_h^0, \quad K \in T_h, \quad e \in E_h \setminus E_h^{\text{in}}. \quad (6.7)$$

We can easily see that the operators Q_2^K defined in (4.11) and (4.15) and the restrictions of $Q_1|_e$, on a side e , defined by (3.2) and (3.6), satisfy all the above assumptions. Next, we will show the following auxiliary lemma.

Lemma 6.1. *Let K be a triangle and e a side of K . Then for $\varphi \in H^1(K)$ there exists a constant C independent of K such that*

$$\left| \int_e \varphi(\chi - Q_2^K \chi) \, ds \right| \leq C(\sigma)h_K |\varphi|_{1,K} |\chi|_{1,K}, \quad \forall \chi \in \mathbb{P}_1(K), \forall \varphi \in H^1(K).$$

Proof. In view of (6.3) and (6.4), we have

$$\int_e \varphi(\chi - Q_2^K \chi) \, ds = \int_e (\varphi - c_1)(\chi - c_2 - Q_2^K(\chi - c_2)) \, ds, \quad \forall \chi \in \mathbb{P}_1(K), \forall \varphi \in H^1(K),$$

for all constants $c_1, c_2 \in \mathbb{R}$, $K \in T_h$ and $e \in E_h(K)$. Using now in the relation above, (6.5) and a local inverse inequality, we get for all constants $c_1, c_2 \in \mathbb{R}$, $\chi \in \mathbb{P}_1(K)$ and $\varphi \in H^1(K)$,

$$\begin{aligned} \left| \int_e \varphi(\chi - Q_2^K \chi) \, ds \right| &\leq \|\varphi - c_1\|_{L^2(e)} \|\chi - c_2 - Q_2^K(\chi - c_2)\|_{L^2(e)} \\ &\leq h_e^{1/2} \|\varphi - c_1\|_{L^2(e)} \|\chi - c_2 - Q_2^K(\chi - c_2)\|_{L^\infty(e)} \\ &\leq Ch_e^{1/2} \|\varphi - c_1\|_{L^2(e)} \|\chi - c_2\|_{L^\infty(e)} \leq C \|\varphi - c_1\|_{L^2(e)} \|\chi - c_2\|_{L^2(e)}, \end{aligned} \quad (6.8)$$

with $h_e = |e|$. In view of the Bramble–Hilbert lemma and a standard homogeneity argument, we can easily show

$$\inf_{c \in \mathbb{R}} \|\varphi - c\|_{L^2(e)} \leq C(\sigma)h_e^{1/2} |\varphi|_{1,K}, \quad \forall \varphi \in H^1(K).$$

Finally, combining this with (6.8) we obtain the desired estimate. \square

In the sequel we will prove Lemmas 5.1 and 5.2.

Proof of Lem. 5.1. Due to (6.1), (5.2) follows easily. Let now $C_K(f) = \int_B f \omega_K \, dx$, where B is a ball in \mathbb{R}^2 , satisfying $B \subset\subset K$ and ω_K a cut-off function supported in \bar{B} , cf. [3, Def. 4.1.3]. Then, according to the Bramble–Hilbert lemma, in the form given in [3, Lem. 4.3.8], we obtain

$$\|f - C_K(f)\|_{L^p(K)} \leq C(\sigma)h_K |f|_{W^{1,p}(K)}, \quad \forall f \in W^{1,p}(K), \quad p \geq 1. \quad (6.9)$$

Since $C_K(f)$ is constant over K , due to (6.2), we have

$$(f, \chi - Q_1 \chi)_K = (f - C_K(f), \chi - Q_1 \chi)_K, \quad \forall \chi \in V_h.$$

Thus, due to this, (6.9) and (6.1), we get for every $\chi \in V_h$

$$|(f, \chi - Q_1 \chi)_K| = |(f - C_K(f), \chi - Q_1 \chi)_K| \leq Ch_K^2 |f|_{W^{1,p}(K)} |\chi|_{W^{1,q}(K)},$$

with $\frac{1}{p} + \frac{1}{q} = 1$, which concludes the proof of (5.3). \square

Proof of Lem. 5.2. First we will show (5.4). In view of Green's formula, we have

$$\varepsilon_a(\psi, \chi) = \sum_K (L\psi, \chi - Q_1 \chi)_K + \sum_K (A\nabla\psi \cdot n, \chi - Q_2^K \chi)_{\partial K} = I + II. \quad (6.10)$$

For the first term we have from (6.1),

$$|I| \leq C \sum_K \|L\psi\|_K \|\chi - Q_1\chi\|_K \leq C \sum_K h_K |\psi|_{1,K} |\chi|_{1,K}.$$

The bound for II follows at once from Lemma 6.1 since $|A\nabla\psi \cdot n|_{1,K} \leq C|\psi|_{1,K}$.

We now turn to (5.5). Let $\psi = u_h$ in (6.10) and $(\nabla A)_K$ be the average over K , then in view of (6.2) we have for every $\chi \in V_h$

$$(Lu_h, \chi - Q_1\chi)_K = ((\nabla A - (\nabla A)_K)\nabla u_h, \chi - Q_1\chi)_K \leq Ch_K^2 |u_h|_{1,K} |\chi|_{1,K},$$

from which we easily obtain the desired bound for I . Let now $\bar{A}_e = A(m_e)$ where m_e is the midpoint of the edge e . We will show that for every $\chi \in V_h$ and $\psi \in X_h^0$

$$\begin{aligned} II &= \sum_K ((A - \bar{A}_e)\nabla(u_h - u) \cdot n, \chi - Q_2^K \chi)_{\partial K} \\ &\quad + \sum_K ((A - \bar{A}_e)\nabla u \cdot n, \chi - \psi - Q_2^K(\chi - \psi))_{\partial K}. \end{aligned} \quad (6.11)$$

Provided that this holds, we may apply Lemma 6.1 and estimates

$$\begin{aligned} |(A - \bar{A}_e)\nabla(u_h - u)|_{1,K} &\leq C(|u_h - u|_{1,K} + h_K|u|_{2,K}), \\ |(A - \bar{A}_e)\nabla u|_{1,K} &\leq C(|u|_{1,K} + h_K|u|_{2,K}), \end{aligned}$$

to obtain

$$\begin{aligned} |II| &\leq C \sum_K h_K (|u_h - u|_{1,K} + h_K|u|_2) |\chi|_{1,K} + \sum_K h_K (|u|_{1,K} + h_K|u|_2) |\chi - \psi|_{1,K} \\ &\leq Ch(|u_h - u|_{1,h} + h|u|_2) |\chi|_{1,h} + Ch|u|_2 |\chi - \psi|_{1,h}, \quad \forall \chi \in V_h, \psi \in X_h^0, \end{aligned}$$

which gives the desired estimate for II . Therefore it remains to prove (6.11). For this we will show, for every $\psi \in X_h^0$

$$\sum_K (A\nabla u \cdot n, \psi - Q_2^K \psi)_{\partial K} = \sum_K (\bar{A}_e \nabla u \cdot n, \psi - Q_2^K \psi)_{\partial K} = 0. \quad (6.12)$$

In the first sum we have by Green's formula for every $\psi \in X_h^0$

$$\sum_K (A\nabla u \cdot n, \psi)_{\partial K} = \sum_K (A\nabla u, \nabla \psi)_K - (Lu, \psi)_K = (A\nabla u, \nabla \psi) - (Lu, \psi) = 0.$$

In addition, $\sum_K (A\nabla u \cdot n, Q_2^K \psi)_{\partial K} = 0$ because $Q_2^K \psi$ is piecewise constant on each interior edge e and $A\nabla u \cdot n$ is continuous across e (in the trace sense), and $Q_2^K \psi = 0$ on $\partial\Omega$. Since the first sum in (6.12) vanishes for each smooth A , and is continuous in A on $L^1(\cup\partial K)$ the second sum is the limit of sums with a smooth A and therefore also vanishes. Finally, since $\bar{A}_e \nabla u_h \cdot n$ is constant on each e , in view of (6.4) we have

$$\sum_K (\bar{A}_e \nabla u_h \cdot n, \chi - Q_2^K \chi)_{\partial K} = 0, \quad \forall \chi \in V_h.$$

□

Remark 6.1. It is known, *cf.*, *e.g.* [17, Chap. 5.2], that if $f \in L^p$, $1 < p < 2$, the solution u of (1.1) is an element of $W^{2,p}$ and by imbedding, $u \in H^s$, where H^s is the fractional order Sobolev space, with $s = 3 - 2/p$, *cf.* [1, Chap. VII, Th. 7.58]. For (1.5) to make sense, we need $s > 3/2$ and thus $p > 4/3$. Therefore, one can prove analogous results to Theorems 5.1 and 5.2, that is,

$$\|u - u_h\|_{1,h} \leq Ch^{2-2/p} \|u\|_{W^{2,p}}, \quad (6.13)$$

and, if in addition $f \in W^{1,p}$, since then $f \in L^2$, we have

$$\|u - u_h\| \leq Ch^2 (\|u\|_2 + \|f\|_{W^{1,p}}). \quad (6.14)$$

To obtain these estimations we need to modify Lemma 6.1 and prove that

$$|(\varphi, \chi - Q_2^K \chi)_e| \leq Ch_K |\varphi|_{W^{1,p}(K)} |\chi|_{W^{1,q}(K)}, \quad \forall \varphi \in W^{1,p}(K), \quad \forall \chi \in \mathbb{P}_1(K), \quad (6.15)$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Also, we need the local inverse inequality

$$|\psi|_{W^{1,q}(K)} \leq Ch_K^{1-2/p} |\psi|_{1,K}, \quad \forall \psi \in V_h. \quad (6.16)$$

Further, using the interpolation theory of Sobolev spaces, *cf.*, *e.g.* [3, Chap. 12], we have

$$\inf_{\chi \in V_h} \|u - \chi\|_{1,h} \leq Ch^{s-1} \|u\|_{H^s}, \quad 1 < s < 2. \quad (6.17)$$

Following the proof of Theorems 5.1 and 5.2 and combining (6.15)–(6.17), we can derive estimations (6.13) and (6.14). \square

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