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FINITE VOLUMES AND FINITE ELEMENTS: TWO 'GOOD FRIENDS'

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SUMMARY

In this paper a comparison between the finite element and the finite volume methods is presented in the context of elliptic, convective–diffusion and fluid flow problems. The paper shows that both procedures share a number of features, like mesh discretization and approximation. Moreover, it is shown that in many cases both techniques are completely equivalent.

1. INTRODUCTION

The Finite Volume Method (FVM) evolved in the early seventies via the Finite Difference Method (FDM) and today has many proponents in the field of fluid mechanics. Several applications of FVM show this method to be specially attractive for the solution of convective dominant equations and many of the problems existing in the solution using the Finite Element Method (FEM) are not explicitly present in the solution via the FVM. On the other hand, the FEM is a more general and mathematically consistent approach, which can provide many other alternatives and possibilities to solve the same problems and where the rate accuracy/computational cost may be easily adjusted to the desired problem.

A general FVM–FEM format has been proposed in Reference 1, in which the differential equation system

$$\mathbf{A}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \tag{1}$$

is solved by a weighted residual method

$$\int_\Omega \mathbf{W}_i(\mathbf{A}\hat{\mathbf{u}} - \mathbf{f})\,\mathrm{d}\Omega = 0 \tag{2}$$

in which $\mathbf{W}_i (i = 1, \ldots, n)$ are weighting functions and $\hat{\mathbf{u}}$ are the approximations to the unknown

$$\mathbf{u} \cong \hat{\mathbf{u}} = \sum \mathbf{N}_j \mathbf{u}_j \quad (j = 1, \ldots, n) \tag{3}$$

where $\mathbf{N}_j$ are some basis functions and $\mathbf{u}_j$ the nodal values of the unknown.

In self-adjoint problems with constant parameters, the optimal weighting for FEM is the Galerkin one, with

$$\mathbf{W}_i = \mathbf{N}_i$$

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whereas in the FVM it is chosen as

$$\mathbf{W}_i = \mathbf{I} \quad \text{in } \Omega_c \quad (\mathbf{W}_i = 0 \text{ elsewhere})$$

where $\mathbf{I}$ is the unity matrix and Ω_c is a control volume which can be defined in different ways. Examples of the applications of these concepts to convective–diffusion and structural problems can be found in References 1 and 2, respectively.

Certainly, the different use of the approximating functions $\mathbf{N}_i$ in the FEM yields different numerical results as well as different control volumes Ω_c will give different results in the FVM. For the same triangular or quadrangular regions, different finite elements may be obtained changing the order of the approximating functions. In parallel, many choices of the finite control volumes involving one, more than one, a part of several triangular or quadrilateral regions, etc. have been considered. Here two main types of formulations have emerged: the cell-centred approach in which the value of the unknown is taken at the centre of the cell over which conservation is imposed[3] and the cell-vertex scheme in which the unknowns are taken at the vertices of the cell. In this last approach, one of the possibilities is to define the control volume associated with each node as the region shared by this node.[3] Another possibility is to take each triangular or quadrangular region as the control volume.[4] The latter approach is difficult to use in non-structured meshes. In this paper, just cell-centred non-structured control volumes will be considered.

Are FVM and FEM formulations completely equivalent in some particular cases? In Reference 1, Zienkiewicz and Oñate have shown that for the one-dimensional convective–diffusion equation solved with a centred scheme (ie. $\mathbf{W}_i = \mathbf{N}_i$ in FEM or $\mathbf{W}_i = \mathbf{I}$ in FVM) both FE and FV formulations are numerically equivalent when linear approximations are used, the mass is lumped at the nodes and cell-centred control volumes are used. Similar results have been recently found by Selmin[5] in the context of the Euler equations for an inviscid gas. Oñate and Idelsohn[6] have succeeded in extending this equivalence to n-dimensional systems of equations. In this paper, we extend the results of Reference 6 to show in detail that the more 'popular' finite element, the triangular (or tetrahedral) element with linear shape functions, is numerically equivalent to the most standard FVM based on cell-centred control volumes.

In fact, most of the prejudices about the advantages of the FVM as computer cost, or the greater accuracy of FEM due to Galerkin optimal approximations for self-adjoint problems, must be definitively buried.

Nevertheless, it is well known that for non-self-adjoint equations, none of the 'centred weighting schemes' give optimal results, and both FVM and FEM propose different ways of solving this handicap.[7–9] Writing both formulations in a general FVM/FEM format as in Reference 1 allows us to understand better and to take further advantages of the solution scheme chosen.

Fluid mechanics equations are typically composed of two parts: the convective terms involving first derivatives of the unknowns, and the diffusive ones, involving second derivatives. The convective part generally introduces non-self-adjoint operators and, thus, non-centred (upwinding) weighting functions $\mathbf{W}_i$ must be used. Several functions of this kind have been proposed for the FEM during the last 10 years. One of the more popular choices is the so-called SUPG approach[10] in which exactly nodal values are obtained in one-dimensional problems and no artificial diffusion transversal to the streamlines is guaranteed for diagonalizable systems. Unfortunately, for non-diagonalizable systems the absence of transverse diffusion cannot be ensured any longer. In parallel, the main feature of the FVM is the integration of the flux on the surface of the control volume. This integration transforms the 3-D problem into a problem on a surface. For the convective terms, the problem becomes 1-D in the direction of the outside normal to the surface. One-dimensional systems of differential equations may be easily diagonalized and

optimal off-centred weighted functions may be chosen in this case. This is one of the most important advantages of the FVM, which must be further exploited in future FE and FV upwinding methods. Indeed, SUPG ideas have also been incorporated in the FV context,[11,12] but here again, this proposition works correctly just for diagonalizable systems.

The detailed comparison of some FV and FE formulations and the way to derive finite elements with the finite volume assumptions and *vice versa* is one of the objectives of this paper. Apparently the FVM is better suited to solve the non-self-adjoint part, while the FEM gives more possibilities of solving the symmetric terms. The equivalence of both methods in some cases introduces the possibility of solving one part of the equations by the FVM and the other by the FEM, obtaining in this manner a 'compromise' approach leading to an improved solution.

The layout of the paper is as follows. Section 2 shows the equivalences between FEM and FVM for the convective–diffusion one-dimensional equation for both the scalar and vectorial cases. Section 3 introduces the comparison of FEM and FVM for the general n-dimensional case. Finally, the conditions under which both procedures coincide for centred weighting and upwinded schemes are discussed in Sections 4 and 5, respectively. It is shown that for a certain class of approximations both procedures yield the same discretized system of equations.

2. THE ONE-DIMENSIONAL CASE

2.1. *The scalar problem*

To fix ideas we shall consider the scalar equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = 0 \tag{4}$$

with constants a and k being the convective and diffusive coefficients, respectively. The temporal term is first integrated by explicit or implicit finite differences. This temporal integration introduces a numerical diffusion proportional to Δt which must be taken into account in the spatial discretization.

Let us use a backward Euler explicit integration:

$$\frac{u^{n+1} - u^n}{\Delta t} + a\left(\frac{\partial u}{\partial x}\right)^n - \frac{\partial}{\partial x}\left(k^*\frac{\partial u}{\partial x}\right)^n = 0 \tag{5}$$

in which the superindex n represents the function evaluated at the nth time step and k^* represents the material diffusion plus the numerical diffusion introduced by the temporal integration. Taylor–Galerkin methods[7] use this numerical diffusion as a balancing term in order to eliminate oscillations of the solution. In our analysis, this numerical diffusion is not considered as a balancing term.

For simplicity, the superindex n in (5) denoting values at the nth time step will be omitted hereafter.

The spatial discretization (see Figure 1) will now be accomplished by three methods: FDM, FVM and FEM.

2.1.1. Finite difference method (FDM). Let the convective flux f and the diffusive flux g be defined by

$$f = au, \quad g = k^*\frac{\partial u}{\partial x}$$

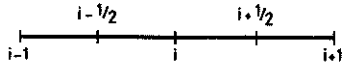

Figure 1. Spatial discretization

Equation (5) evaluated at node i then becomes

$$\frac{\Delta u_i}{\Delta t} + \left(\frac{\partial f}{\partial x}\right)_i - \left(\frac{\partial g}{\partial x}\right)_i = 0 \tag{6}$$

In the FDM, the derivatives $(\partial f/\partial x)_i$ and $(\partial g/\partial x)_i$ are approximated by

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x}, \quad \left(\frac{\partial g}{\partial x}\right)_i = \frac{g_{i+1/2} - g_{i-1/2}}{\Delta x} \tag{7}$$

where

$$f_{i+1/2} = \frac{f_i + f_{i+1}}{2} + \xi \operatorname{sign}(a)\frac{f_i - f_{i+1}}{2}$$

$$g_{i+1/2} = k^* \frac{u^n_{i+1} - u^n_i}{\Delta x} \tag{8}$$

in which

$$\operatorname{sign}(a) = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$$

and ξ is the well-known optimum upwind parameter[10]:

$$\xi = \coth P_e - \frac{1}{P_e}, \quad P_e = \frac{|a|\Delta x}{2k^*} \tag{9}$$

The values $f_{i-1/2}$ and $g_{i-1/2}$ are easily deduced from (8). Equations (8) show the off-centred approach for the convective flux f and the centred one for the diffusive flux g.

2.1.2. Finite volume method (FVM). Equation (5) is now integrated over the control volume Ω_c. Taking a *cell-centred volume* between $i - 1/2$ and $i + 1/2$ (this is sometimes called a 'node centred' scheme[5]), the FVM claims that

$$\frac{1}{\Delta x}\int_{i-1/2}^{i+1/2} \frac{\partial f}{\partial x}\,dx = \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x}, \quad \frac{1}{\Delta x}\int_{i-1/2}^{i+1/2} \frac{\partial g}{\partial x}\,dx = \frac{g_{i+1/2} - g_{i-1/2}}{\Delta x} \tag{10}$$

where $(\ )_{i+1/2}$ and $(\ )_{i-1/2}$ are computed precisely as explained in (8) for the FDM. The coincidence of FVM and FDM is obvious in this case.

It can be shown that the final system of discretized equations can be obtained from the following weighted residual form:

$$\frac{1}{\Delta x}\int_{i-1/2}^{i+1/2} W_i\left(\frac{\Delta u}{\Delta t} + \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\right)dx = 0 \tag{11}$$

where

$$\begin{cases} W_i = 1 + 2p_i & \forall x\ (i - \frac{1}{2} \leqslant x \leqslant i + \frac{1}{2}) \\ W_i = 0 & \text{elsewhere} \end{cases} \tag{12}$$

with

$$p_i = \frac{\Delta x}{2}\xi \operatorname{sign}(a)\frac{\partial N_i}{\partial x} \tag{13}$$

and linear interpolations of the unknown functions as

$$u \cong \hat{u} = \sum_j N_j u_j \quad (j = 1, \ldots, n) \tag{14}$$

where

$$N_i = \frac{1}{\Delta x}x \quad \text{for } i - 1 \leqslant x \leqslant i$$

$$N_i = 1 - \frac{1}{\Delta x}x \quad \text{for } i \leqslant x \leqslant i + 1 \tag{15}$$

is the standard linear shape function of node i shown in Figure 2.

2.1.3. Finite element method (FEM). Equation (5) must be integrated over the volume of the elements surrounding node i (i.e. between $i - 1$ and $i + 1$) weighted by a function W_i:

$$\int_{i-1}^{i+1} W_i\left(\frac{\Delta u}{\Delta t} + \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\right)dx = 0 \tag{16}$$

In Reference 1 the equivalence between (11) and (16) is shown for the centred (not upwinded) case (i.e. $W_i = N_i$ in FEM and $W_i = 1$ in FVM), provided a linear interpolation is used and the mass is lumped at the nodes. For a non-centred scheme with upwinded weighting $W_i = N_i + p_i$, the equivalence between FE and FV expressions may also be shown. Taking the off-centred part of (16)

$$\int_{i-1}^{i+1} p_i\left(\frac{\Delta u}{\Delta t} + \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\right)dx \tag{17}$$

with p_i defined as in (13) and using linear elements with lumped mass, the temporal and diffusive terms of (17) vanish and the convective term becomes

$$\int_{i-1}^{i+1} p_i \frac{\partial f}{\partial x}\,dx = \int_{i-1}^{i} p_i \frac{\partial f}{\partial x}\,dx + \int_{i}^{i+1} p_i \frac{\partial f}{\partial x}\,dx \tag{18}$$

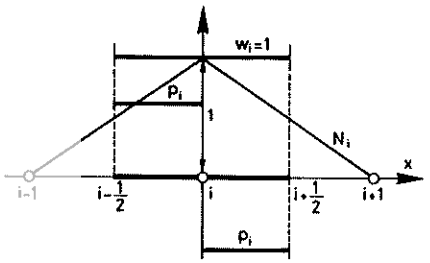

Figure 2. N_i, W_i and p_i functions in FE and cell-centred FV

Taking into account (13) and (15) the second integral of the RHS of (18) is

$$\int_i^{i+1} p_i \frac{\partial f}{\partial x} dx = \xi \, \text{sign}(a) \frac{f_i - f_{i+1}}{2} \tag{19}$$

which is equivalent to the off-centred part of (8). It is easy to show that the first integral of the RHS of (18) gives the off-centred part of the term $f_{i-1/2}$ in (7).

We have therefore shown the equivalence between the three formulations for the scalar one-dimensional problem.

Note that the upwinding FE scheme may also be interpreted as the addition of a balancing diffusion given by Reference 14:

$$\bar{k} = \frac{\Delta x}{2} \xi \, a \, \text{sign}(a) = \frac{\Delta x}{2} \xi |a| \tag{20}$$

2.2. *The vectorial one-dimensional problem*

For a system of m one-dimensional differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{u}}{\partial x} - \frac{\partial}{\partial x}\left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial x}\right) = 0 \tag{21}$$

in which $\mathbf{A}$ is a non-symmetric $(m \times m)$ matrix and $\mathbf{K}$ is a diagonal matrix, the time integration reads

$$\frac{\Delta \mathbf{u}}{\Delta t} + \mathbf{A}\left(\frac{\partial \mathbf{u}}{\partial x}\right)^n - \frac{\partial}{\partial x}\left(\mathbf{K}^* \frac{\partial \mathbf{u}}{\partial x}\right)^n = 0 \tag{22}$$

where $\mathbf{K}^*$ represents the material diffusion plus the numerical diffusion introduced by the temporal integration.[7,16,17]

Let $\mathbf{S}$ be the matrix which diagonalizes $\mathbf{A}$ as

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda} \tag{23}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of matrix $\mathbf{A}$. Making the change of variable $\mathbf{v} = \mathbf{S}^{-1}\mathbf{u}$, (22) becomes

$$\frac{\Delta \mathbf{v}}{\Delta t} + \mathbf{\Lambda}\left(\frac{\partial \mathbf{v}}{\partial x}\right)^n - \frac{\partial}{\partial x}\left(\mathbf{S}^{-1}\mathbf{K}^*\mathbf{S} \frac{\partial \mathbf{v}}{\partial x}\right)^n = 0 \tag{24}$$

For a diagonal $\mathbf{S}^{-1}\mathbf{K}^*\mathbf{S}$ matrix (convective dominant problems) equation (24) is a set of uncoupled one-dimensional problems which may be solved separately with the approach described in Section 2.1. Both FV and FE solutions are described and compared next.

2.2.1. Finite volume method. Taking a cell-centred volume (see Figure 2), the convective part of equation (24) reads (for simplicity, the superindex n will again be dropped from all terms)

$$\frac{1}{\Delta x} \int_{i-1/2}^{i+i/2} \frac{\partial}{\partial x}(\mathbf{\Lambda v}) dx = \mathbf{\Lambda v}_{i+1/2} - \mathbf{\Lambda v}_{i-1/2} \tag{25}$$

with

$$\mathbf{\Lambda v}_{i+1/2} = \mathbf{\Lambda}\left[\frac{(\mathbf{v}_i + \mathbf{v}_{i+1})}{2} + \xi \, \text{sign}(\mathbf{\Lambda}) \frac{(\mathbf{v}_i - \mathbf{v}_{i+1})}{2}\right] \tag{26}$$

and the diffusive term

$$\frac{1}{\Delta x} \int_{i-1/2}^{i+1/2} \frac{\partial}{\partial x}\left(\mathbf{S}^{-1}\mathbf{K}^*\mathbf{S} \frac{\partial \mathbf{v}}{\partial x}\right) dx = \mathbf{S}^{-1}\mathbf{K}^*\mathbf{S}\left[\left(\frac{\partial \mathbf{v}}{\partial x}\right)_{i+1/2} - \left(\frac{\partial \mathbf{v}}{\partial x}\right)_{i-1/2}\right] \tag{27}$$

with

$$\mathbf{S}^{-1}\mathbf{K}^*\mathbf{S}\left(\frac{\partial \mathbf{v}}{\partial x}\right)_{i+1/2} = \mathbf{S}^{-1}\mathbf{K}^*\mathbf{S} \frac{\mathbf{v}_{i+1} - \mathbf{v}_i}{\Delta x} \tag{28}$$

The $(\,)_{i-1/2}$ terms are readily obtained from (27) and (28).

In terms of the initial variables $\mathbf{u}$ gives

$$\int_{i-1/2}^{i+1/2} \frac{1}{\Delta x} \mathbf{A} \frac{\partial \mathbf{u}}{\partial x} dx = \mathbf{f}_{i+1/2} - \mathbf{f}_{i-1/2} \tag{29}$$

with

$$\mathbf{f}_{i+1/2} = \mathbf{A}\left(\frac{\mathbf{u}_i + \mathbf{u}_{i+1}}{2}\right) + \mathscr{T}|\mathbf{A}|\left(\frac{\mathbf{u}_i - \mathbf{u}_{i+1}}{2}\right) \tag{30}$$

and

$$\frac{1}{\Delta x} \int_{i-1/2}^{i+1/2} \mathbf{K}^* \frac{\partial \mathbf{u}}{\partial x} dx = \mathbf{g}_{i+1/2} - \mathbf{g}_{i-1/2} \tag{31}$$

with

$$\mathbf{g}_{i+1/2} = \mathbf{K}^* \frac{\mathbf{u}_{i+1} - \mathbf{u}_i}{\Delta x} \tag{32}$$

In (26)–(32) we have used the following notation:

$$\text{sign}(\mathbf{\Lambda}) = \text{diag}(\text{sign}\,\lambda_1, \text{sign}\,\lambda_2, \ldots, \text{sign}\,\lambda_m) \tag{33}$$

$$\text{sign}(\mathbf{A}) = \mathbf{S}\,\text{sign}(\mathbf{\Lambda})\mathbf{S}^{-1} \tag{34}$$

$$|\mathbf{A}| = \mathbf{A}\,\text{sign}(\mathbf{A}) \tag{35}$$

$$\mathscr{T} = \mathbf{S}\xi\mathbf{S}^{-1} \tag{36}$$

$$\xi = \text{diag}(\xi_1, \xi_2, \ldots, \xi_m) \tag{37}$$

and the optimal upwinding parameter ξ_l is evaluated using (9) with

$$P_l = \frac{\lambda_l \Delta x}{2k_l^*} \tag{38}$$

where k_l^* are the diagonal terms of $\mathbf{S}^{-1}\mathbf{K}^*\mathbf{S}$.

Exact solutions are found at the nodes when $\mathbf{S}^{-1}\mathbf{K}^*\mathbf{S}$ is a diagonal matrix; otherwise some approximation is introduced.

Equations (29)–(32) can also be interpreted from a weighted residual sense. The starting point is the interpolation of vector $\mathbf{u}$ using the standard linear functions N_i defined in (15) as

$$\mathbf{u} \cong \hat{\mathbf{u}} = \sum_j \mathbf{N}_j \mathbf{u}_j \tag{39}$$

in which $\mathbf{N}_j = N_j \mathbf{I}$, and writing

$$\frac{1}{\Delta x}\int_{i-1/2}^{i+1/2} \mathbf{W}_i\left(\frac{\Delta \mathbf{u}}{\Delta t} + \frac{\partial \mathbf{f}}{\partial x} - \frac{\partial \mathbf{g}}{\partial x}\right) = \mathbf{0}$$

with

$$\begin{cases} \mathbf{W}_i = \mathbf{I} + 2\mathbf{P}_i & \forall x \ (i - \frac{1}{2} \leqslant x \leqslant i + \frac{1}{2}) \\ \mathbf{W}_i = 0 & \text{elsewhere} \end{cases} \tag{41}$$

where

$$\mathbf{P}_i = \frac{\Delta x}{2}\mathbf{S}\boldsymbol{\xi}\mathbf{S}^{-1}\,\text{sign}(\mathbf{A})\frac{\partial \mathbf{N}_i}{\partial x} \tag{42}$$

The terms $\int_L \mathbf{P}_i(\Delta \mathbf{u}/\Delta t)\,dx$ and $\int_L \mathbf{P}_i(\partial \mathbf{g}/\partial x)\,dx$ vanish for linear functions $\mathbf{N}_j$ and lumped mass matrices, and the term $\int_L \mathbf{W}_i(\partial \mathbf{f}/\partial x)\,dx$ is equivalent to (29) and (30).

2.2.2. Finite element method. In this method, the approximation (39) is used and the weighted residual equation for the SUPG formulation becomes

$$\int_{i-1/2}^{i+1/2} \mathbf{W}_i\left(\frac{\Delta \mathbf{u}}{\Delta t} + \frac{\partial \mathbf{f}}{\partial x} - \frac{\partial \mathbf{g}}{\partial x}\right)dx \tag{43}$$

with

$$\mathbf{W}_i = \mathbf{N}_i + \mathbf{P}_i \tag{44}$$

where $\mathbf{N}_i$ are the standard one-dimensional linear functions and $\mathbf{P}_i$ the same off-centred functions defined for the FVM in (42).

After diagonalization of (22) the problem is transformed in a system of uncoupled one-dimensional scalar equations and the equivalence with the FVM is obvious following the lines of the previous section.

From a weighted residual point of view this means that

$$\int_{i-1/2}^{i+1/2} (\mathbf{I} + 2\mathbf{P}_i)\left(\frac{\Delta \mathbf{u}}{\Delta t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial x}\right)dx = \int_{i-1}^{i+1} (\mathbf{N}_i + \mathbf{P}_i)\left(\frac{\Delta \mathbf{u}}{\Delta t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial x}\right)dx \tag{45}$$

As a conclusion, we have shown that for the 1-D case both FV and FE formulations can be expressed in a similar weighted residual (Petrov–Galerkin) form. Furthermore, both formulations are equivalent for linear approximations, when cell-centred finite volumes and lumped mass matrices are used. Moreover, it can be shown that both FV and FE approaches can also be derived from a centred weighting scheme with the addition of the following balancing diffusion term:

$$\bar{\mathbf{K}} = \frac{\Delta x}{2}\mathscr{T}\,\text{sign}(\mathbf{A})\,\mathbf{A} = \frac{\Delta x}{2}\mathscr{T}\,|\mathbf{A}| \tag{46}$$

3. THE n-DIMENSIONAL CASE

Consider the system of m differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}_i\frac{\partial \mathbf{u}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij}\frac{\partial \mathbf{u}}{\partial x_j}\right) = \mathbf{Q} \tag{47}$$

where $\mathbf{A}_i$ $(i = 1, 2, 3)$ is a set of three $(m \times m)$ matrices and $\mathbf{K}_{ij}$ a set of nine $(m \times m)$ matrices. After time integration (47) becomes

$$\frac{\Delta \mathbf{u}}{\Delta t} + \mathbf{A}_i\left(\frac{\partial \mathbf{u}}{\partial x_i}\right)^n - \frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij}^{\theta}\frac{\partial \mathbf{u}}{\partial x_j}\right)^n = \mathbf{Q} \tag{48}$$

Equations (47) represent, for instance, the Navier–Stokes equations for a compressible flow. These equations are not, in general, a diagonalizable system of equations even in the case of pure convective problems ($\mathbf{K}_{ij} = 0$). This means that *there is not* a matrix $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\mathbf{A}_i\mathbf{S} = \mathbf{A}_i \quad \text{for } i = 1, 2, 3 \tag{49}$$

This is one of the most important difficulties of these equations. If the system is not diagonalizable, it means that there is no way to transform the problem in an equivalent uncoupled set of equations and, in such a case, the implementation of an upwind scheme becomes difficult.

FEM and FVM take a very different approach in treating the convective terms. The FEM with the SUPG techniques try to write the equations in the 'flow' (streamline) direction. This is only possible with diagonalizable systems in which equation (49) is satisfied. In such a case the SUPG gives a nearly exact solution governed by a parameter related to the element length.[10,7,15] Unfortunately, equations like (47) are diagonalizable for very particular cases only. On the other hand, the FVM solves this problem in a more natural way. By transforming the volume integration in a surface one, the n-dimensional problem is transformed into an infinite number of one-dimensional problems. In a discretized approach, the control volumes may have a finite set of surfaces. On each surface, the one-dimensional upwinding scheme described in Section 2 may be used.

3.1. *Finite volume method*

Let the convective part of equation (48) be written as

$$\int_{\Omega_c} \mathbf{A}_i\frac{\partial \mathbf{u}}{\partial x_i}d\Omega = \oint_{\Gamma_c} \mathbf{A}_i n_i \mathbf{u}\,d\Gamma = \oint_{\Gamma_c} \mathbf{A}_n \mathbf{u}\,d\Gamma = \oint_{\Gamma_c} \mathbf{f}_n\,d\Gamma \tag{50}$$

where Ω_c is the control volume and Γ_c the external surface of this control volume with external normal components n_i. The matrix

$$\mathbf{A}_n = \mathbf{A}_i n_i \tag{51}$$

is a $(m \times m)$ matrix.

In a discretized scheme the control volumes have a finite set of surfaces with normal $\hat{n}$ (see Figure 3) and then equation (50) can be written as

$$\int_{\Omega_c} \mathbf{A}_i\frac{\partial \mathbf{u}}{\partial x}d\Omega = \oint_{\Gamma_c} \mathbf{A}_n \mathbf{u}\,d\Gamma = \sum_i \oint_{\Gamma_{c_i}} \mathbf{A}_{n_i}\mathbf{u}\,d\Gamma \tag{52}$$

The problem is then transformed in a finite set of one-dimensional vectorial problems and thus all the 1-D expressions described in Section 2.2 may be used.

Let us consider, for instance, the surface Γ_{c_1} in which the normal is $\hat{n}_1$ (see Figure 3). The convective terms become

$$\mathbf{A}_i n_{1i} = \mathbf{A}_{n_1} \tag{53}$$

Let $\mathbf{S}_1$ be the matrix which diagonalizes $\mathbf{A}_{n_1}$ as

$$\mathbf{S}_1^{-1}\mathbf{A}_{n_1}\mathbf{S}_1 = \boldsymbol{\Lambda}_1 \tag{54}$$

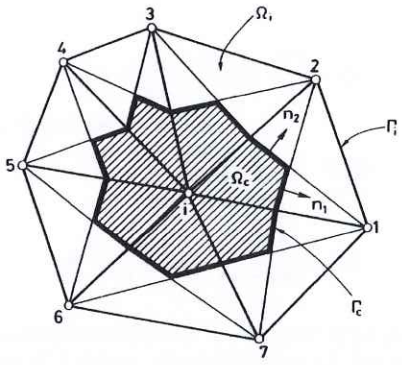

Figure 3. Classical cell-centred control volume and external normal

Making the change of variable $\mathbf{v}_1 = \mathbf{S}_1^{-1}\mathbf{u}$, we can write

$$\mathbf{S}_1^{-1}\oint_{\Gamma_{c_i}} \mathbf{A}_{n_1}\mathbf{u}\,d\Gamma = \oint_{\Gamma_{c_i}} \mathbf{\Lambda}_1\mathbf{v}_1\,d\Gamma = (\mathbf{\Lambda}_1\mathbf{v}_1)|_{\Gamma_{c_i}}\Delta\Gamma_1 \tag{55}$$

The right-hand side of equation (55) is now evaluated using (27) as

$$(\mathbf{\Lambda}_1\mathbf{v}_1)|_{\Gamma_{c_i}} = \mathbf{\Lambda}_1\left[\frac{\mathbf{v}_{1_i}+\mathbf{v}_{1_1}}{2} + \xi\,\mathrm{sign}(\mathbf{\Lambda}_1)\frac{(\mathbf{v}_{1_i}-\mathbf{v}_{1_1})}{2}\right] \tag{56}$$

In terms of the initial variables

$$\oint_{\Gamma_{c_i}} \mathbf{A}_{n_1}\mathbf{u}\,d\Gamma = (\mathbf{A}_{n_1}\mathbf{u})|_{\Gamma_{c_i}}\Delta\Gamma_1 \tag{57}$$

with

$$(\mathbf{A}_{n_1}\mathbf{u})|_{\Gamma_{c_i}} = \mathbf{A}_{n_1}\frac{(\mathbf{u}_i+\mathbf{u}_1)}{2} + \mathscr{T}\,|\mathbf{A}_{n_1}|\frac{(\mathbf{u}_i-\mathbf{u}_1)}{2} \tag{58}$$

where

$$\begin{aligned}
&\mathscr{T} = \mathbf{S}_1\xi\mathbf{S}_1^{-1}\\
&|\mathbf{A}_{n_1}| = \mathbf{A}_{n_1}\,\mathrm{sign}(\mathbf{A}_{n_1})\\
&\mathrm{sign}(\mathbf{A}_{n_1}) = \mathbf{S}_1\,\mathrm{sign}(\mathbf{\Lambda}_1)\mathbf{S}_1^{-1}\\
&\mathrm{sign}(\mathbf{\Lambda}_1) = \mathrm{diag}(\mathrm{sign}\,\Lambda_{11}, \mathrm{sign}\,\Lambda_{12}, \ldots, \mathrm{sign}\,\Lambda_{1m})\\
&\Delta\Gamma_1 = \oint_{\Gamma_1} d\Gamma\\
&\xi = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m)
\end{aligned} \tag{59}$$

ξ_l is evaluated using (9) with $P_l = \Lambda_{1l}|\overline{i1}|/2k_l^*$ and k_l^* is normally taken as the l-diagonal term of

$n_1\mathbf{K}_{ij}^* n_{1j}$. Note that the non-upwinding case simply implies that $\mathscr{T} = 0$ in (58), thus recovering the usual mid-point rule for the surface integral computations.

3.2. *The finite element method*

In Reference 10, Hughes *et al.* generalized the SUPG method for n-dimensional vectorial equations by proposing the following balancing diffusion term:

$$\bar{\mathbf{K}}_{ij} = \mathbf{A}_i\mathbf{\Pi}_{ij}\mathbf{A}_j\frac{h}{2} \tag{60}$$

where h is a characteristic element length and $\mathbf{\Pi}_{ij}$ is defined in such a way that, for a diagonalizable system, the 'exact' upwind solution is found. For instance, for a scalar problem $\mathbf{\Pi}_{ij} = \Pi = \xi(A_iA_i)^{-1/2}$ and for a vectorial one-dimensional problem $\mathbf{\Pi}_{ij} = \mathbf{\Pi} = \mathscr{T}\,|A|^{-1}$. For non-diagonalizable systems the definition of $\mathbf{\Pi}_{ij}$ is not obvious and several possibilities have been analysed.[10,7,15] A simplified version of the SUPG method is the so-called Taylor–Galerkin method in which $\mathbf{\Pi}_{ij} = \Delta t/h$.[16] Other possibilities have been proposed by Zienkiewicz *et al.*[18] in which equations (47) are split into two parts: a diagonal system plus the remaining system involving pressure terms only. Solving in a first step the diagonal system by an exact SUPG method, the second part is solved by a particular time-integration procedure.

In all these approaches, when the system is non-diagonalizable, a numerical diffusion normal to the flow direction is introduced which is very difficult to evaluate.

We can conclude that while the FEM gives an accurate solution for the cases of diagonalizable systems in which a flow direction may be identified and the streamline upwind concept may be applied, the FVM gives a more consistent solution to the discretized space in all situations. The FVM will be more or less 'streamline upwind exact' depending on the surface shape defining the control volume. Moreover, the FVM uses the same methodology for *both* diagonalizable and non-diagonalizable systems, which is one of the handicaps of the FE SUPG based methods.

We shall first show that for a centred weighting scheme, FEM and FVM are equivalent in 2-D problems provided a linear triangular approximation is used, the mass is lumped at the nodes, and cell-centred control volumes are used. The generalization to 3-D problems is obvious. The results of Section 4 are a generalization for n-dimensional vectorial systems of the results presented in Reference 1. In order to better understand the extension of the FVM to upwinding schemes, in Section 5 we will try to derive upwinded finite elements with the finite volume approximation.

4. EQUIVALENCE OF FEM AND FVM FOR NON-UPWINDING SCHEMES

In the FEM, equation (47) is solved approximating the unknown functions $\mathbf{u}$ by

$$\mathbf{u} \cong \hat{\mathbf{u}} = \sum_j \mathbf{N}_j\mathbf{u}_j \tag{61}$$

Choosing now a Galerkin scheme ($\mathbf{W}_i = \mathbf{N}_i$), equation (47) reads

$$\begin{aligned}
&\int_\Omega \mathbf{N}_i\sum_j \mathbf{N}_j\dot{\mathbf{u}}_j\,d\Omega + \int_\Omega \mathbf{N}_i\mathbf{A}_k\sum_j\frac{\partial\mathbf{N}_j}{\partial x_k}d\Omega\,\mathbf{u}_j\\
&\quad - \int_\Omega \mathbf{N}_i\frac{\partial}{\partial x_k}\left(\mathbf{K}_{kl}\sum_j\frac{\partial\mathbf{N}_j}{\partial x_l}\right)d\Omega\,\mathbf{u}_j - \int_\Omega \mathbf{N}_i\mathbf{Q}\,d\Omega = 0
\end{aligned} \tag{62}$$

We will consider a discretization in three-node triangular elements with standard linear shape functions $\mathbf{N}_i$ as shown in Figure 4. The equivalence of FEM and FVM for constant matrices $\mathbf{A}_i$ and $\mathbf{K}_{ij}$ will be demonstrated for the different terms of equation (62).

4.1. The temporal term

If mass lumping is used then

$$\int_{\Omega_i} \mathbf{N}_i \frac{\partial \mathbf{u}}{\partial t} d\Omega = \frac{1}{3}\Omega_i \dot{\mathbf{u}}_i \tag{63}$$

where Ω_i is the area of all elements containing node i (see Figure 4). It is easy to check that equation (66) is equivalent to the temporal term of the FVM when mass lumped cell-centred control volumes defined by the median of the triangles are used (Figure 3), i.e.

$$\int_{\Omega_c} \frac{\partial \mathbf{u}}{\partial t} d\Omega = \Omega_c \dot{\mathbf{u}}_i = \frac{1}{3}\Omega_i \dot{\mathbf{u}}_i \tag{64}$$

4.2. The convective term

Assuming $\mathbf{A}_k$ to be constant we can write

$$\int_{\Omega_i} \mathbf{N}_i \mathbf{A}_k \sum_j \frac{\partial \mathbf{N}_j}{\partial x_k} \mathbf{u}_j d\Omega = \mathbf{A}_k \sum_j \mathbf{u}_j \int_{\Omega_i} \mathbf{N}_i \frac{\partial \mathbf{N}_j}{\partial x_k} d\Omega \tag{65}$$

The contribution to $\sum_j$ of the term $j = i$ vanishes. In fact, for any arbitrary vector b we can write

$$\int_{\Omega_i} b_k N_i \frac{\partial N_i}{\partial x_k} d\Omega = \int_{\Omega_i} \frac{\partial}{\partial x_k}\left(\frac{1}{2} b_k N_i^2\right) d\Omega = \oint_{\Gamma_i} \frac{N_i^2}{2} b_k n_k \, d\Gamma \tag{66}$$

but $N_i = 0$ on Γ, which implies

$$\int_{\Omega_i} \mathbf{N}_i \frac{\partial \mathbf{N}_i}{\partial x_k} d\Omega = 0 \tag{67}$$

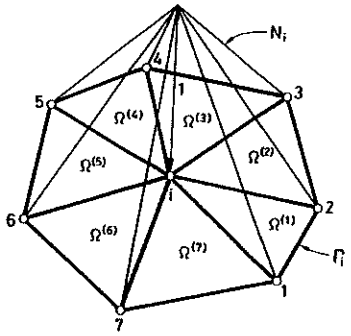

Figure 4. Linear N_i functions defined over a triangular finite element domain

Taking into account (67), we can change the sign of the ith term in (65) to give

$$\int_{\Omega_i} \mathbf{N}_i \mathbf{A}_k \sum_j \left(\frac{\partial \mathbf{N}_j}{\partial x_k} \mathbf{u}_j\right) d\Omega = \mathbf{A}_k \left(\sum_{j \neq i} \mathbf{u}_j \int_{\Omega_i} \mathbf{N}_i \frac{\partial \mathbf{N}_j}{\partial x_k} d\Omega - \mathbf{u}_i \int_{\Omega_i} \mathbf{N}_i \frac{\partial \mathbf{N}_j}{\partial x_k} d\Omega \right) \tag{68}$$

Let us now consider the contribution to Ω_i of an individual element $12i$ of area $\Omega^{(1)}$. In each triangular region we have a relationship of the type

$$\frac{\partial \mathbf{N}_i}{\partial x_k} = -\frac{\partial \mathbf{N}_1}{\partial x_k} - \frac{\partial \mathbf{N}_2}{\partial x_k} \tag{69}$$

Then, the contribution of Ω_1 to (68) is

$$\int_{\Omega^{(1)}} \mathbf{N}_i \mathbf{A}_k \left[(\mathbf{u}_1 - \mathbf{u}_i) \frac{\partial \mathbf{N}_1}{\partial x_k} + (\mathbf{u}_2 - \mathbf{u}_i) \frac{\partial \mathbf{N}_2}{\partial x_k} \right] d\Omega \tag{70}$$

For linear N_i functions, the gradients $\partial \mathbf{N}_i / \partial x_k$ may be written in vectorial notation as

$$\nabla N_1 = \frac{\mathbf{k} \times \overrightarrow{2\mathbf{i}}}{2\Omega_1}, \quad \nabla N_2 = \frac{\mathbf{k} \times \overrightarrow{\mathbf{i}1}}{2\Omega_1} \tag{71}$$

in which k represents a vector normal to the surface (see Figure 5) and $\times$ denotes the vectorial-product.

Introducing (71) in (70) and taking into account that $\int_{\Omega_1} N_i \, d\Omega = \frac{1}{3}\Omega^{(1)}$, the contribution to (68) of $\Omega^{(1)}$ becomes

$$\frac{1}{3}\Omega^{(1)} \left[\mathbf{A} \cdot \frac{\mathbf{k} \times \overrightarrow{2\mathbf{i}}}{2\Omega^{(1)}} (\mathbf{u}_1 + \mathbf{u}_i) + \mathbf{A} \cdot \frac{\mathbf{k} \times \overrightarrow{\mathbf{i}2}}{2\Omega^{(1)}} (\mathbf{u}_2 + \mathbf{u}_i) \right] \tag{72}$$

Repeating this procedure for all the triangular regions which share node i, and grouping the terms in $(\mathbf{u}_1 + \mathbf{u}_i)$, we have

$$\frac{1}{3}\mathbf{A} \cdot \mathbf{k} \times \overrightarrow{2\mathbf{i}} \left(\frac{\mathbf{u}_1 + \mathbf{u}_i}{2}\right) + \frac{1}{3}\mathbf{A} \cdot \mathbf{k} \times \overrightarrow{\mathbf{i}7} \left(\frac{\mathbf{u}_1 + \mathbf{u}_i}{2}\right) = \frac{1}{3}\mathbf{A} \cdot \mathbf{k} \times (\overrightarrow{2\mathbf{i}} + \overrightarrow{\mathbf{i}7}) \left(\frac{\mathbf{u}_1 + \mathbf{u}_i}{2}\right) \tag{73}$$

From the triangles of Figure 6 we can see that

$$\frac{\overrightarrow{2\mathbf{i}} + \overrightarrow{\mathbf{i}7}}{3} = \frac{\overrightarrow{27}}{3} = \overrightarrow{\mathbf{G}_1\mathbf{G}_7} = \overrightarrow{\mathbf{G}_1\mathbf{M}_1} + \overrightarrow{\mathbf{M}_1\mathbf{G}_7} \tag{74}$$

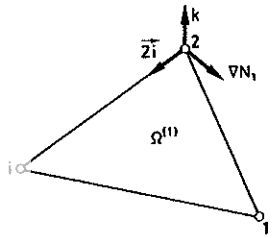

Figure 5. Vector definitions

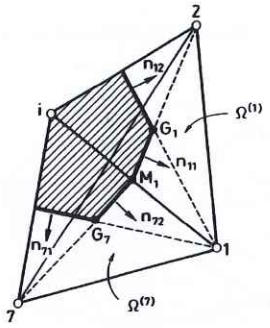

Figure 6. Geometric considerations for elements 1 and 7

Then

$$\frac{1}{3}\mathbf{A}\cdot\mathbf{k}\times(\overrightarrow{2i}+\overrightarrow{i7})\left(\frac{\mathbf{u}_1+\mathbf{u}_i}{2}\right)=\mathbf{A}\cdot\mathbf{k}\times\overrightarrow{\mathbf{G}_1\mathbf{M}_1}+\mathbf{A}\cdot\mathbf{k}\times\overrightarrow{\mathbf{M}_1\mathbf{G}_7} \tag{75}$$

Taking into account that

$$\mathbf{k}\times\overrightarrow{\mathbf{G}_1\mathbf{M}_1}=\mathbf{n}_{11}|\overrightarrow{\mathbf{G}_1\mathbf{M}_1}|=\mathbf{n}_{11}\Delta\Gamma_{11}$$

and

$$\mathbf{k}\times\overrightarrow{\mathbf{M}_1\mathbf{G}_7}=\mathbf{n}_{72}|\overrightarrow{\mathbf{M}_1\mathbf{G}_7}|=\mathbf{n}_{72}\Delta\Gamma_{72}$$

in which $\mathbf{n}_{11}$ represents the normal to the contour $\overrightarrow{\mathbf{G}_1\mathbf{M}_1}$ and $\Delta\Gamma_{11}$ its length.

The total contribution of the convective term to node i becomes

$$\int_{\Omega_i}\mathbf{N}_i\mathbf{A}_k\sum_j\frac{\partial\mathbf{N}_i}{\partial x_k}\mathbf{u}_i\,\mathrm{d}\Omega=\sum_{e=1}^{7}\left(\mathbf{A}_k\left(\frac{\mathbf{u}_e+\mathbf{u}_i}{2}\right)(\mathbf{n}_{e1})_k\Delta\Gamma_{el}+\mathbf{A}_k\left(\frac{\mathbf{u}_{e+1}+\mathbf{u}_i}{2}\right)(n_{e2})_k\Delta\Gamma_{e2}\right) \tag{76}$$

Equation (77) represents the surface integral of the convective flow

$$\oint_{\Omega_c}\mathbf{A}_k\mathbf{u}\,n_k\,\mathrm{d}\Gamma \tag{77}$$

around a contour Γ_c defined by the median of the triangles, which is precisely the finite volume approach of the convective term.

4.3. The diffusive term

Taking into account that $N_i=0$ on Γ we can write

$$\oint_{\Omega_i}\mathbf{N}_i\frac{\partial}{\partial x_k}\mathbf{K}_{kl}\sum_j\left(\frac{\partial\mathbf{N}_j}{\partial x_l}\mathbf{u}_j\right)\mathrm{d}\Omega=-\int_{\Omega_i}\frac{\partial\mathbf{N}_i}{\partial x_k}\mathbf{K}_{kl}\sum_j\left(\frac{\partial\mathbf{N}_j}{\partial x_l}\mathbf{u}_j\right)\mathrm{d}\Omega \tag{78}$$

The contribution of the $\Omega^{(1)}$ triangular region to equation (78) is

$$-\int_{\Omega^{(1)}}\frac{\partial\mathbf{N}_i}{\partial x_k}\mathbf{K}_{kl}\sum_j\left(\frac{\partial\mathbf{N}_j}{\partial x_l}\mathbf{u}_j\right)\mathrm{d}\Omega=-\frac{\partial\mathbf{N}_i}{\partial x_k}\mathbf{K}_{kl}\sum_j\left(\frac{\partial\mathbf{N}_j}{\partial x_l}\mathbf{u}_j\right)\Omega^{(1)} \tag{79}$$

Using vectorial notation we have

$$\nabla N_i=\frac{\mathbf{k}\times\overrightarrow{\mathbf{12}}}{2\Omega_1} \tag{80}$$

and (see Figure 7)

$$\frac{\overrightarrow{\mathbf{12}}}{2}=\overrightarrow{\mathbf{M}_1\mathbf{M}_2}=\overrightarrow{\mathbf{M}_1\mathbf{G}_1}+\overrightarrow{\mathbf{G}_1\mathbf{M}_2} \tag{81}$$

Then

$$\frac{\mathbf{k}\times\overrightarrow{\mathbf{12}}}{2}=-\mathbf{n}_{11}|\overrightarrow{M_1G_1}|-\mathbf{n}_{12}|\overrightarrow{G_1n_2}|=-\mathbf{n}_{11}\Delta\Gamma_{11}-\mathbf{n}_{12}\Delta\Gamma_{12} \tag{82}$$

and the total diffusive contribution to node i becomes

$$\int_{\Omega_i}\mathbf{N}_i\frac{\partial}{\partial x_k}\sum_j\left(\frac{\partial\mathbf{N}_j}{\partial x_k}\mathbf{u}_j\right)\mathrm{d}\Omega=\sum_e\left(\mathbf{K}_{kl}\sum_j\left(\frac{\partial\mathbf{N}_j}{\partial x_l}\mathbf{u}_j\right)n_{e1k}\Delta\Gamma_{e1}+\mathbf{K}_{kl}\sum_j\left(\frac{\partial\mathbf{N}_j}{\partial x_l}\mathbf{u}_j\right)n_{e2k}\Delta\Gamma_{e2}\right)\quad(e=1,2,\ldots,7) \tag{83}$$

Taking into account that $\mathbf{K}_{kl}\sum_j((\partial\mathbf{N}_j/\partial x_l)\mathbf{u}_j)=\mathbf{K}_{kl}(\partial\mathbf{u}/\partial x_l)=\mathbf{g}_k$, equation (83) is the integral of the diffusive term $\mathbf{g}_k n_k$ over a contour Γ_c, which is equivalent to the expression resulting from the cell-centred FVM.

4.4. The source term

It is easy to prove that the expression of the source term is equivalent to that of the cell-centred FVM for constant Q sources, i.e.

$$\int_\Omega\mathbf{N}_i\mathbf{Q}\,\mathrm{d}\Omega=\frac{1}{3}\Omega\mathbf{Q}=\Omega_c\mathbf{Q} \tag{84}$$

Resume. We have shown that for centred weighting (not upwinding) schemes the FEM is numerically *equivalent* to the FVM when the following conditions are satisfied:

(1) Linear triangles with a lumped mass matrix are used in both FE and FV approaches.
(2) Cell-centred control volumes are used in the FVM.
(3) Constant Q sources loads are assumed.

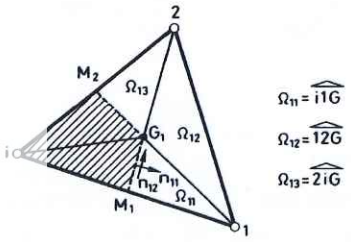

Figure 7. Geometric definitions on element 1

5. FEM EQUIVALENT TO FVM FOR n-DIMENSIONAL UPWINDED PROBLEMS

In this section we study under which conditions FEM and FVM coincide if upwind weighting is used.

Let the field $\mathbf{u}$ be approximated by the standard linear functions $\mathbf{N}_i$ (Figure 4). In order to develop a finite element formulation equivalent to that given by the FVM we must find either a $\mathbf{P}_i$ function such that

$$\int_\Omega (\mathbf{N}_i + \mathbf{P}_i)\mathbf{A}_k \frac{\partial}{\partial x_k}(\sum \mathbf{N}_j \mathbf{u}_j)\,\mathrm{d}\Omega = \oint_{\Gamma_c} \mathbf{A}_n \mathbf{u}\,\mathrm{d}\Gamma \tag{85a}$$

or a balancing diffusive matrix $\bar{\mathbf{K}}_{km}$ such that

$$\int_\Omega (\mathbf{N}_i \mathbf{A}_k \frac{\partial}{\partial x_k}(\sum \mathbf{N}_j \mathbf{u}_j)\,\mathrm{d}\Omega + \int_\Omega \frac{\partial \mathbf{N}_i}{\partial x_k}\bar{\mathbf{K}}_{km}\frac{\partial}{\partial x_m}(\sum \mathbf{N}_j \mathbf{u}_j)\,\mathrm{d}\Omega = \oint_{\Gamma_c} \mathbf{A}_n \mathbf{u}\,\mathrm{d}\Gamma \tag{85b}$$

with the definition of $(\mathbf{A}_n\mathbf{u})|_{\Gamma_c}$ given in (57) and (58). This second approach will be the one followed in this paper.

In the previous section, we have shown that the centred part of (85b), i.e.

$$\int_\Omega \mathbf{N}_i \mathbf{A}_k \frac{\partial}{\partial x_k}\left(\sum_j \mathbf{N}_j \mathbf{u}_j\right)\mathrm{d}\Omega \tag{86}$$

is equal to the first term of the RHS of (58) for some particular FE and FV approaches. In the appendix we develop a very special anisotropic finite element with a diffusive $\bar{\mathbf{K}}_{km}$ matrix which is equal to the contribution of the off-centred convective terms in the FE equations. Although the demonstration is exact for equilateral triangles, it represents an approximation for other element shapes. The underlying idea is not to build up finite elements with this complicated anisotropic matrix, but to try to understand better the finite volume approach from a finite element point of view.

We can conclude that both FEM and FVM will yield approximately the same system of discretized equations for upwinded problems if triangular elements are divided into three subregions, and in each one of these an anisotropic diffusive matrix is added with a diffusion term in the $\tilde{n}$ direction, $\tilde{n}$ being the normal to the median crossing by this subregion (Figure 7). The value of this diffusive matrix must be evaluated as a function of the $\mathbf{A}_n = \mathbf{A}_i n_i$ matrix as in the 1-D case. This equivalence between FVM and FEM is *exact* for equilateral elements.

6. CONCLUDING REMARKS

FVM and FEM have been permanently compared in the CFD literature by numerical tests. The advantages of one over the other and *vice versa* have been proved many times for particular problems. In this paper we have tried to find a particular case in which both methods are coincident. We have seen that for self-adjoint problems, the most popular finite element, the linear triangle, is coincident with the most popular finite volume, the cell-centred triangular control volume.

Several upwinding schemes have been proposed for both methods to deal with non-self-adjoint problems. In FEM the SUPG technique seems to be the more consistent for diagonalizable systems such as the convection–diffusion heat transfer equation. This is not true for non-diagonalizable systems such as the Navier–Stokes or Euler equations for which SUPG methods may introduce spurious transverse diffusion. On the other hand, by changing the 3-D spatial integral into a surface integral over the control volume, the FVM transforms the 3-D problem

into an infinite set of one-dimensional problems which may be easily diagonalized. From this point of view, the FVM is more consistent because both diagonalizable and non-diagonalizable systems are solved with the same approach. A special FE based on the FV approach has been developed in this paper. This can help to understand better the meaning of the FV upwinding scheme. The results of this paper also open the possibility for developing new computational procedures in which finite elements and finite volume can be 'blended' to produce more efficient numerical solutions. In particular, for triangular linear elements, it is possible to use finite elements for the diffusive part and finite volumes for the convective terms. Other possibilities may be explored in which the accuracy of the Galerkin finite elements to solve self-adjoint equations may be combined with the easy way than the FVM deals with the convective terms.

APPENDIX

An anisotropic finite element equivalent to a finite volume for upwinded problems

Let Ω_1 be one of the triangular elements of Ω in Figure 3, for instance $\Omega^{(1)} = \overset{\Delta}{i12}$. Dividing the triangle by the medians we can define the following three triangular subregions (Figure 7):

$$\Omega_{11} = \overset{\Delta}{i1G_1}, \quad \Omega_{12} = \overset{\Delta}{12G_1}, \quad \Omega_{13} = \overset{\Delta}{2iG_1}$$

Then, the integral over the domain Ω_1 may be written as

$$\int_{\Omega_1} \frac{\partial \mathbf{N}_i}{\partial x_k}\bar{\mathbf{K}}_{km}\frac{\partial \mathbf{u}}{\partial x_m}\mathrm{d}\Omega = \sum_{n=1}^{3}\int_{\Omega_{1n}} \frac{\partial \mathbf{N}_i}{\partial x_k}\bar{\mathbf{K}}_{km}\frac{\partial \mathbf{u}}{\partial x_m}\mathrm{d}\Omega \tag{87}$$

In order to derive a diffusive matrix $\bar{\mathbf{K}}_{km}$ that gives the equivalent term of the FVM, we shall rotate in each subregion Ω_{1n} the co-ordinates x_k in three orthogonal directions n_{1_1}, n_{1_2} and n_{n1_3}, where n_{1_1} is normal to the median crossing this subregion (Figure 7).
We can write

$$\int_{\Omega_{11}} \frac{\partial \mathbf{N}_i}{\partial n_{1p}}\frac{\partial n_{1p}}{\partial x_k}\bar{\mathbf{K}}_{km}\frac{\partial \mathbf{u}}{\partial n_{1r}}\frac{\partial n_{1r}}{\partial x_m}\mathrm{d}\Omega = \int_{\Omega_{11}} \frac{\partial \mathbf{N}_i}{\partial n_{1p}}\bar{\mathbf{K}}_{pr}\frac{\partial \mathbf{u}}{\partial n_{1r}} \tag{88}$$

where $\bar{\mathbf{K}}_{pr}$ is the rotated balancing diffusive matrix.
We will now define

$$\begin{aligned} \bar{\mathbf{K}}_{pr} &= \alpha_1 \mathscr{T}_{n_1}|\mathbf{A}_{n_1}| \quad \text{for } p = r = 1 \\ \bar{\mathbf{K}}_{pr} &= 0 \quad \text{for } p = 2 \text{ or } r = 2 \end{aligned} \tag{89}$$

where $\mathscr{T}_{n_1}$ is the upwind matrix defined in (57) in the n_{1_1} direction, and α_1 is a coefficient to be adjusted later
With this definition, equation (88) becomes

$$\int_{\Omega_{11}} \frac{\partial \mathbf{N}_i}{\partial n_{1p}}\bar{\mathbf{K}}_{pr}\frac{\partial \mathbf{u}}{\partial n_{1r}}\mathrm{d}\Omega = \int_{\Omega_{11}} \frac{\partial \mathbf{N}_i}{\partial n_{1_1}}\alpha_1 \mathscr{T}_{n_1}|\mathbf{A}_{n_1}|\frac{\partial \mathbf{u}}{\partial n_{1_1}}\mathrm{d}\Omega$$

but

$$\frac{\partial N_i}{\partial n_{1_1}} \cong -\frac{1}{|\vec{\mathbf{i1}}|}, \quad \frac{\partial N_1}{\partial n_{1_1}} \cong \frac{1}{|\vec{\mathbf{i1}}|}, \quad \frac{\partial N_2}{\partial n_{1_1}} \cong 0 \quad \text{(the equality occurs when } \vec{\mathbf{2M}}_1 \perp \vec{\mathbf{i1}}\text{)}$$

and

$$\frac{\partial \mathbf{u}}{\partial n_{1_1}} = \frac{\partial}{\partial n_{1_1}} \sum_j \mathbf{N}_j \mathbf{u}_j \cong -\frac{1}{|\vec{\mathbf{i1}}|}\mathbf{u}_i + \frac{1}{|\vec{\mathbf{i1}}|}\mathbf{u}_1 = \frac{\mathbf{u}_1 - \mathbf{u}_i}{|\vec{\mathbf{i1}}|} \tag{90}$$

The contribution of the Ω_{11} region is

$$\int_{\Omega_{11}} \frac{\partial \mathbf{N}_i}{\partial x_k} \hat{\mathbb{K}}_{km} \frac{\partial \mathbf{u}}{\partial x_m} \mathrm{d}\Omega = \frac{\alpha_1 \Omega_{11}}{|\vec{\mathbf{i1}}|^2} 2\mathscr{T}_1 |\mathbf{A}_{n_1}| \frac{(\mathbf{u}_i - \mathbf{u}_1)}{2} \tag{91}$$

From (57) and (58) we can define α_1 such that

$$\frac{2\alpha_1 \Omega_{11}}{|\vec{\mathbf{i1}}|^2} = \Delta\Gamma_1 \tag{92}$$

and then

$$\alpha_1 = \frac{\Delta\Gamma_1 |\vec{\mathbf{i1}}|^2}{2\Omega_{11}} \tag{93}$$

Taking

$$\Omega_{11} \cong \frac{|\vec{\mathbf{i1}}| \Delta\Gamma_1}{2} \tag{94}$$

α_1 becomes

$$\alpha_1 = |\vec{\mathbf{i1}}| \tag{95}$$

In the same way the subregion Ω_{13} gives the contributions to the $\vec{\mathbf{i2}}$ direction, and the subregion Ω_{12} gives a contribution nearly zero to the node i due to the face that

$$\frac{\partial \mathbf{N}_i}{\partial n_{21}} \cong 0 \quad \text{(the equality occurs when } \vec{\mathbf{iG}}_1 \perp \vec{\mathbf{12}}) \tag{96}$$

Therefore, the resulting finite element with an anisotropic $\hat{\mathbf{K}}_{pr}$ matrix as defined in (89) leads to the same system of discretized equations for upwinded problems as in the standard cell-centred FVM. This equivalence is exact for equilateral triangular regions and approximated for other shapes.

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