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FINITELY ADDITIVE CONDITIONAL PROBABILITIES, CONGLOMERABILITY AND DISINTEGRATIONS[1]

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For any finitely additive probability measure to be disintegrable, that is, to be an average with respect to some marginal distribution of a system of finitely additive conditional probabilities, it suffices, and is plainly necessary, that the measure be conglomerative, that is, that there be a conditional expectation such that the expectation of no random variable can be negative if that random variable's conditional expectation given each of the marginal events is nonnegative.

With respect to some margins, that is, partitions, there are finitely additive probability measures that are so far from being disintegrable that they cannot be approximated in the total variation norm by those that are. Those partitions which have this property are determined.

Many partially defined conditional probabilities, and in particular, all disintegrations, or, equivalently, strategies, are restrictions of full conditional probabilities $Q = Q(A \mid B)$ defined for all pairs of events A and B with B non-null.

0. Introduction. The three sections of this note treat three aspects of general, that is, of finitely additive, probability measures, and, having no logical interdependencies, the sections can be read in any order.

Section 1 demonstrates the equivalence of two properties, that, surprisingly, not all probability measures possess. Nonconglomerability, according to its discoverer, de Finetti, [4] ([5] page 98), obtains if an event can have a probability larger than the supremum of its conditional probability given each of a mutually exclusive and exhaustive set of events. And a probability measure is disintegrable along a particular margin if it is the average with respect to its marginal distribution of some system of conditional probabilities. As Section 1 explicates and demonstrates, disintegrability is equivalent to a closely related notion of conglomerability.

Section 2 shows that there exists a finitely additive probability measure (on a product space) that not only admits of no disintegration (with respect to one of the coordinate axes) but which cannot be approximated in total variation norm by those that do.

A function defined for all pairs of events A and B with B non-null, $Q(A/B)$,

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the conditional probability of A given B, and satisfying certain simple, natural axioms, is called a full conditional probability. The main purpose of Section 3 is to show that disintegrations of probability measures can always be extended to be full conditional probabilities. That section also establishes by standard analysis, and in slightly greater generality, the basic results obtained by Krauss [6], by nonstandard analysis, that all probability measures, and certain other partially defined conditional probabilities, can be extended to be full conditional probabilities.

1. Conglomerative measures are strategic. Throughout this section, Ω designates a fixed, nonempty set; π a *partition* of Ω, that is, a class of nonemply, pairwise-disjoint subsets of Ω whose set-theoretic union is Ω; a *random variable* X is a bounded, real-valued function on Ω; and an *expectation* E is a nonnegative, linear functional defined on the space of random variables which satisfies

$$\inf X \leqq E(X) \leqq \sup X\,, \tag{1}$$

where $\inf X$ and $\sup X$ are the infimum and the supremum of the values that X assumes.

I adopt the useful suggestion of de Finetti ([5] page xviii) that the same symbol that designates an event also designates the indicator of that event, namely the function that is 1 on the event and 0 off the event.

A *π-strategy* σ is a function of two variables X and h, where X ranges over the space of random variables and h ranges over the elements of π, such that, for each h, $\sigma(\cdot \mid h)$ is an expectation for which $\sigma(h \mid h) = 1$. (This usage of "strategy" is not quite the one adopted in [2].)

For each $S \subset \pi$, let the same letter "S" also designate the union of all $h \in S$ (as well as the indicator of this event). With these useful conventions, the *π-marginal* of an expectation E is that probability measure γ defined for $S \subset \pi$, by $\gamma(S) = E(S)$.

The two properties, conglomerability and disintegrability, that a pair (E, π) may possess can now be defined.

If there is a π-strategy σ such that, for all bounded X,

$$\sigma(X \mid h) \geqq 0 \quad \text{for all} \quad h \in \pi \quad \text{implies} \quad E(X) \geqq 0\,, \tag{2}$$

then E is *π-conglomerative*.

If for some π-strategy σ and all bounded X,

$$E(X) = \int \sigma(X \mid h)\, d\gamma(h)\,, \tag{3}$$

then E is *π-disintegrable*, or *π-strategic*, and σ is a *π-disintegration* or a *π-strategy* for E.

As is evident, if (3) holds then so does (2). Somewhat surprisingly, the converse also obtains.

THEOREM 1. *A necessary and sufficient condition for E to be π-disintegrable is that E be π-conglomerative. More precisely, for any π-strategy σ,* (3) *holds for all bounded X if, and only if,* (2) *does.*

PROOF. Suppose that (2) holds for all bounded X. Then, as is easily verified, for all bounded X and all real numbers u,

$$\sigma(X \mid h) \leqq u \qquad \text{for all} \quad h \in \pi \tag{4}$$

implies $E(X) \leqq u$.

Now let $S \subset \pi$. As will be shown, (4) can be strengthened to:

$$\sigma(X \mid h) \leqq u \qquad \text{for all} \quad h \in S \tag{5}$$

implies $E(XS) \leqq u\gamma(S)$.

For, let X' be X on S and u on S^c. Plainly,

$$\sigma(X' \mid h) \leqq u \qquad \text{for all} \quad h \in \pi \,. \tag{6}$$

Hence, by (4),

$$\begin{aligned} u \geqq E(X') &= E(X'S) + E(X'S^c) \\ &= E(XS) + E(u(1 - S)) \\ &= E(XS) + u - u\gamma(S) \,. \end{aligned} \tag{7}$$

So (5) is established.

Now let $\varepsilon > 0$. For each integer j, let S_j be the set of $h \in \pi$ such that

$$(j - 1)\varepsilon \leqq \sigma(X \mid h) < j\varepsilon \,, \tag{8}$$

and calculate thus,

$$\begin{aligned} \int_{S_j} \sigma(X \mid h)\, d\gamma(h) &\geqq (j - 1)\varepsilon\gamma(S_j) \\ &= j\varepsilon\gamma(S_j) - \varepsilon\gamma(S_j) \\ &\geqq E(XS_j) - \varepsilon\gamma(S_j) \,, \end{aligned} \tag{9}$$

where the last inequality holds by (5) and (8).

Sum over the finite number of j for which S_j is nonempty, note ε is arbitrary, and conclude:

$$\int \sigma(X \mid h)\, d\gamma(h) \geqq E(X) \,. \tag{10}$$

If X is replaced by $-X$, the inequality reverse to (10) is seen to hold, which, together with (10), implies (3).

For simplicity of exposition, Theorem 1 and the relevant definitions were formulated without measurability restraints. However, the theorem, with the analogous proof, holds also if formulated relative to measurability with respect to a distinguished field, not necessarily a sigma-field, of subsets of Ω.

If conglomerability is defined in terms of inequalities on the conditional probabilities of events only, as in [4] and [5], then, for some non-nonatomic probability spaces, the notion is technically weaker than that of this note. Moreover, according to de Finetti ([5] page 99), conglomerability is equivalent to countable additivity. Since there certainly are π-disintegrable expectations that are not countably additive, the observations of this note would seem to come into conflict with his, unless the notion of conglomerability I explicate is, in some essential way, not exactly the one intended by him.

2. Approximability in norm by strategic or disintegrable measures. An *F-gamble* is a finitely additive probability measure defined on all subsets of the nonempty set F. In this section only, a strategy σ is a pair (σ_0, σ_1) where:

(i) σ_0 is an F_1 gamble and

(ii) σ_1 is a mapping that assigns to each $i \in F_1$ an F_2-gamble, $\sigma_1(i)$. If H designates $F_1 \times F_2$, then the strategy σ determines an H-gamble, or, equivalently an H-expectation, here ambiguously, but harmlessly, also designated by σ, and defined for all bounded, real-valued functions g on H, thus,

$$\sigma g = \int \left[\int g(i, j)\, d\sigma_1(j \mid i)\right] d\sigma_0(i) . \tag{11}$$

An H-gamble, probability measure, or expectation, that thus arises from a strategy is *strategic*.

A first question that arises is the existence on H of nonstrategic probability measures. As is evident, if F_1 is a finite set, every probability measure on H is strategic. If it is F_2 that is assumed finite, and F_1 infinite, then there are probability measures on H that are not strategic. For an example: let F_2 contain two elements, $+$ and $-$; let F_1 be the positive integers, and let μ be a probability measure whose marginal on F_2 assigns probability $\frac{1}{2}$ to $+$ and probability $\frac{1}{2}$ to $-$; let the conditional distribution of F_1 given $+$ assign positive probability to each positive integer; and let the conditional distribution of F_1 given $-$ be a *diffuse* gamble, that is, one that assigns to every finite subset of F_1 the probability 0. The proof that μ is not strategic is easily furnished and is, in any event, implicit in the discussion on page 205 of [5].

The question then arises as to whether every H-gamble can be approximated by a strategic gamble. That is, given an H-gamble μ and an $\varepsilon > 0$, is there a strategic gamble σ such that, for every subset A of H

$$|\mu(A) - \sigma(A)| < \varepsilon\,? \tag{12}$$

If F_2 is a finite set, the answer is in the affirmative, as Proposition 1 below implies. So even if F_2 is denumerable and the marginal distribution of μ on F_2 is countably additive, μ can be approximated arbitrarily well by strategic gambles.

It is the main purpose of this section to demonstrate that there exist finitely additive probabilities on H which cannot be approached by strategic ones.

If μ and ν are such that, for every $\varepsilon > 0$, there is a set A of μ-probability exceeding $1 - \varepsilon$ but of ν-probability less than ε, then μ is *singular* with respect to ν, in which event ν is singular with respect to μ, and then one writes $\mu \perp \nu$.

THEOREM 2. *Let J be the set of positive integers. Then there exists a finitely additive probability measure on $J \times J$ which is singular with respect to every strategic measure on $J \times J$.*

PROOF OF THEOREM. A probability measure on J is *remote* if every finite set has probability 0 and if every set that does not have probability 0 has probability 1.

Let α and β be remote probability measures, and define the H-gamble μ for bounded, real-valued functions g on $J \times J$, thus,

$$\mu g = \int \left[\int g(i, j)\, d\alpha(i)\right] d\beta(j)\,. \tag{13}$$

Of course, the μ measure of a rectangle $A \times B$ is the product of $\alpha(A)$ with $\beta(B)$. There is a measure $\hat{\mu}$ that agrees with μ on rectangles which is nevertheless quite different from μ and to which attention is called mainly to avoid confusion. Under $\hat{\mu}$ for each first coordinate i, the second coordinate is distributed according to β whereas, under μ, for each second coordinate j, the first coordinate is distributed according to α. According to the definition of strategic above, $\hat{\mu}$ is strategic whereas, as will be seen, μ is not. Under $\hat{\mu}$, with probability 1, the second coordinate exceeds the first, whereas, under μ, the first coordinate almost certainly exceeds the second. The appearance of paradox to which pairs such as μ and $\hat{\mu}$ give rise has been noted by de Finetti and P. Lévy. For references see [5] page 98.

Lemma 1. *μ is remote on $J \times J$.*

Lemma 2. *If G is the graph of a function g on J into J, then $\mu(G) = 0$.*

Proof of Lemma 2.

$$\mu(G) = \int \alpha(Gj)\, d\beta(j)\,, \tag{14}$$

where Gj is the set of i such that $g(i) = j$. Since the sets Gj are pairwise disjoint, $\sum \alpha(Gj) \leqq 1$, from which it plainly follows that $\alpha(Gj)$ converges to 0. Since every finite set of integers has β-probability 0, the right-hand side of (14) is zero. □

Let T be that subset of $J \times J$ consisting of all (i, j) such that $j \leqq i$.

Lemma 3.

$$\mu(T) = 1\,.$$

Proof. For each j, the set Tj of all i such that $(i, j) \in T$ contains all but a finite number of i. Therefore, $\alpha(Tj) = 1$. As is evident,

$$\begin{aligned} \mu(T) &= \int \alpha(Tj)\, d\beta(j) \\ &= \int 1\, d\beta(j) \\ &= 1\,. \end{aligned} \tag{15}$$

If σ is a strategy such that, for all i, $\sigma_1(i)(Ti) = 1$, where Ti is the i-section of T, that is, the set of j such that $(i, j) \in T$, then σ is a strategy that *lives on T*. Plainly, if σ is a strategy that lives on T, then $\sigma(T) = 1$. The converse does not quite hold. However, one has:

Lemma 4. *If σ is a strategy which determines a measure $\hat{\sigma}$ on H for which $\hat{\sigma}(T) = 1$, then, among the strategies that determine $\hat{\sigma}$, there is one that lives on T.*

Proof. Suppose σ is a strategy for which $\hat{\sigma}(T) = 1$. Let $\sigma_0' = \sigma_0$, and let $\sigma_1'(i)$ agree with $\sigma_1(i)$ for all $j < i$, but let $\sigma_1'(i)$ assign to i the sum of what $\sigma_1(i)$ assigns to i and to the half infinite interval (i, ∞). As is easily verified, the two strategies σ and σ' determine the same measure. □

LEMMA 5. *For a μ which satisfies $\mu(T) = 1$ to be singular with respect to all strategic measures, it suffices that it be singular with respect to all strategic measures that live on T.*

PROOF OF LEMMA 5. Verify that every strategic measure is a convex combination of a strategic measure that assigns to T probability 1 with one that assigns to T probability 0. Plainly, it suffices that μ be singular with respect to the first, so, in view of Lemma 4, the proof is complete.

LEMMA 6. *The measure μ defined by* (13) *is singular with respect to every strategy σ that lives on T.*

PROOF. Let σ be a strategy that lives on T. Since for each i, the i-section of T consists of the i integers $1, \cdots, i$, which is a finite set, $\sigma_1(i)$ is a discrete probability measure that lives on $(1, \cdots, i)$. For each p, $0 \leqq p \leqq 1$, let $g(i) = g_p(i)$ be the p-percentile of the probability measure $\sigma_1(i)$, that is $g_p(i)$ is the smallest integer l such that

$$\sigma_1(i)[1 \leqq j \leqq l] \geqq p\,. \tag{16}$$

For each p, the graph $G(p)$ of g_p has μ-probability 0, as Lemma 2 implies. Now choose a large positive integer N, and let $p_k = k/N$, as k ranges through the $k + 1$ integers 0 to N. The area strictly between $G(p_n)$ and $G(p_{n+1})$ plainly has σ-probability at most $1/N$. Moreover, since μ is remote, as Lemma 1 states, for every partition of H into a finite number of sets, one of these sets has μ-probability 1. Since the complement of T and each graph $G(p_k)$ has μ-probability 0, for some n, $0 \leqq n < N$, the area strictly between $G(p_n)$ and $G(p_{n+1})$ has μ-probability one. This proves Lemma 6.

Together, Lemmas 5 and 6 plainly imply Theorem 2.

Let Σ be the set of all strategic probability measures on $J \times J$ and let Σ' be the set of all probabilities on $J \times J$ that are singular with respect to every element of Σ. As is evident, $\Sigma \subset \Sigma''$, and, according to Theorem 2, Σ' is nonempty. Every probability measure on $J \times J$ can be expressed in one and only one way as a convex combination of an element of Σ' and an element of Σ'', as is implied by Theorems 1 and 3 in [1]. Perhaps the elements of Σ'', as well as those of Σ', have a nice characterization.

Of course, the phenomena reported in this note do not require that H be the Cartesian product of the integers with itself. Obviously, H could be the Cartesian product of any two sets. When H is a product space, the vertical sections determine a partition π of H. But, as in Section 1, the notions of marginal, conditional distribution, and strategy are meaningful relative to any partition π of any set H. A partition π is *simple* if, for some integer n, each of all but a finite number of its elements has cardinality at most n.

For any finitely additive, signed measure μ defined for subsets of H, μ is of *bounded variation* if

$$\sup (A \subset H)(|\mu(A)| + |\mu(A^c)|) < \infty , \tag{17}$$

in which event this sup is called the (total variation) *norm* of μ and designated by $||\mu||$.

Henceforth, measures of bounded variation, and these only, are designated by the term measure.

PROPOSITION 1. *If a partition π is simple, then the π-strategic measures are norm-dense. That is, for each probability P on H, there is, for each $\varepsilon > 0$, a π-strategy σ such that*

$$|P(A) - \int \sigma(A \mid h)\, d\gamma(h)| < \varepsilon \tag{18}$$

for all $A \subset H$, where γ is the π-marginal of P.

*A set A is a π-*selection *if, for each $h \in \pi$, $A \cap h$ contains at most one point.*

LEMMA 7. *Let A be a subset of H, π a partition of H, and P a measure on H. Then, for each $\varepsilon > 0$, there is a function f, defined on π (or defined on H and π-measurable) such that $0 \leqq f \leqq 1$, $f(h) = 0$ if $h \cap A = \emptyset$, $f(h) = 1$ if $h \cap A = h$, and*

$$-\varepsilon < P(A \cap S) - \int_S f\, dP < \varepsilon \tag{19}$$

for every $S \subset \pi$ (or π-measurable set S).

PROOF. Easy in view of a finitely additive Radon–Nikodym theorem [3].

PROOF OF PROPOSITION 1. Suppose that every element of π has cardinality at most n. An application of the axiom of choice yields n pairwise-disjoint sets $A_1, \cdots, A_n$ whose union is H and such that $h \cap A_i$ contains at most one point, $1 \leqq i \leqq n$. Apply Lemma 7 to find, for each $\varepsilon > 0$, nonnegative π-functions $f_1, \cdots, f_n$ such that $\sum f_i = 1$, $f_i(h) = 0$ if $h \cap A_i = \emptyset$, $f_i(h) = 1$ if $h \cap A_i = h$, and

$$-\frac{\varepsilon}{n} < P(A_i \cap S) - \int_S f_i\, dP < \frac{\varepsilon}{n} \tag{20}$$

for every subset S of π and each i. For all $A \subset H$ and each $h \in \pi$, define $\sigma(A \mid h) = \sum f_i(h)$ where the sum is taken over those i for which the intersection $A_i \cap A \cap h \neq \emptyset$.

Verify that σ is a π-strategy, and that (18) holds. This completes the proof when every element of π has cardinality at most n. The easy modifications necessary to handle the case in which there are a finite number of exceptional elements of π of cardinality greater than n are omitted.

The observations of this section can be summarized as saying that the following three conditions are equivalent.

(i) π is simple;

(ii) every probability measure on H is in the norm closure of the set of π-strategic measures;

(iii) there exists no probability measure orthogonal to every π-strategic measure.

Moreover, since π-conglomerative measures are none other than the π-strategic ones, as was explained in Section 1, this section can be viewed as a contribution to the study of conglomerability.

3. Extensions of some partially defined conditional probabilities. A *probability measure* on a Boolean algebra $\mathscr{A}$ with unit Ω is a finitely additive, nonnegative, function, normalized so as to assume the value 1 on Ω.

If $\mathscr{H}$ is a subalgebra of a Boolean algebra $\mathscr{A}$, then P is a *conditional probability* on $(\mathscr{A}, \mathscr{H})$, and $(\mathscr{A}, \mathscr{H}, P)$ is a *conditional probability space*, if P is a function whose domain is $\mathscr{A} \times \mathscr{H}^\circ$ ($\mathscr{H}^\circ$ is $\mathscr{H}$ without the null element of $\mathscr{H}$), satisfying:

$$(21) \qquad P(\cdot \mid H) \quad \text{is a probability measure on} \quad \mathscr{A}, \quad \text{for each} \quad H \in \mathscr{H}^\circ ;$$

$$(22) \qquad P(H \mid H) = 1 \qquad \text{for all} \quad H \in \mathscr{H}^\circ ;$$

$$(23) \qquad P(A \mid C) = P(A \mid B) P(B \mid C) \qquad \text{whenever} \quad A \subset B \subset C , \quad A \in \mathscr{A},\ B \in \mathscr{H}^\circ,\ C \in \mathscr{H}^\circ .$$

In the presence of (21) and (22), condition (23) is equivalent to

$$(23\text{a}) \qquad P(AB \mid C) = P(B \mid C) P(A \mid BC) \qquad \text{for all} \quad A, B, C , \quad \text{with} \quad A \in \mathscr{A},\ B \in \mathscr{A},\ C \in \mathscr{H},\ BC \in \mathscr{H}^\circ .$$

A conditional probability P on $(\mathscr{A}, \mathscr{H})$ is *full* (on $\mathscr{A}$) if $\mathscr{H} = \mathscr{A}$.

LEMMA 8. *Let $\mathscr{A}^\circ$ be the set of nonzero elements of a Boolean algebra $\mathscr{A}$, and let P be defined on a subset $\mathscr{D}$ of $\mathscr{A} \times \mathscr{A}^\circ$. Then these two conditions are equivalent*:

(a) *There is a full conditional probability on $\mathscr{A}$ which agrees with P on $\mathscr{D}$.*

(b) *For every finite Boolean subalgebra $\mathscr{F}$ of $\mathscr{A}$, there is a full conditional probability on $\mathscr{F}$ which agrees with P on the intersection of $\mathscr{D}$ with $\mathscr{F} \times \mathscr{F}^\circ$.*

PROOF OF LEMMA 8. Obviously (a) $\rightarrow$ (b). So assume (b). Let $Q_{\mathscr{F}}$ be the set of all mappings M of $\mathscr{A} \times \mathscr{A}^\circ$ into the closed unit interval which, restricted to $\mathscr{F} \times \mathscr{F}^\circ$, is a conditional probability which agrees with P on $\mathscr{D} \cap (\mathscr{F} \times \mathscr{F}^\circ)$. Plainly, $Q_{\mathscr{F}}$ is compact in the usual product topology and as (b) implies, $Q_{\mathscr{F}}$ is nonempty. Since the family of $Q_{\mathscr{F}}$ plainly possesses the finite intersection property, the intersection of all the $Q_{\mathscr{F}}$ is nonempty, which establishes (a).

THEOREM 3. *For every conditional probability space $(\mathscr{A}, \mathscr{H}, P)$, there is an extension Q of P which is a full conditional probability on $\mathscr{A}$.*

PROOF. When $\mathscr{A}$ *is finite*, define a sequence $H_1, \cdots, H_n$ of nonzero elements of $\mathscr{H}$ which are disjoint and whose union is Ω, thus, H_1 is the least element of $\mathscr{H}$ such that

$$(24) \qquad P(H_1) = P(H_1 \mid \Omega) = 1 .$$

If $H_1 = \Omega$, let $n = 1$. If $H_1 \neq \Omega$, let H_2 be the least element of $\mathscr{H}$ included in H_1^c such that

$$(25) \qquad P(H_2 \mid H_1^c) = 1 .$$

In general, if $H_1, \cdots, H_i$ have been defined and their union is Ω, let $n = i$. Otherwise, let H_{i+1} be the least element of $\mathscr{H}$ such that

$$(26) \qquad P(H_{i+1} \mid (H_1 \vee \cdots \vee H_i)^c) = 1 .$$

For nonzero $B \in \mathscr{A}$, let $i = i(B)$ be the smallest integer such that $BH_i \neq 0$ and define

$$(27) \qquad \begin{aligned} Q(A \mid B) &= P(AB \mid H_i)/P(B \mid H_i) \qquad &\text{if } P(B \mid H_i) > 0 , \\ &= \frac{|ABH_i|}{|BH_i|} &\text{if } P(B \mid H_i) = 0 , \end{aligned}$$

where $|E|$ is the number of atoms comprising E.

To see that Q has the requisite properties, first verify easily that, for each nonzero $B \in \mathscr{A}$, $Q(\cdot \mid B)$ is a probability measure on $\mathscr{A}$ and that $Q(B, B) = 1$. To check that $Q(A \mid H) = P(A \mid H)$ for nonzero $H \in \mathscr{H}$, suppose that $HH_j = 0$ for $j < i$ and $HH_i \neq 0$. Then $P(HH_i \mid H_i) > 0$, so

$$(28) \qquad \begin{aligned} Q(A \mid H) &= P(AH \mid H_i)/P(H \mid H_i) \\ &= P(H \mid H_i)P(A \mid HH_i)/P(H \mid H_i) \\ &= P(A \mid HH_i) \\ &= P(A \mid H) , \end{aligned}$$

where the last equality can be seen to hold for all A.

What must now be seen is that

$$(29) \qquad Q(A \mid C) = Q(A \mid B)Q(B \mid C) ,$$

wherever $A \subset B \subset C$ and B is not null.

Plainly, $i(C) \leqq i(B)$. If $i(C) < i(B)$, then B is disjoint from $H_{i(C)}$, which implies that $Q(B \mid C)$, and *a fortiori*, $Q(A \mid C) = 0$, so (29) holds. It may be assumed henceforth, therefore, that $i(C) = i(B) = i$. Now if $P(B \mid H_i) > 0$, certainly $P(C \mid H_i) > 0$, and (29) is trivially checked. If $P(B \mid H_i) = 0 = P(C \mid H_i)$, then again (29) holds. In the remaining case, $P(B \mid H_i) = 0$ and $P(C \mid H_i) > 0$. Then $Q(B \mid C)$, and, *a fortiori*, $Q(A \mid C)$ are zero.

PROOF OF THEOREM 3 FOR GENERAL $\mathscr{A}$. Let $\mathscr{F}$ be a finite subalgebra of $\mathscr{A}$. Plainly, $\mathscr{H}_1 = \mathscr{F} \cap \mathscr{H}$ is a subalgebra of $\mathscr{F}$, and P, restricted to $\mathscr{F} \times \mathscr{H}_1^\circ$, is a conditional probability. Since $\mathscr{F}$ is finite, there exists a full conditional probability $\mathscr{F}$ which agrees with P on $\mathscr{F} \times \mathscr{H}_1^\circ$, which is the intersection of $\mathscr{D} = \mathscr{A} \times \mathscr{H}^\circ$ with $\mathscr{F} \times \mathscr{F}^\circ$. So (b) of Lemma 8 holds. This completes the proof of Theorem 3.

If $\mathscr{H}$ is the trivial 2-element algebra consisting of the null and universal elements, 0 and Ω, Theorem 3 becomes a result of Krauss', namely

COROLLARY 1 (Krauss). *Every probability measure on a Boolean algebra can be extended to a full conditional probability.*

THEOREM 4. *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{H}$ be Boolean algebras with $\mathscr{A} \supset \mathscr{B} \supset \mathscr{H}$ and let P be a conditional probability on $(\mathscr{B}, \mathscr{H})$. Then there is a full conditional probability on $\mathscr{A}$ which is an extension of P.*

PROOF. Suppose first that $\mathscr{H}$ has only a finite number of atoms, say $h_i, \cdots, h_n$. For each i, the probability measure $P(\cdot \mid h_i)$ defined on $\mathscr{B}$ can be extended to some probability measure $Q(\cdot \mid h_i)$ defined on $\mathscr{A}$.

For each $H \in \mathscr{H}$, define

$$Q(A \mid H) = \textstyle\sum_{j=1}^{n} Q(A \mid h_i)P(h_i \mid H) . \tag{30}$$

As is not difficult to verify, $Q(A \mid H)$ is a conditional probability on $(\mathscr{A}, \mathscr{H})$ which extends P. In view of Theorem 3, Q can further be extended to be a full conditional probability on $\mathscr{A}$.

Now drop the assumption that $\mathscr{H}$ is finite, and let $\mathscr{F}$ be a finite subalgebra of $\mathscr{A}$. Then $P_{\mathscr{F}}$, the restriction of P to $(\mathscr{B} \cap \mathscr{F}, \mathscr{H} \cap \mathscr{F})$, can be extended to a full conditional probability on $\mathscr{F}$. An application of Lemma 8 in which $\mathscr{D} = \mathscr{B} \times \mathscr{H}^\circ$, now completes the proof of Theorem 4.

By specializing $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{H}$ in various ways, special cases of interest are obtained. In particular, if $\mathscr{A} = \mathscr{B}$, Theorem 4 reduces to Theorem 3. If $\mathscr{B} = \mathscr{H}$, one obtains:

COROLLARY 2 (Krauss). *A full conditional probability on a subalgebra of a Boolean algebra $\mathscr{A}$ can be extended to be a full conditional probability on $\mathscr{A}$.*

Henceforth, let $\mathscr{A}$ be the field of all subsets of a set Ω. A *partition* π is a set of nonempty, disjoint subsets of Ω whose union is Ω. A *strategy* σ on $(\mathscr{A}, \pi)$ is a pair of functions (σ_0, σ_1) where σ_0 is the *marginal* of σ on π and σ_1 is a *conditional probability given* π. (This usage of "strategy" differs slightly from that of Sections 1 and 2 and that of [3].) More precisely: σ_1 is a function defined on $\mathscr{A} \times \pi$ such that, for each $h \in \pi$, $\sigma(\cdot \mid h)$ is a gamble defined on $\mathscr{A}$ for which $\sigma(h \mid h) = 1$. And the marginal σ_0 is a probability measure defined on the field of subsets of π.

A *π-measurable* set is a subset of Ω which is a union of elements of π. Thus, to each subset S of π, the union of all $h \in S$ is π-measurable and can and will also be designated by "S," for every π-measurable set is of this form for some unique $S \subset \pi$. Hence, σ_0 determines (and is determined by) a unique probability measure on the π-measurable sets which will also be designated by "σ_0."

THEOREM 5. *For each strategy σ on $(\mathscr{A}, \pi)$, there is a full conditional probability on $\mathscr{A}$ which is an extension of σ.*

PROOF. In accordance with Corollary 1, σ_0 can be extended to be a full conditional probability on π, here also designated by σ_0. Thus, for each nonempty subset S of π, $\sigma_0(\cdot \mid S)$ is a probability measure defined on 2^π, the set of subsets of π.

For each $A \subset \Omega$ and nonempty $S \in 2^\pi$, the function whose value at $h \in \pi$ is $\sigma_1(A \mid h)$ can be integrated with respect to the measure $\sigma_0(dh \mid S)$, obtaining

$$P(A \mid S) = \int \sigma_1(A \mid h)\sigma_0(dh \mid S) . \tag{31}$$

If S is interpreted as ranging over $\hat{\pi}$, the field of π-measurable subsets of Ω, P is easily seen to be a conditional probability on $(\mathscr{A}, \hat{\pi})$ which extends σ. In view of Theorem 3, P can be further extended to be a full conditional probability on $\mathscr{A}$. This completes the proof of Theorem 5.

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