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Finitely-additive invariant measures on Euclidean spaces

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To the memory of V. M. Alexeyev

Abstract. It is shown that for $n \geq 3$ the Lebesgue measure is the unique finitely-additive isometry-invariant measure on the ring of bounded Lebesgue measurable subsets of the n-dimensional Euclidean space.

Let β be the ring of bounded Lebesgue measurable sets in the n-dimensional real space R^n or in S^n, the n-dimensional unit sphere in R^{n-1}, and let λ_n be the Lebesgue measure on β normalized by $\lambda_n(J^n) = 1$, where $J = [0, 1)$, or by $\lambda_n(S^n) = 1$ respectively. It is well known that λ_n is up to proportionality the unique countably-additive isometry-invariant measure on β. In 1923, Banach (see [**1**]) studied the following question of Ruziewicz: is λ_n, up to proportionality, the unique finitely-additive isometry-invariant (positive) measure on β. Banach gave a negative answer to this question for R^1, R^2 and S^1. But for R^n, $n \geq 3$, or S^n, $n \geq 2$, the Ruziewicz question was unanswered for a long time.

Let ν be a finitely-additive isometry-invariant measure on β and let L^0_∞ be the space of bounded measurable functions on R^n or on S^n having a compact support. It is well known (see, for example, [**11**]) that, for R^n, $n \geq 3$, or S^n, $n \geq 2$, ν is absolutely continuous with respect to λ_n i.e. $\nu(X) = 0$ whenever $\lambda_n(X) = 0$, $X \in \beta$. Then ν determines (by the formulae

$$l(f) = \int f\, d\nu)$$

an isometry-invariant linear positive ($l(f) \geq 0$ if $f \geq 0$) functional l on L^0_∞. Conversely, any isometry-invariant linear positive functional l on L^0_∞ determines an isometry-invariant finitely-additive measure ν on β by the formula

$$\nu(X) = l(\chi(X)),$$

where $\chi(X)$ is the characteristic function of $X \in \beta$. Therefore, for R^n, $n \geq 3$, or S^n, $n \geq 2$, the Ruziewicz question is equivalent to the following question:

(A) is any isometry-invariant linear positive functional l on L^0_∞ proportional to the functional α_n given by

$$\alpha_n(f) = \int f\, d\lambda_n, \quad f \in L^0_\infty.$$

In [9] and [13] (independently and simultaneously), it was proved that, for S^n, $n \geq 4$, the answer to (A) is positive. The proof in both [9] and [13] is based on reducing (A) to a question on asymptotically invariant sets (see [2], [5], [11]) and on using Kazhdan's results on property (T).

In this paper, it is proved (see theorem 4) that, for R^n, $n \geq 3$, the answer to the Ruziewicz question (and to the equivalent question (A)) is positive. The proof of theorem 4 is based, roughly speaking, on a reduction (by arguments of Namioka) to a question on almost invariant nets of locally integrable functions and on using methods of representation theory.

Thus the Ruziewicz question remains unanswered only for S^2 and S^3.

1. *Property (T) and almost invariant nets*

Let H be a separable unimodular locally compact group. Let $\tilde{H}$ (respectively $\hat{H}$) denote the set of unitary (respectively unitary irreducible) continuous representations of H in separable Hilbert spaces. Let μ_H be a Haar measure on H and let $A(H)$ be the set of non-negative continuous functions f on H having a compact support such that

$$\int_H f\, d\mu_H = 1.$$

For any $U \in \tilde{H}$ and any μ_H-integrable function f on H, put

$$U(f) = \int_H f(h) U(h)\, d\mu_H(h).$$

Let $L(U)$ denote the space of a representation U. For $f \in A(H)$ and $\varepsilon > 0$, put

$$V(\varepsilon, f) = \{U \in \tilde{H} \mid \text{there exists } x \in L(U), x \neq 0, \text{ such that } \|U(f)x - x\| < \varepsilon \|x\|\}.$$

Let T_0 be the one-dimensional trivial representation of H. One can introduce on $\tilde{H}$ the standard topology (see [4], [6]) for which sets $\mathcal{V}(\varepsilon, f)$ form a basis of neighbourhoods of T_0. We say that $U \in \tilde{H}$ is *close to* T_0 if T_0 belongs to the closure of $\{U\}$. According to Kazhdan (see [4], [6]), we say that H has property (T) if one of two following equivalent conditions is satisfied:

(I) there exists a neighbourhood $\mathcal{V}$ of T_0 such that any $U \in \mathcal{V} \cap \hat{H}$ is equivalent to T_0;

(II) any representation $U \in \tilde{H}$ close to T_0 contains T_0 (i.e. there exists $x \in L(U)$, $x \neq 0$, such that $U(H)x = x$).

Let us give now a slight generalization of this definition.

DEFINITION 1. *Let $B \subset H$. We say that the pair (H, B) has property (T) if one of two following equivalent conditions is satisfied*:

(I′) *there exists a neighbourhood $\mathcal{V}$ of T_0 such that any $U \in \mathcal{V} \cap \hat{H}$ is trivial on B*

$$(i.e. \quad U(B)x = x \quad \text{for any } x \in L(U)).$$

(II′) *if $U \in \tilde{H}$ is close to T_0, then there exists $x \in L(U)$, $x \neq 0$, such that*

$$U(B)x = x.$$

The equivalence of (I′) and (II′) can be proved in the same way as the equivalence of (I) and (II) was proved in [6] (by a decomposition of $U \in \tilde{H}$ into a continuous sum of irreducible representations).

Let us give now one further definition. We make the convention that $0/0 = 0$.

Definition 2. *Let $U \in \tilde{H}$, $M \subset H$ and $\{x_\alpha\}$ be a net (generalized sequence) of elements of $L(U)$. We say the net $\{x_\alpha\}$ is almost M-invariant if, for any compact set $K \subset H$,*

$$\lim_\alpha \left[\sup_{h \in K \cap M} \frac{\|U(h)x_\alpha - x_\alpha\|}{\|x_\alpha\|} \right] = 0.$$

We say the net $\{x_\alpha\}$ is weakly almost M-invariant if, for any $h \in M$,

$$\lim_\alpha \frac{\|U(h)x_\alpha - x_\alpha\|}{\|x_\alpha\|} = 0.$$

Theorem 1. *Let $B \subset H$, Γ be a lattice in H (i.e. Γ be a discrete subgroup of H such that*

$$\mu_H(\Gamma \backslash H) < \infty), \quad U \in \tilde{H} \quad \text{and} \quad \{x_\alpha\}$$

be a net of elements of $L(U)$. Denote by S the closure of the set

$$B \cdot \Gamma \stackrel{\text{def}}{=} \{b\gamma \mid b \in B, \gamma \in \Gamma\}.$$

Let us assume that the pair (H, B) has property (T). If the net $\{x_\alpha\}$ is weakly almost Γ-invariant, then $\{x_\alpha\}$ is almost S-invariant.

Proof. Choose (see [3], chapter VII, § 2, exercise 12]) a Borel subset $Y \subset H$ with the following properties:

(a) Y is a fundamental domain for Γ i.e. $\Gamma \cdot Y = G$ and

$$(\gamma_1 Y) \cap (\gamma_2 Y) = \varnothing \quad \text{if } \gamma_1 \neq \gamma_2,\ \gamma_1, \gamma_2 \in \Gamma;$$

(b) for any compact set $K \subset H$, the set

$$\{\gamma \in \Gamma \mid (\gamma Y) \cap K \neq \varnothing\}$$

is finite.

Let $\rho \in \tilde{H}$ be the representation of H induced in the sense of Mackey by the restriction of U to Γ. The space J of the representation ρ consists of measurable $j : H \to L(U)$ such that

$$\|j\|^2 = \int_Y \|j(h)\|^2 \, d\mu_H(h) < \infty$$

and

$$j(\gamma h) = U(\gamma) j(h), \quad \text{for } \gamma \in \Gamma, h \in H. \tag{1}$$

The scalar product $\langle \cdot, \cdot \rangle$ on J is introduced by

$$\langle j_1, j_2 \rangle = \int_Y \langle j_1(h), j_2(h) \rangle \, d\mu_H(h), \quad \text{for } j_1, j_2 \in J, \tag{2}$$

and the representation ρ is defined by

$$(\rho(h)j)(h_1) = j(h_1 h),\ h, h_1 \in H, j \in J. \tag{3}$$

Since the pair (H, B) has property (T), there exist $f\in A(H)$ and $\varepsilon>0$ such that any $W\in\hat{H}\cap V(\varepsilon,f)$ is trivial on B. Now decompose the unitary representation ρ into a continuous sum of irreducible representations $\rho_z\in\hat{H}$. Let ρ_1 denote the continuous sum of ρ_z such that

$$\rho_z\in V(\varepsilon,f)$$

and let ρ_2 denote the continuous sum of ρ_z such that $\rho_z\notin V(\varepsilon,f)$. Then

$$\rho=\rho_1\oplus\rho_2,$$

ρ_1 is trivial on B and $\rho_2\notin V(\varepsilon,f)$. Let P_r, $r=1,2$, denote the orthogonal projection of J onto $L(\rho_r)$. Since $\rho_2\notin V(\varepsilon,f)$,

$$\|\rho(f)j-j\|\geq\varepsilon\|P_2h\|=\varepsilon\sqrt{\|j\|^2-\|P_1j\|^2} \tag{4}$$

for any $j\in J$.

Let us give the following definition. Two nets $\{u_\alpha\}$ and $\{u'_\alpha\}$ of elements of a Banach space are called equivalent if

$$\lim_\alpha\frac{\|u'_\alpha-u_\alpha\|}{\|u_\alpha\|}=0.$$

Define $j_\alpha\in J$ by

$$j_\alpha(\gamma y)=U(\gamma)x_\alpha,\quad \gamma\in\Gamma,\quad y\in Y. \tag{5}$$

For $h\in H$, define $\gamma(h)\in\Gamma$ by the inclusion $h\in\gamma(h)Y$. Let $K\subset H$. In view of (1), (2), (3) and (5), we have

$$\begin{aligned}\sup_{h\in K}\|\rho(h)j_\alpha-j_\alpha\|^2&=\sup_{h\in K}\int_Y\|j_\alpha(h_1h)-j_\alpha(h_1)\|^2\,d\mu_H(h_1)\\&=\sup_{h\in K}\int_Y\|U(\gamma(h_1h))x_\alpha-x_\alpha\|^2\,d\mu_H(h_1)\\&\leq\int_Y(\sup_{h\in K}\|U(\gamma(h_1h))x_\alpha-x_\alpha\|^2)\,d\mu_H(h_1).\end{aligned} \tag{6}$$

The property (b) of Y implies the set $\gamma(M\cdot K)$ is finite for any compact sets $K\subset H$ and $M\subset H$. On the other hand, the net $\{x_\alpha\}$ is weakly almost Γ-invariant. Therefore,

$$\lim_\alpha\sup_{h\in K,h_1\in M}\frac{\|U(\gamma(h_1h))x_\alpha-x_\alpha\|}{\|x_\alpha\|}=0$$

for any compact sets $K\subset H$ and $M\subset H$. On the other hand, as U is unitary then

$$\|U(\gamma(h_1h))x_\alpha-x_\alpha\|\leq2\|x_\alpha\|$$

for any h, $h_1\in H$. Therefore, (6) implies the net $\{j_\alpha\}$ is almost H-invariant. On the other hand, as $f\in A(H)$ then

$$\|\rho(f)j_\alpha-j_\alpha\|\leq\sup_{h\in\operatorname{supp}f}\|\rho(h)j_\alpha-j_\alpha\|,$$

where $\operatorname{supp}f$ denotes as usual the (compact) support of f. Therefore, the nets $\{j_\alpha\}$ and $\{\rho(f)j_\alpha\}$ are equivalent. From this and (4) we conclude that the nets $\{j_\alpha\}$ and $\{P_1j_\alpha\}$ are equivalent.

Let us now fix a compact set $K \subset H$ containing e. For any $j \in J$, we have

$$\begin{aligned}\sup_{h\in K}\|(\rho(f)j)(h)\| &= \sup_{h\in K}\left\|\int_H f(h_1)j(hh_1)\,d\mu_H(h_1)\right\|\\ &\le \sup_{h\in K}\int_H f(h_1)\|j(hh_1)\|\,d\mu_H(h_1)\\ &\le (\sup_{h'\in H} f(h'))\cdot\int_{K\cdot\operatorname{supp} f}\|j(\tilde h)\|\,d\mu_H(\tilde h)\\ &\le (\sup_{h'\in H} f(h'))\cdot\sqrt{\mu_H(K\cdot\operatorname{supp} f)}\cdot\sqrt{\int_{K\cdot\operatorname{supp} f}\|j(\tilde h)\|^2\,d\mu_H(\tilde h)} \qquad (7)\end{aligned}$$

(the last inequality is implied by the Schwartz inequality). Since the set $K\cdot\operatorname{supp} f$ is compact, it belongs to $M\cdot Y$ for some finite set $M\subset\Gamma$. Then in view of (2),

$$\int_{K\cdot\operatorname{supp} f}\|j(\tilde h)\|^2\,d\mu_H(\tilde h)\le|M|\cdot\|j\|^2.$$

This inequality and (7) imply

$$\sup_{h\in K}\|(\rho(f)j)(h)\|\le C\|j\| \qquad (8)$$

for any $j\in J$, where

$$C=(\sup_{h'\in H} f(h'))\cdot\sqrt{\mu_H(K\cdot\operatorname{supp} f)}\cdot\sqrt{|M|}.$$

Further, put

$$j'_\alpha=\rho(f)P_1 j_\alpha=P_1\rho(f)j_\alpha.$$

Then in view of (8),

$$\sup_{h\in K}\|j'_\alpha(h)-(\rho(f)j_\alpha)(h)\|\le C\|P_1 j_\alpha-j_\alpha\|. \qquad (9)$$

In view of (1), (3) and (5) we have

$$\begin{aligned}\sup_{h\in K}\|x_\alpha-(\rho(f)j_\alpha)(h)\| &= \sup_{h\in K}\left\|x_\alpha-\int_H f(h_1)j_\alpha(hh_1)\,d\mu_H(h_1)\right\|\\ &= \sup_{h\in K}\left\|\int_H f(h_1)(x_\alpha-j_\alpha(hh_1))\,d\mu_H(h_1)\right\|\\ &\le \sup_{\tilde h\in K\cdot\operatorname{supp} f}\|x_\alpha-j_\alpha(\tilde h)\|\\ &= \sup_{\gamma\in\gamma(K\cdot\operatorname{supp} f)}\|x_\alpha-U(\gamma)x_\alpha\|. \qquad (10)\end{aligned}$$

As the nets $\{j_\alpha\}$ and $\{P_1 j_\alpha\}$ are equivalent, the set $\gamma(K\cdot\operatorname{supp} f)$ is finite (in view of the property (b) of Y), the net $\{x_\alpha\}$ is weakly almost Γ-invariant, and

$$\|j_\alpha\|=\sqrt{\mu_H(Y)}\|x_\alpha\|,$$

we conclude from (9) and (10) that

$$\lim_\alpha\sup_{h\in K}\frac{\|j'_\alpha(h)-x_\alpha\|}{\|x_\alpha\|}=0. \qquad (11)$$

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Let $\tilde{B}$ be a normal subgroup of H generated by B. Since ρ_1 is trivial on B, it is also trivial on $\tilde{B}$. On the other hand

$$j'_\alpha \in L(\rho_1).$$

Therefore for any $h \in H$, we have

$$\begin{aligned} j'_\alpha(\tilde{B}h) = j'_\alpha(h\tilde{B}) &= (\rho(\tilde{B})j'_\alpha)(h) \\ &= (\rho_1(\tilde{B})j'_\alpha)(h) = j'_\alpha(h). \end{aligned} \tag{12}$$

The equalities (11) and (12) imply

$$\lim_\alpha \sup_{h \in \tilde{B} \cdot K} \frac{\|j'_\alpha(h) - x_\alpha\|}{\|x_\alpha\|} = 0. \tag{13}$$

Since U is unitary, we get using (1) that

$$\begin{aligned} \|U(\gamma)x - x\| &\le \|U(\gamma)x - j(\gamma)\| + \|j(\gamma) - x\| \\ &= \|U(\gamma)(x - j(e))\| + \|j(\gamma) - x\| \\ &= \|j(e) - x\| + \|j(\gamma) - x\| \end{aligned} \tag{14}$$

for any $\gamma \in \Gamma$, $x \in L(U)$ and $j \in J$. Using (13) and (14), we get

$$\lim_\alpha \sup_{\gamma \in (\tilde{B} \cdot K) \cap \Gamma} \frac{\|U(\gamma)x_\alpha - x_\alpha\|}{\|x_\alpha\|} = 0. \tag{15}$$

As K is an arbitrary compact subset of H the representation U is continuous and $\tilde{B} \supset B$, then (15) implies the assertion of the theorem.

LEMMA 1. *Let K be a non-discrete locally compact field and let $H = SL_2(k) \propto B$ be a semi-direct product of* $SL_2(k)$ *and a separable commutative locally compact group B (as usual* $SL_2(k)$ *denotes the group of unimodular matrices of order* 2 *with coefficients in k). Consider the natural action of H on the character group B^* of B*

$$(h\chi)(b) = \chi(h^{-1}bh),\ h \in H, \chi \in B^*, b \in B).$$

Let us assume that (I) *any orbit under this action is locally closed,* (II) *the subgroup*

$$G_\chi = \{h \in SL_2(k) | h\chi = \chi\}$$

is commutative for any $\chi \in B^*$, $x \ne 0$. Then the pair (H, B) has property (T).

Proof. Let $U \in \hat{H}$ and suppose U is not trivial on B. Then in view of Mackey's results (see [**8**]), U is induced by an irreducible unitary representation ρ of a subgroup

$$H_\chi = \{h \in H | h\chi = \chi\}, \chi \in B^*, \chi \ne 0.$$

On the other hand since G_χ is commutative, H_χ is solvable and, consequently, ρ belongs to the closure of the regular representation of H_χ. Therefore, U belongs to the closure of the regular representation τ of H. Now it remains to note that (see [**6**]) the regular representation of $SL_2(k)$ is not close to T_0 and so τ is not close to T_0. □

Remark. Some arguments of [**6**] were in fact used in the proof of lemma 1.

Let us prove now the following theorem using theorem 1 and lemma 1.

THEOREM 2. *Let $n \geq 3$, H_n be the group of isometries of the n-dimensional Euclidean space R^n, and let $B_n \subset H_n$ be the group of parallel translations of R^n. Then there exists a countable subgroup Γ of H_n with the following property: if $U \in \tilde{H}_n$ and $\{x_\alpha\}$ is a weakly almost Γ-invariant net of elements of $L(U)$, then $\{x_\alpha\}$ is almost B_n-invariant.*

Proof. For a commutative ring L with identity, denote by $O_n(L)$ the group of matrices of order n with coefficients in L which preserve the quadratic form $C = x_1^2 + \cdots + x_n^2$ and put

$$G_n(L) = \left\{\begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}\right\}, \qquad F_n(L) = \left\{\begin{pmatrix} E & B \\ 0 & 1 \end{pmatrix}\right\},$$

where $A \in O_n(L)$ and B is a column vector with coefficients in L. It is well known that there exists a topological isomorphism of groups $G_n(R)$ and H_n which maps $F_n(R)$ onto B_n. Therefore, one can replace H_n by $G_n(R)$ and B_n by $F_n(R)$ in the formulation of the theorem.

Let $Z(1/5)$ denote the subring of the field Q generated by $1/5$ and let Q_5 denote the field of 5-adic numbers. Put

$$\Gamma = G_n(Z(1/5)).$$

The diagonal embedding of $Z(1/5)$ into $R \times Q_5$ induces the diagonal embedding of Γ into

$$H \stackrel{\text{def}}{=} G_n(R) \times G_n(Q_5).$$

Let us suppose we have proved that the pair $(G_n(Q_5), F_n(Q_5))$ has property (T). Then the pair $(H, F_n(Q_5))$ also has property (T). (We think of the subgroup

$$F_n(Q_5) \subset G_n(Q_5)$$

as naturally embedded in H). On the other hand in view of the Borel reduction theorem (see [**2**]), Γ is a lattice in H. Therefore applying theorem 1 to the pair $(H, F_n(Q_5))$, the lattice Γ, the net $\{x_\alpha\}$, and the representation $U' = p \cdot U$, where $p: H \to G_n(R)$ is the natural mapping, we get the net $\{x_\alpha\}$ is almost S-invariant, where S is the closure in $F_n(R)$ of $F_n(Z(1/5))$. On the other hand as $Z(1/5)$ is dense in R, then $S = F_n(R)$.

Thus, it remains to prove that the pair $(G_n(Q_5), F_n(Q_5))$ has property (T). For this we consider three cases: (1) $n \geq 5$; (2) $n = 4$; (3) $n = 3$.

Case (1). Let a representation $\rho \in \widetilde{G(Q_5)}$ be close to T_0. As -1 is a square in Q_5, the Q_5-rank (i.e. the dimension of a maximal Q_5-split torus) of $O_n(Q_5)$ is equal to the integer part of $n/2$ and, consequently, is not less than $[5/2] = 2$. On the other hand if $n \neq 4$, then O_n is an absolutely (almost) simple algebraic group. Therefore in view of Kazhdan's results on property (T) (see [**4**], [**6**]), the group $O_n(Q_5)$ has property (T). Hence, there exists $x \in L(\rho)$, $x \neq 0$, such that $\rho(P)x = x$, where

$$P = \left\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right\}, \qquad A \in O_n(Q_5).$$

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Let

$$D=\{x\in Q_5^n \mid C(x)=0\}.$$

For any $d\in D$, one can find a matrix $A_d\in O_n(Q_5)$ such that

$$\lim_{n\to+\infty} A_d^n(d)=0.$$

Then

$$\lim_{n\to+\infty}\begin{pmatrix}A_d & 0\\ 0 & 1\end{pmatrix}^n\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\begin{pmatrix}A_d & 0\\ 0 & 1\end{pmatrix}^{-n}=E. \tag{16}$$

Since $\rho(P)x=x$ and the representation ρ is continuous and unitary, (16) and the generalization of the Mautner lemma given by Prasad (see [**10**]) imply

$$\rho\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}x=x \quad \text{for any } d\in D.$$

On the other hand as -1 is a square in Q_5, then D generates Q_5 as a group. Therefore, $\rho(F_n(Q_5))x=x$.

Case (2). Let $\mathrm{i}\in Q_5$ such that $\mathrm{i}^2=-1$. For $A\in \mathrm{SL}_2(Q_5)$, put

$$X_A=\begin{pmatrix}E & \mathrm{i}E\\ E & -\mathrm{i}E\end{pmatrix}^{-1}\begin{pmatrix}A & 0\\ 0 & {}^tA^{-1}\end{pmatrix}\begin{pmatrix}E & \mathrm{i}E\\ E & -\mathrm{i}E\end{pmatrix}.$$

where ${}^tA$ is the matrix transposed to A and E is the identity matrix of the second order. One can directly check that $X_A\in O_4(Q_5)$. Put

$$M=\left\{\begin{pmatrix}X_A & 0\\ 0 & 1\end{pmatrix}\right\}\subset G_4(Q_5).$$

Then applying lemma 1, we obtain the pair $(M\cdot F_4(Q_5), F_4(Q_5))$ and, consequently, the pair $(G_4(Q_5), F_4(Q_5))$, has property (T).

Case (3). As -1 is a square in Q_5, then C can be reduced to the form $x_1^2+x_2x_3$. Then identifying a vector (x_1, x_2, x_3) with the matrix

$$\begin{pmatrix}x_1 & x_2\\ x_3 & -x_1\end{pmatrix}$$

and conjugating this matrix by $A\in\mathrm{SL}_2(Q_5)$, we get the homomorphism $\varphi:\mathrm{SL}_2(Q_5)\to O_3(Q_5)$. Now, we can define a semi-direct product

$$T=\mathrm{SL}_2(Q_5)\propto F_3(Q_5)$$

and a homomorphism

$$\tilde\varphi: T\to G_3(Q_5)=O_3(Q_5)\propto F_3(Q_5)$$

such that

$$\tilde\varphi|\mathrm{SL}_2(Q_5)=\varphi \quad\text{and}\quad \tilde\varphi|F_3(Q_5)=Id.$$

Applying lemma 1, we get the pair $(T, F_3(Q_5))$ and, consequently, the pair $(G_3(Q_5), F_3(Q_5))$, has property (T).

2. *Almost invariant nets of locally integrable functions and finitely-additive invariant measures*

Let β be the ring of bounded Lebesgue measurable sets in R^n, and let λ_n be the Lebesgue measure on R^n normalized by

$$\lambda_n(J^n) = 1, \text{ where } J = [0, 1).$$

Let L denote the space of locally integrable functions on R^n. If $X \in \beta$, $p \geq 1$ and $f^p \in L$, put

$$\|f\|_{p,X} = \left(\int_X |f|^p \, d\lambda_n\right)^{1/p};$$

for $X \in \beta$ and $f \in L$, put

$$X(f) = \int_X f \, d\lambda_n.$$

Put $P = \{f \in L | f \geq 0, J^n(f) = 1\}$. Let H_n and B_n be the same as in the formulation of theorem 2. For $h \in H_n$ and $f \in L$, define $hf \in L$ by

$$(hf)(x) = f(h^{-1}x), x \in R^n.$$

THEOREM 3. *Let $n \geq 3$ and $\{f_\alpha\}$ be a net of elements of P. Assume that*

(a) $\lim_\alpha |X(hf_\alpha) - X(f_\alpha)| = 0$ *for any $h \in H_n$ and $X \in \beta$;*

(b) *for any $X \in \beta$, there exists a limit*

$$\lim_\alpha X(f_\alpha) \overset{\text{def}}{=} \nu(X).$$

Then $\nu(X) = \lambda_n(X)$ for any $X \in \beta$.

Proof. Introduce on the linear space L the standard topology for which sets

$$\{f \in L | \|f\|_X < \varepsilon\}, x \in \beta, \varepsilon > 0,$$

form a basis of neighbourhoods of zero. Put $L_h = L$ and $R_X = R$ for $h \in H$ and $X \in \beta$ and consider the Tychonoff product

$$M = \prod_{h \in H} L_h \times \prod_{X \in \beta} R_X.$$

The space M can be considered as the set of pairs (a, b), where $a: H \to L$ and $b: \beta \to R$ are arbitrary mappings. Define a linear mapping $T: L \to M$ by

$$Tf = (T_1 f, T_2 f), \quad (T_1 f)(h) = hf - f, \quad (T_2 f)(X) = X(f),$$
$$f \in L, h \in H, X \in \beta.$$

As $L_\infty(Y, \mu)$ is the dual of $L_1(Y, \mu)$ for any space Y with a countably-additive finite measure μ, then the bounded measurable space L_∞^0 having compact support functions on R^n is the dual of L. Therefore, the condition (a) implies the net $\{hf_\alpha - f_\alpha\}$ weakly converges to zero for any $h \in H$. From this and condition (b) we conclude that the weak closure of $T(P)$ in M contains $(0, \nu)$. On the other hand as the space M is locally convex and $T(P)$ is a convex set, the weak and strict closures of $T(P)$ coincide. Therefore, there exists a net $\{g_\delta\}$ of elements of P such that, for any $h \in H$ and $X \in \beta$, we have

$$\lim_{\delta} \|hg_\delta - g_\delta\|_{1,X} = 0 \tag{1}$$

and

$$\lim_{\delta} X(g_\delta) = \nu(X). \tag{2}$$

Let $\Omega \subset B_n$ denote the subgroup of parallel translations by vectors with integer coefficients. Let $\Gamma \subset H_n$ be the countable subgroup defined in theorem 2 and let Γ' be the subgroup generated by Γ and Ω. Let us fix now $X_0 \in \beta$ and denote by N^+ the set of positive integer numbers. As Γ' is countable, the space R^n is σ-compact, and

$$\|f\|_{1,X} \leq \|f\|_{1,X'} \quad \text{if } X \subset X',$$

then, in view of (1) and (2), one can choose a countable subsequence $\{g_m\}$, $m \in N^+$, of the net $\{g_\delta\}$ such that

$$\lim_{m\to\infty} \|\gamma g_m - g_m\|_{1,X} = 0 \tag{3}$$

for any $\gamma \in \Gamma'$, $X \in \beta$, and

$$\lim_{m\to\infty} X_0(g_m) = \nu(X_0). \tag{4}$$

Any point $x \in R^n$ is uniquely represented in the form

$$x = \omega j,\ \omega \in \Omega,\ j \in J^n.$$

Define functions $\tilde{g}_m \in L$ by

$$\tilde{g}_m(\omega j) = g_m(j),\ \omega \in \Omega,\ j \in J^n.$$

Then

$$\|\tilde{g}_m - g_m\|_{1,\omega J^n} = \|\omega^{-1} g_m - g_m\|_{J^n}$$

for any $m \in N^+$ and $\omega \in \Omega$. On the other hand, any $X \in \beta$ can be covered by a finite number of sets of the type ωJ^n, $\omega \in \Omega$. Therefore, (3) implies

$$\lim_{m\to\infty} \|\tilde{g}_m - g_m\|_{1,X} = 0$$

for any $X \in \beta$. On the other hand,

$$\|\gamma f\|_{1,X} = \|f\|_{1,\gamma^{-1}X} \quad \text{for any } f \in L.$$

Therefore, we can replace g_m by $\tilde{g}_m$ in (3) and (4) and suppose

$$g_m(\omega x) = g_m(x) \tag{5}$$

for any $m \in N^+$, $\omega \in \Omega$, $x \in R^n$.

Let $U(r)$ denote the ball in R^n centered at 0 of radius r. As Γ' is countable, then, in view of (3), one can choose a sequence $\{r_m > 0\}$ such that

$$\lim_{m\to\infty} r_m = \infty \tag{6}$$

and

$$\lim_{m\to\infty} \|\gamma g_m - g_m\|_{1,U(r_m)} = 0 \tag{7}$$

for any $\gamma \in \Gamma'$.

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Let $\chi(V)$ denote the characteristic function of a set $V \subset R^n$. Put

$$\chi_m = \chi(U(r_m)) \text{ and } \varphi_m = (g_m\chi_m)^{\frac{1}{2}} = g_m^{\frac{1}{2}}\chi_m.$$

Then, for any $m \in N^+$ and $h \in H_n$,

$$\begin{aligned} h\varphi_m - \varphi_m &= (hg_m^{\frac{1}{2}})(h\chi_m) - (g_m^{\frac{1}{2}}\chi_m) \\ &= ((hg_m^{\frac{1}{2}}) - g_m^{\frac{1}{2}})\chi_m + (hg_m^{\frac{1}{2}})(h\chi_m - \chi_m) \\ &= ((hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}})\chi(U(r_m)) \\ &\quad + (hg_m^{\frac{1}{2}})[\chi(hU(r_m) - U(r_m)) - \chi(U(r_m) - hU(r_m))] \end{aligned}$$

and, consequently,

$$\begin{aligned} \big| \|h\varphi_m - \varphi_m\|_{2,R^n} - \|hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}}\|_{2,U(r_m)} \big| &\le \|hg_m^{\frac{1}{2}}\|_{2,hU(r_m)\Delta U(r_m)} \\ &= \|g_m^{\frac{1}{2}}\|_{2,U(r_m)\Delta h^{-1}U(r_m)}. \end{aligned} \tag{8}$$

For $V \subset R^n$, put

$$F(V) = \{\omega \in \Omega | \omega J^n \subset V\}$$

and

$$F'(V) = \{\omega \in \Omega | \omega J^n \cap V \ne \varnothing\}.$$

Classical theorems on the number of integer points imply

$$\lim_{r\to\infty} \frac{|F(U(r))|}{\lambda_n(U(r))} = \lim_{r\to\infty} \frac{|F'(U(r))|}{\lambda_n(U(r))} = 1. \tag{9}$$

It is clear that

$$F'(V_1 - V_2) \subset F'(V_1) - F(V_2)$$

for any V_1, $V_2 \subset R^n$. Therefore, (9) implies

$$\lim_{r\to\infty} \frac{|F'(U(r) - U(r-d))|}{\lambda_n(U(r))} = 0 \tag{10}$$

for any $d > 0$.

The equality (5) and the inclusion $g_m \in \rho$ imply that

$$\|g_m^{\frac{1}{2}}\|_{2,\omega J^n} = \|g_m^{\frac{1}{2}}\|_{2,J^n} = (\|g_m\|_{1,J^n})^{\frac{1}{2}} = 1$$

for any $m \in N^+$ and $\omega \in \Omega$. Therefore, for any $m \in N^+$ and $X \in \beta$,

$$|F(X)| \le \|g_m^{\frac{1}{2}}\|_{2,X} \le |F'(X)|. \tag{11}$$

For $h \in H_n$, denote by d_h the distance between 0 and $h(0)$. Then

$$U(r)\Delta h^{-1}U(r) \subset U(r + d_h) - U(r - d_h)$$

for any $r > 0$, $h \in H_n$. Therefore, (6), (10) and (11) imply, for any $h \in H_n$,

$$\lim_{m\to\infty} \frac{\|g_m^{\frac{1}{2}}\|_{2,U(r_m)\Delta h^{-1}U(r_m)}}{\lambda_n(U(r_m))} = 0. \tag{12}$$

As $|a-b|^2 \le a^2 - b^2$ for any $a, b > 0$, then

$$\|f_1^{\frac{1}{2}} - f_2^{\frac{1}{2}}\|_{2,X} \le \|f_1 - f_2\|_{1,X}^{\frac{1}{2}}$$

for any $f_1, f_2 \in L$ and $X \in \beta$. Therefore, (6), (7), (8) and (12) imply,

$$\lim_{m\to\infty} \frac{\|\gamma\varphi_m - \varphi_m\|_{2,R^n}}{\lambda_n(U(r_m))} = 0 \tag{13}$$

for any $\gamma \in \Gamma'$. Further, in view of (11),

$$\|\varphi_m\|_{2,R^n} = \|g_m^{\frac{1}{2}}\|_{2,U(r_m)} \geq |F(U(r_m))|.$$

From this and (9) we conclude that

$$\lim_{m\to\infty} \frac{\|\varphi_m\|_{2,R^n}}{\lambda_n(U(r_m))} = 1. \tag{14}$$

The equalities (13) and (14) imply $\{\varphi_m\}$ is a weakly almost Γ'-invariant sequence of elements of $L_2(R^n, \lambda_n)$. Therefore, in view of theorem 2, $\{\varphi_m\}$ is almost B_n-invariant. From this and (8), (12) and (14) we conclude that, for any compact set $K \subset B_n$,

$$\lim_{m\to\infty} \sup_{h\in K} \frac{\|hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}}\|_{2,U(r_m)}}{\lambda_n(U(r_m))} = 0. \tag{15}$$

It follows from (5) and the commutativity of B_n that

$$(hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}})(\omega x) = (hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}})(x)$$

for any $m \in N^+$, $h \in B_n$, $\omega \in \Omega$, $x \in R^n$. Therefore,

$$\|hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}}\|_{2,\omega J^n} = \|hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}}\|_{2,J^n}$$

for any $m \in N^+$, $h \in B_n$. Hence,

$$\|hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}}\|_{2,X} \geq |F(X)| \, \|hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}}\|_{2,J^n} \tag{16}$$

for any $m \in N^+$, $h \in B_n$ and $X \in \beta$. It follows from (9), (15) and (16) that

$$\lim_{m\to\infty} \sup_{h\in K} \|hg_m^{\frac{1}{2}} - g_m^{\frac{1}{2}}\|_{2,J^n} = 0 \tag{17}$$

for any compact set $K \subset B_n$.

The Schwartz inequality implies that

$$\|f_1 - f_2\|_{1,X} \leq \|f_1^{\frac{1}{2}} - f_2^{\frac{1}{2}}\|_{2,X} \cdot (\|f_1^{\frac{1}{2}}\|_{2,X} + \|f_2^{\frac{1}{2}}\|_{2,X})$$

for any $f_1, f_2 \in L$ and $X \in \beta$. On the other hand as any $x \in R^n$ is uniquely represented in the form $x = \omega j$, $\omega \in \Omega$, $j \in J^n$, then (5) and the inclusion $g_m \in P$ imply

$$\|hg_m^{\frac{1}{2}}\|_{2,J^n} = \|g_m^{\frac{1}{2}}\|_{2,h^{-1}J^n} = \|g_m^{\frac{1}{2}}\|_{2,J^n} = 1$$

for any $h \in H_n$ and $m \in N^+$. Therefore, (17) implies

$$\lim_{m\to\infty} \sup_{h\in K} \|hg_m - g_m\|_{1,J^n} = 0 \tag{18}$$

for any compact set $K \subset B_n$. Identifying R^n and B_n, we get using (5), (18) and the inclusion $g_m \in P$ that

$$\lim_{m\to\infty} \|g_m - 1\|_{1,J^n} = \lim_{m\to\infty} \|g_m - \int_{J^n} (hg_m)\, d\lambda_n(h)\|_{1,J^n}$$

$$\leq \lim_{m\to\infty} \int_{J^n} \|g_m - hg_m\|_{1,J^n}\, d\lambda_n(h) = 0. \tag{19}$$

It follows from (19) and (5) that

$$\lim_{m\to\infty} \|g_m - 1\|_{1,X_0} = 0$$

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and, consequently,

$$\lim_{m\to\infty} X_0(g_m) = \lambda_n(X_0).$$

From this and (4) we conclude that $\nu(X_0) = \lambda_n(X_0)$. The theorem has been proved. □

Now, we can prove the main result of the paper.

THEOREM 4. *Let $n \geq 3$ and let ν be a finitely-additive H_n-invariant measure on β such that $\nu(J^n) = 1$. Then $\nu = \lambda_n$.*

Proof. As $n \geq 3$ then (see the beginning of the paper) ν is absolutely continuous with respect to λ_n and, consequently, ν determines H_n-invariant linear positive functional l on L^0_∞ by

$$l(f) = \int f\,d\nu, \qquad f \in L^0_\infty.$$

As for any space Y with a countably-additive finite measure μ, the set

$$\{f \in L_1(Y, \mu) | f \geq 0\}$$

is dense in the weak* topology in the set of positive linear functionals on $L_\infty(Y, \mu)$, it follows that ρ is dense in weak* topology in the set of linear positive functionals b on L^0_∞ such that $b(\chi(J^n)) = 1$. Therefore, there exists a net $\{f_\alpha\}$ of elements of ρ such that

$$\lim_\alpha \int_{R^n} f_\alpha\,d\lambda_n = l(f) \tag{20}$$

for any $f \in L^0_\infty$. It follows from (20) and the H_n-invariance of l that

$$\lim_\alpha |X(hf_\alpha) - X(f_\alpha)| = 0$$

and

$$\lim_\alpha X(f_\alpha) = \nu(X)$$

for any $h \in H_n$ and $X \in \beta$. Therefore, in view of theorem 3, $\nu = \lambda_n$. □

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