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FINITELY GENERATED GROUPS, p-ADIC ANALYTIC GROUPS, AND POINCARÉ SERIES

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INTRODUCTION

Igusa [I 1, I 2] was the first to exploit p-adic integration with respect to the Haar measure on $\mathbf{Q}_p$ in the study of Poincaré series arising in number theory and developed a method using Hironaka's resolution of singularities to evaluate a limited class of such integrals. Denef [D 1, D 2] and, more recently, Denef and van den Dries [DvdD] have applied results from logic, profiting from the flexibility of the concept of definable, greatly to enlarge the class of integrals amenable to Igusa's method. In [DvdD] these results are employed to answer questions posed by Serre [S] and Oesterlé [O] concerning the rationality of various Poincaré series associated with the p-adic points of a closed analytic subset of $\mathbf{Z}_p^m$. In this note we apply these techniques to prove that various Poincaré series associated with finitely generated groups and p-adic analytic groups are rational in p^{-s}, extending results of [GSS].

RESULTS

Let G be a group and denote by $a_n(G)$ the number of subgroups of index n in G. We are interested in groups for which $a_n(G)$ is finite for every $n \in \mathbf{N}$. For each prime p, we can then associate the following Poincaré series with this arithmetical function:

$$(1) \qquad \zeta_{G,p}(s) = \sum_{n=0}^{\infty} a_{p^n}(G)p^{-ns} = \sum_{H \in \mathbf{X}_p} |G : H|^{-s}$$

where $\mathbf{X}_p = \{H \leq G : H \text{ has finite } p\text{-power index in } G\}$.

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§1. p-ADIC ANALYTIC GROUPS

We consider firstly the case where G is a compact p-adic analytic group—that is, a compact topological group with the underlying structure of a p-adic analytic manifold with respect to which the group operations are analytic (see [Lz] and [DduSMS]). For such groups, $a_n(G)$ is finite for every n. We wish to announce the following:

Theorem 1. *If G is a compact p-adic analytic group, then $\zeta_{G,p}(s)$ is rational in p^{-s}.*

The philosophy behind the proof is to express our Poincaré series as a p-adic integral

$$\int_M |f(\mathbf{x})|^s |g(\mathbf{x})|\,|d\mathbf{x}|, \tag{2}$$

where $|d\mathbf{x}|$ is the normalized Haar measure on $\mathbf{Z}_p^n$ and the functions f, $g: \mathbf{Z}_p^n \to \mathbf{Z}_p$ and the subset M are definable in the language describing the analytic theory of the p-adic numbers. We can then evaluate such definable integrals applying the techniques developed by Denef and van den Dries [DvdD] (which include quantifier elimination results for the analytic theory of $\mathbf{Z}_p$) to prove our theorem.

The translation from our Poincaré series to such a definable p-adic integral makes full use of Lazard's results on the close relationship between the structure of compact p-adic analytic groups and filtrations defined on such groups [Lz]. In answer to "Hilbert's 5th problem for p-adic analytic groups," Lazard has shown that a compact topological group has the structure of a p-adic analytic group if and only if there exists a normal subgroup G_1 of finite index in G which is *p-saturable*—that is, there exists a filtration on G_1

$$G_1 > G_2 > \cdots > G_i > \cdots$$

such that: (i) G_1 is a pro-p group with a fundamental system of neighborhoods of the identity given by $\{G_i : i \in \mathbf{N}\}$; (ii) for all $i \geq 1$, G_i/G_{i+1} is an elementary Abelian p-group of finite rank; and (iii) for all $i \geq 1$, the map $P_i: G_i/G_{i+1} \to G_{i+1}/G_{i+2}$ defined by $xG_{i+1} \to x^p G_{i+2}$ is an isomorphism of $\mathbf{F}_p$ vector spaces.

A p-saturable group has the underlying structure of a pro-p, p-adic analytic group with a global coordinate system $\mathbf{Z}_p^r$ given by p-adic powers of elements $x_1, \ldots, x_r$ where $x_1G_2, \ldots, x_rG_2$ is an $\mathbf{F}_p$ vector space basis for G_1/G_2.

We first prove Theorem 1 in the case where G is a p-saturable group. Recall that subgroups of finite index in a pro-p group have p-power index. The idea is to associate with every subgroup H of finite index in G, a subset $M(H)$ of $r \times r$ matrices over $\mathbf{Z}_p$ whose rows form coordinates for a *good basis* for H. Every subgroup of finite index in a compact p-adic analytic group is open. So, there exists m such that $H \geq G_m$. We define the concept of a *good basis* for H as a set of elements $h_1, \ldots, h_r$ such that, for each $i = 1, \ldots, m$, if we let $v_j = h_j^{p^{e(i,j)}} \in H \cap G_i$ for $\omega(h_j) \leq i$, where $e(i, j) = i - \omega(h_j)$ and $\omega(g) = n$ if $g \in G_n \backslash G_{n+1}$, then

$$\{v_j G_{i+1} : j \text{ such that } \omega(h_j) \leq i\}$$

is an $\mathbf{F}_p$ vector space basis for $(H \cap G_i)G_{i+1}/G_{i+1}$. The index of H in G is encoded in the measure of the subset $M(H)$ and we identify functions $f, g \colon \mathbf{Z}_p^n \to \mathbf{Z}_p$ such that, for all subgroups H of finite index in G

$$|G : H|^{-s} = \int_{M(H)} |f(\mathbf{x})|^s \, |g(\mathbf{x})| \, |d\mathbf{x}|.$$

Summing over all subgroups of finite (necessarily p-power) index in G, we can express our Poincaré series $\zeta_{G,p}(s)$ as a p-adic integral of the form (2) where $M \subseteq \mathbf{Z}_p^{r^2}$ is the (disjoint) union of subsets $M(H)$ for all $H \in \mathbf{X}_p$.

The problem now is to show that this integral is definable in the sense of [DvdD]. The set of r-tuples of elements of G which form a good basis for some subgroup of finite index in G is definable by a filtered group theoretic statement. We show how to translate such statements into statements about coordinates of elements of G definable now in the language describing the analytic theory of the p-adic numbers. Using this translation we can show that the subset M is definable. With regards to the functions f and g, we show that there exists a finite partition of $\mathbf{Z}_p^{r^2}$ into definable subsets such that, on each subset, f and g are defined by polynomial functions. Thus f and g are definable functions. We are then in a position to apply the techniques of [DvdD] to this definable integral and thus prove Theorem 1 in the case where G is a p-saturable group.

We extend this to a proof of Theorem 1 using the following ideas. Let G be a compact p-adic analytic group and G_1 a normal p-saturable subgroup of finite index in G. If H is subgroup of

p-power index in G, then it is determined by $H_1 = H \cap G_1$ and a transversal for H_1 in H. We associate with each H a subset $N(H)$ consisting of coordinates both for a good basis for H_1 and coordinates for a transversal for H_1 in H. Extending our integral associated with the p-saturable group G_1, we can express $\zeta_{G,p}(s)$ as a definable p-adic integral over the union of the subsets $N(H)$. We can therefore apply [DvdD] to prove that our Poincaré series associated with the compact p-adic analytic group G is rational in p^{-s}.

§2. Finitely generated groups

Theorem 1 has various corollaries for finitely generated groups. If Γ is a finitely generated group, then $a_n(\Gamma)$ is finite for all n. We can therefore consider the Poincaré series defined in (1). We consider first a variant of this Poincaré series. Define $a_n^{\triangleleft\triangleleft}(\Gamma)$ to be the number of subnormal subgroups of index n in Γ. For each prime p, we associate with the finitely generated group Γ, the following Poincaré series:

$$\zeta_{\Gamma,p}^{\triangleleft\triangleleft}(s) = \sum_{n=0}^{\infty} a_{p^n}^{\triangleleft\triangleleft}(\Gamma)p^{-ns} = \sum_{H \in \mathbf{Y}_p} |\Gamma : H|^{-s},$$

where $\mathbf{Y}_p = \{H \leq \Gamma : H \text{ is subnormal of } p\text{-power index in } \Gamma\}$.

We say that $a_{p^n}(\Gamma)$ grows polynomially if there exists $c \in \mathbf{N}$ such that $a_{p^n}(\Gamma) \leq p^{nc}$ for all n. Similarly for $a_{p^n}^{\triangleleft\triangleleft}(\Gamma)$. We then have the following:

Theorem 2. *Let* Γ *be a finitely generated group and* p *a prime. If* $a_{p^n}^{\triangleleft\triangleleft}(\Gamma)$ *grows polynomially, then* $\zeta_{\Gamma,p}^{\triangleleft\triangleleft}(s)$ *is rational in* p^{-s}.

There is a one-to-one correspondence between subnormal subgroups of finite p-power index in Γ and subgroups of finite index in the pro-p completion G of Γ. So $\zeta_{G,p}(s) = \zeta_{\Gamma,p}^{\triangleleft\triangleleft}(s)$. By Lubotzky and Mann's characterization of pro-p groups which have the underlying structure of a p-adic analytic group [LM], if $a_{p^n}(G) = a_{p^n}^{\triangleleft\triangleleft}(\Gamma)$ grows polynomially, then G is a p-adic analytic pro-p group. By Theorem 1, $\zeta_{\Gamma,p}^{\triangleleft\triangleleft}(s)$ is rational in p^{-s}.

We recall the definition of an *upper p-chief factor* of Γ—that is, a chief factor of some finite quotient of Γ whose order is divisible by p. We then have the following:

Theorem 3. *Let* Γ *be a finitely generated group and* p *a prime such that the order of all* p*-chief factors of* Γ *is bounded. If* $a_{p^n}(\Gamma)$ *grows polynomially, then* $\zeta_{\Gamma,p}(s)$ *is rational in* p^{-s}.

The bound on the order of p-chief factors in Γ implies that there exists a normal subgroup Γ_0 of finite index in Γ whose subgroups of p-power index are all subnormal. We construct a finite extension G of the pro-p completion of Γ_0 whose subgroups of finite index are in one-to-one correspondence with subgroups of p-power index in Γ. If $a_{p^n}(\Gamma)$ grows polynomially, then, by [LM], G is a finite extension of a p-adic analytic pro-p group. So G is a compact p-adic analytic group and by Theorem 1, $\zeta_{G,p}(s) = \zeta_{\Gamma,p}(s)$ is rational in p^{-s}.

Theorem 3 includes a large class of examples, some of which we collect together in the following corollary. We recall that the *upper p-rank* of Γ is the supremum of $r(P)$ as P ranges over all p-subgroups of finite quotients of Γ, where $r(P)$ is the rank of P.

Corollary 4. *If Γ is a finitely generated group of finite upper p-rank, then $\zeta_{\Gamma,p}(s)$ is rational in p^{-s}.*

This follows from a remark in [MS].

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