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(10) Pradeep Dubey

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FINITENESS AND INEFFICIENCY OF NASH EQUILIBRIA*

by

Pradeep Dubey

1. Introduction

Our aim is to explore efficiency, as well as strongness, of the Nash Equilibria (N.E.) of finite-person noncooperative games in strategic form. We show that--with "smooth" strategy sets and payoff functions--it is "almost always" the case that (a) the N.E. are finite in number; (b) efficient N.E. are "extremal" points; and (c) strong N.E. are "inactive" points. Extremal (inactive) points are points in the Cartesian product of the players' strategy sets at which at least one (at most one) of the players is (is not) at a "vertex" of her/his strategy set. Both sets of points are therefore "thin" (they are nonexistent if the strategy-sets are vertex-free). This result is not very surprising because efficiency or strongness is generally an outcome of cooperation. And, indeed, it has been part of the "folklore" of Game Theory (witness: the Prisoners' Dilemma).

Then we turn to an analysis of multi-matrix games. The same generic result holds, but "inactive points" in this instance correspond to pure-strategy Nash Equilibria.

Our results also raise the question of how this inefficiency depends

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upon the number of players. We will discuss this elsewhere [4].

Methods from Differential Topology--in particular the use of Thom's Transversal Density Theorem--that were introduced by Debreu [2] and Smale [7] in examining finiteness of the competitive equilibria of economies, are also the key tools used here. Needless to say, the debt to their work is enormous.

2. The Manifold of Noncooperative Games

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the set of players, and $S^i \subset R^{k(i)}$ $(k(i) \geq 1)$ the strategy set of player i. We will assume that $S^i = \{x \in R_+^{k(i)} : \sum_{j=1}^{k(i)} x_j \leq 1\}$. (This can be relaxed. In fact our theorem remains true if the S^i are assumed to be stratified sets. We spell this out in Remark 1. But for the moment we will make the simplifying assumption in order to keep the notation down to a minimum.) Put $S = S^1 \times \ldots \times S^n$. Let U be the linear space of all C^2 functions* from S to the reals endowed with the C^2-norm, i.e., for all u in U, $\|u\| = \sup\{\|u(s)\|, \|Du(s)\|, \|D^2u(s)\| : s \in S\}$. Our space of noncooperative games will be $(U)^n$; for any $u = (u^1, \ldots, u^n) \in (U)^n$, u^i is the payoff function of player i.

For any $s = \{s^i : i \in N\} \in S$, $J \subset N$, and $e = \{e^i : i \in J\} \in \times_{i \in J} S^i$, let $(s|e)$ denote the element of S obtained from s by replacing s^i by e^i for each $i \in J$.

Consider any $u \in (U)^n$ and $s \in S$. s is called a <u>Nash Equilibrium</u> (N.E.) of u if, for each $i \in N$,

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*See Appendix, Part 1, for definition of these.

$$u^i(s) \geq u^i(s|e) \ , \quad e \in S^i \ .$$

Next define s to be a J-efficient point of u if there does not exist any $e \in \bigtimes_{i \in J} S^i$ such that

$$u^i(s|e) \geq u^i(s) \ , \quad \text{all} \quad i \in J \ ;$$

$$u^j(s|e) > u^j(s) \ , \quad \text{some} \quad j \in J \ .$$

If s is N-efficient, we call it simply efficient. If s is J-efficient for all $J \subset N$, we call it a strong Nash Equilibrium.

Let us denote the set of all N.E., strong N.E., and efficient points of the game u by $\eta(u)$, $\gamma(u)$ and $\varepsilon(u)$ respectively. Note that $\gamma(u) \subset \eta(u) \cap \varepsilon(u)$.

It will be useful for us to distinguish the sets S' and S'' of extremal points and inactive points in S. Put

$$S' = \{(s^1, \ldots, s^n) \in S : s^i \text{ is a vertex of } S^i \text{ for at least one } i \in N\}$$

$$S'' = \{(s^1, \ldots, s^n) \in S : s^i \text{ is not a vertex of } S^i \text{ for at most one } i \in N\} \ .$$

We are prepared to state our result.

[General] Theorem. There is an open dense subset 0 of $(U)^n$ such that, for any $u \in 0$,

(a) $\eta(u)$ is a finite set

(b) $\eta(u) \cap \varepsilon(u) \subset S'$

(c) $\gamma(u) \subset S''$.

If the S^i did not have vertices--for instance if they were taken to be unit spheres--then we could replace (b) and (c) by: " $\eta(u) \cap \varepsilon(u) = \emptyset$." But we chose the S^i to be simplices for two reasons. First they occur in various classes of games that have traditionally been examined, e.g. [6], [9], [10]. Second--and, in our opinion, no less importantly--they enable us to give a generic description of efficient (strong) N.E. as extremal (inactive) points in S .

3. Preparation for the Proof

3.1. Notation

V^i = the set of all the $k(i)+1$ vertices of S^i ,

$\tilde{V}^i$ = the set of all nonempty subsets of V^i ,

$\tilde{V} = \tilde{V}^1 \times \dots \times \tilde{V}^n$.

For any $T^i \in \tilde{V}^i$, $T_0^i = T^i \setminus \{0^i\}$, where 0^i is the zero vertex of S^i .

For any $T = \{T^1, \dots, T^n\} \in \tilde{V}$ and $i \in N$:

$S^i(T)$ = convex hull of T^i , $\dot{S}^i(T)$ = relative interior of $S^i(T)$,

$S(T) = S^1(T) \times \dots \times S^n(T)$, $\dot{S}(T) = \dot{S}^1(T) \times \dots \times \dot{S}^n(T)$,

$N(T) = \{i \in N : |T^i| > 1\}$,

$\hat{T} = \cup\{T_0^i : i \in N(T)\}$,

$t^i = |T_0^i|$,

$\hat{t} = \sum_{i \in N(T)} t^i$,

$t = |N(T)|$,

$R^{N\hat{T}}$ = Euclidean space of dimension $t\hat{t}$ whose axes are indexed by pairs $(i,j) \in N(T) \times \hat{T}$.

For any $v \in R^{N\hat{T}}$, v_j^i will be its $(i,j)^{th}$ component. Also for any $L \subset \hat{T}$, v_L^i will be the vector in R^L (whose axes are indexed by elements of L) with components $\{v_j^i : j \in L\}$.

Note that there is a natural correspondence between elements of V_0^i and the variables $\{s_j^i : 1 \leq j \leq k(i)\}$. Thus, without confusion, we will speak sometimes of the variable x_ℓ , $\ell \in \hat{T}$.

We construct a mapping:

$$_T\psi : (U)^n \times S(T) \to R^{N\hat{T}}$$

which will enable us to study $\eta(u)$ and $\varepsilon(u)$. Letting $u = (u^1, \ldots, u^n) \in (U)^n$, $s \in S(T)$:

$$_T\psi_\ell^i(u,s) = \left(\frac{\partial u^i}{\partial x_\ell}\right)(s) \ , \quad i \in N(T) \ , \quad \ell \in \hat{T} \ .$$

For a fixed $u \in (U)^n$,

$$^u_T\psi : S(T) \to R^{N\hat{T}}$$

is given by $^u_T\psi(s) = {}_T\psi(u,s)$.

Finally we need to define two subsets of $R^{N\hat{T}}$. To this end, first let

$$T_a = \{i \in N(T) : 0^i \in T^i\}$$

$$T_b = \{i \in N(T) : 0^i \notin T^i\} \ .$$

Then let

$$\Delta^1(T) = \{v \in R^{N\hat{T}} : \text{ the projections of the } v^i,\ i \in N(T), \text{ on } S(T) \text{ are linearly dependent}\},$$

$$\Delta^2(T) = \{v \in R^{N\hat{T}} : v^i_{T^i_0} = 0 \text{ for } i \in T_a;\quad v^j_j = v^i_\ell \text{ for } i \in T_b,\ j \in T^i_0,\ \ell \in T^i_0\}.$$

3.2. The General Lemma

The following lemma is critical to our proof.

[General] Lemma. Let M be any submanifold of $R^{N\hat{T}}$. Then ${}_T\psi$ is transversal to M (${}_T\psi \pitchfork M$). (See Appendix, Part 1, for definition of "transversal".)

Proof. Consider any $(u,z) \in (U)^n \times S(T)$ such that ${}_T\psi(u,z) = y \in M$. Take any $v \in R^{N\hat{T}}$. We will show that there is a smooth path $\{(u_t, z_t)\}_{t=0}^1$ in $(U^n) \times S(T)$ such that:

$$(u_0, z_0) = (u,z)$$

$$\frac{d}{dt}[{}_T\psi(u_t, z_t)]\bigg|_{t=0} = v .$$

To do this, define $u_t = (u^1_t, \ldots, u^n_t)$ and z_t for $0 \le t \le 1$, and $s \in S(T)$, by:

$$z_t = z_0$$

$$u^i_t(s) = u^i(s) + \sum_{\ell \in \hat{T}} v^i_\ell t s^p_k(\ell), \quad \text{where } s^p_k(\ell) \text{ corresponds to } \ell .$$

Then

$$_T\psi^i_j(u_t, z_t) = \left(\frac{\partial u^i}{\partial x_j}\right)(z_0) + tv^i_j .$$

And thus the path constructed has the requisite properties.

To complete the proof, we need to check that $(T_{(u,z)}\ T^{\psi})^{-1}(T_y M)$ splits. For convenience denote the tangent space of (u,z) by W, the tangent space of y by V, and the derivative map by f. We must show that there is a closed subspace W' of W such that $W' \oplus f^{-1}(V) = W$, where $\oplus$ denotes direct sum. Let $\alpha_1, \ldots, \alpha_p$ be a basis for V and augment it by the set $\beta_1, \ldots, \beta_q$ to get a basis for $R^{N\hat{T}}$ ($p+q = t\acute{t}$ here). As we saw earlier, f is surjective, hence each of $f^{-1}(\beta_1), \ldots, f^{-1}(\beta_q)$ is nonempty. Pick $w_1 \in f^{-1}(\beta_1), \ldots, w_q \in f^{-1}(\beta_q)$; and set $W' = \text{Span}\{w_1, \ldots, w_q\}$.

Now consider any $w \in W$. Let $f(w) = \sum_{i=1}^{p} a_i\alpha_i + \sum_{i=1}^{q} b_i\beta_i$. Put $w' = \sum_{i=1}^{q} b_i w_i$, $w'' = w - w'$. Since f is linear, $f(w'') \in V$, i.e., $w'' \in f^{-1}(V)$. By construction, $w' \in W'$ and $w = w' + w''$. It is also clear that $W' \cap f^{-1}(V) = 0$.

Q.E.D.

4. Proof of the General Theorem

Note that $\{\dot{S}(T) : T \in \tilde{V}\}$ is a partition of S . It will be convenient for us to partition $\eta(u)$ into $\{\eta(u,T) : T \in \tilde{V}\}$, where $\eta(u,T) = \eta(u) \cap \dot{S}(T)$. Clearly $\eta'(u) \subset \cup\{\eta(u,T) : T \in \tilde{V}, t > 1\}$.

<u>The case</u> $t = n$

(1) $\Delta^1(T)$ is a finite union of submanifolds in $R^{N\hat{T}}$ each of which has codimension ≥ 1 ; say these are $\hat{\Delta}^3(T), \ldots, \hat{\Delta}^{r(T)}(T)$.

(2) $\Delta^2(T)$ is a submanifold in $R^{N\hat{T}}$ of codimension $t\hat{t} - t_b$ (where $t_b = |T_b|$).

(3) $\Delta^i(T) = \hat{\Delta}^i(T) \cap \Delta^2(T)$ is a submanifold in $R^{N\hat{T}}$ of codimension $> \hat{t}t - t_b$ $[i = 3, \ldots, r(T)]$.

(4) By the transversal density and openness theorems (see [1] and Appendix, Part 1)--which applies in the light of the Lemma and the above facts--there is a dense open set $O^i(T)$ in $(U)^n$ such that for any $u \in O^i(T)$, ${}^u_T\psi \pitchfork \Delta^i(T)$ $[i = 2, \ldots, r(T)]$.

Set $O(T) = \cap\{O^i(T) : i = 2, \ldots, r(T)\}$. Then $O(T)$ is dense open, and for any $u \in O(T)$ we have ${}^u_T\psi \pitchfork \Delta^i(T)$ $[i = 2, \ldots, r(T)]$.

Consider any $u \in O(T)$. Then $\eta(u,T) \subset {}^u_T\psi^{-1}(\Delta^2(T))$. Since ${}^u_T\psi \pitchfork \Delta^2(T)$, $\operatorname{codim} {}^u_T\psi^{-1}(\Delta^2(T)) = \operatorname{codim} \Delta^2(T) = t\hat{t} - t_b$. But $\dim S(T) = t\hat{t} - t_b$. Hence ${}^u_T\psi^{-1}(\Delta^2(T))$ has dimension zero, which means it is a discrete set. Being bounded, it must be a finite set. We have shown: if $u \in O(T)$, then $\eta(u,T)$ is finite.

Next let us consider $\varepsilon(u) \cap \dot{S}(T)$. Points in $\varepsilon(u)$ are efficient in S ; <u>a fortiori</u> they must be efficient in $S(T) \subset S$. But by Proposition A in [8] we get (as explained in Appendix, Part 2)

$\varepsilon(u) \cap \dot{S}(T) \subset \bigcup_{i=3}^{r(T)} {}^u_T\psi^{-1}(\hat{\Delta}^i(T))$. This implies that

$\eta(u,T) \cap \varepsilon(u) \subset \bigcup_{i=3}^{r(T)} {}^u_T\psi^{-1}(\Delta^i(T))$. Take $u \in O(T)$. Then ${}^u_T\psi \pitchfork \Delta^i(T)$, $i = 3, \ldots, r(T)$. Also $\operatorname{codim} \Delta^i(T) > t\hat{t} - t_b = \dim S(T)$. This shows that ${}^u_T\psi^{-1}(\Delta^i(T)) = \phi$ $[i = 3, \ldots, r(T)]$. Hence $\eta(u,T) \cap \varepsilon(u) = \phi$ for $u \in O(T)$.

The case $2 \leq t \leq n-1$

The argument used in the case $t = n$ carries over to show that $\eta(u,T)$ is finite and $\gamma(u) = \emptyset$ for any u in an open dense set $O(T)$. (Replace "efficient" by "N(T)-efficient" and $\varepsilon(u)$ by $\gamma(u)$ throughout.)

The case $t = 1$

Again the argument used in the case $t > 1$ carries over to show that $\eta(u,T)$ is finite for any u in an open dense set $O(T)$.

The case $t = 0$

Obviously $|\eta(u,T)| = 1$, for any u, in this case.

Completion of the Proof

Put $O = \cap\{O(T) : T \in \tilde{V},\ t \geq 1\}$. Then O is open dense in $(U)^n$, and for any $u \in O$, (a) $\eta(u)$ is finite and (b) $\eta(u) \cap \varepsilon(u) \subset S'$; (c) $\gamma(u) \subset S''$. Q.E.D.

5. Interlude

Our space of games $(U)^n$ is in some sense too big. Several interesting classes of games may form submanifolds M of $(U)^n$ without being open sets of $(U)^n$. But then the question of the generic behavior of N.E. must be raised within the context of M alone. This question is clearly independent of the question for $(U)^n$, i.e., the results of either case yield no information about the other. However our basic approach will often help analyze the situation for various kinds of M. To illustrate this we consider an example: mixed extensions of games with finite pure strategy sets (henceforth "Multi-Matrix Games"). A similar theorem holds in this context. But we are pleasantly surprised to find that inactive points in this case correspond to pure-strategy Nash Equilibria. In other words, (generically) only pure strategy Nash Equilibria can be strong.

Another interesting submanifold of games are those that arise from markets (see [9] for a description of the model). This submanifold is analyzed elsewhere [3]. Here both finiteness and inefficiency of the Nash Equilibria are generic.

6. Multi-Matrix Games

First let us quickly recall their definition (see [6]). Each player $i \in N = \{1, \ldots, n\}$ has a finite number $k(i)$ of pure strategies, $k(i) \geq 2$. Let

$$\tilde{K}^i = \{1, \ldots, k(i)\} ,$$

$$\tilde{K} = \tilde{K}^1 \times \ldots \times \tilde{K}^n ,$$

$$\tilde{K}^{-i} = \tilde{K}^1 \times \ldots \times \tilde{K}^{i-1} \times \tilde{K}^{i+1} \times \ldots \times \tilde{K}^n .$$

For any $K^{-i} \in \tilde{K}^{-i}$, $j \in \tilde{K}^i$, jK^{-i} = sequence of n integers in $\tilde{K}$ whose i^{th} element is j and whose other elements are according to K^{-i}.

We can specify a multi-matrix game by assigning payoffs to each $i \in N$ on the domain $\tilde{K}$. Thus we can think of it as a vector in $R^N \times R^{\tilde{K}} = R^{N\tilde{K}}$, say. For any $a \in R^{N\tilde{K}}$, $i \in N$, $K \in \tilde{K}$, the component a^i_K of a is the payoff to i when all players (including himself) use pure strategies given by K. The set of mixed strategies of player i is $S^i = \{x^i \in R^{k(i)}_+ : \sum_{j=1}^{k(i)} x^i_j = 1\}$. (Thus $\tilde{K}^i$ can be thought of as the vertices of S^i.) For any $x = (x^1, \ldots, x^n) \in S = S^1 \times \ldots \times S^n$, and

$K \in \tilde{K}$, we will denote by x_K the sequence $x^1_{j(1)}, \ldots, x^n_{j(n)}$ where $\{j(1), \ldots, j(n)\} = K$.

Now fix $a \in R^{N\tilde{K}}$. We define the mixed extension $\tilde{a}$ of the game a by specifying payoff functions ${}^a\Pi^i : S \to R$, $i \in N$, as follows:

$$ {}^a\Pi^i(x) = \sum_{K \in \tilde{K}} x_K a^i_K \ . $$

Define $\eta(\tilde{a})$, $\gamma(\tilde{a})$, $\varepsilon(\tilde{a})$, S' and S'' as before. Also define $S^* = \{(s^1, \ldots, s^n) \in S : s^i$ is a vertex of S^i for all $i \in N\}$.

[Multi-Matrix] Theorem. There is a dense and open set 0^* of $R^{N\tilde{K}}$ such that, for all $a \in 0^*$,

(a) $\eta(\tilde{a})$ is a finite set;

(b) $\eta(\tilde{a}) \cap \varepsilon(\tilde{a}) \subset S'$;

(c) $\gamma(\tilde{a}) \subset S^*$.

Here (c) has a nice interpretation. It says that (generically) only pure-strategy N.E. can be strong.

In order to prove this we first establish an analogue of the lemma of the general case.

First let us carry over the notation of Section 4.1, with the obvious modifications: $V^i = \tilde{K}^i$, $T^i_0 = T^i$, etc. Now for any $T = \{K^1, \ldots, K^n\}$, where $K^i \subset \tilde{K}^i$, $K^i \neq \phi$, define

$$ {}_T\hat{\Pi} : R^{N\tilde{K}} \times S(T) \to R^{N\hat{T}} $$

by

$$_T\hat{\Pi}^i_\ell(a,x) = \left(\frac{\partial\, {}^a\Pi^i}{\partial x_\ell}\right)(x)$$

for $i \in N$, $\ell \in \hat{T}$.

[Multi-Matrix] Lemma. Let M be any submanifold of $R^{N\hat{T}}$. Then $_T\hat{\Pi} \pitchfork M$.

Proof. W.l.o.g. take $T = \{\tilde{K}^1, \ldots, \tilde{K}^n\}$, in other words, $S(T) = S^1 x..x S^n$.(The proof for other T is exactly the same.)

Consider any $(a,x) \in R^{N\tilde{K}} \times S(T)$ such that $_T\hat{\Pi}(a,x) = y \in M$. It will suffice to evince a smooth path $\{({}^ta, {}^tx)\}_{t=0}^1$ in $R^{N\tilde{K}} \times S(T)$ such that

$$({}^0a, {}^0x) = (a,x)$$

$$\frac{d}{dt}[_T\hat{\Pi}({}^ta, {}^tx)]\bigg|_{t=0} = v$$

for any $v \in R^{N\hat{T}}$.

Let $\ell \in \hat{T}$ correspond to the j^{th} pure strategy of player i [in notation: $\ell \equiv \binom{i}{j}$]. Observe* that, for any $k \in N$,

$$_T\hat{\Pi}^k_\ell(a,x) = \sum_{K^{-i} \in \tilde{K}^i} x_{K^{-i}}\, a^k_{jK^{-i}} \, .$$

To construct the path, fix some ${}^*K^{-i} \in \tilde{K}^i$ for each $i \in N$ such that $x_{{}^*K^{-i}} \neq 0$. Then put

$${}^tx = x$$

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*$x_{K^{-i}}$ denotes $x^1_{j(1)} \cdots x^{i-1}_{j(i-1)} \cdot x^{i+1}_{j(i+1)} \cdots x^n_{j(n)}$ where $\{j(1), \ldots, j(i-1), j(i+1), \ldots, j(n)\} = K^{-i}$.

$$ {}^{t}a_K^i = \begin{cases} [a_K^i + (tv_\ell^i)]/x_{*K^{-k}} & , \text{ if } K = j*K^{-k} \text{ and } \ell = \binom{k}{j} ; \\ a_K^i & , \text{ otherwise.} \end{cases} $$

It is easily checked that

$$ {}_T\hat{\Pi}_\ell^k({}^t a,\ {}^t x) = {}_T\hat{\Pi}_\ell^k(a,x) + tv_\ell^k . $$

Thus the path has the desired property. The rest of the proof is finished as in the general case.

Q.E.D.

Proof of the Multi-Matrix Theorem

In the light of the above lemma, we can go through all the steps of the proof in the general case (replacing $(U)^n$ by $R^{N\tilde{K}}$, ψ by $\hat{\Pi}$, u by a) to arrive at the result: there is an open dense set 0 of $R^{N\tilde{K}}$ such that, for any $a \in 0$,

(a) $\eta(\tilde{a})$ is a finite set;

(b) $\eta(\tilde{a}) \cap \varepsilon(\tilde{a}) \subset S'$;

(c) $\gamma(\tilde{a}) \subset S''$.

Now set $\hat{0} = \{a \in R^{N\tilde{K}}$: no two numbers in $(a_K^i : K \in \tilde{K})$ are equal, for each $i \in N\}$. It is well known that 0 is also open dense in $R^{N\tilde{K}}$. Let $0^* = 0 \cap \hat{0}$. Clearly if $s \in \eta(\tilde{a})$ then $s \in \eta(\tilde{a}) \cap S''$. If, moreover $a \in 0^* \subset \hat{0}$, we immediately see that $s \in S^*$.

Q.E.D.

8. Remarks

1) It is clear that our general theorem can be proved if the S^i are assumed to be stratified sets. T would now vary to pick out various simplices in the triangulations of $S^1, \ldots, S^n$. An inactive (extremal) point would be one which places at least n-1 (one) players' strategies on vertices of their (triangulated- upto-diffeomorphism) strategy sets.

2) Instead of keeping the strategy sets fixed we could vary them in a "nice and differentiable" manner with parameters lying in a C'-manifold. Then the space of games would be the product of this manifold with $(U)^n$. The same theorem can be shown to hold here (with the same proof). But this is an uninteresting and arcane generalization.

3) Existence of N.E. is assured for multi-matrix games by [6]. For the general case also we can display an open set Q in $(U)^n$ for which N.E. exist. Let Q consist of all those $(u^1, \ldots, u^n)$ for which:

$$u^i(s^1, \ldots, s^{i-1}, t, s^{i+1}, \ldots, s^n), \quad t \in S^i,$$

is strictly concave in t for each i, for every choice of $s^j \in S^j$, $j \neq i$.

4) Consider multimatrix games. Let W be a bounded cube in $R^{N\tilde{K}}$. Define $W'' \subset W' \subset W$ by:

$$W' = \{a \in W : \tilde{a} \text{ has at least one pure strategy N.E.}\}$$

$$W'' = \{a \in W' : \gamma(\tilde{a}) \neq \phi\}.$$

The volume of W' has been computed in [11]. It overestimates the "probability" that a game in W has strong N.E. We feel that it vastly overestimates, i.e., that the volume of W'' is much smaller than that of W'. But we have not done any computation in this direction.

5) The generic finiteness of N.E. of multi-matrix games (indeed oddness) was established (by different methods) in [5] and [10].

6) It is worth pointing out that--as in [7]--the analysis here rests ultimately upon the indifference surfaces $\{(u^i)^{-1}(c) : c \in R\}$ of the players' payoff functions. The u^i serve merely as a convenient description of these.

APPENDIX

We recall the results used in this paper, and explain their use in our proofs.

Part 1 (the quotation is from [1])

"Let X and Y be C^1 manifolds, $f : X \to Y$ a C^1 map, and $W \subset Y$ a submanifold. We say that f is transversal to W at a point $x \in X$, in symbols: $f \pitchfork_x W$, iff, where $y = f(x)$, either $y \notin W$ or $y \in W$ and

(1) the inverse image $(T_x f)^{-1}(T_y W)$ splits, and

(2) the image $(T_x f)(T_x X)$ contains a closed complement to $T_y W$ in $T_y Y$.

We say f is transversal to W, in symbols: $f \pitchfork W$, iff $f \pitchfork_x W$ for every $x \in X$.

Let A, X, and Y be C^r manifolds, $C^r(X,Y)$ the set of C^r maps from X to Y, and $\rho : A \to C^r(X,Y)$ a map. For $a \in A$ we write ρ_a instead of $\rho(a)$; i.e., $\rho_a : X \to Y$ is a C^r map. We say ρ is a C^r representation iff the evaluation map

$$ev_\rho : A \times X \to Y$$

given by

$$ev_\rho(a,x) = \rho_a(x)$$

for $a \in A$ and $x \in X$ is a C^r map from $A \times X$ to Y.

Transversal Density Theorem. Let A, X, Y be C^r manifolds, $\rho : A \to C^r(X,Y)$ a C^r representation, $W \subset Y$ a submanifold (not necessarily closed), and $ev_\rho : A \times X \to Y$ the evaluation map. Define $A_W \subset A$ by

$$A_W = \{a \in A | \rho_a \pitchfork W\} .$$

Assume that:

(1) X has finite dimension n and W has finite codimension q in Y;

(2) A and X are second countable;

(3) $r > \max(0, n-q)$;

(4) $ev_\rho \pitchfork W$.

Then A_W is residual (and hence dense) in A.

Openness of Transversal Intersection. Let A, X, and Y be C^1 manifolds with X finite dimensional, $W \subset Y$ a closed C^1 submanifold, $K \subset X$ a compact subset of X, and $\rho : A \to C^1(X,Y)$ a C^1 pseudorepresentation. Then the subset $A_{KW} \subset A$ defined by

$$A_{KW} = \{a \in A | \rho_a \pitchfork_x W \text{ for } x \in K\}$$

is open. This holds even if X is not finite dimensional, provided that ρ is a C^1 representation."

For our purposes, it is enough to note that every C^1 representation is a C^1 pseudorepresentation. Also T_yW is the tangent space to W at y; $T_xf : T_xX \to T_yY$ is the derivative map of f at x. See [1] for detailed definitions.

Since our $S(T)$ are not exactly manifolds, some thought must be bestowed on how we use these theorems.

Let P be an open set in $R^{n[k(1)+\ldots+k(n)]}$ which contains S. For any $T = (T^1, \ldots, T^n) \in \tilde{V}$, define $P^i(T) = P^i \cap \{\text{Affine span of } T^i\}$, $P(T) = \bigtimes_{i \in N} P^i(T)$. When we say that a function

$$f : S(T) \to M$$

or

$$f : A \times S(T) \to M$$

is C^r, where A and M are manifolds, we mean that there is a function $\hat{f}$

$$\hat{f} : P(T) \to M$$

or

$$\hat{f} : A \times P(T) \to M$$

such that $\hat{f}$ is C^r, and the restriction of $\hat{f}$ equals* f. Similarly, for any submanifold Δ of M, when we say

$$f : A \times S(T) \to M$$

is transversal to Δ, we mean that there is an $\hat{f}$ as above such that $\hat{f}$ is transversal to Δ, etc.

Consider the manifold $\tilde{U}$ of all C^2 functions $\tilde{u}$ from P to R such that $\|\tilde{u}\| = \sup_{x \in P}\{\|\tilde{u}(x)\|, \|D\tilde{u}(x)\|, \|D^2\tilde{u}(x)\|\}$ is finite. For any $\tilde{u} \in \tilde{U}$, we denote by ${}_r\tilde{u}$ the member of U obtained by restricting $\tilde{u}$ to S. Also for any $\tilde{Q} \subset \tilde{U}$ denote by ${}_r\tilde{Q}$ the subset of U given

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*Any two such extensions will yield the same derivatives on the (relative) boundary of $S(T)$.

by ${}_r\tilde{Q} = \{{}_r\tilde{u} : \tilde{u} \in \tilde{Q}\}$. Similarly, for $\tilde{Q} \subset (\tilde{U})^n$, we can define ${}_r\tilde{Q} \subset (\tilde{U})^n$. Note

(vi) $\|\tilde{u}\| < \varepsilon \Longrightarrow \|{}_r\tilde{u}\| < \varepsilon$, for any $\tilde{u} \in (\tilde{U})^n$.

(vii) As is well known, there exists a $K > 0$ such that: given $u \in (U)^n$ with $\|u\| < \varepsilon$, we can find $\tilde{u} \in (\tilde{U})^n$ with $\|\tilde{u}\| < K\varepsilon$ and ${}_r\tilde{u} = u$.

We now prove a fact used repeatedly by us. Recall the mapping ${}_T\psi : (U)^n \times S(T) \to R^{N\hat{T}}$. The (general) lemma stated ${}_T\psi \pitchfork M$ (where M is any manifold in $R^{N\hat{T}}$). This may be rephrased more accurately. First define ${}_T\tilde{\psi} : (\tilde{U})^n \times P(T) \to R^{N\hat{T}}$ exactly as on page 6.* Then ${}_T\tilde{\psi} \pitchfork M$ (the proof of the general lemma really shows this--substitute $P(T)$ for $S(T)$, and $s \in S(T)$ by $s \in P(T)$). Now define $\tilde{Q}^1 \subset (\tilde{U})^n$, $\tilde{Q}^2 \subset (\tilde{U})^n$ as follows:

$$\tilde{Q}^1 = \{\tilde{u} \in (\tilde{U})^n : {}^{\tilde{u}}_T\psi \pitchfork M \text{ , i.e. } {}^{\tilde{u}}_T\psi \pitchfork_x M \text{ , } x \in P(T)\}$$

$$\tilde{Q}^2 = \{\tilde{u} \in (\tilde{U})^n : {}^{\tilde{u}}_T\psi \pitchfork_x M \text{ , } x \in S(T)\} .$$

By the density theorem, $\tilde{Q}^1$ is dense in $(\tilde{U})^n$; by the openness theorem, $\tilde{Q}^2$ is open in $(\tilde{U})^n$. Clearly $\tilde{Q}^1 \subset \tilde{Q}^2$, so $\tilde{Q}^2$ is open dense in $(\tilde{U})^n$. Consider $Q \subset (U)^n$ obtained by setting $Q = {}_r\tilde{Q}^2$. Then, by (vi) and (vii), Q is open dense in $(U)^n$, and--by construction--if $u \in Q$, ${}^u_T\psi \pitchfork M$.

Let us also note that our argument for the finiteness of $\eta(u,T)$ was a little slipshod. A rigorous version would go like this. Take $\tilde{Q}^2$ and Q as above with $M = \Delta^2(T)$. Then for any $\tilde{u} \in \tilde{Q}^2$, we have

$$\eta({}_r\tilde{u}, T) \subset [{}^{\tilde{u}}_T\tilde{\psi}^{-1}(\Delta^2(T))] \cap S(T) .$$

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*Note that for any $\tilde{u} \in (\tilde{U})^n$, and $x \in S(T)$, ${}_T\tilde{\psi}(\tilde{u},x) = {}_T\psi({}_r\tilde{u}, x)$.

As in the proof, ${}^{\tilde{u}}_{T}\tilde{\psi}^{-1}(\Delta^2(T))$ is a submanifold of dimension 0 . Since $S(T)$ is compact, the above intersection must be finite. Thus $\eta(u,T)$ is finite for any u in the open dense set $Q \subset (U)^n$.

In general, for any argument involving $S(T)$ which requires it to be a manifold, we embed it in the manifold $P(T)$ and use the above technique.

We wind up this part of the Appendix with a comment for the Multi-Matrix Case. Take P^i to be the affine hull of $\tilde{K}^i$, and $P = P^1 \times \ldots \times P^n$. For any $a \in R^{N\tilde{K}}$ define the extension ${}^{a}\tilde{\Pi}^i : P \to R$ in the "natural" manner, i.e. by the formula on page 12. Note however that here the manifold of games stays the same (i.e $R^{N\tilde{K}}$) even when we extend ${}^{a}_{T}\Pi^i$ or ${}^{a}_{T}\hat{\Pi}^i$ onto $P(T)$.

Part 2 (the quotation, with minor modifications, is from [8])

"Let $u^i : W \to R$ be smooth (i.e., C^r, $r \geq 1$) functions ($i = 1, \ldots, n$) where W is a manifold in some finite dimensional Euclidean space. Assume that $\dim W \geq n$ throughout. Consider $u = (u^1, \ldots, u^n)$, $u : W \to R^n$. At any $x \in W$, the derivative of u at x, $Du(x)$, is a linear map from T_xW to R^n (made up of $Du^i(x) : T_xW \to R$, $i = 1, \ldots, n$).

Proposition. Given W and u as above, $x \in W$ is an efficient point of u iff $\exists\, \lambda_i \geq 0$, $i = 1, \ldots, n$, not all zero with

$$\sum_{i=1}^{n} \lambda_i Du_i(x) = 0 \quad \text{for all } x \in T_xW ."$$

Using the proposition it follows easily that

$\varepsilon(u) \cap S(T) \subset \bigcup_{i=3}^{r(T)} {}_{T}^{u}\psi^{-1}(\hat{\Delta}^i(T))$ (see Proof of General Theorem, the case $t = n$).

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