FINITENESS OF INFINITESIMAL DEFORMATIONS OF CR MAPPINGS OF CR MANIFOLDS OF NONDEGENERATE LEVI FORM

Chung-Ki Cho and Chong-Kyu Han

Abstract. Let M and N be CR manifolds with nondegenerate Levi forms of hypersurface type of dimension $2m+1$ and $2n+1$, respectively, where $1 \leq m \leq n$. Let $f : M \to N$ be a CR mapping. Under a generic assumption we construct a complete system of finite order for the infinitesimal deformations of f. In particular, we prove the space of infinitesimal deformations of f forms a finite dimensional Lie algebra.

0. Introduction. We are concerned in this paper with the finite-dimensionality of the space of infinitesimal deformations of embeddings of a CR manifold M of nondegenerate Levi form into a CR manifold N, $\dim M \leq \dim N$, of nondegenerate Levi form (see Section 2 for definitions). Let $f : M \to N$ be a CR embedding. The finite dimensionality of infinitesimal deformations of f may be regarded as rigidity of embedding. Let d be the dimension of the linear space of the infinitesimal deformations of f and d' the dimension of CR automorphism group of N. Every 1-parameter group of local CR automorphisms of N gives rise to an infinitesimal deformation of f, thus $d \geq d'$. If $d = d'$ the embedding f is infinitesimally rigid. The following example shows that the infinitesimal deformations of embeddings of a Levi flat CR manifold can be infinite dimensional.

Example 0.1. Let $M := \mathbb{C} \times \mathbb{R}$ with the usual complex structure on $\mathbb{C}$ and $N = \mathbb{C}^2 \times \mathbb{R}$. Let $f(z,t) = (f_1(z), f_2(z), t)$, where f_j, $j = 1, 2$, are holomorphic and the complex derivative of at least one of f_j is nonvanishing. Then for any smooth real valued function $\phi(t)$ the real vector field

2000 Mathematics Subject Classification: 32V05, 32V40.

Key words and phrases: CR manifold, CR mapping, tangential Cauchy-Riemann equations, infinitesimal deformation, complete system.

Second author was partially supported by Korea Research Foundation of yr 2000

$V := \phi(t)\frac{\partial}{\partial t}$ is an infinitesimal deformation of CR embedding f. The set of such vector fields is infinite dimensional.

Generically, an overdetermined system admits prolongation to a complete system of finite order, that is, we can solve for all the partial derivatives of the unknown functions of certain order, say k, as functions of derivatives of the unknown functions of order less than k after differentiating the original equations sufficiently many times. This occurs when the coefficients to the k-th order partial derivatives satisfy the nondegeneracy condition of the implicit function theorem.

In this paper we shall show that the linearized system (2.5)-(2.6) at a CR embedding f admits prolongation to a complete system of finite order under certain generic conditions on the embedding, which implies the finite-dimensionality of the space of infinitesimal deformations.

1. Prolongation and complete system. Let $m, n \in \mathbb{N}$. Let X be an open subset of $\mathbb{R}^n$ and let $\mathbb{R}^{(q)}$ be a euclidean space whose coordinates represent all the partial derivatives of $\mathbb{R}^m$-valued smooth maps defined on X of all orders from 0 to q. A multi-index of order r is an unordered r-tuple of integers $J = (j_1, \cdots, j_r)$, with $1 \leq j_s \leq n$. The order of a multi-index J is denoted by $|J|$. By u^α_J we denote the $|J|$-th order partial derivative of u^α with respect to $x^{j_1}, \cdots, x^{j_{|J|}}$, and we often drop the parentheses and commas in writing multi-indices, thus $u^\alpha_j = u^\alpha_{(j)} = \partial u^\alpha/\partial x^j$, $u^\alpha_{jk} = u^\alpha_{(j,k)} = (\partial^2 u^\alpha)/(\partial x^j \partial x^k)$, and so forth. A point in $\mathbb{R}^{(q)}$ will be denoted by $u^{(q)}$, so that $u^{(q)} = (u^\alpha_J)_{1\leq\alpha\leq m,\, 0\leq|J|\leq q}$.

The product space $\mathrm{J}^q(X, \mathbb{R}^m) = X \times \mathbb{R}^{(q)}$ is called the q-th order jet space of the space $X \times \mathbb{R}^m$. If $f = (f^1, \cdots, f^m) : X \to \mathbb{R}^m$ is smooth, let $(j^q f)(x) = (x, \partial_J f^\alpha(x) : 1 \leq \alpha \leq m, |J| \leq q)$, then $j^q f$, called the q-graph of f, is a smooth section of $\mathrm{J}^q(X, \mathbb{R}^m)$.

Consider a system of partial differential equations of order q, $q \geq 1$, for unknown functions $u = (u^1, \cdots, u^m)$ of independent variables $x = (x^1, \cdots, x^n)$:

$$\Delta_\nu(x, u^{(q)}) = 0, \quad \nu = 1, \cdots, \ell, \tag{1.1}$$

where each $\Delta_\nu(x, u^{(q)})$ is a smooth function in its arguments. Then $\Delta = (\Delta_1, \cdots, \Delta_\ell)$ is a smooth map from $\mathrm{J}^q(X, \mathbb{R}^m)$ into $\mathbb{R}^\ell$.

Then the subset $\mathcal{S}_\Delta$ of $\mathrm{J}^q(X, \mathbb{R}^m)$ defined by $\Delta = 0$ is called the solution subvariety of (1.1). Thus, a smooth solution of (1.1) is a smooth map $f : X \to \mathbb{R}^m$ whose q-graph is contained in $\mathcal{S}_\Delta$.

A differential function $P(x, u^{(q)})$ of order q is a smooth function defined on an open subset of $\mathrm{J}^q(X, \mathbb{R}^m)$. The total derivatives of $P(x, u^{(q)})$ with respect to x^i is the differential function of order $q+1$ defined by

$$D_i P(x, u^{(q+1)}) := \frac{\partial P}{\partial x^i} + \sum_{a=1}^{m} \sum_{|J| \leq q} u^a_{J,i} \frac{\partial P}{\partial u^a_J},$$

where J, i denotes the multi-index $(j_1, \cdots, j_{|J|}, i)$. For each nonnegative integer r, the rth-prolongation $\Delta^{(r)}$ of the system (1.1) is the system consisting of all the total derivatives of (1.1) of order up to r. Let $(\Delta^{(r)})$ be the ideal generated by $\Delta^{(r)}$ of the ring of differential functions on $\mathrm{J}^{q+r}(X, \mathbb{R}^m)$. If $\widetilde{\Delta} \in (\Delta^{(r)})$ for some r, the equation

$$\widetilde{\Delta}(x, u^{(q+r)}) = 0 \tag{1.2}$$

is called a prolongation of (1.1). Note that any smooth solution of (1.1) must satisfy (1.2). If k is the order of the highest derivative involved in $\widetilde{\Delta}$, we call (1.2) a prolongation of order k. We now define the complete system.

Definition 1.1. We say that (1.1) admits prolongation to a complete system of order k if there exist prolongations of (1.1) of order k

$$\widetilde{\Delta}_\nu(x, u^{(k)}) = 0, \quad \nu = 1, \cdots, N \tag{1.3}$$

which can be solved for all the k-th order partial derivatives as smooth functions of lower order derivatives of u, namely, for each $a = 1, \cdots, m$ and for each multi-index J with $|J| = k$,

$$u^a_J = H^a_J(x, u^{(k-1)}) \tag{1.4}$$

for some function H^a_J which is smooth in its arguments. (1.4) is called a complete system of order k.

The complete system (1.4) is obtained from (1.3) when the coefficients to u^a_J, $|J| = k$, in (1.3) satisfies the nondegeneracy condition of the implicit function theorem, therefore, generically an overdetermined system admits prolongation to a complete system of order k for some sufficiently large k. Prolongation to a complete system for overdetermined systems seems to have wide application. Very recently A. Hayashimoto, D. Zaitsev and S.Y. Kim use the method of prolongation or consequences of the complete system, see [Hay1], [Hay2], [Zait] and [Kim].

We find the idea of the complete system in the equivalence problem of E. Cartan(see [BS], [HY]): Let G be a Lie-subgroup of $\mathbf{GL}(n;\mathbb{R})$. Suppose that a manifold E of dimension n has a G-structure and $\pi : Y \to E$ is the associated principal bundle of G-frames. The equivalence problem is finding canonically a system of differential 1-forms

$$\omega^1, \cdots, \omega^N, \quad \text{where} \quad N = n + \dim G, \tag{1.5}$$

on Y so that a mapping $f : E \to \tilde{E}$ preserves the G-structure if and only if there exists a mapping $F : Y \to \tilde{Y}$, which satisfies that $\tilde{\pi} \circ F = f \circ \pi$, and that

$$F^* \tilde{\omega}^i = \omega^i, \quad i = 1, \cdots, N, \tag{1.6}$$

where $\tilde{E}$ is a manifold of dimension n with a G-structure and $\tilde{\pi} : \tilde{Y} \to \tilde{E}$ is the associated principal bundle and $\tilde{\omega}^i$ are the corresponding 1-forms on $\tilde{Y}$. (1.5) is called a complete system of invariants of the G-structure and (1.6) is a complete system of order 1 for F in the sense of Definition 1.1. It turns out that (1.6) is equivalent to a complete system of order 2 for f .

Now we recall that solving the given system of partial differential equations (1.1) is equivalent to finding an integral manifold of the corresponding exterior differential system

$$du^a_I - \sum_{i=1}^n u^a_{I,i} dx^i = 0$$

for all multi-index I with $|I| < q$ and $a = 1, \cdots, m$, with an independence condition $dx_1 \wedge \cdots \wedge dx_n \neq 0$ on $\mathcal{S}_\Delta$(see [BCGGG]). If (1.1) admits prolongation to a complete system of order k then we have the following Pfaffian system on $\mathrm{J}^{k-1}(X, \mathbb{R}^m)$:

$$\begin{cases} du^a - \sum_{j=1}^n u^a_j dx^j = 0, & \\ \quad \vdots & \\ du^a_I - \sum_{j=1}^n u^a_{I,j} dx^j = 0, & |I| = k-2, \\ du^a_I - \sum_{i=1}^n H^a_{I,i} dx^i = 0, & |I| = k-1 \end{cases} \tag{1.7}$$

with an independence condition $dx^1 \wedge \cdots \wedge dx^n \neq 0$, where $H^a_{I,i}$ are as in (1.4). Therefore, if (1.1) admits a prolongation to a complete system (1.4) a C^k-function $u = f(x)$ is a solution of (1.1) if and only if

$$(x) \mapsto (x, \partial_J f(x) : |J| \leq k-1)$$

is an integral manifold of the Pfaffian system (1.7). In particular, we have

PROPOSITION 1.2. *Suppose that (1.1) admits a complete system (1.4), then a solution is uniquely determined by its* $(k-1)$ *jet at a point and is* C^∞ *provided that it is* C^k*. Furthermore, if (1.1) is real analytic in its arguments then each* H^a_J *is real analytic and every* C^k *solution of (1.1) is real analytic.*

Now let $\omega^1, \ldots, \omega^N$ be the differential 1-form on $\mathrm{J}^{k-1}(X, \mathbb{R}^m)$ given in the left hand side of (1.7) and let $\omega = (\omega^1, \ldots, \omega^N)^T$. Then

$$d\omega = \Theta \wedge \omega + \Omega \tag{1.8}$$

for some $N \times N$ matrix-valued 1-form Θ and a 2-form Ω which is determined uniquely modulo the ideal generated by $\omega^1, \ldots, \omega^N$. Ω is the obstruction to the existence of solutions of (1.7). If Ω is identically zero then (1.7) is involutive and by the Frobenius theorem there exists a unique solution for any initial condition. Little further is known about the existence of solutions.

2. Embedding of Cauchy-Riemann (CR) manifolds. Let M be a differentiable manifold of dimension $2m+1$. A CR structure on M is a subbundle $\mathcal{V}$ of the complexified tangent bundle $T_{\mathbb{C}}M$ having the following properties :

(a) each fiber is of complex dimension m,
(b) $\mathcal{V} \cap \bar{\mathcal{V}} = \{ 0 \}$,
(c) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (integrability).

Given a CR structure $\mathcal{V}$ we define the Levi form $\mathcal{L}$ by

$$\mathcal{L}(L_1, L_2) := \sqrt{-1}[L_1, \bar{L_2}], \quad \mathrm{mod}\ (\mathcal{V} + \bar{\mathcal{V}}).$$

$\mathcal{L}$ is a hermitian form on $\mathcal{V}$ with values in $T_{\mathbb{C}}M/(\mathcal{V} + \bar{\mathcal{V}})$. M is said to be strictly pseudoconvex if $\mathcal{L}$ is positive or negative definite. A real hypersurface in a complex manifold has natural CR structure induced from the complex structure of the ambient space.

A complex valued function f is called a CR function if f is annihilated by $\mathcal{V}$. Let $\{L_1, \cdots, L_m\}$ be a set of complex vector fields that generates $\mathcal{V}$. Then f is a CR function if and only if

$$\bar{L}_i f = 0, \quad i = 1, \cdots, m \quad \text{(tangential Cauchy-Riemann equations)}.$$

Let $(N, \mathcal{V}')$ be a CR manifold of dimension $2n+1$, $n \geq m$, with the CR structure bundle $\mathcal{V}'$. A mapping $f : M \to N$ is called a CR mapping if f preserves the CR structure, that is,

$$f_* \mathcal{V} \subset \mathcal{V}'.$$

A real analytic (C^ω) CR manifold of dimension $2n+1$ is realizable as a (C^ω) real hypersurface in $\mathbb{C}^{n+1}$, see [Ja2]. Hence, let N be a real hypersurface in $\mathbb{C}^{n+1}$ defined by $r(z, \bar{z}) = 0$, where r is a real valued function with $dr \neq 0$ and $z = (z^1, \cdots, z^{n+1})$. Let $z^A = (z^1)^{a_1} \cdots (z^{n+1})^{a_{n+1}}$ if $A = (a_1, \cdots, a_{n+1})$ is an $(n+1)$-tuple of nonnegative integers. After a holomorphic change of coordinates $r(z, \bar{z})$ takes the normal form as in [ChM].

$$r(z, \bar{z}) = z^{n+1} + \bar{z}^{n+1} + \sum_{j=1}^{n} \lambda_j z^j \bar{z}^j + \sum_{A,B} c_{AB} z^A \bar{z}^B, \tag{2.1}$$

where each λ_j is either 1 or -1 and each term in the last summand is of weight ≥ 4. Weight of a term $c_{AB} z^A \bar{z}^B$ is $\sum_{j=1}^{n} (a_j + b_j) + 2(a_{n+1} + b_{n+1})$.

We denote by $\mathcal{C}$ the last summand of (2.1) and write $\mathcal{C}$ in Moser's normal form (see [ChM]) as

$$\begin{aligned} \mathcal{C} &:= \sum_{A,B} c_{AB} z^A \bar{z}^B \\ &= F_{11}(z', \bar{z'}, z^{n+1}, \bar{z}^{n+1}) + \sum_{min(k,\ell) \geq 2} F_{k\ell}(z', \bar{z'}, z^{n+1}, \bar{z}^{n+1}), \end{aligned} \tag{2.2}$$

where $z' = (z^1, \cdots, z^n)$ and $F_{k\ell}$ satisfies

$$F_{k\ell}(tz', s\bar{z'}, z^{n+1}, \bar{z}^{n+1}) = t^k s^\ell F_{k\ell}(z', \bar{z'}, z^{n+1}, \bar{z}^{n+1})$$

and

$$F_{11}(z', \bar{z}', 0, 0) = 0.$$

A system of complex valued functions $f = (f^1, \cdots, f^{n+1})$ on M is a CR mapping of M into N if and only if

$$\bar{L}_i f^j = 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n+1 \tag{2.3}$$

(tangential Cauchy-Riemann equations), and

$$r \circ f = 0. \tag{2.4}$$

A mapping $g = (g^1, \cdots, g^{n+1}) : M \to \mathbb{C}^{n+1}$ is an infinitesimal deformation of f if and only if g satisfies the linearization of (2.3)-(2.4), namely,

$$\bar{L}_i g^j = 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n+1 \tag{2.5}$$

(tangential Cauchy-Riemann equations), and

$$\sum_{j=1}^{n+1} \{ (\frac{\partial r}{\partial z^j} \circ f) g^j + (\frac{\partial r}{\partial \bar{z}^j} \circ f) \bar{g}^j \} = 0. \tag{2.6}$$

For an m-tuple of non-negative integers $\alpha = (\alpha_1, \cdots, \alpha_m)$ let $L^\alpha = L_1^{\alpha_1} \cdots L_m^{\alpha_m}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. We have

Theorem 2.1. *Let M^{2m+1}, $m \geq 1$, be a germ of C^ω CR manifold of nondegenerate Levi form. Let $\{L_1, \cdots, L_m\}$ be C^ω independent sections of the CR structure bundle $\mathcal{V}$. Let N be a germ of C^ω real hypersurface at the origin of $\mathbb{C}^{n+1}$, $n \geq m$, defined by $r(z, \bar{z}) = 0$, where $r(z, \bar{z})$ is normalized as in (2.1) and (2.2). Let $f : M \to N$ be an analytic (C^ω) CR mapping. Suppose that for some positive integer k the vectors $\{L^\alpha f : |\alpha| \leq k\}$ evaluated at the reference point together with $(0, \cdots, 0, 1)$ span $\mathbb{C}^{n+1}$ over $\mathbb{C}$. Then the system (2.5)-(2.6) admits prolongation to a complete system of order $2k+1$. Thus, the space of infinitesimal deformations of f is finite dimensional and an infinitesimal deformation is uniquely determined by its $2k$-jet at a point.*

In [Han3] it is shown that the CR embeddings of M into N satisfies a complete system of finite order and that if a CR embedding is sufficiently smooth then it is real analytic. The integer k in the hypotheses of Theorem 2.1 is independent of the choice of basis $\{L_1, \cdots, L_m\}$.

Now we consider the special case of $m = n$, and f is the identity map. In this case an infinitesimal deformation is an infinitesimal CR automorphism of M. Since $\{L_1 f, \cdots, L_m f\}$ together with $(0, \cdots, 0, 1)$ span $\mathbb{C}^{m+1}$, $k = 1$, and thus we have

Corollary 2.2. *Let M^{2m+1}, $m \geq 1$, be a C^ω CR manifold of nondegenerate Levi form. Then the infinitesimal CR automorphisms satisfy a complete system of order 3. Thus, the space of infinitesimal CR automorphisms forms a finite dimensional Lie algebra and an infinitesimal CR automorphism is uniquely determined by its 2-jet at a point.*

Corollary 2.2 is a well known basic fact of CR geometry. Recently N. Stanton studied holomorphic vector fields tangent to M: let $hol(M)$ be the set of holomorphic vector fields on an open subset of $\mathbb{C}^{n+1}$ that are tangent to an analytic real hypersurface M. M is said to be holomorphically nondegenerate at O if there are no such vector fields. In [Sta2] it is shown that $hol(M)$ is finite dimensional if and only if M is holomorphically nondegenerate. Relevant results are found in [BER1] and [BER2].

Proof of Theorem 2.1. We shall construct a complete system by differentiating (2.6) repeatedly and by reducing the order of derivatives using (2.5). First we obtain $\frac{\partial r}{\partial z^j}$ and $\frac{\partial r}{\partial \bar{z}^j}$ from (2.1) and substitute in (2.6), to get

$$g^{n+1}+\bar{g}^{n+1}+\sum_{j=1}^{n}\lambda_j(\bar{f}^j g^j+f^j\bar{g}^j)+\sum_{j=1}^{n+1}(\frac{\partial \mathcal{C}}{\partial z^j}\circ f)g^j+(\frac{\partial \mathcal{C}}{\partial \bar{z}^j}\circ f)\bar{g}^j=0. \tag{2.7}$$

Apply $\bar{L}^\alpha$ to (2.7) for each α with $|\alpha| \leq k$. Then by (2.5) we have

$$\bar{L}^\alpha\bar{g}^{n+1}+\sum_{j=1}^{n}\lambda_j(\bar{L}^\alpha\bar{f}^j g^j+f^j\bar{L}^\alpha\bar{g}^j)+\sum_{j=1}^{n+1}a_j(x,\bar{L}^\beta\bar{g}:|\beta|\leq k)g^j=0, \tag{2.8}$$

where a_j is analytic in its arguments, x is the local coordinates of M. We observe that (2.2) implies that each a_j vanishes at the reference point $0 \in M$. Then by the hypothesis of the theorem we can solve (2.7)-(2.8) for g^j, $j = 1, \cdots, n+1$, to get

$$g^j = H^j(x, \bar{L}^\alpha\bar{g} : |\alpha| \leq k), \quad j = 1, \cdots, n+1, \tag{2.9}$$

where each H^j is an analytic function of the arguments in the parenthesis.

Let $\beta = (\beta_1, \cdots, \beta_m)$ be any multi-index. Apply L^β to (2.9). Then we have

$$L^\beta g^j = L^\beta\left(H^j(x, \bar{L}^\alpha\bar{g} : |\alpha| \leq k)\right). \tag{2.10}$$

Now let T be a C^ω real vector field on M which is transversal to the $\mathcal{V} \oplus \bar{\mathcal{V}}$, so that the set $\{T, L_j, \bar{L}_j, j = 1, \cdots, m\}$ forms a basis of the complexified tangent space of M. Let $[L_j, \bar{L}_k] = \sqrt{-1}\rho_{j\bar{k}} T \mod (\mathcal{V}, \bar{\mathcal{V}})$. Then $(\rho_{j\bar{k}})$, $j, k = 1, \cdots, m$, is a non-degenerate hermitian matrix. We may assume that $[\rho_{j\bar{k}}(0)]$ is diagonal at the refence point $0 \in M$. In the right hand side of (2.10), each time we apply L_i to $H^j(x, \bar{L}^\alpha \bar{g} : |\alpha| \leq k)$, computations by chain rule show that T-directional derivatives occurs when commuting L and $\bar{L}$, and by (2.5) the total order of the derivatives remains $\leq k$, for example,

$$\begin{aligned} \bar{L}_1 L_1 g^j &= (L_1 \bar{L}_1 - [L_1, \bar{L}_1]) g^j \\ &= \{L_1 \bar{L}_1 - (\sqrt{-1}\rho_{1\bar{1}} T + \sum_{i=1}^m (a_i L_i + b_i \bar{L}_i))\} g^j \\ &\qquad \text{for some functions } a_i \text{ and } b_i \\ &= -\sqrt{-1}\rho_{1\bar{1}} T g^j + \sum_{i=1}^m a_i L_i g^j \quad \text{by (2.5).} \end{aligned} \tag{2.11}$$

Now we introduce notations : for each pair of non-negative integers (p, q) with $p \geq q$, let C_p be the set of C^ω functions in the arguments

$$T^t L^\alpha g^j : t + |\alpha| \leq p, \ j = 1, \cdots, n+1$$

and $C_{p,q}$ be the subset of C_p of all the C^ω functions in the arguments

$$T^t L^\alpha g^j : t + |\alpha| \leq p, \ t \leq q, \ j = 1, \cdots, n+1,$$

and let $\bar{C}_p, \bar{C}_{p,q}$ be the complex conjugate of C_p and $C_{p,q}$, respectively. Then (2.10) implies that $L^\beta g^j \in \bar{C}_k$, for any multi-index $\beta = (\beta_1, \cdots, \beta_m)$.

In particular, for each $i = 1, \cdots, m$

$$L_i g^j \in \bar{C}_k. \tag{2.12}$$

Apply $\bar{L}_i$ to (2.12), then by the same calculation as in (2.11) we have

$$T g^j \in \bar{C}_{k+1,k}. \tag{2.13}$$

Similarly, for each $i, k = 1, \cdots, m$, we have

$$L_k L_i g^j \in \bar{C}_k. \tag{2.14}$$

Apply $\bar{L}_k$ to (2.14), then by (2.12), (2.13) and (2.14) we have

$$T L_i g^j \in \bar{C}_{k+1,k}. \tag{2.15}$$

Then by induction on $|\alpha|$, we have

$$TL^{\alpha}g^{j} \in \bar{C}_{k+1,k}. \tag{2.16}$$

Now apply $\bar{L}_i\bar{L}_k$ to (2.14), then by (2.12) – (2.16) we have

$$T^2g^j \in \bar{C}_{k+2,k}, \tag{2.17}$$

and by induction on $|\alpha|$, we have

$$T^2L^{\alpha}g^j \in \bar{C}_{k+2,k}. \tag{2.18}$$

Then by induction on t, we have

$$T^tL^{\alpha}g^j \in \bar{C}_{k+t,k}, \quad \text{for each} \quad j = 1, \cdots, n+1, \tag{2.19}$$

which shows that

$$C_{p,q} \subset \bar{C}_{k+q,k}, \quad \text{for any pair } (p,q) \text{ with } p \geq q. \tag{2.20}$$

Taking the complex conjugate of (2.20), we have

$$\bar{C}_{p,q} \subset C_{k+q,k}, \quad \text{for any pair } (p,q) \text{ with } p \geq q. \tag{2.21}$$

In paticular, if $q = k$

$$\bar{C}_{p,k} \subset C_{2k,k} \quad \text{for all} \quad p \geq k. \tag{2.22}$$

Substitute (2.22) in (2.20), to get

$$C_{p,q} \subset C_{2k,k}, \quad \text{for any pair } (p,q) \text{ with } p \geq q. \tag{2.23}$$

In particular, we have

$$C_{2k+1} \subset C_{2k}. \tag{2.24}$$

Now consider the derivatives $T^tL^{\alpha}\bar{L}^{\beta}g^j$, where $t+|\alpha|+|\beta| = 2k+1$. If $|\beta| \neq 0$, this is zero by (2.3). If $|\beta| = 0$, then (2.24) shows that $T^tL^{\alpha}g^j, t+|\alpha| = 2k+1$, can be expressed as a C^{ω} function in the arguments $T^tL^{\beta}g^j$: $t+|\beta| \leq 2k$, thus, g satisfies a complete system of order $2k+1$, which completes the proof. □

References

[BER1] M.S. Baouendi, P. Ebenfelt & L. P. Rothschild, *Infinitesimal CR automorphisms of real analytic manifolds in complex space*, Comm. Anal. Geom. **6** (1998), 291-315.

[BER2] ———, *Real submanifolds in complex space and their mappings*, Princeton University Press, Priceton N.J., 1999.

[Bog] A. Bogges, *CR manifolds and the tangential Cauchy-Riemann complex*, CRC Press, Boca Raton, 1991.

[BCGGG] R. Bryant, S.-S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior differential systems*, Springer-Verlag, Berlin, 1991.

[BS] D. Burns & S. Shnider, *Real hypersurfaces in complex manifolds*, Proc. Symp. Pure Math. **30** (1976), 141–167.

[ChM] S.S. Chern & J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219-271.

[Han1] C.-K. Han, *Analyticity of CR equivalence between real hypersurfaces in $\mathbb{C}^n$ with degenerate Levi form*, Invent. Math. **73** (1983), 51-69.

[Han2] ———, *Regularity of isometric immersions of positively curved Riemannian manifolds and its analogy with CR geometry*, J. Diff. Geom. **28** (1988), 477-484.

[Han3] ———, *Complete differential system for the mappings of CR manifolds of nondegenerate Levi forms*, Math. Ann **309** (1997), 401-409.

[Hay1] A. Hayashimoto, *On the complete system of finite order for CR mappings and its application*, Osaka J. Math. **35** (1998), 617-628.

[Hay2] ———, *On the relation between the holomorphic extendability theorems and the finiteness properties*, Contemp. Math. **222** (1999), 219-226.

[HY] C.-K. Han & J.-N. Yoo, *A method of prolongation of tangential Cauchy-Riemann equations*, Advanced Studies in Pure Math. **25** (1997), 158-166.

[Ja1] H. Jacobowitz, *Deformation leaving a hypersurface fixed*, Proc. Symp. Pure Math. **23** (1971), 343–351.

[Ja2] ———, *Introduction to CR structures*, Amer. math. Soc., Providence, 1990.

[Kim] S.Y. Kim, *Finiteness and analyticity of pseudo-conformal embeddings*, Nagoya Math. J., to appear.

[Olv] P. J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York, 1993.

[Sta1] N. Stanton, *Infinitesimal CR automorphisms of rigid hypersurfaces*, Amer. J. Math. **117** (1995), 141–167.

[Sta2] ———, *Infinitesimal CR automorphisms of real hypersurfaces*, Amer. J. Math. **118** (1996), 209–233.

[Web] S. M. Webster, *The rigidity of CR hypersurfaces in a sphere*, Indiana Univ. Math. J. **28** (1979), 405-416.

[Zait] D. Zaitsev, *Germs of local automorphism of real-analytic CR structures and analytic dependence on k-jets*, Math. Research Letters **4** (1997), 823-842.

C.K. Cho : Department of Mathematics, Soonchunhyang University, Asan, Choongnam 336-745, Korea. E-mail address : ckcho@sch.ac.kr

C.K. Han : Department of Mathematics, Seoul National University, Seoul 151-742, Korea. E-mail address : ckhan@math.snu.ac.kr