

## FINITENESS OF $\pi_1$ AND GEOMETRIC INEQUALITIES IN ALMOST POSITIVE RICCI CURVATURE

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ABSTRACT. – We show boundedness of the diameter and finiteness of the fundamental group under a global  $L^p$  control (for  $p > n/2$ ) of the Ricci curvature. Conversely, metrics with similar  $L^{\frac{n}{2}}$ -control of their Ricci curvature are dense in the set of complete metrics of any compact differentiable manifold.

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RÉSUMÉ. – Les théorèmes de Myers sur le diamètre et le groupe fondamental s'étendent aux variétés dont la courbure de Ricci est presque supérieure à  $n - 1$  en norme  $L^p$  pour  $p > n/2$ . Inversement, les métriques à courbure de Ricci presque supérieure à  $n - 1$  au sens  $L^{\frac{n}{2}}$  sont denses dans l'ensemble des métriques riemanniennes de toute variété compacte.

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### 1. Introduction

Finding the topological, geometrical or analytical properties induced by curvature bounds is a classical problem in Riemannian geometry. For instance, S. Myers showed that any complete  $n$ -manifold with  $\text{Ric} \geq k(n-1)$ ,  $k > 0$ , is compact with diameter bounded by  $\frac{\pi}{\sqrt{k}}$  and has a finite  $\pi_1$ . Conversely, J. Lohkamp showed in [11] that on every  $n$ -manifold with  $n \geq 3$  there exists a metric with negative Ricci curvature. This paper is devoted to the study of Riemannian manifolds satisfying only an  $L^p$  bound on the negative lower part of their Ricci curvature tensors.

NOTATIONS. – For any positive real  $p$ , we set  $\rho_p = \int_M (\underline{\text{Ric}} - (n-1))_-^p$ , where  $\underline{\text{Ric}}(x) = \inf_{X \in T_x M \setminus \{0\}} \text{Ric}_x(X, X)/g(X, X)$  is the lowest eigenvalue of the Ricci tensor at  $x \in M$  and for any function  $f$ , we set  $f_-(x) = \max(-f(x), 0)$ .

Our first result is the following Bishop type theorem.

THEOREM 1.1. – *Let  $(M^n, g)$  be a complete manifold and  $p > \frac{n}{2}$ ; then we have*

$$\text{Vol}(M, g) \leq \text{Vol} \mathbb{S}^n (1 + \rho_p^{\frac{9}{10}}) (1 + C(p, n) \rho_p^{\frac{1}{10}}),$$

where  $C(p, n)$  denotes a constant that depends only on  $p$  and  $n$ . In particular, if  $\rho_p < \infty$ , then  $(M^n, g)$  has finite volume.

In the case  $\text{Ric} \geq n - 1$ , our estimate implies the classical Bishop theorem. However note that the manifolds with  $\text{Ric} \geq n - 1$  are automatically compact, with finite  $\pi_1$ , and form a precompact

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family for the Gromov–Hausdorff distance on the length spaces, whereas the manifolds with finite  $\rho_p$  form a family that contains every compact Riemannian manifold and some noncompact ones (for instance hyperbolic manifolds with finite volume), and is Gromov–Hausdorff dense among all the length spaces (Prop. 9.1). Note also that the condition  $p > n/2$  is optimal since for any  $V > 0$  and any  $\epsilon > 0$ , there exists a large (actually dense for the Gromov–Hausdorff distance among the length spaces) family of Riemannian manifolds with volume  $V$  and  $\rho_{\frac{n}{2}} \leq \epsilon$  (Prop. 9.2). P. Petersen and G. Wei have shown a similar estimate (Th. 1.1 of [15]) for compact manifolds but with an upper bound that depends also on the diameter of  $M$ .

Our second main result is the following Myers type theorem.

**THEOREM 1.2.** – *Let  $(M^n, g)$  be a complete manifold and  $p > n/2$ . If  $\frac{\rho_p}{\text{Vol } M} \leq \frac{1}{C(p, n)}$ ; then  $M$  is compact with finite  $\pi_1$  and*

$$\text{Diam}(M, g) \leq \pi \times \left( 1 + C(p, n) \left( \frac{\rho_p}{\text{Vol } M} \right)^{\frac{1}{10}} \right).$$

COMMENTS AND REMARKS. –

- 1) Such a diameter bound was obtained in [14] under stronger curvature assumptions but the finiteness of the  $\pi_1$  was a conjecture (see also [19]). As noticed in [14], although  $L^\infty$  bounds on the curvature transfer readily to the universal cover, the situation is different for integral controls since we cannot assume the compactness of the universal cover. The main point of this paper is to pass over this difficulty and to get information on the fundamental group from purely integral control on the Ricci curvature.
- 2) For any  $k > 0$ , a renormalization argument readily shows that we can replace  $\rho_p$  by  $\rho_p^k = \int_M (\text{Ric} - k(n-1))_-^p$  in Theorems 1.2 and 1.1 provided we replace  $C(p, n)$  by  $C(p, n, k)$ , and also  $\text{Vol } \mathbb{S}^n$  by  $\frac{\text{Vol } \mathbb{S}^n}{k^{\frac{n}{2}}}$  and  $\pi$  by  $\frac{\pi}{\sqrt{k}}$ . The space  $\mathbb{R}^n$  makes obvious that it does not generalize to  $k \leq 0$ .
- 3) The Cartesian product of a small  $\mathbb{S}^1$  with a finite volume hyperbolic manifold shows that the compactness and the  $\pi_1$ -finiteness cannot be obtained if we only assume that  $\rho_p$  is small (or that  $\frac{\rho_p}{\text{Vol } M}$  is finite). We can also slightly modify the example A.2 of [8] to get a manifold with infinite topology, finite volume and finite  $\rho_p$ , which, when multiplied by a small  $\mathbb{S}^1$ , gives a manifold with infinite topology, finite volume and  $\rho_p$  as small as we want.
- 4) In the case  $p = 1$  and  $n = 2$  the theorem is still valid ( $\pi_1$ -finiteness obviously follows from the Gauss–Bonnet theorem), but in case  $p = n/2$  and  $n \geq 3$  no generalization of the classical results known under  $\text{Ric} \geq n - 1$  can be expected, as shown by the following theorem.

**THEOREM 1.3.** – *Let  $(M^n, g)$  be any compact Riemannian  $n$ -manifold ( $n \geq 3$ ). There exists a sequence of complete Riemannian metrics  $(g_m)$  on  $M$  that converges to  $g$  in the Gromov–Hausdorff distance and such that*

$$\frac{\rho_{n/2}(g_m)}{\text{Vol } g_m} \rightarrow 0.$$

Since 1941 several generalizations of Myers’ theorem appeared, under roughly three different kinds of hypothesis:

- a) the control on integrals of the Ricci curvature along minimizing geodesics ([1,5,3,10,12]),
- b) the Ricci curvature is almost bounded from below by  $n - 1$  (in many different senses) but not allowed to take values under a given negative number ([7,20,17,19]),

c) the  $L^\infty$  lower bound on the Ricci curvature of the case b) is replaced by bounds on other Riemannian invariants (for example the volume bounded from below or the diameter bounded from above or the sectional curvature bounded).

The proofs of Myers' theorem of type a) and b) are essentially based on the second variation formula for the length of geodesics, but we cannot use this formula under our curvature assumptions. To prove Myers' theorems of type c), where the use of the second variation formula also fails, alternative techniques, based on Moser iteration schemas, have been developed (see [4,14,7,17]). But, until this present article (Prop. 8.1), only two bounds on Sobolev constants were known under an integral control of the Ricci curvature: one by S. Gallot requiring a bound on the diameter [8], one by D. Yang requiring a lower bound on the volume of the small balls [21]. Such extra hypotheses are natural (and necessary) for manifolds with almost nonnegative Ricci curvature, but are not relevant in our context: for instance a lower bound on the volume would bound the cardinality of  $\pi_1$  whereas the set of  $n$ -manifolds with Ricci curvature bounded from below by  $n - 1$  has finite but not bounded cardinalities of  $\pi_1$ .

To avoid these unnatural extra hypotheses, we develop another technique based on measure concentration estimates, which allows to prove the following local version of the Myers' theorem.

LEMMA 1.4. – *Let  $(M^n, g)$  be a manifold (not necessarily complete). If  $M$  contains a subset  $T$  satisfying the following conditions:*

1.  $T$  is star-shaped at a point  $x$  (see Definition p. 679),
2.  $B(x, R_T) \supset T \supset B(x, R_0)$  for some  $R_T \geq R_0 > \pi$ ,
3.  $\epsilon = R_T^2 \left[ \frac{1}{\text{Vol} T} \int_T (\underline{\text{Ric}} - (n - 1))_-^p \right]^{\frac{1}{p}} \leq B(p, n) \left( 1 - \frac{\pi}{R_0} \right)^{100}$ ,

then  $\text{Diam}(M^n, g) \leq \pi(1 + C(p, n)\epsilon^{1/20})$  (and  $M \subset T$ ).

The connected sum of an  $n$ -sphere of diameter  $2R_0 - \pi$  with a Euclidean  $n$ -space by a sufficiently small cylinder shows that to get the compactness of  $M$ , we need that  $T$  contain a ball of radius  $R_0 > \pi$  and also that  $\frac{1}{\text{Vol} T} \int_T (\underline{\text{Ric}} - (n - 1))_-^p$  tend to 0 when  $R_0$  tends to  $\pi$ .

To prove Lemma 1.4, we show that if the  $L^p$  norm on  $T$  of  $(\underline{\text{Ric}} - (n - 1))_-$  tends to 0, then the probability Riemannian measure of  $T$  concentrates in  $B(x, \pi)$  while that of any  $B(y, r) \subset T$  remains uniformly bounded from below by a positive increasing function of  $r$ . These two contradicting behaviours of the measure in  $T$  prevent the manifold from having points too far away from  $x$ .

To prove Theorem 1.1, we decompose  $M$  into star-shaped subsets and show that either  $M$  has small volume or Lemma 1.4 applies to at least one of these subsets. The bound on the volume is then inferred by the volume estimates developed for the proof of Lemma 1.4. To show the  $\pi_1$ -finiteness of Theorem 1.2, we construct a star-shaped domain in the universal Riemannian cover of  $(M^n, g)$  which satisfies the assumptions of Lemma 1.4.

Under our curvature assumptions, we also get the following generalization of the Lichnerowicz theorem (which becomes false with  $p = \frac{n}{2}$  when  $n \geq 3$ ).

PROPOSITION 1.5. – *Let us denote by  $\lambda_1$ ,  $\lambda_1^1$  and  $\tilde{\lambda}_1^1$  respectively the first nonzero eigenvalue of the Laplacian on functions, the first eigenvalue on 1-forms and on co-closed 1-forms of  $(M^n, g)$ . Then:*

$$\lambda_1(M^n, g) = \lambda_1^1(M^n, g) \geq n \times \left( 1 - C(p, n) \left( \frac{\rho_p}{\text{Vol} M} \right)^{\frac{1}{p}} \right),$$

$$\tilde{\lambda}_1^1(M^n, g) \geq 2(n-1) \times \left( 1 - C(p, n) \left( \frac{\rho_p}{\text{Vol } M} \right)^{\frac{1}{p}} \right).$$

By adapting the proofs of Lemmas 5.1 and 4.1 (see [2] for details), we further obtain a Bishop–Gromov type result:

**PROPOSITION 1.6.** – *If  $\eta^{10} = \frac{\rho_p}{\text{Vol } M} \leq \frac{1}{C(p, n)}$  then, for all  $x \in M$  and all radii  $0 \leq r \leq R$ , we have:*

$$\begin{aligned} \left( \frac{\text{Vol}_{n-1} S(x, R)}{L_{1-\eta}(R)} \right)^{\frac{1}{2p-1}} - \left( \frac{\text{Vol}_{n-1} S(x, r)}{L_{1-\eta}(r)} \right)^{\frac{1}{2p-1}} &\leq \eta^2 (R-r)^{\frac{2p-n}{2p-1}}, \\ \frac{\text{Vol } B(x, r)}{\text{Vol } B(x, R)} &\geq (1-\eta) \frac{A_1(r)}{A_1(R)}, \end{aligned}$$

where  $L_k(t)$  (resp.  $A_k(t)$ ) stands for the volume of a geodesic sphere (resp. ball) of radius  $t$  in  $(\mathbb{S}^n, \frac{can}{k})$ , hence also:

$$\begin{aligned} \text{Vol}_{n-1} S(x, R) &\leq (1+\eta^2) L_{1-\eta}(R), \\ \text{Vol } B(x, R) &\leq (1+\eta) A_1(R). \end{aligned}$$

Note that, in contrast to the case  $\text{Ric} \geq (n-1)$ , we cannot expect, under our assumptions, an upper bound on the quotient  $\frac{\text{Vol}_{n-1} S(x, \cdot)}{L_1}$  for all possible values of  $r \leq \text{Diam}(M)$  since the diameter of our manifolds can be greater than  $\pi$ . These results are similar to the results obtained in [15] and [14] under stronger curvature assumptions.

Finally, Theorem 1.2 and Proposition 1.6 imply that the set of  $n$ -manifolds satisfying  $\frac{\rho_p}{\text{Vol } M} \leq C(p, n)$ , for a  $p > n/2$ , is pre-compact for the Gromov–Hausdorff distance. We show in the last section that this property is false in the case  $n \geq 3$  and  $p = n/2$ , even for the pointed Gromov–Hausdorff distance.

This article is organised as follows. To prove Lemma 1.4, we need to improve the estimates on volume established in [14] (see also [8], [21] and [15] for other similar estimates and techniques). Section 2 is devoted to a brief survey on the volume of star-shaped domains. In Section 3, we establish a comparison lemma (see Lemma 3.1) à la Petersen–Sprouse–Wei which provides a bound from above of the mean curvature of geodesic spheres of radius  $r$  by a curvilinear integral of  $(\text{Ric} - (n-1))_-$ . This lemma is used in Sections 4 and 5 to establish some estimates on the concentration of the Riemannian probability measure. The diameter and volume bounds of Theorems 1.1 and 1.2 are proved in Section 6. Section 7 is devoted to the proof of the  $\pi_1$ -finiteness of Theorem 1.2, and Section 8 to the proof of Proposition 1.5. Finally, we discuss in Section 9 the case  $p = n/2$ .

## 2. Volume and mean curvature of spheres

**NOTATIONS.** – Let  $x \in M$ . We denote by  $U_x$  the *injectivity domain* of the exponential map at  $x$  and we identify points of  $U_x \setminus \{0_x\}$  with their polar coordinates  $(r, v) \in \mathbb{R}_+^* \times \mathbb{S}_x^{n-1}$  (where  $\mathbb{S}_x^{n-1}$  is the set of normal vectors at  $x$ ). We write  $v_g$  for the Riemannian measure and set  $\omega = \exp_x^* v_g = \theta(r, v) dr dv$ , where  $dv$  and  $dr$  are the canonical measures of  $\mathbb{S}_x^{n-1}$  and  $\mathbb{R}_+^*$ . Henceforth, we extend  $\theta$  to  $(\mathbb{R}_+^* \times \mathbb{S}^{n-1}) \setminus U_x$  by 0.

For all  $(r, v)$  in  $U_x \setminus \{0\}$ , we denote by  $h(r, v)$  the mean curvature at  $\exp_x(rv)$  (for the exterior normal  $\frac{\partial}{\partial r}$ ) of the sphere centered at  $x$  and of radius  $r$ . This function  $h$  is defined on  $U_x$  and satisfies the formula  $\frac{\partial \theta}{\partial r}(t, v) = h(t, v)\theta(t, v)$  (cf. [18], p. 329).

For all real  $k$ , we set  $h_k = (n - 1) \frac{s'_k(r)}{s_k(r)}$  for the corresponding function on the model space  $(S^n_k, g_k)$  ( $n$ -dimensional, simply connected, with sectional curvature  $k$ ) where,

$$s_k(r) = \frac{\sinh(\sqrt{|k|}r)}{\sqrt{|k|}} \text{ when } k < 0, \quad s_k(r) = r \text{ when } k = 0,$$

$$s_k(r) = \begin{cases} \frac{\sin(\sqrt{k}r)}{\sqrt{k}} & \text{if } r \leq \frac{\pi}{\sqrt{k}} \\ 0 & \text{if } r > \frac{\pi}{\sqrt{k}}, \end{cases} \text{ when } k > 0.$$

On  $U_x$  (resp. on  $U_x \cap B(0, \frac{\pi}{\sqrt{k}})$  if  $k > 0$ ), we set  $\psi_k = (h_k - h)_-$ . Following [15], we will use the following lemma:

LEMMA 2.1. – *Let  $u$  be an element of  $S_x^{n-1}$  and  $I_u = ]0, r(u)[$  the interval of values  $t$  such that  $(t, u) \in U_x$ . The function  $t \mapsto \psi_k(t, u)$  is continuous, right and left differentiable everywhere in  $I_u \cap ]0, \frac{\pi}{\sqrt{k}}[$  and it satisfies:*

$$\begin{cases} 1) \lim_{t \rightarrow 0^+} \psi_k(t, u) = 0, \\ 2) \frac{\partial \psi_k}{\partial r} + \frac{\psi_k^2}{n-1} + \frac{2\psi_k h_k}{n-1} \leq \rho_k \end{cases}$$

(where this differential inequality is satisfied by the right and left derivatives of  $\psi_k$  and where  $\rho_k = (\text{Ric} - k(n - 1))_-$ ).

*Proof.* – We apply the well known Bochner formula

$$g(\nabla \Delta f, \nabla f) = \frac{1}{2} \Delta |\nabla f|^2 + |Ddf|^2 + \text{Ric}(\nabla f, \nabla f)$$

to the distance to  $x$  function  $d_x$  (where we have used the convention  $\Delta f = -\text{tr} Ddf$ ). Since  $|\nabla d_x| = 1$  and the Hessian  $Dd(d_x)$  is zero on  $\mathbb{R}\nabla d_x$  and equal to the second fundamental form of the geodesic sphere of center  $x$  on  $\nabla d_x^\perp$ , we infer that  $h$  satisfies the following Riccati inequality

$$\frac{\partial h}{\partial r} + \frac{h^2}{n-1} + \text{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq 0.$$

This inequality becomes an equation on the model spaces  $(S^n_k, g_k)$ , which easily gives inequality 2) of Lemma 2.1. Since  $h \sim (n - 1)/r + o(1)$  (see [18] for details), we also easily get 1).  $\square$

### 2.1. Volume of star-shaped domains

DEFINITION. – *Let  $x \in M$  and  $T \subset M$ . We say that  $T$  is star-shaped at  $x$  if for all  $y \in T$  there exists a minimizing geodesic from  $x$  to  $y$  contained in  $T$ . Equivalently, we may assume that  $T = \exp_x(T_x)$ , where  $T_x$  is an affine star-shaped subset of  $\overline{U}_x \subset T_x M$ .*

Given  $T$ , a subset of  $M$  star-shaped at  $x$ , let  $A_T(r)$  denote the volume of  $B(x, r) \cap T$ . We set  $L_T(r)$  the  $(n - 1)$ -dimensional volume of  $(rS_x^{n-1}) \cap U_x \cap T_x$  for the measure  $\theta(r, \cdot) dv$ . Note that  $L_T(r) = \int_{S_x^{n-1}} \mathbb{1}_{T_x} \theta(r, v) dv$  and  $A_T(r) = \int_0^r L_T(t) dt$ . Finally, the functions corresponding to  $\theta$ ,  $A$  and  $L$  on the model manifold  $(S^n_k, g_k)$  will be denoted by  $\theta_k$ ,  $A_k$  and  $L_k$  respectively.

The regularity properties of the functions  $L_T$  and  $A_T$  used subsequently are summarized in the following lemma:

LEMMA 2.2. – *Let  $T$  be a star-shaped subset of  $(M, g)$ .*

- (i)  $L_T$  is a right continuous, left lower semi-continuous function,
- (ii)  $A_T$  is a continuous, right differentiable function of derivative  $L_T$ .
- (iii) Given  $\alpha \in ]0, 1]$ , the function

$$f(r) = \left( \frac{L_T(r)}{L_k(r)} \right)^\alpha - \frac{\alpha}{\text{Vol } \mathbb{S}^{n-1}} \int_0^r \int_{\mathbb{S}_x^{n-1}} \left( \frac{L_T(s)}{L_k(s)} \right)^{\alpha-1} \mathbb{1}_{T_x} \psi_k \frac{\theta}{\theta_k}$$

is decreasing either on  $\mathbb{R}_+^*$  (if  $k \leq 0$ ) or on  $]0, \frac{\pi}{\sqrt{k}}[$  (if  $k > 0$ ).

*Proof.* – To prove (i), note that  $\theta(r, v)\mathbb{1}_{T_x}$  is the product of  $r^{n-1}\mathbb{1}_{T_x}(r, v)$  by the Jacobian of  $\exp_x$ , hence  $r \mapsto \theta(r, v)\mathbb{1}_{T_x}$  is positive on an interval  $]0, r(v)[$ , vanishes on  $[r(v), +\infty[$ , and so is right continuous and left lower semi-continuous on  $\mathbb{R}$ . We infer also that  $\mathbb{1}_{T_x}\theta$  is bounded on every compact of  $T_x M$ . This yields the boundedness of  $L_T$  on every compact subset of  $]0, +\infty[$ . We infer (i) from the Lebesgue dominated convergence theorem and the Fatou lemma. Property (ii) now follows from (i) by the definition of  $A_T$ .

To complete the proof of Lemma 2.2, we note that, by definition of  $L_T$ , and since  $\text{Vol } M \setminus \exp_x(U_x) = 0$ , we may assume that  $T_x \subset U_x$ . For all integers  $m \geq 1$ , let  $T_x^{(m)} = (1 - \frac{1}{m})T_x \subset T_x$  be the image of  $T_x$  by the homothety of center 0 and factor  $(1 - \frac{1}{m})$  in  $T_{x_0}M$  and set  $T^{(m)} = \exp_x(T_x^{(m)})$ . By the monotone convergence theorem, we have  $A_T = \lim_{m \rightarrow \infty} A_{T^{(m)}}$  and  $L_T = \lim_{m \rightarrow \infty} L_{T^{(m)}}$ . Hence, it only remains to show (iii) for  $T^{(m)}$ . We will use the following elementary lemma:

LEMMA 2.3. – *A function  $f : [a, b] \rightarrow \mathbb{R}$  is decreasing if and only if it satisfies the two conditions*

- (a) for all  $x \in [a, b]$ ,  $\limsup_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \leq 0$ ,
- (b) for all  $x \in ]a, b]$ ,  $\liminf_{h \rightarrow 0^-} f(x+h) \geq f(x)$ .

As for  $L_T$  and  $A_T$ , the function  $r \mapsto \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x^{(m)}} \psi_k \frac{\theta}{\theta_k}(r, v) dv$  is right continuous, left lower semi-continuous on  $I_k = ]0, +\infty[$  if  $k \leq 0$  (resp. on  $I_k = ]0, \frac{\pi}{\sqrt{k}}[$  if  $k > 0$ ), and  $r \mapsto \int_0^r \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x^{(m)}} \psi_k \frac{\theta}{\theta_k}$  is continuous, right differentiable on  $I_k$ ; so the function  $f$  satisfies inequality (b) of Lemma 2.3. We now prove (a):

For all  $r > 0$ , let  $\mathbb{S}_{T^{(m)}}^r = \{v \in \mathbb{S}_x^{n-1} \mid rv \in T_x^{(m)}\}$ . We denote by  $\tilde{L}(r+t)$  the volume of  $(r+t).\mathbb{S}_{T^{(m)}}^r$  for the measure  $\theta(r+t, \cdot)dv$ . Since  $T_x^{(m)}$  is star-shaped at  $x$ , we have  $\tilde{L}(r+t) \geq L_{T^{(m)}}(r+t)$  (with equality if  $t = 0$ ). Hence

$$\lim_{t \rightarrow 0^+} \frac{L_{T^{(m)}}(r+t) - L_{T^{(m)}}(r)}{t} \leq \lim_{t \rightarrow 0^+} \frac{\tilde{L}(r+t) - \tilde{L}(r)}{t}.$$

Since  $\tilde{L}(r+t) = \int_{\mathbb{S}_{T^{(m)}}^r} \theta(r+t, v) dv$  and  $\frac{\partial \theta}{\partial r} = h\theta$ , we obtain, by differentiating this integral expression of  $\tilde{L}$  (note that  $h\theta$  and  $\psi_k\theta$  are integrable on the set  $\mathbb{S}_{T^{(m)}}^r$ , which could be false for  $T$  and this is why we introduced the sets  $T^{(m)}$ ): for any  $t \in [0, \frac{1}{m-1}r[$ , the closure of  $(r+t).\mathbb{S}_{T^{(m)}}^r$  in  $T_x M$  is compact and belongs to  $U_x \setminus \{0_x\}$  because the cut-radius is continuous on  $\mathbb{S}_x^{(n-1)}$

(see [18]) and bounded from below by  $\frac{m}{m-1}r > r + t$  on  $\mathbb{S}_{T^{(m)}}^r$ ; but, the function  $h = \frac{1}{\theta} \frac{\partial \theta}{\partial r}$  is smooth on  $U_x \setminus \{0_x\}$ , and so uniformly bounded on every set  $(r + t) \cdot \mathbb{S}_T^{(m)}(r)$ ,

$$\lim_{t \rightarrow 0^+} \frac{\tilde{L}(r+t) - \tilde{L}(r)}{t} = \int_{\mathbb{S}_x^{n-1}} h \mathbb{1}_{T^{(m)}} \theta \, dv \leq \int_{\mathbb{S}_x^{n-1}} (\psi_k + h_k) \mathbb{1}_{T^{(m)}} \theta \, dv.$$

Combining the last two inequalities, we get:

$$\overline{\lim}_{t \rightarrow 0^+} \frac{L_{T^{(m)}}(r+t) - L_{T^{(m)}}(r)}{t} \leq h_k(r) L_{T^{(m)}}(r) + \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T^{(m)}} \psi_k \theta.$$

The case  $\alpha = 1$  of (a) easily follows, noting that  $L_k$  has derivative  $h_k L_k$  and that:

$$\begin{aligned} & \limsup_{t \rightarrow 0^+} \frac{\frac{L_{T^{(m)}}(r+t)}{L_k(r+t)} - \frac{L_{T^{(m)}}(r)}{L_k(r)}}{t} \\ &= \limsup_{t \rightarrow 0^+} \frac{L_{T^{(m)}}(r+t) - L_{T^{(m)}}(r)}{t L_k(r)} + \lim_{t \rightarrow 0^+} \left[ L_{T^{(m)}}(r+t) \frac{1}{t} \left( \frac{1}{L_k(r+t)} - \frac{1}{L_k(r)} \right) \right] \\ &= \frac{1}{\text{Vol } \mathbb{S}^{n-1} \theta_k(r)} \left[ \limsup_{t \rightarrow 0^+} \frac{L_{T^{(m)}}(r+t) - L_{T^{(m)}}(r)}{t} - h_k(r) L_{T^{(m)}}(r) \right]. \end{aligned}$$

To prove inequality (b) for any  $\alpha \in ]0, 1]$ , we first set  $B = \frac{1}{\text{Vol } \mathbb{S}^{n-1}} \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{\mathbb{S}_T^{(m)}} \psi_k \frac{\theta}{\theta_k} \, dv$ . For all  $\epsilon > 0$ , there exists  $t_\epsilon > 0$  such that for all  $t \in ]0, t_\epsilon[$ , we have  $\frac{L_{T^{(m)}}(r+t)}{L_k(r+t)} \leq \frac{L_{T^{(m)}}(r)}{L_k(r)} + t(B + \epsilon)$ . So, by concavity we get:

$$\left( \frac{L_{T^{(m)}}(r)}{L_k(r)} + t(B + \epsilon) \right)^\alpha - \left( \frac{L_{T^{(m)}}(r)}{L_k(r)} \right)^\alpha \leq \alpha \left( \frac{L_{T^{(m)}}(r)}{L_k(r)} \right)^{\alpha-1} t(B + \epsilon).$$

It follows that  $\limsup_{t \rightarrow 0^+} \frac{f(r+t) - f(r)}{t} \leq \alpha \epsilon \left( \frac{L_{T^{(m)}}(r)}{L_k(r)} \right)^{\alpha-1}$ , which gives (b) by letting  $\epsilon$  tend to 0.  $\square$

### 3. Comparison lemma on mean curvature

The following lemma improves Lemma 2.2 in [15] and Theorem 2.1 in [14]. We provide a pointwise bound on  $\psi_k$  which, in case  $k > 0$ , admits a sharp polynomial blow-up when  $r \rightarrow \frac{\pi}{\sqrt{k}}$ ; these both improvements are necessary for our proof of Theorem 1.2 (see the proof of Lemma 4.1).

LEMMA 3.1. – *Let  $k \in \mathbb{R}$ ,  $p > n/2$  and  $r > 0$ ; assume  $r \leq \frac{\pi}{2\sqrt{k}}$  if  $k > 0$ . We have:*

$$\psi_k^{2p-1}(r, v) \theta(r, v) \leq (2p-1)^p \left( \frac{n-1}{2p-n} \right)^{p-1} \int_0^r \rho_k^p(t, v) \theta(t, v) \, dt.$$

Moreover if  $k > 0$  and  $\frac{\pi}{2\sqrt{k}} < r < \frac{\pi}{\sqrt{k}}$ , then

$$\sin^{4p-n-1}(\sqrt{kr}) \psi_k^{2p-1}(r, v) \theta(r, v) \leq (2p-1)^p \left( \frac{n-1}{2p-n} \right)^{p-1} \int_0^r \rho_k^p(t, v) \theta(t, v) dt.$$

These two inequalities hold for all normal vector  $v \in \mathbb{S}_x^{n-1}$ , even if we replace  $\theta$  everywhere by  $\mathbb{1}_{[0, s_v]} \theta$  (for any  $s_v \geq 0$ ).

*Remark.* – The bounds diverge when  $p$  tends to  $n/2$  except in the case  $n = 2$  (which then yields a control of  $\psi_k$  by the  $L_1$ -norm of  $\rho_k$ ).

*Proof.* – Let  $\phi$  be a nonnegative  $C^1$  function on  $U_x \setminus \{0\}$ , bounded in the neighborhood of 0. By Lemma 2.1, the function  $r \mapsto \phi(r, v) \psi_k^{2p-1}(r, v) \theta(r, v)$  is continuous and right differentiable on  $I_v$ , and its derivative satisfies:

$$\begin{aligned} \frac{\partial}{\partial r} (\phi \psi_k^{2p-1} \theta) &\leq (2p-1) \rho_k \phi \psi_k^{2p-2} \theta - \left( \frac{2p-n}{n-1} \right) \phi \psi_k^{2p} \theta \\ &\quad + \left( \frac{4p-n-1}{n-1} h_k - \frac{1}{\phi} \frac{\partial \phi}{\partial r} \right) \phi \psi_k^{2p-1} \theta \end{aligned}$$

where we used  $\frac{\partial \theta}{\partial r} = h\theta \leq h_k \theta + \psi_k \theta$ . Setting  $X = \left( \int_0^r \phi \psi_k^{2p} \theta dt \right)$  and integrating, we get:

$$(1) \quad 0 \leq \phi \psi_k^{2p-1} \theta(r) \leq (2p-1) \left( \int_0^r \phi \rho_k^p \theta dt \right)^{1/p} X^{1-\frac{1}{p}} - \left( \frac{2p-n}{n-1} \right) X \\ + \left[ \int_0^r \left( \frac{4p-n-1}{n-1} h_k - \frac{1}{\phi} \frac{\partial \phi}{\partial r} \right)^{2p} \phi \theta dt \right]^{1/2p} X^{1-\frac{1}{2p}}$$

where we used  $\lim_{t \rightarrow 0} \phi(t, v) \psi_k^{2p-1}(t, v) \theta(t, v) = 0$ . Dividing out by  $X^{1-\frac{1}{p}}$ , we obtain a quadratic polynomial that takes a nonnegative value at  $X^{\frac{1}{2p}}$  and we infer:

$$\begin{aligned} \left( \int_0^r \phi \psi_k^{2p} \theta dt \right)^{\frac{1}{2p}} &\leq \sqrt{\frac{(n-1)(2p-1)}{2p-n}} \left( \int_0^r \phi \rho_k^p \theta dt \right)^{1/2p} \\ &\quad + \frac{n-1}{2p-n} \left( \int_0^r \left( h_k \frac{2p-1+(2p-n)}{n-1} - \frac{\partial \phi / \partial r}{\phi} \right)^{2p} \phi \theta dt \right)^{1/2p}. \end{aligned}$$

We prove the first inequality of Lemma 3.1 by taking  $\phi(r, v) = 1$ . Indeed then, the above inequality and the positivity of  $h_k$  yield:

$$\int_0^r \psi_k^{2p} \theta dt \leq \left( \frac{(2p-1)(n-1)}{2p-n} \right)^p \int_0^r \rho_k^p \theta dt.$$

Plugging this into inequality (2), we obtain

$$\psi_k^{2p-1} \theta(r) \leq (2p-1)^p \left( \frac{n-1}{2p-n} \right)^{p-1} \left( \int_0^r \rho_k^p \theta dt \right).$$



For the second inequality, we set  $\phi = \sin^{4p-n-1}(\sqrt{k}r)$  and observe that, in this case, the last term of inequality (2) vanishes. So we get for all  $r < \frac{\pi}{\sqrt{k}}$ :

$$\sin^{4p-n-1}(\sqrt{k}r)\psi_k^{2p-1}\theta \leq (2p-1)^p \left(\frac{n-1}{2p-n}\right)^{p-1} \int_0^{\frac{\pi}{\sqrt{k}}} \rho_k^p \theta dt. \quad \square$$

#### 4. Hyper-concentration of the measure

In this section, we prove the first volume estimate required in our proof of Theorem 1.2. It says that, if the Ricci curvature concentrates sufficiently above  $n - 1$  on a star-shaped subset  $T$  of  $M$  at  $x$ , then the Riemannian measure of  $T$  is almost contained in  $B(x, \pi) \cap T$ .

LEMMA 4.1. – *There exists an explicit constant  $C(p, n)$  such that if  $(M^n, g)$  contains a subset  $T$ , star-shaped at a point  $x$ , on which:*

$$\epsilon = R_T^2 \left[ \frac{1}{\text{Vol}T} \int_T (\text{Ric} - (n-1))_-^p \right]^{\frac{1}{p}} \leq \left(\frac{\pi}{6}\right)^{2-\frac{1}{p}},$$

where  $R_T$  is such that  $T \subset B(x, R_T)$ , then, for all radius  $R_T \geq r \geq \pi$ :

$$L_T(r) \leq \frac{C(p, n)}{r} \epsilon^{\frac{p(n-1)}{2p-1}} \text{Vol}T.$$

*Remark.* – The same conclusion holds in case  $n = 2$  and  $p = 1$  by letting  $n = 2$  and  $p \rightarrow 1$  in the proof below.

*Proof.* – Lemma 2.2 (with  $0 < t \leq r < \frac{\pi}{\sqrt{k}}$ ,  $\alpha = \frac{1}{2p-1}$  and  $k > 0$  fixed) yields:

$$(2) \quad \left(\frac{L_T(r)}{L_k(r)}\right)^{\frac{1}{2p-1}} - \left(\frac{L_T(t)}{L_k(t)}\right)^{\frac{1}{2p-1}} \leq \frac{1}{2p-1} \int_t^r \left(\frac{L_T}{L_k}\right)^{\frac{1}{2p-1}-1} \frac{1}{\text{Vol}\mathbb{S}^{n-1}} \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x} \psi_k \frac{\theta}{\theta_k}.$$

Since

$$\frac{(L_T/L_k)^{\frac{2(1-p)}{2p-1}}}{\text{Vol}\mathbb{S}^{n-1}} \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x} \psi_k \frac{\theta}{\theta_k} \leq \frac{1}{(L_k)^{\frac{1}{2p-1}}} \left( \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x} \psi_k^{2p-1} \theta \right)^{\frac{1}{2p-1}},$$

we get that:

$$(3) \quad \frac{1}{\text{Vol}\mathbb{S}^{n-1}} \int_t^r \left(\frac{L_T(s)}{L_k(s)}\right)^{\frac{2-2p}{2p-1}} \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x} \psi_k \frac{\theta}{\theta_k} dv ds \leq \int_t^r \frac{(\sqrt{k})^{\frac{n-1}{2p-1}}}{\sin^2(\sqrt{k}s)} \left( \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x} \frac{\sin^{4p-n-1}(\sqrt{k}s)\psi_k^{2p-1}\theta}{\text{Vol}\mathbb{S}^{n-1}} dv \right)^{\frac{1}{2p-1}} ds.$$

If we combine inequalities (2), (3), Lemma 3.1 and the equality  $L_k(s) = \text{Vol}\mathbb{S}^{n-1} \frac{\sin^{n-1}(\sqrt{k}s)}{(\sqrt{k})^{n-1}}$ , we get that:

$$\begin{aligned} & \left( \frac{L_T(r)}{\sin^{n-1}(\sqrt{k_r}r)} \right)^{\frac{1}{2p-1}} - \left( \frac{L_T(t)}{\sin^{n-1}(\sqrt{k_r}t)} \right)^{\frac{1}{2p-1}} \\ & \leq \left( \frac{(n-1)}{(2p-1)(2p-n)} \right)^{\frac{p-1}{2p-1}} \left( \int_{T \cap B(x,r)} \rho_k^p \right)^{\frac{1}{2p-1}} \int_t^r \frac{1}{\sin^2(\sqrt{k_r}s)} ds. \end{aligned}$$

We set  $\epsilon' = \epsilon^{\frac{p}{2p-1}}$ ,  $k_r = \frac{(\pi-\epsilon')^2}{r^2}$  and we assume that  $t \in [\frac{\pi}{2(\pi-\epsilon')}r, r]$ . By concavity of the sine function on  $[\frac{\pi}{2}, \pi]$ , we have that

$$\int_t^r \frac{1}{\sin^2(\sqrt{k_r}s)} ds \leq \frac{\pi^2(r-t)}{4(\pi - \sqrt{k_r}t)(\pi - \sqrt{k_r}r)} \leq \frac{\pi^2(r-t)}{4(\pi - \sqrt{k_r}t)\epsilon'} \leq \frac{\pi r}{2\epsilon'}$$

(we have used that  $t \mapsto \frac{r-t}{\pi - \sqrt{k_r}t}$  is decreasing) and so we get that:

$$\frac{L_T(r)^{\frac{1}{2p-1}}}{(\sin(\sqrt{k_r}r))^{\frac{n-1}{2p-1}}} - \frac{L_T(t)^{\frac{1}{2p-1}}}{(\sin(\sqrt{k_r}t))^{\frac{n-1}{2p-1}}} \leq \frac{\pi}{2R_T^{\frac{1}{2p-1}}} \left( \frac{n-1}{(2p-1)(2p-n)} \right)^{\frac{p-1}{2p-1}} \text{Vol}(T)^{\frac{1}{2p-1}}.$$

Multiplying this inequality by  $(\sin(r\sqrt{k_r}))^{\frac{n-1}{2p-1}} \leq (\epsilon')^{\frac{n-1}{2p-1}}$ , we infer that for all  $t \in [\frac{\pi}{2(\pi-\epsilon')}r, r]$

$$\begin{aligned} L_T(r)^{\frac{1}{2p-1}} & \leq L_T(t)^{\frac{1}{2p-1}} \left( \frac{\epsilon'}{\sin((\pi - \epsilon')\frac{t}{r})} \right)^{\frac{n-1}{2p-1}} \\ & \quad + \frac{\pi}{2R_T^{\frac{1}{2p-1}}} \left( \frac{n-1}{(2p-1)(2p-n)} \right)^{\frac{p-1}{2p-1}} \text{Vol} T^{\frac{1}{2p-1}} \epsilon'^{\frac{n-1}{2p-1}}. \end{aligned}$$

Using the inequality  $(a+b)^\alpha \leq 2^{\alpha-1}(a^\alpha + b^\alpha)$  (for all  $a, b \geq 0$ ), with  $\alpha = 2p-1$ , and the fact that  $\sin[(\pi - \epsilon')\frac{t}{r}] \geq \sin(\frac{\pi}{6}) = \frac{1}{2}$  when  $t \in [\frac{\pi}{2(\pi-\epsilon')}r, \frac{5\pi}{6(\pi-\epsilon')}r]$ , we get that

$$L_T(r) \leq 2^{2p+n-3} \epsilon^{\frac{p(n-1)}{2p-1}} L_T(t) + \frac{\pi^{2p-1}}{2R_T} \text{Vol}(T) \epsilon^{\frac{p(n-1)}{2p-1}} \left( \frac{n-1}{(2p-1)(2p-n)} \right)^{p-1},$$

for all  $t \in [\frac{\pi}{2(\pi-\epsilon')}r, \frac{5\pi}{6(\pi-\epsilon')}r]$  (note that  $\frac{5\pi}{6(\pi-\epsilon')}r \leq r$ , hence  $\frac{t}{r} \leq 1$ ).

By the mean value property, there exists  $t \in [\frac{\pi}{2(\pi-\epsilon')}r, \frac{5\pi}{6(\pi-\epsilon')}r]$  such that  $L_T(t)$  is bounded from above by  $\frac{3(\pi-\epsilon')}{\pi r} \int_{\frac{\pi}{2(\pi-\epsilon')}r}^{\frac{5\pi}{6(\pi-\epsilon')}r} L_T(s) ds$  which is less than  $\frac{3}{r} \int_0^{R_T} L = \frac{3}{r} \text{Vol}(T)$ . In summary, we conclude that

$$L_T(r) \leq \left[ 3 \cdot 2^{2p+n-3} + \frac{\pi^{2p-2}}{2} \left( \frac{n-1}{(2p-1)(2p-n)} \right)^{p-1} \right] \frac{\text{Vol}(T)}{r} \epsilon^{\frac{p(n-1)}{2p-1}}. \quad \square$$

## 5. Lower bound on the volume of geodesic balls

In this section, we bound from below the relative volume of the geodesic balls. The following lemma, which is the second step of the proof of Theorem 1.2, contains generalizations of Theorem 2.1 of [16] to star-shaped domains or nonconcentric balls.

LEMMA 5.1. – Let  $n \geq 2$  be an integer and  $p > n/2$  be a real. There exist (computable) constants  $C(p, n) > 0$  and  $B(p, n)$  such that when  $(M^n, g)$  contains a star-shaped subset  $T$  which satisfies

$$\epsilon = R_T^2 \left[ \frac{1}{\text{Vol } T} \int_T (\text{Ric})_-^p \right]^{\frac{1}{p}} \leq B(p, n),$$

then we have

- (i) for all  $0 < r \leq R \leq R_T$ ,  $\frac{A_T(r)}{A_T(R)} \geq (1 - C(p, n)\epsilon^{\frac{p}{2p-1}}) \frac{r^n}{R^n}$ ;
- (ii) if  $T = B(x, R_0)$ ,  $y \in T$  and  $r \geq 0$  satisfy  $d(x, y) + r \leq R_0$ , then

$$\begin{aligned} & \left( \frac{\text{Vol } B(y, r)}{\text{Vol } B(x, R_0)} \right)^{\frac{1}{2p'-1}} \\ & \geq \left( \frac{r}{R_0} \right)^{\frac{n}{2p'-1}} \left[ \left( \frac{2}{3} - C(p, n)\epsilon^{\frac{p'}{2p'-1}} \right) \left( \frac{r}{R_0} \right)^{\frac{2n}{2p'-1}} - C(p, n)\epsilon^{\frac{p'}{2p'-1}} \right], \end{aligned}$$

where  $p' = \max(n, p)$ .

Proof. – Lemma 2.2 (with  $k = 0$  and  $\alpha = 1$ ) and the Hölder inequality yield, for all  $t \leq r \leq R_T$ :

$$\frac{L_T(r)}{r^{n-1}} - \frac{L_T(t)}{t^{n-1}} \leq \int_t^r \frac{L_T(s)^{1-\frac{1}{2p-1}}}{s^{n-1}} \left( \int_{\mathbb{S}_x^{n-1}} \mathbb{1}_{T_x} \psi_0^{2p-1} \theta \, dv \right)^{\frac{1}{2p-1}} ds.$$

Then Lemma 3.1 implies that

$$\begin{aligned} \frac{L_T(r)}{r^{n-1}} - \frac{L_T(t)}{t^{n-1}} & \leq C(p, n) \int_t^r \frac{L_T(s)^{1-\frac{1}{2p-1}}}{s^{n-1}} \left( \int_{B(x,s) \cap T} \rho_0^p \right)^{\frac{1}{2p-1}} \\ & \leq \frac{C(p, n)}{t^{n-1}} \left( \int_T \rho_0^p \right)^{\frac{1}{2p-1}} \int_t^r L_T^{1-\frac{1}{2p-1}}. \end{aligned}$$

Multiplying this inequality by  $nr^{n-1}t^{n-1}$ , using the inequality

$$\int_t^r L_T^{1-\frac{1}{2p-1}} \leq (r-t)^{\frac{1}{2p-1}} (A_T(r) - A_T(t))^{1-\frac{1}{2p-1}},$$

and integrating the result with respect to  $t$  from 0 to  $r$ , we get that

$$(4) \quad \frac{d}{dr} \left( \frac{A_T}{r^n} \right) \leq \left( \frac{A_T(r)}{r^n} \right)^{1-\frac{1}{2p-1}} C(p, n) \left( \int_T \rho_0^p \right)^{\frac{1}{2p-1}} nr^{\frac{1-n}{2p-1}}$$

(since  $\frac{A_T(r)}{r^n}$  is right differentiable). Integrating once again yields

$$\left[ \frac{A_T(R)}{R^n} \right]^{\frac{1}{2p-1}} - \left[ \frac{A_T(r)}{r^n} \right]^{\frac{1}{2p-1}} \leq C(p, n) \left( \int_T \rho_0^p \right)^{\frac{1}{2p-1}} R^{\frac{2p-n}{2p-1}} (E_T^{r,R}).$$

Inequality  $(E_T^{R, R_T})$  implies

$$\left[ \frac{A_T(R)}{A_T(R_T)R^n} \right]^{\frac{1}{2p-1}} \geq R_T^{\frac{-n}{2p-1}} (1 - C(p, n)\epsilon^{\frac{p}{2p-1}}) \geq \frac{1}{2} R_T^{\frac{-n}{2p-1}}$$

as soon as  $B(p, n)$  is sufficiently small. This and  $(E_T^{r, R})$  imply (i).

To show (ii), we may assume, by the Hölder inequality, that  $p \in ]n/2, n]$ . Let  $z \in B(x, R_0)$  and  $(r, R)$  such that  $0 < r \leq R \leq R_0 - d(x, z)$ . Multiplying  $(E_{B(z, R)}^{r, R})$  by  $(\frac{1}{A_x(R_0)})^{\frac{1}{2p-1}}$  and since  $B(z, R) \subset B(x, R_0)$ , we get that

$$\left( \frac{A_z(R)}{A_x(R_0)} \right)^{\frac{1}{2p-1}} \leq C(p, n) \left( \frac{R^2 \epsilon}{R_0^2} \right)^{\frac{p}{2p-1}} + \left( \frac{R}{r} \right)^{\frac{n}{2p-1}} \left( \frac{A_z(r)}{A_x(R_0)} \right)^{\frac{1}{2p-1}}.$$

We now construct a sequence of increasing balls  $B_i = B(y_i, R_i)$  such that  $B_1 = B(y, r)$ ,  $B_k$  is concentric to  $B(x, R_0)$ , and  $B_i$  contains a ball centered at  $y_{i+1}$  and of radius  $r_{i+1}$  close to  $R_i$ .

Let  $\gamma : [0, d(x, y)] \rightarrow M$  be a minimizing geodesic from  $x$  to  $y$  and  $\alpha = \alpha(p, n) < 1$  close enough to 1 such that we have  $-\text{Log } \alpha \leq 2 \text{Log}(2 - \alpha)$ ,  $(2 - \alpha)^{2p-n} \alpha^n < 1$  and  $\alpha^{\frac{n}{2p-1}} \geq \frac{2}{3}$ .

For all integers  $1 \leq i \leq k = E[2 + \frac{\text{Log}(1 + \frac{d(x, y)}{r})}{\text{Log}(2 - \alpha)}]$ , we define

$$y_i = \gamma(d(x, y) + r - (2 - \alpha)^{i-1}r) \quad \text{if } i \leq k - 1, \quad y_k = x, \\ r_i = \alpha(2 - \alpha)^{i-2}r, \quad R_i = (2 - \alpha)^{i-1}r.$$

We have  $B(y_{i+1}, r_{i+1}) \subset B(y_i, R_i) \subset B(x, R_0)$  for any  $i \leq k - 1$ . We set  $z = y_{i+1}$ ,  $R = R_{i+1}$  and  $r = r_{i+1}$ , in the inequality above and get

$$\left( \frac{A_{y_{i+1}}(R_{i+1})}{A_x(R_0)} \right)^{\frac{1}{2p-1}} \\ \leq C(p, n) \left( \frac{r^2 \epsilon}{R_0^2} \right)^{\frac{p}{2p-1}} (2 - \alpha)^{\frac{2pi}{2p-1}} + \left( \frac{(2 - \alpha)^n A_{y_{i+1}}(r_{i+1})}{\alpha^n A_x(R_0)} \right)^{\frac{1}{2p-1}}, \\ \leq C(p, n) \left( \frac{r^2 \epsilon}{R_0^2} \right)^{\frac{p}{2p-1}} (2 - \alpha)^{\frac{2pi}{2p-1}} + \left( \frac{(2 - \alpha)^n A_{y_i}(R_i)}{\alpha^n A_x(R_0)} \right)^{\frac{1}{2p-1}},$$

If we set  $a_i = (\frac{A_{y_{i+1}}(R_{i+1})}{A_x(R_0)})^{\frac{1}{2p-1}}$ ,  $C = C(p, n)(\frac{r^2 \epsilon}{R_0^2})^{\frac{p}{2p-1}}$ ,  $\beta = (2 - \alpha)^{\frac{2p}{2p-1}}$ ,  $d = (\frac{2 - \alpha}{\alpha})^{\frac{n}{2p-1}}$ , then the last inequalities is  $a_{i+1} \leq C\beta^i + da_i$  for any  $0 \leq i \leq k - 1$ , hence  $a_i \leq C\frac{d^i - \beta^i}{d - \beta} + d^{i-1}a_1 \leq C\frac{d^i}{d - \beta} + d^{i-1}a_1 \leq d^{i-1}(a_1 + C\frac{1}{1 - \beta/d})$ , that is to say

$$\left( \frac{A_{y_{k-1}}(R_{k-1})}{A_x(R_0)} \right)^{\frac{1}{2p-1}} \leq \left( \frac{2 - \alpha}{\alpha} \right)^{\frac{n(k-2)}{2p-1}} \left[ \left( \frac{A_y(r)}{A_x(R_0)} \right)^{\frac{1}{2p-1}} + \frac{C(p, n)(\frac{r^2 \epsilon}{R_0^2})^{\frac{p}{2p-1}}}{1 - (2 - \alpha)^{\frac{2p-n}{2p-1}} \alpha^{\frac{n}{2p-1}}} \right].$$

By inequality (i) of Lemma 5.1 we get

$$\left( \frac{A_{y_{k-1}}(R_{k-1})}{A_x(R_0)} \right)^{\frac{1}{2p-1}} \geq \left( \frac{A_{y_k=x}(r_k)}{A_x(R_0)} \right)^{\frac{1}{2p-1}} \\ \geq (1 - C(p, n)\epsilon^{\frac{p}{2p-1}}) \alpha^{\frac{n}{2p-1}} (2 - \alpha)^{\frac{n(k-2)}{2p-1}} \left( \frac{r}{R_0} \right)^{\frac{n}{2p-1}}.$$

These two estimates on  $\frac{A_{y_{k-1}}(R_{k-1})}{A_x(R_0)}$ , and the fact that by assumption  $\alpha^{\frac{n(k-1)}{2p-1}} \geq \alpha^{\frac{n}{2p-1}} \times \left(\frac{r}{r+d(x,y)}\right)^{\frac{-n \text{Log } \alpha}{(2p-1) \text{Log}(2-\alpha)}} \geq \frac{2}{3} \left(\frac{r}{R_0}\right)^{\frac{2n}{2p-1}}$ , imply that there exist constants  $C(p, n) > 0$  and  $B(p, n) > 0$  such that when  $\epsilon \leq B(p, n)$ ,

$$\left(\frac{A_y(r)}{A_x(R_0)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{r}{R_0}\right)^{\frac{n}{2p-1}} \left[ \left(\frac{2}{3} - C(p, n)\epsilon^{\frac{p}{2p-1}}\right) \left(\frac{r}{R_0}\right)^{\frac{2n}{2p-1}} - C(p, n)\epsilon^{\frac{p}{2p-1}} \right]. \quad \square$$

In the case  $(n, p) = (2, 1)$ , the following lemma holds

LEMMA 5.2. – *There exists constants  $B > 0$  and  $C > 0$  such that if a surface  $(S^2, g)$  contains a star-shaped subset  $T$  on which the sectional curvature  $K$  satisfies  $\epsilon = \frac{R_T^2}{\text{Vol} T} \int_T K_- \leq B$ , then*

$$(i) \quad \frac{A_T(r)}{\text{Vol} T} \geq \left(\frac{r}{R_T}\right)^2 \left(1 - \epsilon \ln\left(\frac{R_T}{r}\right)\right),$$

for all  $r \leq R_T$ . In addition, if  $T = B(x, R_0)$ ,  $y \in T$  and  $d(x, y) + r \leq R_0$ , then

$$(ii) \quad \frac{\text{Vol} B(y, r)}{\text{Vol} B(x, R_0)} \geq \left(\frac{r}{R_0}\right)^4 \left(1 - 3\epsilon \left(\frac{R_0}{r}\right)^2\right).$$

*Proof.* – The constant  $C(p, n)$  involved in inequality (4) is  $\frac{2p-1}{2p} \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p}}$ . In case  $n = 2$ , we can let  $p$  tend to 1 in that differential inequality and get  $\frac{d}{dr} \left(\frac{A}{r^2}\right) \leq \frac{1}{r} \int_T K_-$ . This implies  $\frac{A(R)}{R^2} - \frac{A(r)}{r^2} \leq \ln \frac{R}{r} \int_T K_-$  and so (i).

(ii) is proved as in Lemma 5.1 (note that, in this case, we may let  $\alpha$  tend to 1, which simplifies the final formula).  $\square$

## 6. Diameter bound

### 6.1. Proof of Lemma 1.4

Note that if  $B(p, n)$  is sufficiently small then Lemma 5.1 implies  $\frac{A_T(R)}{\text{Vol} T} \geq \frac{R^n}{2R_T^n}$ . Hence we may assume that  $T = B(x, R_0)$  and  $\pi < R_0 \leq 2\pi$ . Fix  $\delta \in ]0, \frac{R_0 - \pi}{2}[$ . If  $y \in M$  is at a distance greater than  $(\pi + \delta)$  from  $x$ , then we have  $B(y, \delta) \subset B(x, \pi + 2\delta) \setminus B(x, \pi)$ . Lemma 4.1 now yields the bounds

$$\text{Vol} B(y, \delta) \leq \int_{\pi}^{\pi+2\delta} L \leq 2C(p, n)A(R_0)\delta\epsilon^{\frac{p(n-1)}{2p-1}}$$

(where  $A(R_0) = \text{Vol} B(x, R_0)$ ). On the other hand, Lemma 5.1 (ii) provides:

$$\text{Vol} B(y, \delta) \geq \left(\frac{\delta}{2\pi}\right)^n \left[ \frac{1}{2} \left(\frac{\delta}{2\pi}\right)^{\frac{2n}{2p'-1}} - C(p, n)\epsilon^{\frac{p'}{2p'-1}} \right]^{2p'-1} A(R_0)$$

by taking  $B(p, n)$  small enough (still setting  $p' = \max(p, n)$ ).

At this stage, we can distinguish two cases:

- either  $(\frac{\delta}{2\pi})^{\frac{2n}{2p'-1}} \leq 4C(p, n)\epsilon^\beta$ , where  $\beta = \frac{2np(n-1)}{(2p-1)(2p'-1)(3n-1)}$ ,
- or the above inequality becomes (since  $\beta \leq \frac{p'}{2p'-1}$ )

$$\text{Vol } B(y, \delta) \geq C(p, n) \left(\frac{\delta}{2\pi}\right)^n A(R_0)\epsilon^{(2p'-1)\beta}.$$

These two estimates on  $\text{Vol } B(y, \delta)$  imply a bound on  $\delta$ :

$$\pi + \delta \leq \pi + C(p, n)\epsilon^{\beta\frac{2p'-1}{2n}} \leq \pi + C(p, n)\epsilon^{\frac{1}{10}} < R_0.$$

We infer that  $M \subset B(x, R_0)$ . Let  $z$  be any point of  $M$ . We have  $\rho_{B(z, R_0)}^p = (\frac{R_0^{2p}}{\text{Vol } B(z, R_0)}) \times \int_{B(z, R_0)} (\text{Ric} - (n-1))_-^p \frac{1}{p} \leq (\frac{\text{Vol } B(x, R_0)}{\text{Vol } B(z, R_0)})^{\frac{1}{p}} \epsilon$ . But  $B(x, R_0 - \pi - C(p, n)\epsilon^{\frac{1}{10}}) \subset B(z, R_0)$  and so Lemma 5.1 (i) implies:

$$\frac{\text{Vol } B(z, R_0)}{\text{Vol } B(x, R_0)} \geq \frac{(R_0 - \pi - C(p, n)\epsilon^{\frac{1}{10}})^n}{2(2\pi)^n} \geq \frac{(R_0 - \pi)^n}{4(2\pi)^n}.$$

What is done above for  $x$  can be done for any  $z \in M$  (just replace  $\epsilon$  by  $\frac{4(2\pi)^{n/p}}{(R_0 - \pi)^{n/p}} \epsilon$ , for  $\rho_{B(z, R_0)}^p \leq \frac{4(2\pi)^{n/p}}{(R_0 - \pi)^{n/p}} \epsilon$ ), which completes the proof.

### 6.2. Proof of the geometric inequalities of Theorem 1.2

Let  $(M^n, g)$  be a complete manifold such that  $\int_M (\text{Ric} - (n-1))_-^p$  is finite and let  $(B(x_i, 2\pi))_{i \in I}$  be a maximal family of disjoint balls in  $M$ . The Dirichlet domains  $T_i = \{y \mid d(x_i, y) < d(x_j, y), \forall j \neq i\}$  satisfy the three following classical facts:

- 1)  $B(x_i, 4\pi) \supset T_i \supset B(x_i, 2\pi)$ ,
- 2)  $T_i$  is star-shaped at the  $x_i$  and
- 3) except for a set of zero measure,  $M$  is the disjoint union of the sets  $T_i$ .

Thus, setting  $\alpha = \inf_{i \in I} [\frac{1}{\text{Vol } T_i} \int_{T_i} (\text{Ric} - (n-1))_-^p]^{\frac{1}{p}}$ , we have that

$$\begin{aligned} \int_M (\text{Ric} - (n-1))_-^p &= \sum_{i \in I} \int_{T_i} (\text{Ric} - (n-1))_-^p \\ &\geq \alpha^p \sum_{i \in I} \text{Vol } T_i = \alpha^p \text{Vol } M. \end{aligned}$$

If  $\alpha > [\frac{B(p, n)}{2^{101} 16 \pi^2}]^p$  (where  $B(p, n)$  is the constant of Lemma 1.4), then  $\text{Vol } M \leq C(p, n)\rho^{(p)}(M)$  (where  $C(p, n)$  is a universal constant). Elsewhere, there exists a star-shaped set  $T_i$  satisfying the assumptions of Lemma 1.4. In the latter case (which is the only possible one under the stronger assumption  $\rho_M^{(p)} \leq \frac{\text{Vol } M}{C(p, n)}$ , with  $C(p, n)$  sufficiently large) we bound the diameter of  $M$  with Lemma 1.4 and the volume of  $M$  using Lemma 5.1.

### 7. Finiteness of the fundamental group

To show the  $\pi_1$ -finiteness of the manifolds satisfying  $\frac{\rho_p}{\text{Vol } M} \leq \frac{1}{C(p, n)}$ , we just have to show that their universal covers are compact. To apply Lemma 1.4 to the universal Riemannian cover

$(\widetilde{M}, \widetilde{g})$ , we have to construct a *good* star-shaped subset of  $\widetilde{M}$  on which the Ricci curvature is controlled by  $\frac{\rho_p}{\text{Vol} M}$ .

The fundamental group acts freely and isometrically on  $\widetilde{M}$ . For all  $\tilde{x} \in \widetilde{M}$  and any subset  $T$  of  $\widetilde{M}$ , we denote by  $m_T(\tilde{x})$  the cardinality of  $T \cap \pi_1.\tilde{x}$ . Set  $\tilde{x}_0 \in \widetilde{M}$  and  $\tilde{x} \in B(\tilde{x}_0, 2\pi)$  that maximizes  $m_{B(\tilde{x}_0, 2\pi)}$ . Since we can assume  $\text{Diam} M \leq 2\pi$ , we have  $1 \leq m_{B(\tilde{x}_0, 2\pi)}(y) \leq N$  and  $m_{B(\tilde{x}_0, 6\pi)}(y) \geq N$  for all  $y \in B(\tilde{x}_0, 2\pi)$  (where  $N = m_{B(\tilde{x}_0, 2\pi)}(\tilde{x})$ ). For all  $y$  in  $B(\tilde{x}_0, 2\pi)$ , we choose  $N$  distinct points  $y_1, \dots, y_N$  in  $\pi_1.y$  such that for any  $1 \leq i \leq N$  we have  $d(y_i, x_0) \leq \inf_{z \in \pi_1.y \setminus \{y_1, \dots, y_N\}} d(z, x_0)$ , and let  $T$  be the union of these  $\{y_1, \dots, y_N\}$  for all  $y \in B(\tilde{x}_0, 2\pi)$ . Hence  $B(\tilde{x}_0, 6\pi) \supset T \supset B(\tilde{x}_0, 2\pi)$  and  $m_T \equiv N$  on  $\widetilde{M}$ . We infer

$$\frac{1}{\text{Vol} T} \int_T (\text{Ric} - (n - 1))_-^p dv_{\tilde{g}} = \frac{1}{\text{Vol} M} \int_M (\text{Ric} - (n - 1))_-^p dv_g.$$

It only remains to show that  $T$  is a star-shaped subset of  $(\widetilde{M}, \widetilde{g})$ . Set  $y \in T$  and let  $\gamma$  be a minimizing geodesic from  $y$  to  $\tilde{x}_0$ . Assume there exists  $z \in \gamma \setminus T$ . Since  $m_T(z) = N$ , there exist  $(\sigma_1, \dots, \sigma_N)$  in  $\pi_1(M) \setminus \{id\}$  such that  $\sigma_i.z \in T$  for all  $1 \leq i \leq N$ . But every element of  $\pi_1(M) \setminus \{id\}$  acts without fixed point on  $\widetilde{M}$ , thus there exists  $1 \leq i_0 \leq N$  such that  $\sigma_{i_0}.y \notin T$ . Since  $\sigma_{i_0}$  acts isometrically, we have

$$d(\tilde{x}_0, y) \leq d(\tilde{x}_0, \sigma_{i_0}.y), \quad d(\tilde{x}_0, z) \geq d(\tilde{x}_0, \sigma_{i_0}.z),$$

$$d(z, y) = d(\sigma_{i_0}.z, \sigma_{i_0}.y).$$

The relations above combined with  $d(\tilde{x}_0, y) = d(\tilde{x}_0, z) + d(z, y)$  and the triangle inequality provide

$$d(\tilde{x}_0, y) = d(\tilde{x}_0, \sigma_{i_0}.y) = d(\tilde{x}_0, \sigma_{i_0}.z) + d(\sigma_{i_0}.z, \sigma_{i_0}.y).$$

We infer that there exists a minimizing geodesic segment from  $\sigma_{i_0}.y$  to  $\tilde{x}_0$  which contains  $\sigma_{i_0}.z$ . But  $d(\sigma_{i_0}.z, \sigma_{i_0}.y) = d(z, y) < d(\tilde{x}_0, y) \leq d(\tilde{x}_0, \sigma_{i_0}.y)$ , so there is only one geodesic minimizing the distance between  $\sigma_{i_0}.z$  and  $\sigma_{i_0}.y$ , which implies that the geodesic  $\sigma_{i_0}(\gamma)$  contains  $\tilde{x}_0$ . Since  $d(z, \tilde{x}_0) = d(\sigma_{i_0}.z, \tilde{x}_0)$ , we have  $\sigma_{i_0}.x_0 = x_0$ , contradicting the fact that  $\sigma_{i_0}$  has no fixed point.

### 8. Spectral lower bounds

To prove Proposition 1.5 we need bounds on some Sobolev constants. In [8], S. Gallot provides such bounds under the control  $\text{Diam}(M)^2 \left(\frac{1}{\text{Vol} M} \int_M (\text{Ric})_-^p\right)^{\frac{1}{p}} \leq \epsilon(p, n)$ , where  $p > n/2$  and  $\epsilon(p, n) > 0$  is a universal constant. Combined with Theorem 1.2 this yields

PROPOSITION 8.1. – *Let  $(M^n, g)$  be a complete Riemannian manifold. If  $\frac{\rho_p}{\text{Vol} M} \leq \frac{1}{C(p, q, n)}$  (for  $p > n/2$  and  $q > n$ ), then we have that*

- (i) for all  $u \in H^{1,2}(M)$ ,  $\|u\|_{\frac{2q}{q-2}} \leq \text{Diam}(M)C(p, q, n)\|du\|_2 + \|u\|_2$ .
- (ii) for all  $u \in H^{1,q}(M)$ ,  $\sup u - \inf u \leq \text{Diam}(M)C(p, q, n)\|du\|_q$ .

We now prove Proposition 1.5. Let  $\alpha$  be a 1-form on  $M$  such that  $\|\alpha\|_2^2 = 1$  and  $\Delta\alpha = \lambda\alpha$ . The Bochner formula (see [18]) yields

$$\int_M \frac{g(\Delta\alpha, \alpha)}{\text{Vol} M} = \|D\alpha\|_2^2 + \int_M \frac{(\text{Ric} - (n - 1))(\alpha, \alpha)}{\text{Vol} M} + (n - 1).$$

Combined with Hölder's inequality, this implies:

$$\lambda \geq \|D\alpha\|_2^2 - \left(\frac{\rho_p}{\text{Vol } M}\right)^{\frac{1}{p}} \|\alpha\|_{\frac{2p}{p-1}}^2 + (n-1).$$

Since we may assume  $\text{Diam } M \leq 2\pi$ , Proposition 8.1 gives that

$$\|\alpha\|_{\frac{2p}{p-1}}^2 \leq C(p, n) \|D\alpha\|_2^2 + 2\|\alpha\|_2^2.$$

We infer

$$(5) \quad (\lambda - (n-1) + 2\epsilon) \geq \left(1 - C(p, n) \left(\frac{\rho_p}{\text{Vol } M}\right)^{\frac{1}{p}}\right) \|D\alpha\|_2^2.$$

Splitting orthogonally the 2-tensor  $D\alpha$  into antisymmetric part  $\frac{d\alpha}{2}$ , traceless symmetric part and scalar part  $-\frac{\delta\alpha}{n}g$ , we obtain  $\|D\alpha\|_2^2 \geq \frac{1}{n}\|\delta\alpha\|_2^2 + \frac{1}{2}\|d\alpha\|_2^2$ . Combining the splitting with inequality (5), and distinguishing the cases  $d\alpha = 0$  (where  $\|\delta\alpha\|_2^2 = \lambda$ ) and  $\delta\alpha = 0$  (where  $\|d\alpha\|_2^2 = \lambda$ ), we easily get Proposition 1.5.

### 9. $L^{\frac{n}{2}}$ -pinching on the Ricci curvature

In the case  $n = 2$  and  $p = 1$ , the  $\pi_1$ -finiteness follows readily from the Gauss–Bonnet theorem. The proofs of Theorems 1.1 and 1.2, Lemma 1.4, and Propositions 1.5 and 1.6 may be easily adapted. For instance, to prove Lemma 1.4 we just use Lemma 5.2 in place of Lemma 5.1. To prove Proposition 1.5, we may assume  $\lambda \leq 2n$  and use the Sobolev inequality  $\|u\|_4 \leq C\|du\|_2 + \|u\|_2$  to show by Moser's iteration that  $\|\alpha\|_\infty \leq C'$ ; this implies that inequality (5) still holds and then we finish the proof as in the case  $p > 1$ .

We now focus on counter-examples announced in the introduction. Let  $\sigma$  (resp.  $\underline{\sigma}(x)$ ) stand for the sectional curvature (resp. the smallest sectional curvature of tangent planes at  $x$ ).

**PROPOSITION 9.1.** – *Set  $n \geq 3$ . For any  $p, \epsilon > 0$ , the  $n$ -Riemannian manifolds with  $\int_M |\sigma|^p \leq \epsilon$  and  $\text{Vol}(M) \leq \epsilon$  are dense in (pointed) Gromov–Hausdorff distance among all the (noncompact) length spaces.*

*Proof.* – The  $(n-1)$ -Riemannian manifolds are obviously GH-dense among all the finite graphs (perform some connected sums of spheres  $\mathbb{S}^{n-1}$  to get slightly thickened graphs). Then, just take Riemannian product of these manifolds with a sufficiently small  $\mathbb{S}^1$  factor.  $\square$

The next density results are more interesting since we want to keep a control on the volume of our family of manifolds.

**PROPOSITION 9.2.** – *For any reals  $K$  and  $V_0 > 0$ , any integer  $n \geq 3$  and real any  $\epsilon > 0$  the compact Riemannian  $n$ -manifolds  $(M^n, g)$  that satisfy*

$$\int_M (\underline{\sigma} - K)_-^{\frac{n}{2}} < \epsilon \quad \text{and} \quad \text{Vol } M = V_0$$

*are dense in (pointed) Gromov–Hausdorff distance among all the (noncompact) length spaces.*

*We can also replace  $\int_M (\underline{\sigma} - K)_-^{\frac{n}{2}}$  by  $\int_M |\underline{\sigma}|^{\frac{n}{2}}$  or by  $\int_M |\sigma|^p$  for any  $p < n/2$ .*

With the same kind of gluing techniques, it is not difficult to construct complete  $n$ -manifolds with infinite volume which satisfy  $\rho_{n/2} \leq \epsilon$  (for any  $n \geq 3$  and any  $\epsilon > 0$ ).



PROPOSITION 9.3. – Let  $(M^n, g)$  be any compact Riemannian  $n$ -manifold ( $n \geq 3$ ). There exists a sequence of complete Riemannian metrics  $(g_m)$  that converge to  $g$  in the Gromov-Hausdorff distance and such that

$$\frac{\rho_{n/2}(g_m)}{\text{Vol } g_m} \rightarrow 0, \quad \text{Vol}(g_m) \rightarrow \infty, \quad \forall l \in \mathbb{N}, \quad \lambda_l(g_m) \rightarrow 0$$

where  $\lambda_l$  denotes the  $l$ -th eigenvalue of the Laplacian on functions.

*Proof.* – We define five families of warped-product metrics  $dt^2 + b(t)^2 g_{\mathbb{S}^{n-1}}$  on  $I \times \mathbb{S}^{n-1}$

- $\mathbf{C}_\nu^{-1} = [0, \sqrt{\nu}] \times \mathbb{S}^{n-1}$  with  $b(t) = \eta(t^2 + \nu^2)^{\alpha/2}$ , where  $\alpha = 1 + \frac{1}{\sqrt{-\text{Log}(\nu)}}$  and  $\eta = \frac{\sqrt{1+\nu}}{\alpha(\nu+\nu^2)^{\frac{\alpha-1}{2}}}$  for all  $\nu > 0$ .
- $\mathbf{F}_\nu = [\theta - \frac{\pi}{2}, 0] \times \mathbb{S}^{n-1}$  with  $b(t) = \eta' \cos t$ ,  $\theta = \tan^{-1}(\frac{\sqrt{\nu}}{\alpha}(1 + \nu))$  and  $\eta' = \frac{\sqrt{\alpha^2 + \nu(1+\nu)^2}}{\alpha} = \frac{1}{\cos \theta}$ .
- $\overline{\mathbf{F}}_\nu = [0, \frac{\eta'\pi}{2}] \times \mathbb{S}^{n-1}$  with  $b(t) = \eta' \cos \frac{t}{\eta'}$ .
- $\mathbf{C}_\nu^0 = [0, \frac{\sqrt{\nu}(1+\nu)}{2\alpha}] \times \mathbb{S}^{n-1}$  with  $b(t) = t + \frac{\sqrt{\nu}(1+\nu)}{2\alpha}$ .
- $\overline{\mathbf{C}}_{\nu,L}^0 = [0, L] \times \mathbb{S}^{n-1}$  with  $b(t) = \frac{\nu^{\frac{\alpha+1}{2}}}{\alpha(1+\nu)^{\frac{\alpha-1}{2}}}$ .

If  $(X, Y)$  is an orthonormal family of tangent vectors to  $\mathbb{S}^{n-1}$ , then the sectional curvatures  $\sigma(X, Y)$  of the manifolds  $\mathbf{F}_\nu, \overline{\mathbf{F}}_\nu, \mathbf{C}_\nu^{-1}$  and  $\mathbf{C}_\nu^0$  are equal to

$$\frac{1}{b^2} - \left(\frac{b'}{b}\right)^2 = \begin{cases} 0 & \text{on } \mathbf{C}_\nu^0 \text{ or } \overline{\mathbf{C}}_{\nu,L}^0, \\ \frac{\nu^2 \alpha^2}{(t^2 + \nu^2)^2} - \frac{\alpha^2}{t^2 + \nu^2} \left(1 - \frac{1}{1 + \nu} \left(\frac{\nu + \nu^2}{t^2 + \nu^2}\right)^{\alpha-1}\right) & \text{on } \mathbf{C}_\nu^{-1}, \\ 1 - \frac{\sin^2 \theta}{\cos^2 t} & \text{on } \mathbf{F}_\nu, \\ \frac{1}{\eta'^2} & \text{on } \overline{\mathbf{F}}_\nu. \end{cases}$$

If  $X$  is a unit vector tangent to  $\mathbb{S}^{n-1}$ , then

$$\sigma\left(X, \frac{\partial}{\partial r}\right) = -\frac{b''}{b} = \begin{cases} 0 & \text{on } \mathbf{C}_\nu^0 \text{ or } \overline{\mathbf{C}}_{\nu,L}^0, \\ -\frac{\alpha(2-\alpha)\nu^2}{(t^2 + \nu^2)^2} - \frac{\alpha(\alpha-1)}{t^2 + \nu^2} & \text{on } \mathbf{C}_\nu^{-1}, \\ 1 & \text{on } \mathbf{F}_\nu, \\ \frac{1}{\eta'^2} & \text{on } \overline{\mathbf{F}}_\nu. \end{cases}$$

We now obtain readily the following upper bounds ( $\forall \nu \leq \frac{1}{C(n)}$ )

$$\int_{\mathbf{F}_\nu} (\underline{\sigma} - 1)^{\frac{n}{2}} \leq C(n) \int_0^{\frac{\pi}{2} - \theta} \frac{\sin^n \theta}{\cos t} dt \leq C(n) \sin^n \theta \leq \frac{C(n)}{(-\ln \nu)^{\frac{n-2}{4}}},$$

$$\begin{aligned} \int_{\overline{\mathbf{F}}_\nu} (\underline{\sigma} - 1)_-^{\frac{n}{2}} &\leq C(n) \int_0^{\frac{\eta'\pi}{2}} \frac{\sin^n \theta}{\cos^n \theta} \cos^{n-1} \frac{t}{\eta'} dt \\ &\leq C(n) \sin^n \theta \leq \frac{C(n)}{(-\ln \nu)^{\frac{n-2}{4}}}, \\ \int_{\mathbf{C}_\nu^0} (\underline{\sigma} - 1)_-^{\frac{n}{2}} &\leq C(n) \nu^{\frac{n}{2}} \leq \frac{C(n)}{(-\ln \nu)^{\frac{n-2}{4}}}. \end{aligned}$$

Concerning  $\mathbf{C}_\nu^{-1}$ , first note that  $\sigma(X, Y)$  is decreasing on  $[0, \sqrt{\nu}]$  and so  $\sigma(X, Y) \geq 0$  for  $\nu$  small enough. Hence, using  $\sqrt{\frac{a}{2}} + \sqrt{\frac{b}{2}} \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ , we have

$$\begin{aligned} \int_{\mathbf{C}_\nu^{-1}} (\underline{\sigma} - 1)_-^{\frac{n}{2}} &\leq C(n) \eta^{n-1} \left[ \nu^n \int_0^{\sqrt{\nu}} (t^2 + \nu^2)^{\frac{\alpha(n-1)}{2} - n} dt + \int_0^{\sqrt{\nu}} (t^2 + \nu^2)^{\frac{\alpha(n-1)}{2}} dt \right. \\ &\quad \left. + (\alpha - 1)^{n/2} \int_0^{\sqrt{\nu}} (t^2 + \nu^2)^{\frac{\alpha(n-1)}{2} - \frac{n}{2}} dt \right] \\ &\leq C(n) \eta^{n-1} [\nu^{(\alpha-1)(n-1)} + (\nu + \sqrt{\nu})^{(n-1)\alpha+1} \\ &\quad + (\alpha - 1)^{\frac{n}{2}-1} (\nu + \sqrt{\nu})^{(\alpha-1)(n-1)}] \leq \frac{C(n)}{(-\ln \nu)^{\frac{n-2}{4}}}. \end{aligned}$$

The metrics of these cylinders are normalized to yield a  $C^1$  metric when the small (resp. the large) connected component of the boundary of  $\mathbf{F}_\nu$  is identified with the large connected component of the boundary of  $\mathbf{C}_\nu^0$  (resp. with the boundary of  $\overline{\mathbf{F}}_\nu$ ). Similarly, note that for any  $\nu > 0$  small enough, there exists  $\beta < 1$  such that we get a  $C^1$  metric by identifying a connected component of the boundary of  $\mathbf{C}_{\beta\nu}^{-1}$  with the small connected component of  $\mathbf{C}_\nu^0$ . Let  $\overline{\mathbf{B}}_\nu$  be the manifold  $\mathbf{C}_\nu^0 \# \mathbf{C}_{\beta\nu}^{-1} \# \mathbf{C}_{\beta\nu}^{-1} \# \mathbf{C}_\nu^0 \# \mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$  (Fig. 1)

We then have  $\int_{\overline{\mathbf{B}}_\nu} (\underline{\sigma} - 1)_-^{\frac{n}{2}} \leq \frac{C(n)}{(-\ln \nu)^{\frac{n-2}{4}}}$ , and  $\text{Diam } \overline{\mathbf{B}}_\nu \leq 2\pi$  and  $\text{Vol } \overline{\mathbf{B}}_\nu \geq \frac{1}{C(n)}$  for any  $\nu$  small enough. For all  $N \in \mathbb{N}$ , there exists a  $\nu' > 0$  small enough to have  $\mathbf{C}_\nu^0$  containing at least  $N$  disjoint balls of radius  $\frac{\sqrt{\nu'(1+\nu')}}{\alpha(\nu')}$ . Excise these balls from one of the  $\mathbf{C}_\nu^0$  part of  $\overline{\mathbf{B}}_\nu$  and glue the resulting manifold to  $N$  manifolds  $\overline{\mathbf{B}}_{\nu'}$  along the spheres of radius  $\frac{\sqrt{\nu'(1+\nu')}}{\alpha(\nu')}$  of their boundaries. Taking  $N = (-\ln \nu)^{\frac{n-2}{8}}$  and multiplying the metric by  $\frac{1}{(-\ln \nu)^{\frac{n-2}{8n}}}$ , we get a manifold  $B_\nu$

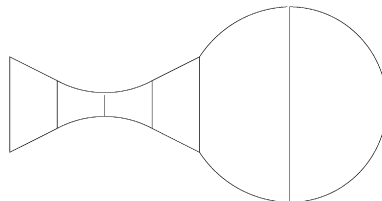


Fig. 1.

which is diffeomorphic to  $B^n$  and satisfies  $\text{Diam } B_\nu \leq \frac{4\pi}{(-\ln \nu)^{\frac{n-2}{16}}}$ ,  $\text{Vol } B_\nu \geq \frac{(-\ln \nu)^{\frac{n-2}{16}}}{C(n)}$  and

$$\int_{B_\nu} (\underline{\sigma} - 1)^{\frac{n}{2}} \leq \frac{C(n)}{(-\ln \nu)^{\frac{n-2}{8}}}.$$

To prove Proposition 9.3, fix a point  $x_0$  in the compact manifold  $M$ . For any  $m \in \mathbb{N}$ , there exists  $r \in ]0, \text{inj}(M, g)[$  and a metric  $g'$  on  $M$  which is equal to  $g$  on  $M \setminus B(x_0, 2r)$ , is flat on  $B(x_0, r)$  and is at Gromov–Hausdorff distance from  $g$  bounded from above by  $\frac{1}{2m}$ . For any  $\nu > 0$  such that  $\frac{\sqrt{\nu}(1+\nu)}{\alpha} < r$  we obtain a new metric  $g'_\nu$  on  $M$  by replacing the flat metric on  $B(x_0, \frac{\sqrt{\nu}(1+\nu)}{\alpha})$  by the metric of  $B_\nu$ . We can find  $\nu_m$  small enough to have a Gromov–Hausdorff distance between  $g$  and  $g'_{\nu_m}$  less than  $\frac{1}{m}$ , and also  $\text{Vol}(g'_{\nu_m}) \geq mC(n)$  and  $\frac{1}{\text{Vol } g'_{\nu_m}} \int_{(M, g'_{\nu_m})} (\underline{\sigma} - 1)^{\frac{n}{2}} \leq \frac{1}{m}$ . We then set  $g_m = g'_{\nu_m}$ . It only remains to show the collapsing of the eigenvalues of the metrics  $g_m$ . In that purpose, first consider on  $\overline{B}_\nu$  the continuous function  $f$  that is equal to 1 on the part  $\mathbf{C}_\nu^0 \# \mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$ , equal to 0 on the part  $\mathbf{C}_\nu^0 \# \mathbf{C}_\nu^{-1}$  and equal to  $f(t) = \frac{t}{\sqrt{\beta\nu}}$  on the remaining part  $\mathbf{C}_{\beta\nu}^{-1}$ . For this function  $f$ , we have that

$$\frac{\int_{\overline{B}_\nu} |\nabla f|^2}{\int_{\overline{B}_\nu} |f|^2} \leq \frac{\eta^{n-1}}{C(n)\eta^{n-1}} \int_0^{\sqrt{\nu}} \left| \frac{\partial f}{\partial t} \right|^2 (t^2 + \nu^2)^{\frac{\alpha(n-1)}{2}} dt \leq C(n)\nu^{\frac{n-2}{2}}.$$

Now,  $(M^n, g_m)$  contains  $(-\ln \nu_m)^{\frac{n-2}{8}}$  manifolds  $\overline{B}_{\nu'_m}$  (with metric multiplied by  $\frac{1}{(-\ln \nu_m)^{\frac{n-2}{8n}}}$ ) and so, if we extend by zero to  $M$  the function  $f$  corresponding to each of these  $\overline{B}_{\nu'_m}$ , we obtain  $(-\ln \nu_m)^{\frac{n-2}{8}}$   $L^2$ -orthogonal functions on  $(M^n, g_m)$ , whose Rayleigh quotients are bounded from above by  $C(n)\nu_m^{\frac{n-2}{2}} (\ln \frac{1}{\nu_m})^{\frac{n-2}{8n}}$ . As we can suppose that  $\nu_m$  tends to 0, the min-max principle implies the collapsing of all eigenvalues to 0 (this collapsing implies that the  $g_m$  do not tend to  $g$  in the  $C^0$  sense and that the Sobolev constants are not bounded under  $L^{\frac{n}{2}}$  control, otherwise the proof of Proposition 1.5 would hold).

We now adapt the above construction to prove Proposition 9.2. Note that on  $\mathbf{C}_\nu^{-1}$  we have  $-\frac{\alpha(2-\alpha)\nu^2}{(t^2+\nu^2)^2} - \frac{\alpha(\alpha-1)}{t^2+\nu^2} \leq \sigma(t) \leq \frac{\nu^2\alpha^2}{(t^2+\nu^2)^2} + \frac{\alpha^2\nu^{\alpha-1}}{(1+\nu)^{2-\alpha}(t^2+\nu^2)^\alpha}$ , and so we have, for any  $p < n/2$ ,  $\int_{\mathbf{C}_\nu^{-1}} |\sigma|^p \leq C(n, p)\nu^{\frac{n}{2}-p}$ . There exists  $\beta < 1$  such that a connected component of the boundary of  $\mathbf{C}_{\beta\nu}^{-1}$  glue metrically in a  $C^1$ -way with the small connected component of  $\mathbf{C}_\nu^0$ . Let  $\mathbf{B}_{\nu, L}^2$  be the manifold  $\overline{\mathbf{F}}_\nu \# \mathbf{F}_\nu \# \mathbf{C}_\nu^0 \# \mathbf{C}_{\beta\nu}^{-1} \# \overline{\mathbf{C}}_{\beta\nu, L}^0 \# \mathbf{C}_{\beta\nu}^{-1} \# \mathbf{C}_\nu^0 \# \mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$  (Fig. 2), we set also  $\mathbf{B}_{\nu, L}^1 = \mathbf{C}_\nu^0 \# \mathbf{C}_{\beta\nu}^{-1} \# \overline{\mathbf{C}}_{\beta\nu, L}^0 \# \mathbf{C}_{\beta\nu}^{-1} \# \mathbf{C}_\nu^0 \# \mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$  (Fig. 3) and  $\mathbf{B}_{\nu, L}^0 = \mathbf{C}_\nu^0 \# \mathbf{C}_{\beta\nu}^{-1} \# \overline{\mathbf{C}}_{\beta\nu, L}^0 \# \mathbf{C}_{\beta\nu}^{-1} \# \mathbf{C}_\nu^0$  (Fig. 4).

It is now easy to see that for any  $L > 0$ ,  $\epsilon > 0$  and  $K \in \mathbb{R}$  we can choose two sequences  $(L_l)$  and  $(\lambda_l)$  such that the sequence  $\overline{B}_{l, \epsilon}^{0, L} = (\lambda_l B_{1/l, L_l}^0)$  (resp.  $\overline{B}_{n, \epsilon}^{1, L} = (\lambda_l B_{1/l, L_l}^1)$  or  $\overline{B}_{l, \epsilon}^{2, L} = (\lambda_l B_{1/l, L_l}^2)$ ) is at Gromov–Hausdorff distance from the segment  $[0, L]$  less than  $\epsilon$  and the

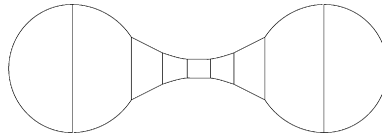


Fig. 2.

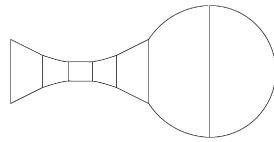


Fig. 3.

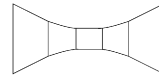


Fig. 4.

integrals  $\int_{\overline{B}_{l,\epsilon}^{i,L}} (\underline{\sigma} - K)_-^{\frac{n}{2}}$  tend to 0 (resp. and the volume of  $\overline{B}_{l,\epsilon}^{i,L}$  tends to any given real in  $]0, C(\epsilon, K, L)[$ ). Note also that, if we take  $m$  large enough, we can glue a number as large as needed of manifolds  $\overline{B}_{m,\epsilon}^{1,L}$  or  $\overline{B}_{m,\epsilon}^{0,L}$  to one of the  $C^0$  part of  $\overline{B}_{l,\epsilon}^{i,L}$ . We deduce that, for any finite graph, we can glue a family  $\overline{B}_{l_k,\epsilon^2}^{i_k,L_k}$  (with the  $n_p$  large enough) to get a manifold which is at Gromov–Hausdorff distance from the graph less than  $\epsilon/2$  and which satisfies  $\int (\underline{\sigma} - K)_-^{\frac{n}{2}} \leq \epsilon/2$  and with volume less than  $V_0/2$ . To get a volume equal to  $V_0$  we glue enough copies of  $\overline{B}_{l,\epsilon^4}^{1,\epsilon^2}$  (for  $K = \frac{1}{\epsilon^8}$ ): the small change on the distance to the graph does not depend on the number of these copies and we can choose the volume of each copy of these  $\overline{B}_{l,\epsilon^4}^{1,\epsilon^2}$  equal to any number in  $]0, C(\epsilon^2, \frac{1}{\epsilon^8}, L)[$ . Since the finite graphs are dense in Gromov–Hausdorff distance, this ends the proof of Theorem 9.2.

To prove the version of Theorem 9.2 with the control on  $\int_M |\underline{\sigma}|^{\frac{n}{2}}$  or  $\int_M |\sigma|^p$  ( $p < n/2$ ) we just have to replace the parts  $\mathbf{F}_\nu \# \overline{\mathbf{F}}_\nu$  in the above definition of the  $\overline{B}_{l,\epsilon}^{i,L}$  by some small flat  $n$ -torus and remark that the metrics constructed this way satisfy  $\underline{\sigma} \leq 0$ .  $\square$

Note that in the proof of Proposition 9.3, we only need that  $\text{Vol } M$  and  $\int_M (\underline{\sigma} - 1)_-^{\frac{n}{2}}$  are finite. It is classical that any manifold supports a complete metric with finite volume but we do not know if both finitenesses above are always fulfilled for at least one complete metric on any (noncompact) manifold. Note also that the finiteness of  $\int_M (\underline{\sigma} - 1)_-^{\frac{n}{2}}$  does not imply  $\text{Vol } M < \infty$  since, for any  $\epsilon > 0$ , we can start from  $\mathbf{B}_{\nu,1}^2$  and then iteratively glue some  $\mathbf{B}_{\nu_k,1}^1$  to the remaining free  $\overline{\mathbf{C}}_{\beta\nu_{k-1}}^0$  element with a sequence  $\nu_k$  chosen so as to get a complete manifold with infinite volume and  $\int_M (\underline{\sigma} - 1)_-^{\frac{n}{2}} \leq \epsilon$ .

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