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FINITENESS THEOREMS FOR VANISHING CYCLES OF FORMAL SCHEMES

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ABSTRACT

Let k be a non-Archimedean field with nontrivial valuation, and k° its ring of integers. In this paper we prove constructibility of vanishing cycles sheaves for arbitrary formal schemes locally finitely presented over k° as well as special formal schemes over k° (for discretely valued k). This allows us to extend continuity results, established earlier for locally algebraic formal schemes, to the whole classes of formal schemes.

Introduction

Let k be a non-Archimedean field with nontrivial valuation, k° its ring of integers, and $\widetilde{k}$ its residue field. A formal scheme $\mathfrak{X}$ over k° is said to be locally finitely presented if it is a locally finite union of open affine subschemes of the form $\mathrm{Spf}(A)$ with A isomorphic to a quotient of $k^\circ\{T_1, \ldots, T_m\}$ by a finitely generated ideal. If the valuation on k is discrete, a formal scheme $\mathfrak{X}$ over k° is said to be special if it is a locally finite union of open affine subschemes $\mathrm{Spf}(A)$ with A isomorphic to a quotient of $k^\circ\{T_1, \ldots, T_m\}[[S_1, \ldots, S_n]]$. In both cases, the generic fiber $\mathfrak{X}_\eta$ of $\mathfrak{X}$ is a paracompact strictly k-analytic space, and the closed fiber $\mathfrak{X}_s$ of $\mathfrak{X}$ is a scheme of locally finite type over $\widetilde{k}$. The class of locally

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finitely presented formal schemes $\mathfrak{X}$ is preserved under the formal completion $\mathfrak{X}_{/\mathcal{Y}}$ along an open subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, and the class of special formal schemes is preserved under the formal completion along an arbitrary subscheme of $\mathfrak{X}_s$.

In [Ber3] and [Ber6], we defined for both classes of formal schemes, respectively, a vanishing cycles functor Ψ_η from the category of étale sheaves on $\mathfrak{X}_\eta$ to the category of étale sheaves on $\mathfrak{X}_{\overline{s}} = \mathfrak{X}_s \otimes_{\widetilde{k}} \widetilde{k}^{\mathrm{s}}$ provided with an action of the Galois group of k. The comparison theorem [Ber3, 5.3] (resp. [Ber6, 3.1]) implies that if $\mathfrak{X}$ is the formal completion $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ of a scheme $\mathcal{X}$ of finite type over k° along an open (resp. arbitrary) subscheme $\mathcal{Y} \subset \mathcal{X}_s$, then for any constructible sheaf $\mathcal{F}$ on $\mathcal{X}_\eta$ with torsion orders prime to $\mathrm{char}(\widetilde{k})$ there is a canonical isomorphism $R\Psi_\eta(\mathcal{F})\big|_{\overline{\mathcal{Y}}} \widetilde{\to} R\Psi_\eta(\widehat{\mathcal{F}}_{/\mathcal{Y}})$, where $\widehat{\mathcal{F}}_{/\mathcal{Y}}$ is the pullback of $\mathcal{F}$ on $(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta$. In particular, by Deligne's theorem on constructibility of the vanishing cycles sheaves of schemes [SGA4$\frac{1}{2}$, Th. finitude, 3.2], the sheaves $R^q\Psi_\eta(\widehat{\mathcal{F}}_{/\mathcal{Y}})$ are constructible. It follows also that the étale cohomology groups of a quasi-algebraic compact k-analytic space with coefficients in a locally constant sheaf of order prime to $\mathrm{char}(\widetilde{k})$ are finite.

The purpose of this paper is to extend the latter facts to the whole classes of compact k-analytic spaces (Theorem 1.1.1) and of locally finitely presented and of special formal schemes (Theorems 1.1.2 and 3.1.1). We notice that, if the characteristic of k is zero, finiteness of the cohomology groups and constructibility of the vanishing cycles sheaves for locally finitely presented formal schemes and for formal completion of them along a subscheme of the closed fiber were proved by Roland Huber in [Hub1] and [Hub2].

The main ingredients of our proof of the above results are Gabber's weak uniformization theorem [ILO, Exp. VII, 1.1] and its version, Theorem 2.1.3. These theorems together with Deligne's cohomological descent theory [SGA4, Exp. Vbis] allow one to deduce the general case of the required facts from the particular cases considered in [Ber3] and [Ber6].

In §1, we introduce for a k-analytic space X a class of étale sheaves, called constructible, which includes finite locally constant sheaves and the pullbacks of constructible sheaves on a scheme $\mathcal{X}$ of finite type over k for every morphism $X \to \mathcal{X}^{\mathrm{an}}$. Theorem 1.1.1 states that, if k is algebraically closed and X is compact, then for any abelian constructible sheaf F on X with torsion orders prime to $\mathrm{char}(\widetilde{k})$, the groups $H^q(X, F)$ are finite. The proof is an application of the cohomological descent theory, which is recalled in §1.2, and a uniformization property for strictly k-analytic spaces, Theorem 1.3.1, which is a direct consequence of Gabber's weak uniformization theorem. Theorem 1.1.2, which states that for any formal scheme $\mathfrak{X}$ locally finitely presented over k° and any abelian constructible sheaf F on $\mathfrak{X}_\eta$ with torsion orders prime to $\mathrm{char}(\widetilde{k})$ the sheaves $R^q\Psi_\eta(F)$ are constructible, is deduced from Theorem 1.1.1. Theorem 1.1.2 allows one to extend the continuity results [Ber3, 8.1. and 8.6] to arbitrary formal schemes finitely presented over k° (Corollaries 1.1.3 and 1.1.4).

In §2, we prove a version of Gabber's theorem. This version, Theorem 2.1.3, is about quasi-excellent noetherian schemes over a complete discrete valuation ring, and it is related to Gabber's theorem in a similar way as de Jong's semi-stable reduction theorem [deJ1, 6.5] is related to his theorem [deJ1, 4.1] for varieties over a field. The proof closely follows the proof of Gabber's result.

In §3, we introduce for a special formal scheme $\mathfrak{X}$ over k° a class of étale sheaves on $\mathfrak{X}_\eta$, called $\mathfrak{X}$-constructible, which is more restrictive than the class of constructible sheaves. Theorem 3.1.1 states that, for any $\mathfrak{X}$-constructible sheaf F with torsion orders prime to $\mathrm{char}(\widetilde{k})$, the sheaves $R^q\Psi_\eta(F)$ are constructible, and Theorem 3.1.5 states that the formation of those sheaves is compatible with extensions of the ground field. We also introduce a modified version of the vanishing cycles functor which is better than that from [Ber6] in the sense that it takes values in the category of étale sheaves provided with a continuous action of the Galois group of k. Theorem 3.1.6 states that the values of the high direct images of both vanishing cycles functors on $\mathfrak{X}$-constructible sheaves coincide. The proof of Theorems 3.1.1, 3.1.5 and 3.1.6 is based on a uniformization property for special formal schemes, Theorem 3.2.1, which is a consequence of the above version of Gabber's result. Theorems 1.1.1 and 3.1.1 allow one to extend the continuity results [Ber6, 4.1 and 4.5] to arbitrary quasicompact special formal schemes over k° (Corollaries 3.1.3 and 3.1.4).

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1. Finiteness theorem for compact analytic spaces

1.1. Formulation of results. Let k be a non-Archimedean field. We say that an étale sheaf F on a k-analytic space X is **constructible** if every point of X has a neighborhood of the form $V_1 \cup \cdots \cup V_n$ such that each V_i is an affinoid domain that contains the point x and the restriction of F to $V_i = \mathcal{M}(\mathcal{A}_{V_i})$ is the pullback of a constructible sheaf on the affine scheme $\mathrm{Spec}(\mathcal{A}_{V_i})$. For example, any finite locally constant sheaf on X is constructible. The category of abelian constructible sheaves on X is abelian, and the inverse image $F\big|_Y = \varphi^*(F)$ of a constructible sheaf F on X with respect to a morphism $\varphi : Y \to X$ is a constructible sheaf on Y.

Theorem 1.1.1: *Suppose that k is algebraically closed, and let X be a compact k-analytic space. Then for any abelian constructible sheaf F on X with torsion orders prime to* $\mathrm{char}(\widetilde{k})$ *the groups $H^q(X, F)$, $q \geq 0$, are finite.*

We now recall the definition (from [Ber3]) of the nearby cycles and vanishing cycles functors Θ and Ψ_η for a formal scheme $\mathfrak{X}$ locally finitely presented over k°. By [Ber3, Lemma 2.1], the correspondence $\mathfrak{Y} \mapsto \mathfrak{Y}_s$ induces an equivalence between the category of formal schemes étale over $\mathfrak{X}$ and the category of schemes étale over $\mathfrak{X}_s$. We fix an inverse functor $\mathfrak{Y}_s \mapsto \mathfrak{Y}$. Then for an étale sheaf F on $\mathfrak{X}_\eta$ and a scheme $\mathfrak{Y}_s$ étale over $\mathfrak{X}_s$, one has $\Theta(F)(\mathfrak{Y}_s) = F(\mathfrak{Y}_\eta)$, where the right-hand side is the global section set of the pullback of F to $\mathfrak{Y}_\eta$. Furthermore, let $\mathfrak{X}_{\overline{s}}$ (resp. $\mathfrak{X}_{\overline{\eta}}$) denote the closed (resp. generic) fiber of the formal scheme $\overline{\mathfrak{X}} = \mathfrak{X}\widehat{\otimes}_{k^\circ}(\widehat{k^{\mathrm{s}}})^\circ$ over $(\widehat{k^{\mathrm{s}}})^\circ$. One has $\mathfrak{X}_{\overline{s}} = \mathfrak{X}_s \otimes_{\widetilde{k}} \widetilde{k^{\mathrm{s}}}$ and $\mathfrak{X}_{\overline{\eta}} = \mathfrak{X}_\eta \widehat{\otimes}_k \widehat{k^{\mathrm{s}}}$. Then $\Psi_\eta(F) = \Theta_{\widehat{k^{\mathrm{s}}}}(\overline{F})$, where $\overline{F}$ is the pullback of F to $\mathfrak{X}_{\overline{\eta}}$. There is a canonical continuous action of the Galois group $G = \mathrm{Gal}(k^{\mathrm{s}}/k)$ on $\Psi_\eta(F)$ compatible with the action of G on $\mathfrak{X}_{\overline{s}}$ and, if F is abelian, it induces a continuous action of G on the sheaves $R^q\Psi_\eta(F)$.

For a prime integer l, we set $s_l(k) = \dim_{\mathbf{F}_l}(|k^*|/|k^*|^l)$.

Theorem 1.1.2: *Let $\mathfrak{X}$ be a formal scheme locally finitely presented over k°, and let F be an abelian constructible sheaf on $\mathfrak{X}_\eta$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *Suppose that $s_l(k) < \infty$ for every prime l that divides a torsion order of F. Then the nearby cycles sheaves $R^q\Theta(F)$ are constructible for all $q \geq 0$.*

Foe example, the assumption is satisfied if the group $|k^*|$ is finitely generated, or if $|k^*|$ is divisible (see also Remarks 1.1.5). This implies constructibility of the vanishing cycles sheaves $R^q\Psi_\eta(F)$.

The following two statements were proved in [Ber3, 8.1 and 8.6] under the assumption that all of the formal schemes considered are formal completions of schemes of finite type over k° along their closed fiber. This assumption was necessary because constructibility of the nearby cycles sheaves was available only for those formal schemes. The same proof works in general due to Theorem 1.1.2.

Let $\mathfrak{T}$ be a fixed formal scheme finitely presented over k°, and let F be an étale abelian sheaf on $\mathfrak{T}_\eta$. Given formal schemes $\mathfrak{X}$ and $\mathfrak{Y}$ finitely presented over $\mathfrak{T}$, any $\mathfrak{T}$-morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ gives rise to homomorphisms of sheaves on $\mathfrak{Y}_s$ and $\mathfrak{Y}_{\overline{s}}$, respectively,

$$\theta^q(\varphi, F) : \varphi_s^*(R^q\Theta(F|_{\mathfrak{X}_\eta})) \to R^q\Theta(F|_{\mathfrak{Y}_\eta}),$$
$$\theta^q_\eta(\varphi, F) : \varphi_{\overline{s}}^*(R^q\Psi_\eta(F|_{\mathfrak{X}_\eta})) \to R^q\Psi_\eta(F|_{\mathfrak{Y}_\eta}).$$

Corollary 1.1.3: *Let F be an abelian constructible sheaf on $\mathfrak{T}_\eta$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$, *and suppose that $s_l(k) < \infty$ for every prime l that divides a torsion order of F. Given a formal scheme $\mathfrak{X}$ finitely presented over $\mathfrak{T}$, there exists a nonzero ideal $\mathbf{a} \subset k^\circ$ such that, for any formal scheme $\mathfrak{Y}$ finitely presented over $\mathfrak{T}$ and any pair of $\mathfrak{T}$-morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ that coincide modulo $\mathbf{a}$, one has $\theta^q(\varphi, F) = \theta^q(\psi, F)$ for all $q \geq 0$.* ∎

For a prime l, a **constructible $\mathbf{Z}_l$-sheaf** on a k-analytic space X is a projective system

$$F = (F_m)_{m\geq 0}$$

of constructible $\mathbf{Z}/l^{m+1}\mathbf{Z}$-modules F_m such that, for each $m \geq 1$, the canonical homomorphism $F_m \to F_{m-1}$ induces an isomorphism $F_m \otimes_{\mathbf{Z}/l^{m+1}\mathbf{Z}} \mathbf{Z}/l^m\mathbf{Z} \widetilde{\to} F_{m-1}$. Furthermore, for a formal scheme $\mathfrak{X}$ over $\mathfrak{T}$ and a nonzero ideal $\mathbf{a} \subset k^\circ$, we denote by $\mathcal{G}(\mathfrak{X}/\mathfrak{T})$ the group of $\mathfrak{T}$-automorphisms of $\mathfrak{X}$ and by $\mathcal{G}_{\mathbf{a}}(\mathfrak{X}/\mathfrak{T})$ its subgroup of the automorphisms trivial modulo $\mathbf{a}$.

Corollary 1.1.4: *Let F be a constructible $\mathbf{Z}_l$-sheaf on $\mathfrak{T}_\eta$, $l \neq$* $\mathrm{char}(\widetilde{k})$, *and suppose that $s_l(k) < \infty$. Given a formal scheme $\mathfrak{X}$ finitely presented over $\mathfrak{T}$, there exists a nonzero ideal $\mathbf{a} \subset k^\circ$ such that the group $\mathcal{G}_{\mathbf{a}}(\mathfrak{X}/\mathfrak{T})$ acts trivially on all of the sheaves $R^q\Theta(F_m|_{\mathfrak{X}_\eta})$.* ∎

Remarks 1.1.5: (i) The assumption $s_l(k) < \infty$ in Theorem 1.1.2 is necessary since, for $\mathfrak{X} = \mathrm{Spf}(k^\circ)$ with algebraically closed residue field $\widetilde{k}$, the sheaf

$$R^1\Theta(\mathbf{Z}/l\mathbf{Z})$$

is associated to the group $H^1(G, \mathbf{Z}/l\mathbf{Z})$, which is isomorphic to $|k^*|/|k^*|^l$.

(ii) If $s_l(k) < \infty$, then $s_l(\mathcal{H}(x)) < \infty$ for any point x of a k-analytic space X. Indeed, in order to show this, we may assume that $X = \mathcal{M}(\mathcal{A})$ is k-affinoid. Furthermore, taking a closed immersion of X in a closed polydisc of dimension n, we may assume that X is the polydisc. The latter is an affinoid domain in the affine space $\mathbf{A}^n$, and so we may assume that $X = \mathbf{A}^n$. Finally, induction reduces the situation to the case $n = 1$. If x is a point of type (1) or (4), then the quotient group $|\mathcal{H}(x)^*|/|k^*|$ is torsion, and it is easy to see that $s_l(\mathcal{H}(x)) \le s_l(k)$. If x is of type (2) or (3), then $s_l(\mathcal{H}(x)) = s_l(k)$ or $s_l(\mathcal{H}(x)) = s_l(k) + 1$, respectively.

1.2. Cohomological descent on non-Archimedean analytic spaces. Let k be a non-Archimedean field. In this subsection we recall and apply some definitions and facts from the cohomological descent theory ([SGA4, Exp. Vbis] and [Del]) to k-analytic spaces. This theory introduces for a given Grothendieck topology a stronger topology that does not change cohomology groups but allows one to calculate them using hypercoverings [SGA4, Exp. V, §7] by objects from a special class.

Let $\mathcal{E}$ be a category that admits finite projective limits and finite sums which are universally disjoint (in the sense of [SGA4, Exp. II, Definition 4.5]). A simplicial object of $\mathcal{E}$ is an object of the category $\Delta^\circ\mathcal{E}$ of contravariant functors from Δ to $\mathcal{E}$, where Δ is the category whose objects are the sets $[n] = \{0, 1, \dots, n\}$, $n \ge 0$, and morphisms are nondecreasing maps. Such an object is denoted by $X_\bullet = (X_n)_{n\ge 0}$, where X_n is the image of $[n]$, and $X_\bullet(f)$ denotes the morphism $X_m \to X_n$ that corresponds to a morphism $f : [n] \to [m]$. If δ_i, $0 \le i \le n+1$, denotes the injective morphism $[n] \to [n+1]$ that omits i, and s_i, $0 \le i \le n$, denotes the surjective morphism $[n+1] \to [n]$ with $s_i(i) = s_i(i+1)$, the corresponding morphisms $X_{n+1} \to X_n$ and $X_n \to X_{n+1}$ are denoted in the same way.

For example, every object S of $\mathcal{E}$ defines a constant simplicial object $S_\bullet$ which corresponds to the functor on Δ that takes the constant value S. An augmentation of a simplicial object $X_\bullet$ to S is a morphism

$$a = (a_n)_{n\ge 0} : X_\bullet \to S_\bullet$$

which is briefly denoted by $a : X_\bullet \to S$. The functor

$$\Delta^\circ(\mathcal{E}/S) \to \mathcal{E}/S : (X_\bullet \stackrel{a}{\to} S) \mapsto (X_0 \stackrel{a_0}{\to} S)$$

from the category of simplicial objects augmented to S to the category of morphisms $X \stackrel{\varphi}{\to} S$ has right adjoint $(X \stackrel{\varphi}{\to} S) \mapsto (\mathrm{cosk}(\varphi) \to S)$. Here $\mathrm{cosk}(\varphi)_n$ is the fiber product of $n+1$ copies of X over S, and morphisms $\mathrm{cosk}(\varphi)_n \to \mathrm{cosk}(\varphi)_m$ that correspond to morphisms $[m] \to [n]$ are defined in the evident way.

Every simplicial k-analytic space $X_\bullet$ defines an **étale site** $X_{\bullet\text{ét}}$ as follows. Let $\mathrm{\acute{E}t}(X_\bullet)$ be the category whose objects are pairs (n, f) consisting of $n \geq 0$ and an étale morphism $f : U \to X_n$ and morphisms $(m, g) \to (n, f)$ with $g : V \to X_m$ are pairs consisting of morphisms $[n] \to [m]$ and $V \to U$ compatible with the corresponding morphism $X_m \to X_n$. The étale site $X_{\bullet\text{ét}}$ is the Grothendieck topology on the category $\mathrm{\acute{E}t}(X_\bullet)$ generated by the pretopology for which the set of coverings of (n, f) with $f : U \to X_n$ is formed by families

$$\{(n, f_i) \stackrel{g_i}{\to} (n, f)\}_{i \in I}$$

over the identity morphism on $[n]$ and with $f_i : U_i \to X_n$ such that

$$U = \bigcup_{i \in I} g_i(U_i).$$

The category of sheaves on $X_{\bullet\text{ét}}$ is denoted by $\widetilde{X_{\bullet\text{ét}}}$ (the **étale topos** of $X_{\bullet\text{ét}}$). For $d \geq 0$, the category of étale $\mathbf{Z}/d\mathbf{Z}$-modules on $X_\bullet$ is denoted by $\mathbf{S}(X_\bullet, \mathbf{Z}/d\mathbf{Z})$, and its derived category is denoted by $D(X_\bullet, \mathbf{Z}/d\mathbf{Z})$.

An étale sheaf on $X_\bullet$ is nothing else than a family $F^\bullet$ of étale sheaves F^n on X_n and a family of $X_\bullet(f)$-morphisms $F^\bullet(f) : F^n \to F^m$ (i.e., morphisms of sheaves $X_\bullet(f)^*(F^n) \to F^m$ on X_m) for all morphisms $f : [n] \to [m]$. One also requires that $F^\bullet(f \circ g) = F^\bullet(f) \circ F^\bullet(g)$ for pairs of composable morphisms. The global sections functor is the functor $\Gamma : F^\bullet \mapsto \mathrm{Ker}(F^0(X_0) \rightrightarrows F^1(X_1))$. The values of its high direct images on an abelian sheaf $F^\bullet$ are denoted by $H^q(X_\bullet, F^\bullet)$.

Examples 1.2.1: (i) For a simplicial k-analytic space $X_\bullet$, the structural sheaves $\mathcal{O}_{X_n}$ form the structural sheaf of rings on $X_\bullet$.

(ii) For an augmented k-analytic space $X_\bullet \to S$, the sheaves of relative differentials $\Omega^m_{X_n/S}$ form a complex of sheaves $(\Omega^{\cdot}_{X_n})_{n \geq 0}$, the de Rham complex of $X_\bullet$.

(iii) If $F^\bullet$ is an étale abelian sheaf on $X_\bullet$, the flabby Godement resolutions $\mathcal{C}^{\cdot}(F^n)$ (see [Ber4, §1]) form a complex of sheaves on $X_\bullet$. Using it, one can construct for every complex of étale abelian sheaves $F^{\cdot,\bullet} \in D^+(X_\bullet, \mathbf{Z})$ a quasi-isomorphism $F^{\cdot,\bullet} \to K^{\cdot,\bullet}$ such that all of the sheaves $K^{m,n}$ are flabby.

(iv) A morphism of simplicial k-analytic spaces $\varphi = (\varphi_n)_{n\geq 0} : Y_\bullet \to X_\bullet$ gives rise to a morphism of sites $Y_{\bullet_{\text{ét}}} \to X_{\bullet_{\text{ét}}}$. The adjoint functors φ^* and φ_* take sheaves $F^\bullet$ on $X_\bullet$ and $G^\bullet$ on $Y_\bullet$ to the sheaves $(\varphi_n^* F^n)_{n\geq 0}$ on $Y_\bullet$ and $(\varphi_{n*} G^n)_{n\geq 0}$ on $X_\bullet$, respectively. The high direct image functor

$$R\varphi_* : D^+(Y_\bullet, \mathbf{Z}) \to D^+(X_\bullet, \mathbf{Z})$$

is calculated componentwise. Namely, if $G^{\cdot,\bullet}$ is a complex of étale abelian sheaves on $Y_\bullet$, one can take a quasi-isomorphism $G^{\cdot,\bullet} \to K^{\cdot,\bullet}$ as in (iii), and one has $R\varphi_* G^{\cdot,\bullet} = \varphi_* K^{\cdot,\bullet}$. In particular, $(R\varphi_* G^{\cdot,\bullet})_n = R\varphi_{n*} G^{\cdot,n}$.

(v) An étale sheaf on a constant k-analytic space $S_\bullet$ is a cosimplicial étale sheaf on S (i.e., a covariant functor $\Delta \to \widetilde{S_{\text{ét}}}$). Such an abelian sheaf $F^\bullet$ defines a differential complex $(F^n, d = \sum_i (-1)^i \delta_i)$ of sheaves on S. More generally, a complex of étale abelian sheaves $F^{\cdot,\bullet} \in D^+(S_\bullet, \mathbf{Z})$ defines a simple complex $(\mathrm{s}F^{\cdot,\bullet})^n = \bigoplus_{p+q=n} F^{p,q}$ with the differential

$$d(f^{pq}) = d_F(f^{pq}) + (-1)^p \sum_i (-1)^i \delta_i(f^{pq}).$$

The functor s takes acyclic complexes to acyclic ones and, therefore, it defines an exact functor $\mathrm{s} : D^+(S_\bullet, \mathbf{Z}) \to D(S, \mathbf{Z})$.

(vi) Let $a : X_\bullet \to S$ be an augmented k-analytic space. If F is an étale sheaf on S, then $a^*(F) = (a_n^* F)_{n\geq 0}$ is an étale sheaf on $X_\bullet$, and the functor $F \mapsto a^*(F)$ has a right adjoint a_* defined by

$$a_*(F^\bullet) = \operatorname{Ker}(a_{0*}(F^0) \rightrightarrows a_{1*}(F^1)),$$

where the two arrows are induced by δ_0^* and δ_1^*. One has $a_* = \varepsilon_* a_{\bullet *}$, where $a_\bullet$ is the corresponding morphism $X_\bullet \to S_\bullet$ and ε is the canonical augmentation $S_\bullet \to S$. It follows from (v) that $R\varepsilon_* \widetilde{\to} \mathrm{s}$ and, therefore, $Ra_* = \mathrm{s}Ra_{\bullet *}$.

One says that an augmented k-analytic space $a : X_\bullet \to S$ is of **$\mathbf{Z}/d\mathbf{Z}$-cohomological descent** if the adjunction morphism of functors $\mathrm{Id} \to Ra_* a^*$ from $D^+(S, \mathbf{Z}/d\mathbf{Z})$ to itself is an isomorphism or, equivalently, for every $F \in \mathbf{S}(S, \mathbf{Z}/d\mathbf{Z})$, one has $F \widetilde{\to} \operatorname{Ker}(a_{0*} a_0^* F \to a_{1*} a_1^* F)$ and $R^q a_*(a^* F) = 0$ for all $q \geq 1$. One says that a morphism of k-analytic spaces $\varphi : X \to S$ is of **$\mathbf{Z}/d\mathbf{Z}$-cohomological descent** if the augmentation morphism $\mathrm{cosk}(\varphi) \to S$ is. One says that a morphism of k-analytic spaces $\varphi : X \to S$ is of **universal $\mathbf{Z}/d\mathbf{Z}$-cohomological descent**, if any base change of φ is of $\mathbf{Z}/d\mathbf{Z}$-cohomological descent. For example, if $\{U_i \to S\}_{i \in I}$ is a covering in the étale topology, then the morphism $\coprod_{i \in I} U_i \to S$ is of universal $\mathbf{Z}$-cohomological descent. By the cohomological descent criterion [SGA4, Exp. Vbis, 3.2.4] and the base change theorem for cohomology with compact support [Ber2, 7.7.1], any surjective compact morphism $\varphi : X \to S$ is of universal $\mathbf{Z}/d\mathbf{Z}$-cohomological descent for all $d \geq 1$ prime to $\mathrm{char}(\widetilde{k})$.

By a general result [SGA4, Exp. Vbis, 3.3.1], the category of universal $\mathbf{Z}/d\mathbf{Z}$-cohomological descent morphisms $X \to S$ provided with the families of morphisms $\{X_i \to X\}_{i \in I}$, for which $\coprod_{i \in I} X_i \to X$ is a universal $\mathbf{Z}/d\mathbf{Z}$-cohomological descent morphism, is a Grothendieck pretopology. The Grothendieck topology generated by it is called the **étale topology of universal $\mathbf{Z}/d\mathbf{Z}$-cohomological descent**. Of course, this topology is stronger than the étale topology, and it follows from the definition that the cohomology groups of any étale $\mathbf{Z}/d\mathbf{Z}$-module F on S calculated in both topologies are the same. Recall now the definition of a hypercovering.

Let Δ_m, $m \geq 0$, be the full subcategory of Δ formed by the objects $[n]$ with $n \leq m$. An m-truncated simplicial object of $\mathcal{E}$ is an object of the category $\Delta_m^\circ \mathcal{E}$. The skeleton is the restriction functor $\mathrm{sk}_m : \Delta^\circ \mathcal{E} \to \Delta_m^\circ \mathcal{E}$, and the coskeleton is the functor $\mathrm{cosk}_m : \Delta_m^\circ \mathcal{E} \to \Delta^\circ \mathcal{E}$ right adjoint to sk_m. Given an object S in $\mathcal{E}$, the same functors applied to the category $\mathcal{E}/S$ are denoted by sk_m^S and cosk_m^S. If now $\mathcal{E}$ admits universally disjoint sums (not necessarily finite) and is provided with a Grothendieck topology, a hypercovering of an object S is an augmented simplicial object $a : X_\bullet \to S$ such that the morphisms $a_0 : X_0 \to S$ and $X_{n+1} \to (\mathrm{cosk}_n^S \mathrm{sk}_n^S X_\bullet)_{n+1}$ for all $n \geq 0$ are coverings in the topology of $\mathcal{E}$.

In [SGA4, Exp. Vbis, §5.1], one describes a general procedure for constructing simplicial objects which is used for constructing hypercoverings with required properties. We briefly recall it.

First of all, a simplicial object $X_\bullet$ of $\mathcal{E}$ is said to be σ-split if there exists a family of subobjects NX_n of X_n, $n \geq 0$, such that the morphism $\coprod NX_m \to X_n$, where the sum is taken over all morphisms $[n] \to [m]$ with $m \leq n$, is an isomorphism.

Suppose we are given a functor $\pi : \mathcal{E} \to \mathcal{B}$ such that, for every object $S \in \mathcal{B}$, the category $\mathcal{E}_S$ (the fiber of $\mathcal{E}$ at S) admits finite projective limits and arbitrary universally disjoint sums. One introduces a property Q of an object of $\mathcal{E}$ which is stable by isomorphisms and satisfies the conditions (a)–(c) from loc. cit., 5.1.4. One also introduces a property P of a morphism in $\mathcal{E}$ over the identity isomorphism of an object of $\mathcal{B}$. This property is assumed to be stable under isomorphisms and to satisfy the conditions (d)–(f) from loc. cit.. One says that a simplicial object $X_\bullet$ of $\mathcal{E}$ satisfies the condition (APQ) if

(1) $\pi(X_\bullet)$ is a constant simplicial object $S_\bullet$ for some $S \in B$;
(2) as a simplicial object of $\mathcal{E}_S$, $X_\bullet$ is σ-split;
(3) X_0 possesses the property Q;
(4) for all $n \geq 0$, the morphism $NX_{n+1} \to (\mathrm{cosk}_n^S X_\bullet)_{n+1}$ possesses the property P.

Finally, let $\mathcal{E}_{(APQ)}$ be the category of simplicial objects $X_\bullet$ of $\mathcal{E}$ which satisfy the condition (APQ) with morphisms of simplicial objects $Y_\bullet \to X_\bullet$ such that $\pi(Y_n) \to \pi(X_n)$ is the same morphism in $\mathcal{B}$ for all $n \geq 0$. Then there is a canonical functor $\overline{\pi} : \mathcal{E}_{(APQ)} \to \mathcal{B}$. Proposition 5.1.7 from loc. cit. asserts that the functor $\overline{\pi}$ is surjective on objects and morphisms. Moreover, for any object S of $\mathcal{B}$ and any pair $(X_\bullet, Y_\bullet)$ of objects of $\mathcal{E}_{(APQ),S}$, there exist two morphisms $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$ from the same object of $\mathcal{E}_{(APQ),S}$.

Examples 1.2.2: (i) Examples of the previous situation we are going to consider are of the following type. Suppose $\mathcal{B}$ is a category that admits finite projective limits and arbitrary universally disjoint unions, Q' is a family of objects in $\mathcal{B}$ which is stable by isomorphisms and disjoint unions, and P' is a family of morphisms which is stable under isomorphisms, compositions and base change and such that, for every object S of $\mathcal{B}$, there exists a morphism $X \to S$ in P' with X in Q'. Let $\mathcal{E}$ be the category of morphisms $X \to S$ in P' with X in Q', and $\pi : \mathcal{E} \to \mathcal{B}$ the canonical functor $(X \to S) \mapsto S$. Then the properties Q of objects and P of morphisms in $\mathcal{E}$ that correspond to the families Q' and P' in $\mathcal{B}$ satisfy the conditions (a)–(f) from loc. cit., 5.1.4, and each object of $\mathcal{E}_{(APQ),S}$ gives rise to an augmented simplicial object $X_\bullet \to S$ of $\mathcal{B}$ with all X_n in Q'.

(ii) Let $\mathcal{B}$ be the category of paracompact k-analytic spaces, Q' the family of all objects of $\mathcal{B}$, and P' the family of compact surjective morphisms. These families evidently satisfy the conditions of (i). The corresponding hypercoverings will be said to be **compact**. Every compact hypercovering is a hypercovering in the étale topology of universal $\mathbf{Z}/d\mathbf{Z}$-cohomological descent for all $d \geq 1$ prime to $\mathrm{char}(\widetilde{k})$.

1.3. A uniformization theorem for paracompact strictly k-analytic spaces. Let $\mathcal{S}$ be a scheme. A morphism $\varphi : \mathcal{Y} \to \mathcal{S}$ is said to be **maximally finitely dominant** if the image of the generic point y of every irreducible component of $\mathcal{Y}$ is the generic point of an irreducible component of $\mathcal{S}$ and the extension of residue fields $\kappa(y)/\kappa(\varphi(y))$ is finite. One denotes by $\mathrm{alt}/\mathcal{S}$ the category of maximally finitely dominant morphisms $\mathcal{Y} \to \mathcal{S}$. The alteration topology on $\mathcal{S}$ is the Grothendieck topology on $\mathrm{alt}/\mathcal{S}$ defined by the pretopology which is generated by coverings in the Zariski topology and by proper surjective and maximally finitely dominant morphisms. Notice that any covering in the étale topology of S has a refinement in the alteration topology of S ([ILO, Exp. II, 3.1.1]).

Gabber's "weak" uniformization theorem [ILO, Exp. VII, Theorem 1.1] states that, given a noetherian quasi-excellent scheme $\mathcal{S}$ and a nowhere dense closed subset $\mathcal{Z} \subset \mathcal{S}$, there exists a finite family of morphisms $\{\mathcal{S}_i \to \mathcal{S}\}_{i\in I}$ which is a covering in the alteration topology of $\mathcal{S}$ and such that each $\mathcal{S}_i$ is regular and connected and the support of the preimage of $\mathcal{Z}$ in $\mathcal{S}_i$ is a divisor with strict normal crossings. (The empty set is considered as a divisor with strict normal crossings.)

Theorem 1.3.1: *Suppose that the field k is perfect and its valuation is non-trivial. Then for every paracompact strictly k-analytic space X, there exists a compact surjective morphism $\coprod_{i\in I} Y_i \to X$ with each Y_i of the form $\widehat{\mathcal{Y}}_\eta$, where $\mathcal{Y}$ is an affine scheme finitely presented over k° with $\mathcal{Y}_\eta$ smooth over k.*

Proof. Since X is paracompact, it admits a locally finite covering by strictly affinoid domains $\{U_i\}_{i\in I}$. If we can find, for every $i \in I$, a surjective morphism $\coprod_{j=1}^{n_i} Y_{ij} \to U_i$ with each Y_{ij} of the form as in the formulation, then $\coprod_{i\in I, 1\leq j\leq n_i} Y_{ij} \to X$ is a required morphism. This reduces the situation to the case when $X = \mathcal{M}(\mathcal{A})$ is strictly k-affinoid.

By a result of Kiehl [Kiehl] (see also [Duc]), the affine scheme $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$ is excellent. We can therefore apply Gabber's theorem. It follows that there exists a finite family of morphisms $\{\mathcal{X}_i \to \mathcal{X}\}_{i\in I}$ which is a covering in the alteration topology of $\mathcal{X}$ and such that each $\mathcal{X}_i$ is regular and connected. By [Ber2, §2.6], each $\mathcal{X}_i$ defines a good strictly k-analytic space $\mathcal{X}_i^{\mathrm{an}}$, and from [Ber2, 2.6.3] it follows that all strictly affinoid subdomains of $\mathcal{X}_i^{\mathrm{an}}$ are regular.

We claim that one can find, for every $i \in I$, a finite family of strictly affinoid domains $\{X_{ij} \subset \mathcal{X}_i^{\mathrm{an}}\}_{j \in J_i}$ such that $X = \bigcup_{i \in I, j \in J_i} \varphi_i(X_{ij})$. Indeed, it suffices to consider the two types of coverings that generate the alteration topology of $\mathcal{X}$. If such a covering is a proper surjective and maximally finitely dominant morphism $\mathcal{Y} \to \mathcal{X}$, then $\mathcal{Y}^{\mathrm{an}}$ is compact and the induced map $\mathcal{Y}^{\mathrm{an}} \to X$ is surjective, by [Ber2, 2.6.8], and this implies the required fact. Suppose that $\{\mathcal{X}_i \to \mathcal{X}\}_{i \in I}$ is a covering in the Zariski topology of $\mathcal{X}$. Replacing it by a refinement, we may assume that each $\mathcal{X}_i$ is a principal open subset $D(f_i)$ with $f_i \in \mathcal{A}$ such that the ideal generated by all f_i's coincides with $\mathcal{A}$. It follows that $1 = \sum_{i \in I} g_i f_i$ for some elements $g_i \in \mathcal{A}$. We set $r = \max_{i \in I} \max_{x \in X} |g_i(x)|$. Then for every point $x \in X$ there exists $i \in I$ with $|f_i(x)| \geq r^{-1}$. This implies that $X = \bigcup_{i \in I} V_i$ for the strictly affinoid domains $V_i = \{x \in X \big| |f_i(x)| \geq r^{-1}\}$, and the claim follows.

The above claim reduces the situation to the case when the strictly k-affinoid space X and the scheme $\mathcal{X}$ are regular. Since the field k is perfect, it follows from [Ber5, §5], that X is smooth in the sense of rigid geometry (called rig-smooth in [Ber7, §1.1]).

LEMMA 1.3.2: *Let k be a non-Archimedean field with nontrivial valuation, and let R be a local henselian subring of k° that contains a nonzero element $a \in k^{\circ\circ}$ and whose a-adic completion $\widehat{R}$ coincides with k° (e.g., $R = k^\circ$). Then every rig-smooth k-affinoid space $X = \mathcal{M}(\mathcal{A})$ is of the form $\widehat{\mathcal{Y}}_\eta$, where $\mathcal{Y}$ is an affine scheme finitely presented over R with $\mathcal{Y}_\eta$ smooth over the fraction field of R.*

The lemma implies that X is isomorphic to the intersection of the analytification of a smooth closed subset of an affine space with the unit closed polydisc. Indeed, if $\mathcal{Y} = \mathrm{Spec}(B)$, a surjective homomorphism $k^\circ[T_1, \ldots, T_n] \to B$ gives rise to a closed immersion of the smooth k-scheme $\mathcal{Y}_\eta = \mathrm{Spec}(B')$ in the n-dimensional affine space, where $B' = B \otimes_{k^\circ} k$, such that the intersection of the analytification $\mathcal{Y}_\eta^{\mathrm{an}}$ with the unit closed polydisc coincides with X.

Proof. One can find a complete a-adic algebra A topologically finitely presented and flat over k° with $A \otimes_{k^\circ} k = \mathcal{A}$. By [Tem, Proposition 3.3.2] (see Remark 1.3.4), A is formally smooth over $\widehat{R} = k^\circ$ outside $V(a)$ in the sense of [Elk, p. 581] and, by Elkik's theorem [Elk, Theorem 7 and Remark 2, p. 587], there exists a finitely presented R-algebra B which is smooth over the fraction field of R and whose a-adic completion $\widehat{B}$ is isomorphic to A. It follows that the affine scheme $\mathcal{Y} = \mathrm{Spec}(B)$ possesses the required property. ∎

Corollary 1.3.3: *If the field k is perfect and its valuation is nontrivial, every paracompact strictly k-analytic space X admits a compact hypercovering $a : Y_\bullet \to X$ in which all Y_n are disjoint unions of strictly affinoid domains in the analytifications of smooth affine schemes over k.* ∎

Remark 1.3.4: In [Tem, Proposition 3.3.2], the setting is assumed to be the same as that before the formulation and, in particular, the ground field is assumed to be discretely valued. But the proof in fact does not use this assumption.

Proof of Theorem 1.1.1. First of all, it suffices to consider the case when $X = \mathcal{M}(\mathcal{A})$ is k-affinoid and F is a pullback of an abelian constructible sheaf $\mathcal{F}$ to $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$. By [SGA4, Exp. IX, 2.14(ii)], one can find a finite family of finite morphisms $\{\mathcal{Y}_i \stackrel{p_i}{\to} \mathcal{X}\}_{i\in I}$ and a monomorphism

$$\mathcal{F} \to \mathcal{F}' = \bigoplus_{i\in I} p_{i*}(\mathcal{G}_i),$$

where $\mathcal{G}_i$ is an abelian constant sheaf on $\mathcal{Y}_i = \mathrm{Spec}(\mathcal{B}_i)$. If F' is the pullback of $\mathcal{F}'$ to X, then

$$H^q(X, F') = \bigoplus_{i\in I} H^q(Y_i, G_i),$$

where G_i is the pullback of $\mathcal{G}_i$ to $Y_i = \mathcal{M}(\mathcal{B}_i)$. This implies validity of the required fact for $q = 0$ since the sets $\pi_0(Y_i)$ are finite. Furthermore, the pullback F'' of $\mathcal{F}'' = \mathrm{Coker}(\mathcal{F} \to \mathcal{F}')$ is a constructible sheaf on X and, therefore, if we know finiteness of cohomology groups with coefficients in finite constant sheaves, finiteness of the group $H^q(X, F)$ is reduced to that of the group $H^{q-1}(X, F'')$. Thus, by induction on q, we may assume that $F = \Lambda_X$ is the constant sheaf associated with a finite abelian group Λ of order prime to $\mathrm{char}(\widetilde{k})$.

Furthermore, by the invariance of cohomology under extensions of the ground field [Ber2, 7.6.1], we may assume that the valuation on k is nontrivial and X is strictly k-affinoid. Since the finiteness statement is true for compact quasi-algebraic k-analytic spaces ([Ber3, 5.6]), it suffices to show that X admits a compact hypercovering $a : Y_\bullet \to X$ in which all Y_n are quasi-algebraic. But this follows from Corollary 1.3.3. ∎

Proof of Theorem 1.1.2. In Step 1 we prove that the vanishing cycles sheaves $R^q\Psi_\eta(F)$ are constructible and, in Step 2, we deduce constructibility of the nearby cycles sheaves $R^q\Theta(F)$.

STEP 1. We follow the proof of [SGA4$\frac{1}{2}$, Th. finitude, Theorem 3.2]. Since the statement is local with respect to $\mathfrak{X}$, we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine with topologically finitely presented A and, in particular, that $d = \dim(\mathfrak{X}_\eta) < \infty$. We may also assume that A is flat over k°. If $d = 0$, then $\mathfrak{X}$ is the completion of the scheme $\mathrm{Spec}(A)$ finite over k°, and the required fact is known. Suppose that $d \geq 1$ and that the theorem is true for schemes whose generic fibre has dimension at most $d-1$. We claim that, for every $q \geq 0$, there exists a constructible subsheaf $\mathcal{G}^q \subset R^q\Psi_\eta(F)$ such that the supports of local sections of the quotient $R^q\Psi_\eta(F)/\mathcal{G}_q$ are finite. First of all, we recall a result from [Ber3] which is an analog of Lemma 3.4 from loc. cit.

Suppose that the canonical morphism $\mathfrak{X} \to \mathrm{Spf}(k^\circ)$ goes through a morphism $\mathfrak{X} \to \mathfrak{A}^1 := \mathrm{Spf}(k^\circ\{T\})$. Notice that $\mathfrak{A}^1$ is the formal completion of the affine line over k°, and $\mathfrak{A}^1_\eta$ is the one-dimensional unit disc over k. Let t be the maximal point of $\mathfrak{A}^1_\eta$ (it corresponds to the norm of the k-affinoid algebra $k\{T\}$). Then the image s' of t under the reduction map $\pi : \mathfrak{A}^1_\eta \to \mathfrak{A}^1_s$ is the generic point of $\mathfrak{A}^1_s$, $\pi^{-1}(s') = \{t\}$ and $\widetilde{\mathcal{H}(t)} = k(s') = \widetilde{k}(T)$. We set

$$\mathfrak{X}' = \mathfrak{X} \times_{\mathfrak{A}^1} \mathrm{Spf}(\mathcal{H}(t)^\circ).$$

This is a formal scheme finitely presented over $\mathcal{H}(t)^\circ$. Let $\mathfrak{X}'_{s'}$ and $\mathfrak{X}'_{\eta'}$ denote the closed and the generic fibres of $\mathfrak{X}'$, and let $\Psi_{\eta'}$ denote the corresponding vanishing cycles functor. The canonical morphism of formal schemes $\lambda : \mathfrak{X}' \to \mathfrak{X}$ induces morphisms $\lambda_s : \mathfrak{X}'_{s'} \widetilde{\to} (\mathfrak{X}_s)_{s'} \to \mathfrak{X}_s$ and $\lambda_\eta : \mathfrak{X}'_{\eta'} \widetilde{\to} (\mathfrak{X}_\eta)_t \to \mathfrak{X}_\eta$, where $(\mathfrak{X}_s)_{s'}$ (resp. $(\mathfrak{X}_\eta)_t$) is the fibre of the morphism $\mathfrak{X}_s \to \mathfrak{A}^1_s$ (resp. $\mathfrak{X}_\eta \to \mathfrak{A}^1_\eta$) at the point s' (resp. t). Let F' denote the pullback of the sheaf F with respect to λ_η, and fix an embedding of fields $k^s \hookrightarrow \mathcal{H}(t)^s$. It induces a morphism $\lambda_{\overline{s}} : \mathfrak{X}'_{\overline{s'}} \widetilde{\to} (\mathfrak{X}_{\overline{s}})_{\overline{s'}} \to \mathfrak{X}_{\overline{s}}$. By [Ber3, 4.6(ii)], there is a canonical isomorphism

$$\lambda^*_{\overline{s}}(R^q\Psi_\eta(F)) \widetilde{\to} R^q\Psi_{\eta'}(F')^P,$$

where $P = G(\mathcal{H}(t)^s/\mathcal{H}(t)^{\mathrm{nr}}k^s)$ (it is a pro-p-group, where $p = \mathrm{char}(\widetilde{k})$).

By [Ber2, 2.5.2], $\dim(\mathfrak{X}'_{\eta'}) = d-1$, and therefore, by the induction hypothesis, the sheaf on the right-hand side is constructible. We now take a closed immersion $\mathfrak{X} \to \mathfrak{A}^n = \mathrm{Spf}(k^\circ\{T_1, \ldots, T_n\})$ to the formal affine space $\mathfrak{A}^n$. It gives rise to a closed immersion of the affine schemes $\mathfrak{X}_s$ to the affine space $\mathfrak{A}^n_s$

over $\widetilde{k}$. By Lemma 3.5 from [SGA4$\frac{1}{2}$, Th. finitude], applied to the vanishing cycles sheaf $R^q\Psi_\eta(F)$ on $\mathfrak{X}_{\overline{s}}$ and to the above closed immersion, there exists a constructible subsheaf $\mathcal{G}^q \subset R^q\Psi_\eta(F)$ such that the supports of local sections of the quotient $\mathcal{H}^q = R^q\Psi_\eta(F)/\mathcal{G}_q$ are finite, i.e., the claim is true.

It follows that $H^p(\mathfrak{X}_{\overline{s}}, \mathcal{H}^q) = 0$ for all $p \geq 1$ and, in order to prove that the sheaves $R^q\Psi_\eta(F)$ are constructible, it suffices to verify that the groups $H^0(\mathfrak{X}_{\overline{s}}, \mathcal{H}^q)$ are finite. For this we use the spectral sequence [Ber3, 4.5(iii)]

$$E_2^{p,q} = H^p(\mathfrak{X}_{\overline{s}}, R^q\Psi_\eta(F)) \Longrightarrow H^{p+q}(\mathfrak{X}_{\overline{\eta}}, \overline{F}).$$

As at the end of the proof of [SGA4$\frac{1}{2}$, Th. finitude, 3.2], we consider the images of the above abelian groups in the quotient of the category of abelian groups by the thick subcategory of finite abelian groups. By the finiteness result loc. cit., Corollary 1.10, one has $E_2^{p,q} \sim H^p(\mathfrak{X}_{\overline{s}}, \mathcal{H}^q)$ and, therefore, $E_2^{p,q} \sim 0$ for all $p \geq 1$. It follows that $E_2^{0,q} \sim H^0(\mathfrak{X}_{\overline{s}}, \mathcal{H}^q) \sim H^q(\mathfrak{X}_{\overline{\eta}}, \overline{F})$. The latter group is finite, by Theorem 1.1.1.

Step 2 (cf. the proof of [SGA4$\frac{1}{2}$, Th. finitude, 3.11]). First of all, for a subfield $k \subset K \subset k^{\rm s}$, let $\mathfrak{X}_{s_K}$ and $\mathfrak{X}_{\eta_K}$ denote the closed and the generic fibres of the formal scheme $\mathfrak{X}_K = \mathfrak{X}\widehat{\otimes}_{k^\circ}\widehat{K}^\circ$ over $\widehat{K}^\circ$. (One has $\mathfrak{X}_{s_K} = \mathfrak{X}_s \otimes \widetilde{K}$ and $\mathfrak{X}_{\eta_K} = \mathfrak{X}_\eta\widehat{\otimes}\widehat{K}$.) Let also Θ_K denote the nearby cycles functor for $\mathfrak{X}_K$ and, for an étale sheaf F, let F_K denote the pullback of F to $\mathfrak{X}_{\eta_K}$. It follows from [Ber3, 4.1] that the higher direct images of the functor $F \mapsto \Theta_K(F_K)$ are $R^q\Theta_K(F_K)$ (they will be denoted by $R^q\Theta_K(F)$), and that the latter are associated with the presheaves $\mathfrak{Y}_s \mapsto H^q(\mathfrak{Y}_\eta, F_K)$ (for $\mathfrak{Y}$ étale over $\mathfrak{X}_K$). It follows that, for a pair $K \subset L$ of Galois extensions of k, there is a Hochschild–Serre spectral sequence

$$E_2^{pq} = \mathcal{H}^p(\mathrm{Gal}(L/K), R^q\Theta_L(F)) \Longrightarrow R^{p+q}\Theta_K(F),$$

where $\mathcal{H}^p(\mathrm{Gal}(L/K), R)$ is the value of the p-th derived functor of the functor $R \mapsto R^{\mathrm{Gal}(L/K)}$ on the category of abelian sheaves on $\mathfrak{X}_{s_L}$, provided with a continuous action of the group $\mathrm{Gal}(L/K)$ compatible with its action on $\mathfrak{X}_{s_L}$, to the category of abelian sheaves on $\mathfrak{X}_{s_K}$. (Given an étale morphism $\mathcal{U} \to \mathfrak{X}_{s_K}$, one has $R^{\mathrm{Gal}(L/K)}(\mathcal{U}) = R(\mathcal{U} \times_{\mathfrak{X}_{s_K}} \mathfrak{X}_{s_L})^{\mathrm{Gal}(L/K)}$, and therefore $\Theta_K(F) = \Theta_L(F)^{\mathrm{Gal}(L/K)}$.)

We now turn to the proof of our theorem. We may assume that the residue field $\widetilde{k}$ is algebraically closed, and F is l-torsion for a prime integer l not divisible by $\mathrm{char}(\widetilde{k})$. Suppose that the order of the quotient group $|k^*|/|k^*|^l$ is l^m. Then the quotient of the Galois group G of k by the maximal invariant subgroup P

of profinite order prime to l is isomorphic to $\mathbf{Z}_l(1)^m$ (e.g., see [Ber2, 2.4.4]). It follows that there is an increasing chain of abelian extensions

$$k_0 = k \subset k_1 \subset \cdots \subset k_m \subset k^{\mathrm{s}}$$

with $\mathrm{Gal}(k^{\mathrm{s}}/k_m) = P$ and each $\mathrm{Gal}(k_i/k_{i-1})$ isomorphic to $\mathbf{Z}_l(1)$. Applying the above spectral sequence to the pair $k_m \subset k^{\mathrm{s}}$, we get $R^q\Theta_{k_m}(F) = R^q\Psi_\eta(F)^P$. By Step 1, the sheaves $R^q\Psi_\eta(F)$ are constructible and, in particular, the group G acts on them through a finite quotient. If $\sigma_1, \ldots, \sigma_n$ are representatives of the similar quotient of the group P, then the sheaf $R^q\Psi_\eta(F)^P$ is the kernel of the homomorphism from $R^q\Psi_\eta(F)$ to the product of n copies of itself defined by the element $\sigma_1 - 1, \ldots, \sigma_n - 1$ and, therefore, this sheaf is constructible. Suppose now that, for $1 \le i \le m$, the sheaves $R^q\Theta_{k_i}(F)$ are constructible. If σ is a generator of the Galois group $\mathrm{Gal}(k_i/k_{i-1})$, then for the above spectral sequence, applied to the pair $k_{i-1} \subset k_i$, we get $E_2^{0q} = \mathrm{Ker}(\sigma - 1, R^q\Theta_{k_i}(F))$, $E_2^{1q} = \mathrm{Coker}(\sigma - 1, R^q\Theta_{k_i}(F))$, and $E_2^{pq} = 0$ for $p \ne 0, 1$. These sheaves are constructible and, therefore, so are the sheaves $R^q\Theta_{k_{i-1}}(F)$. ∎

2. A version of Gabber's weak uniformization theorem

2.1. Formulation of the result. For a scheme $\mathcal{X}$, the underlying reduced scheme is denoted by $\mathcal{X}_{\mathrm{red}}$. For a discrete valuation ring R, we denote by $\widetilde{R}$ its residue field, i.e., the quotient $R/\mathbf{m}_R$ of R by its maximal ideal $\mathbf{m}_R$. For a scheme $\mathcal{X}$ over R, the closed subscheme $\widetilde{\mathcal{X}} = \mathcal{X} \otimes_R \widetilde{R}$ is said to be the ***R*-special fiber of $\mathcal{X}$**. Notice that every closed subset $\mathcal{Z} \subset \mathcal{X}$ (considered as a reduced scheme) is a union $\mathcal{Z}_f \cup \mathcal{Z}'$, where $\mathcal{Z}_f$ is flat over R and $\mathcal{Z}' \subset (\widetilde{\mathcal{X}})_{\mathrm{red}}$.

Definition 2.1.1: (i) A scheme $\mathcal{X}$ over R is said to be an ***R*-strict normal crossings scheme** (or, for brevity, an ***R*-snc scheme**) if $\mathcal{X}$ is a separated noetherian connected regular scheme flat over R and its R-special fiber $\widetilde{\mathcal{X}}$ is a divisor with strict normal crossings.

(ii) A pair $(\mathcal{X}, \mathcal{Z})$ consisting of scheme $\mathcal{X}$ over R and a closed subset $\mathcal{Z} \subset \mathcal{X}$ is said to be an ***R*-snc pair** if

(1) $\mathcal{X}$ is an R-snc scheme;
(2) $\mathcal{Z}$ is a divisor with strict normal crossing;
(3) if $\mathcal{Z}_f = \bigcup_{i \in I} \mathcal{Z}_i$ is the decomposition of $\mathcal{Z}_f$ into irreducible components then, for every subset $J \subset I$, the intersection $\mathcal{Z}_J = \bigcap_{i \in J} \mathcal{Z}_i$ is a disjoint union of R-snc schemes.

Remarks 2.1.2: (i) If $\mathcal{X}$ is defined over the fraction field of R or, equivalently, $\widetilde{\mathcal{X}} = \emptyset$, then $\mathcal{X}$ is an R-snc scheme if and only if it is a noetherian connected regular scheme. (Recall that the empty subset of $\mathcal{X}$ is considered as a divisor with strict normal crossing.)

(ii) If a pair $(\mathcal{X}, \mathcal{Z})$ possesses the properties (1) and (2) and $\mathcal{Z}$ contains $(\widetilde{\mathcal{X}})_{\rm red} = \widetilde{\mathcal{X}}$, then $(\mathcal{X}, \mathcal{Z})$ also possesses the property (3), i.e., it is an R-snc pair. Indeed, let x be a point in $\widetilde{\mathcal{Z}_J} = \mathcal{Z}_J \cap \widetilde{\mathcal{X}}$. Shrinking $\mathcal{X}$, we may assume that x lies in the intersection of all of the irreducible components of $\mathcal{Z}$. By the assumption, we can find a regular system of local parameters $\{c_1, \ldots, c_d\}$ at x such that $c_1, \ldots, c_m$ with $1 \le m \le d-1$ define the irreducible components of $\mathcal{Z}_f$ at x and $c_{m+1}, \ldots, c_n$ with $m+1 \le n \le d$ define the irreducible components of $\widetilde{\mathcal{X}}$ at x. Suppose also $c_1, \ldots, c_k$ with $1 \le k \le m$ define the irreducible components for the set J. Multiplying c_{m+1} by a function invertible in a neighborhood of x, we may assume that $\pi = c_{m+1} \cdots c_n$ is a uniformizing element of R. Then $\{c_{k+1}, \ldots, c_n\}$ is a regular system of local parameters of $\mathcal{Z}_J$ at x, the elements $c_{m+1}, \ldots, c_n$ define the irreducible components of $\widetilde{\mathcal{Z}_J}$ at x, and their product is π. This implies that $\widetilde{\mathcal{Z}_J}$ is a divisor in $\mathcal{Z}_J$ with strict normal crossings.

Recall that an extension of fields K/k is said to possess the property of Epp if, in the case they are of positive characteristic p, all elements of the maximal perfect subfield $K^{p^\infty} = \bigcap_{n=1}^{\infty} K^{p^n}$ of K are algebraic and separable over k. The residue field of a point x of a scheme is denoted by $\kappa(x)$.

THEOREM 2.1.3: *Let $\mathcal{X}$ be a noetherian quasi-excellent scheme flat over a complete discrete valuation ring R, and let $\mathcal{Z}$ be a nowhere dense closed subset of $\mathcal{X}$ that contains $(\widetilde{\mathcal{X}})_{\rm red}$. Suppose that the residue field $\widetilde{R}$ is perfect and, for every closed point $x \in \widetilde{\mathcal{X}}$, the extension $\kappa(x)/\widetilde{R}$ possesses the property of Epp. Then there exists a finite covering $\{\mathcal{X}_i \overset{\varphi_i}{\to} \mathcal{X}\}_{i \in I}$ in the alteration topology of $\mathcal{X}$ such that each morphism $\mathcal{X}_i \to$ Spec(R) factors through a morphism $\mathcal{X}_i \to$ Spec(R_i) that makes $(\mathcal{X}_i, \varphi_i^{-1}(\mathcal{Z})_{\rm red})$ an R_i-snc pair, where R_i is the ring of integers of a finite extension of the fraction field of R.*

Remarks 2.1.4: (i) Gabber's weak uniformization theorem [ILO, Exp. VII, Theorem 1.1] guarantees existence of a finite covering $\{\mathcal{X}_i \overset{\varphi_i}{\to} \mathcal{X}\}_{i \in I}$ in the alteration topology of $\mathcal{X}$ such that each $\mathcal{X}_i$ is a regular connected scheme in which the closed subset $\varphi_i^{-1}(\mathcal{Z})_{\rm red}$ is a divisor with strict normal crossings. By Remark 2.1.2(ii), Theorem 2.1.3 actually states that one can find such a covering with the additional property that the R-algebra $\mathcal{O}(\mathcal{X}_i)$ of each $\mathcal{X}_i$ contains the ring of integers R_i of a finite extension of R and the R_i-special fiber $\mathcal{X}_i \otimes_{R_i} \widetilde{R}_i$ is reduced. Notice that for $\mathcal{X}$ defined over the fraction field of R, Theorem 2.1.3 follows from Gabber's theorem.

(ii) The assumption on the property of Epp is necessary because in the case, when $\mathcal{X}$ is the spectrum of a complete discrete valuation ring, the statement of Theorem 2.1.3 is equivalent to Epp's theorem [Epp, 1.9] which requires that assumption. This will in fact serve as a basis of induction on the dimension of $\mathcal{X}$ in the proof. On the other hand, the assumption on perfectness of the field $\widetilde{R}$ can be, probably, removed.

(iii) While restricting to R-schemes of finite type, our definitions of an R-snc scheme and of an R-snc pair are weaker than de Jong's definitions of a semi-stable scheme and of a semi-stable pair over R ([deJ1, 2.16 and 6.3]). Namely, $\mathcal{X}$ is semi-stable over R if, in addition to being R-snc, the generic fiber $\mathcal{X}_\eta$ is smooth over the fraction field of R and all of the joint intersections of the irreducible components of $\widetilde{\mathcal{X}}$ are smooth over $\widetilde{R}$. His Theorem 6.5 from loc. cit. (which does not require perfectness of $\widetilde{R}$) implies that there exists an alteration $\varphi : \mathcal{X}' \to \mathcal{X}$ for which the pair $(\mathcal{X}', \varphi^{-1}(\mathcal{Z})_{\mathrm{red}})$ is strictly semi-stable over the ring of integers of a finite extension of the fraction field of R.

(iv) The proof of Theorem 2.1.3 follows the proof of Gabber's theorem. We briefly recall the setting, constructions and facts of each step from the proof of the latter, and describe the necessary changes that should be done to get the statement of Theorem 2.1.3.

2.2. Reduction to the local noetherian complete normal case.

Step 1. Suppose that Theorem 2.1.3 is true for affine schemes Spec(A), where A is a local noetherian henselian excellent normal R-flat algebra of dimension at most $d \geq 1$. Then Theorem 2.1.3 is true for $\mathcal{X}$ of dimension d. Indeed, the canonical morphism from the disjoint union of its irreducible components to $\mathcal{X}$ is a covering in the alteration topology and, therefore, we may assume that $\mathcal{X}$ is integral and $\widetilde{\mathcal{X}} \neq \emptyset$ (see Remark 2.1.4(i)). Furthermore, since $\mathcal{X}$ is quasi-excellent, the canonical morphism from the normalization of $\mathcal{X}$ to $\mathcal{X}$ is also a covering in the alteration topology and, therefore, we may assume that $\mathcal{X}$ is normal. Let x be a closed point of $\mathcal{X}$ lying in $\widetilde{\mathcal{X}}$. By [EGAIV, 18.7.6], the henselization $\mathcal{X}_{(x)}$ of $\mathcal{X}$ at x is an excellent scheme. Since x is the only closed point in the special fibre of $\mathcal{X}_{(x)}$ and has the same residue field as in

$\mathcal{X}$, we can apply the assumption. It follows that there exists a finite covering $\{\mathcal{X}_i \xrightarrow{\varphi_i} \mathcal{X}_{(x)}\}_{i\in I}$ in the alteration topology of $\mathcal{X}_{(x)}$ such that, for every $i \in I$, $(\mathcal{X}_i, \varphi_i^{-1}(\mathcal{Z})_{\mathrm{red}})$ is an R_i-snc pair, where R_i is the ring of integers of a finite extension of the fraction field of R. By [ILO, Exp. II, 4.1.2], this covering is the base change of a covering $\{\mathcal{V}_i \to \mathcal{U}\}_{i\in I}$ in the alteration topology of an étale neighborhood $\mathcal{U}$ of x in $\mathcal{X}$. Since the R-algebra $\mathcal{O}(\mathcal{X}_i)$ is the inductive limit of the R-algebras $\mathcal{O}(\mathcal{V}_i)$ taken over $\mathcal{U}$'s, we can find a sufficiently small $\mathcal{U}$ such that each $\mathcal{O}(\mathcal{V}_i)$ contains R_i. The remaining part of the proof of [ILO, Exp. II, 4.3.3] is applicable to our case.

Step 2. Suppose that the theorem is true for affine schemes $\mathrm{Spec}(A)$, where A is a local noetherian complete normal flat R-algebra of dimension at most d. Then Theorem 2.1.3 is true for $\mathcal{X} = \mathrm{Spec}(A)$, where A is a local noetherian henselian excellent normal flat R-algebra of dimension d. Indeed, let $\mathcal{X} = \mathrm{Spec}(A)$, where A is such an R-algebra, and let $\mathcal{Z}$ be a nowhere dense closed subset of $\mathcal{X}$ that contains $(\widetilde{\mathcal{X}})_{\mathrm{red}}$. We follow the proof of [ILO, Exp. III, Proposition 6.2]. First of all, since A is excellent, its completion $\widehat{A}$ with respect to the maximal ideal $\mathbf{m}$ is also normal and, since $\widehat{A}$ is flat over A, the preimage $\widehat{\mathcal{Z}}$ of $\mathcal{Z}$ in $\widehat{\mathcal{X}} = \mathrm{Spec}(\widehat{A})$ is nowhere dense. For $n \geq 0$, one sets $\mathcal{X}_n = \mathrm{Spec}(A/\mathbf{m}^{n+1})$ and, for a scheme $\mathcal{Y}$ over $\mathcal{X}$, one sets $\mathcal{Y}_n = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_n$. Notice that there is a canonical isomorphism $\widehat{\mathcal{X}}_n \widetilde{\to} \mathcal{X}_n$. Two morphisms $\widehat{\mathcal{X}} \to \mathcal{Y}$ and $\mathcal{X} \to \mathcal{Y}$ are said to be n-close if their restrictions to $\widehat{\mathcal{X}}_n = \mathcal{X}_n$ coincide. Notice also that there is a canonical closed immersion $\mathcal{Y}_0 \to \widetilde{\mathcal{Y}}$.

The R-algebra $\widehat{A}$ is a filtered inductive limit $\varinjlim A_\alpha$ of finitely generated A-algebras A_α, $\alpha \in E$. If $\mathcal{X}_\alpha = \mathrm{Spec}(A_\alpha)$, there is an isomorphism of schemes $\widehat{\mathcal{X}} \widetilde{\to} \varprojlim \mathcal{X}_\alpha$. Recall (see [EGAIV, §8]) that every scheme $\mathcal{Y}$ of finite type over $\widehat{\mathcal{X}}$ is a base change of a scheme $\mathcal{Y}_\alpha$ of finite type over $\mathcal{X}_\alpha$ for some α (it is called a model of $\mathcal{Y}$ over $\mathcal{X}_\alpha$). Moreover, given two models $\mathcal{Y}_\alpha$ and $\mathcal{Y}_\beta$ of $\mathcal{Y}$, there exist $\gamma \geq \alpha, \beta$ and an $\mathcal{X}_\gamma$-isomorphism $\mathcal{Y}_\alpha \times_{\mathcal{X}_\alpha} \mathcal{X}_\gamma \widetilde{\to} \mathcal{Y}_\beta \times_{\mathcal{X}_\beta} \mathcal{X}_\gamma$. Similar facts hold for morphisms between schemes of finite type over $\widehat{\mathcal{X}}$. The canonical morphisms $\widehat{\mathcal{X}} \to \mathcal{X}_\alpha$ and $\mathcal{X}_\alpha \to \mathcal{X}$ will be denoted by s_α and t_α, respectively.

By the theorem of Popescu [Pope, 1.3], the excellent henselian local ring A possesses the Artin approximation property. This implies that, for every $\alpha \in E$, the set $\mathcal{X}_\alpha(A) = \mathrm{Hom}(\mathcal{X}, \mathcal{X}_\alpha)$ is dense in $\mathcal{X}_\alpha(\widehat{A}) = \mathrm{Hom}(\widehat{\mathcal{X}}, \mathcal{X}_\alpha)$ in the $\mathbf{m}$-adic topology. It follows that, for every $n \geq 0$ and $\alpha \in E$, there exists a section $u : \mathcal{X} \to \mathcal{X}_\alpha$ of the morphism $t_\alpha : \mathcal{X}_\alpha \to \mathcal{X}$ which is n-close to the morphism $s_\alpha : \widehat{\mathcal{X}} \to \mathcal{X}_\alpha$. One then defines, for a scheme $\mathcal{Y}$ of finite type over $\widehat{\mathcal{X}}$ with a model $\mathcal{Y}_\alpha$ over $\mathcal{X}_\alpha$, a scheme $\mathcal{Y}^{(u)} = \mathcal{Y}_\alpha \times_{\mathcal{X}_\alpha,u} \mathcal{X}$ (it is denoted by $\mathcal{Y}_u$ in [ILO, Exp. III]). Since u is n-close to s_α, it follows that there is a canonical isomorphism $\mathcal{Y}_n^{(u)} = (\mathcal{Y}^{(u)})_n \widetilde{\to} \mathcal{Y}_n$. Similarly, any morphism $\varphi : \mathcal{Y}' \to \mathcal{Y}$ between schemes of finite type over $\widehat{\mathcal{X}}$ with a model $\varphi_\alpha : \mathcal{Y}'_\alpha \to \mathcal{Y}_\alpha$ gives rise to a morphism $\varphi^{(u)} : \mathcal{Y}'^{(u)} \to \mathcal{Y}^{(u)}$, and the induced $\mathcal{X}_n$-morphisms $\varphi_n^{(u)} : \mathcal{Y}_n'^{(u)} \to \mathcal{Y}_n^{(u)}$ and $\varphi_n : \mathcal{Y}'_n \to \mathcal{Y}_n$ are compatible.

For a dominant morphism $\mathcal{Y}' \to \mathcal{Y}$, $\mathcal{Y}'_{\mathrm{r}}$ denotes the closed reduced subscheme of $\mathcal{Y}'$ which is the union of all irreducible components of $\mathcal{Y}'$ that dominate an irreducible component of $\mathcal{Y}$. By [ILO, Exp. III, 3.1(ii)], given a finite covering $\{\mathcal{Y}_i \to \widehat{\mathcal{X}}\}_{i\in I}$ in the alteration topology of $\widehat{\mathcal{X}}$, there exist $\alpha_0 \in E$ and $n_0 \geq 1$ such that, for every $\alpha \geq \alpha_0$, every $n \geq n_0$, every section $u : \mathcal{X} \to \mathcal{X}_\alpha$ of t_α which is n-close to s_α, and every family of models $\mathcal{Y}_{i,\alpha}$ of $\mathcal{Y}_i$ over $\mathcal{X}_\alpha$, the family composition morphism $\{(\mathcal{Y}_i^{(u)})_{\mathrm{r}} \to \mathcal{Y}_i^{(u)} \to \mathcal{X}\}_{i\in I}$ is a covering in the alteration topology of $\mathcal{X}$. Furthermore, by loc. cit. 5.4(iii), given a morphism of finite type $\mathcal{Y} \to \widehat{\mathcal{X}}$ with $\mathrm{r}\mathcal{Y}$ regular (resp. of dimension m), there exist $\alpha_0 \in E$ and $n_0 \geq 1$ such that for every $\alpha \geq \alpha_0$, every $n \geq n_0$, every section $u : \mathcal{X} \to \mathcal{X}_\alpha$ of t_α which is n-close to s_α, and every model $\mathcal{Y}_\alpha$ of $\mathcal{Y}$ over $\mathcal{X}_\alpha$, the scheme $\mathcal{Y}^{(u)}$ is regular (resp. of dimension m) in an open neighborhood of $\mathcal{Y}_0^{(u)}$.

Let now $\{\mathcal{Y}_i \stackrel{\varphi_i}{\to} \widehat{\mathcal{X}}\}_{i\in I}$ be a finite covering in the alteration topology of $\widehat{\mathcal{X}}$ such that, for every $i \in I$, $(\mathcal{Y}_i, \varphi_i^{-1}(\widehat{\mathcal{Z}})_{\mathrm{red}})$ is an R_i-snc pair, where R_i is the ring of integers of a finite extension of the fraction field of R. Since $\mathcal{O}(\mathcal{Y}_i) = \varinjlim \mathcal{O}(\mathcal{Y}_{i,\alpha})$, we can find $\alpha_0 \in E$ such that the R-algebra $\mathcal{O}(\mathcal{Y}_{i,\alpha})$ contains R_i for all $\alpha \geq \alpha_0$ and $i \in I$. Replacing E by the subset $\{\alpha | \alpha \geq \alpha_0\}$, we may assume that $R_i \subset \mathcal{O}(\mathcal{Y}_{i,\alpha})$ for all $\alpha \in E$ and $i \in I$. Then, for every $i \in I$, there is an isomorphism of R_i-special fibers $\widetilde{\mathcal{Y}_i} \widetilde{\to} \varprojlim \widetilde{\mathcal{Y}}_{i,\alpha}$. Let $\mathcal{W}_i$ be the regularity locus of $\widetilde{\mathcal{Y}}_i$; it is an open subscheme of $\widetilde{\mathcal{Y}}_i$. Its complement $\Sigma_i = \widetilde{\mathcal{Y}}_i \backslash \mathcal{W}_i$, provided with the reduced structure, is a closed subscheme of $\widetilde{\mathcal{Y}}_i$. Since $\widetilde{\mathcal{Y}}_i$ is a divisor with strict normal crossings, $\mathcal{W}_i$ consists of the points that lie in only one irreducible component of $\widetilde{\mathcal{Y}}_i$ (all of them are of dimension $d_i - 1$, where $d_i = \dim(\mathcal{Y}_i)$) and $\dim(\Sigma_i) \leq d_i - 2$. Replacing E by the subset $\{\alpha | \alpha \geq \alpha_0\}$ for α_0 large enough, we may assume that $\mathcal{W}_i \widetilde{\to} \varprojlim \mathcal{W}_{i,\alpha}$ and $\Sigma_i \widetilde{\to} \varprojlim \Sigma_{i,\alpha}$ for complementary open and closed subschemes $\mathcal{W}_{i,\alpha}, \Sigma_{i,\alpha} \subset \widetilde{\mathcal{Y}}_i$, respectively. By the results stated in the previous paragraph and the proof of [ILO, III, 6.2], there exist $\alpha_0 \in E$, $n_0 \geq 1$

such that for every $\alpha \geq \alpha_0$, every $n \geq n_0$, every section $u : \mathcal{X} \to \mathcal{X}_\alpha$ of t_α which is n-close to s_α, and every $i \in I$, the following is true:

(1) the scheme $\mathcal{Y}_i^{(u)}$ is regular of dimension d_i in an open neighborhood of $(\mathcal{Y}_i^{(u)})_0$;

(2) the scheme $\mathcal{W}_i^{(u)}$ is regular $(d_i - 1)$-equidimensional and

$$\dim(\Sigma_i^{(u)}) \leq d_i - 2;$$

(3) the support of the preimage of $\mathcal{Z}$ in $\mathcal{Y}_i^{(u)}$ is a divisor with strict normal crossings in an open neighborhood of $(\mathcal{Y}_i^{(u)})_0$;

(4) the family $\{(\mathcal{Y}_i^{(u)})_{\mathrm{r}} \to \mathcal{X}\}_{i \in I}$ is a covering in the alteration topology of $\mathcal{X}$.

For $i \in I$, let $\{\mathcal{Y}'_j\}_{j \in J_i}$ be the connected components of an open neighborhood of $(\mathcal{Y}_i^{(u)})_0$ in which the above properties (1) and (3) hold. We also set $R_j = R_i$ for $j \in J_i$. By (4) and [ILO, Exp. II, 4.1.1], $\{\mathcal{Y}'_j \overset{\psi_j}{\to} \mathcal{X}\}_{j \in J}$ with $J = \bigcup_{i \in I} J_i$ is a covering in the alteration topology of $\mathcal{X}$. By the construction, each $\mathcal{Y}'_j$ is connected and regular, and the support of $\psi_j^{-1}(\mathcal{Z})$ is a divisor with strict normal crossings. By (2), if $j \in J_i$, the intersection $\mathcal{W}_i^{(u)} \cap \widetilde{\mathcal{Y}}'_j$ is dense in $\widetilde{\mathcal{Y}}'_j$ and, therefore, $\widetilde{\mathcal{Y}}'_j$ is reduced at the generic points of all of its irreducible components. Since $(\widetilde{\mathcal{Y}}'_j)_{\mathrm{red}}$ is a divisor with strict normal crossing, it follows that $\widetilde{\mathcal{Y}}'_j$ is reduced (see [ILO, Exp. IV, 4.2.4]). Thus, each $(\mathcal{Y}'_j, \psi_j^{-1}(\mathcal{Z}))$ is an R_j-snc pair.

2.3. A version of the Cohen–Gabber theorem. Let R be a complete discrete valuation ring with fraction field K. An open subset $\mathcal{U}$ of a scheme $\mathcal{X}$ over R is said to be R-*dense* if $\widetilde{\mathcal{U}}$ is dense in $\widetilde{\mathcal{X}}$.

Proposition 2.3.1: *Let $\mathcal{X} =$ Spec(A), where A is a local noetherian complete normal and flat R-algebra of dimension $d \geq 1$ with residue field k and such that $\mathbf{m}_R A \neq A$. Suppose that the residue field $\widetilde{R}$ is perfect and the extension $k/\widetilde{R}$ possesses the property of Epp. Then*

(1) *there exists a finite surjective morphism $\mathcal{X}' =$ Spec$(A') \to \mathcal{X}$, where A' is a local normal flat R-algebra which contains a complete discrete valuation ring S whose residue field coincides with that of A' and $\mathbf{m}_S = \mathbf{m}_{R'} S$ for the ring of integers R' of a finite extension of K in the fraction field of S;*

(2) *there exists a finite surjective morphism* $\mathcal{X}' \to \mathrm{Spec}(S[[T_1, \dots, T_{d-1}]])$ *étale at an* R'*-dense open subset of* $\mathcal{X}'$.

Proof. Step 1. We claim that, for every prime ideal $\mathfrak{p} \subset \widetilde{A}$, the extension $L/\widetilde{R}$, where L is the fraction field of the local noetherian complete ring $\widetilde{A}/\mathfrak{p}$, possesses the property of Epp. Indeed, since the field $\widetilde{R}$ is perfect, the residue field k of $\widetilde{A}$ (which coincides with that of A) is separable and, therefore, formally smooth over $\widetilde{R}$. By [EGAIV, Ch. 0, 19.6.2], there exists a subfield k_1 of $\widetilde{A}$ that contains $\widetilde{R}$ and such that the canonical homomorphism $\widetilde{A} \to k$ induces an isomorphism $k_1 \widetilde{\to} k$. By the assumption, the extension $k_1/\widetilde{R}$ possesses the property of Epp, and by [ILO, Exp. IV, 3.2.3(iv)], the extension L/k_1 possesses the same property. The claim now follows from loc. cit., 3.2.3(i).

Step 2. By Step 1, we can apply Epp's theorem [Epp, 1.9] to the minimal prime ideals of $\widetilde{A}$. It follows that one can find a finite extension K' of K such that for the integral closure R' of R in K' the special fiber $\widetilde{\mathcal{X}}''$ of the normalization $\mathcal{X}'' = \mathrm{Spec}(A'')$ of (the reduction of) $\mathcal{X} \otimes_R R'$ is reduced at the generic points of the irreducible components. By [ILO, Exp. IV, 4.2.4], $\widetilde{\mathcal{X}}''$ is reduced. The ring A'' is semi-local, and so for any connected component $\mathcal{X}' = \mathrm{Spec}(A')$ of $\mathcal{X}''$ the ring A' is local and the canonical morphism $\mathcal{X}' \to \mathcal{X}$ is finite surjective. We choose such a connected component $\mathcal{X}'$.

Step 3. Let k' be the residue field of A'. Since the field $\widetilde{R}'$ is finite over $\widetilde{R}$, it is also perfect, and so k' is formally smooth over $\widetilde{R}'$. It follows from [EGAIV, Ch. 0, 19.7.2], that there exist a local noetherian complete ring S, a formally smooth homomorphism $R' \to S$, and an isomorphism $S \otimes_{R'} \widetilde{R}' \widetilde{\to} k'$. (In particular, the residue field of S is k'.) Since the maximal ideal of S is generated by the maximal ideal of R', S is a discrete valuation field.

Step 4. The quotient ring $A'/\mathbf{m}_{R'}A'$ is reduced and its residue field is k'. Since A' is excellent and flat over R', the quotient ring $A'/\mathbf{m}_{R'}A'$ is $(d-1)$-equidimensional. By the Cohen-Gabber theorem [ILO, Exp. IV, 2.1.1], there exist a $\widetilde{R}'$-lifting $k' \hookrightarrow A'/\mathbf{m}_{R'}A'$ and elements $a_1, \dots, a_{d-1}$ in the maximal ideal of $A'/\mathbf{m}_{R'}A'$ such that the induced homomorphism

$$k'[[T_1, \dots, T_{d-1}]] \to A'/\mathbf{m}_{R'}A' : T_i \mapsto a_i$$

is finite and generically étale.

Step 5. The homomorphism $R' \to S$ is formally smooth. It follows that the composition homomorphism $S \to k' \to A'/\mathbf{m}_{R'}A'$ can be lifted to an R'-homomorphism $S \to A'$. Lifting the elements a_i's to A', we get an injective homomorphism $S[[T_1, \dots, T_{d-1}]] \to A'$ which is finite, by [Bou, Ch. III, §2, n° 11, Prop. 14], and étale over the generic point of the R'-special fiber of $\mathrm{Spec}(S[[T_1, \dots, T_{d-1}]])$, by Step 4. ∎

2.4. Partial algebraization.

Proposition 2.4.1: *Let R be a complete discrete valuation ring, and A a local noetherian complete normal faithfully flat R-algebra of dimension $d \geq 2$ and with residue field k. Suppose that the residue field $\widetilde{R}$ is perfect and the extension $k/\widetilde{R}$ possesses the property of Epp. Then there exists a local normal finite A-algebra A' faithfully flat over R and such that, for any finite family of ideals $\{\mathbf{a}'_i\}_{i\in I}$ of A', one can find a local noetherian regular complete and flat R-algebra B of dimension $d-1$ and an R-homomorphism $\alpha : C \to A'$ from a finitely generated B-algebra C of dimension d for which*

1. $\mathbf{n} = \alpha^{-1}(\mathbf{m}_{A'})$ *is a maximal ideal of* C;
2. α *induces an isomorphism* $\widehat{C}_{\mathbf{n}} \widetilde{\to} A'$;
3. *all of the ideals* $\mathbf{a}'_i$ *are induced by ideals of* $C_{\mathbf{n}}$.

In geometric terms, Proposition 2.4.1 states that there exists a finite morphism $f : \mathcal{X}' = \mathrm{Spec}(A') \to \mathcal{X} = \mathrm{Spec}(A)$ with A' as above such that, for any finite family $\{\mathcal{Z}'_i\}_{i\in I}$ of closed subschemes of $\mathcal{X}'$, one can find morphisms $\mathcal{X}' \xrightarrow{g} \mathcal{Y} = \mathrm{Spec}(C) \xrightarrow{\varphi} \mathcal{S} = \mathrm{Spec}(B)$ with B and C as above and $\widehat{\mathcal{X}}' = \mathrm{Spf}(A') \widetilde{\to} \widehat{\mathcal{Y}}_{/y}$ for a closed point $y \in \widetilde{\mathcal{Y}}$ and $\mathcal{Z}'_i = g^{-1}(\mathcal{W}_i)$ for closed subschemes $\mathcal{W}_i \subset \mathcal{Y}$.

Proof. Let $\mathcal{X}' = \mathrm{Spec}(A') \to \mathcal{X} = \mathrm{Spec}(A)$ be a finite surjective morphism with the properties (1) and (2) of Proposition 2.3.1. It suffices to show that the R-algebra A' and every family $\mathbf{a}'_1, \dots, \mathbf{a}'_n$ of primary ideals of A' possess the required properties. Let $\mathbf{a}$ be one of the ideals $\mathbf{a}'_i$. There are the following two cases.

1. $\dim(A'/(\mathbf{a} + \mathbf{m}_{R'}A') = d-1$. In this case the prime ideal $\mathfrak{p}$ which is the radical of $\mathbf{a}$ has height one. Since A' is normal, it follows that $\mathbf{a} = A' \cap \mathfrak{p}^m A'_{\mathfrak{p}}$ for some $m \geq 1$ (see Corollary of Proposition 9 in [Serre, Ch. III, C, §1]).

(2) $\dim(A'/(\mathbf{a}+\mathbf{m}_{R'}A')) < d-1$. In this case, the image of the corresponding closed subset $V(\mathbf{a})$ in $\mathrm{Spec}(S[[T_1,\dots,T_{d-1}]])$ is contained in the closed subset $V(g_{\mathbf{a}})$ for some element $g_{\mathbf{a}} \in S[[T_1,\dots,T_{d-1}]]$, which is not equal to zero modulo $\mathbf{m}_{R'}$, where S is from Proposition 2.3.1.

Let g be the product $\prod g_{\mathbf{a}}$ taken over all $\mathbf{a} \in \{\mathbf{a}'_1,\dots,\mathbf{a}'_n\}$ of type (2). (If there are no such ideals, then $g=1$.) Let also f be an element of $S[[T_1,\dots,T_{d-1}]]$, which is not zero modulo $\mathbf{m}_{R'}$ and such that the morphism

$$\mathcal{X}' \to \mathrm{Spec}(S[[T_1,\dots,T_{d-1}]])$$

is étale outside $V(f)$. We set $h=fg$. Replacing each T_i for $1 \le i \le d-2$ by an element of the form $T_i + T_{d-1}^{N_i}$ with $N_i \ge 1$ and multiplying h by an invertible element, we may assume that h is a T_{d-1}-distinguished element of $S[[T_1,\dots,T_{d-1}]]$, i.e., $h = T_{d-1}^l + \sum_{i=0}^{l-1} p_i T_{d-1}^i$ for elements p_i from the maximal ideal of $B = S[[T_1,\dots,T_{d-2}]]$ (see [Bou, Ch. VII, §3, n° 7, Lemma 3]). We notice that in this case each $g_{\mathbf{a}}$ as above is a product of a T_{d-1}-distinguished element and an invertible element.

Let $B\{T_{d-1}\}$ be the henselization of the ring $B[T_{d-1}]$ at the maximal ideal $(\mathbf{m}_B, T_{d-1})$. The element h lies in the maximal ideal of $B\{T_{d-1}\}$ and, therefore, the pair $(B\{T_{d-1}\},(h))$ is henselian, by [ILO, Ch. V, 1.2.3]. Since h is a T_{d-1}-distinguished element, the Weierstrass preparation theorem [Bou, Ch. VII, §3, n° 8, Prop. 5] implies that the h-adic completion of $B\{T_{d-1}\}$ coincides with $B[[T_{d-1}]] = S[[T_1,\dots,T_{d-1}]]$ (see [ILO, Ch. V, 1.1.3]). Since $\mathcal{X}'$ is finite over $\mathrm{Spec}(B[[T_{d-1}]])$ and étale outside $V(h)$, it follows from Elkik's theorem [Elk, Theorem 5] that $A' \widetilde{\to} C' \otimes_{B\{T_{d-1}\}} B[[T_{d-1}]]$ for a finite local $B\{T_{d-1}\}$-algebra C' étale outside $V(h)$. It follows that the local ring C' is henselian and of dimension d, the preimage of its maximal ideal $\mathbf{m}_{C'}$ in A' coincides with $\mathbf{m}_{A'}$, and A' coincides with the $\mathbf{m}_{C'}$-adic completion of C'. (Notice that residue fields of A' and C' coincide with that of S.)

By [ILO, Exp. V, 2.2.4], each ideal $\mathbf{a}$ of type (1) comes from an ideal of C'. If $\mathbf{a}$ is of type (2), then $g_{\mathbf{a}}$ is a product of a T_{d-1}-distinguished element and an invertible element and, by the Weierstrass preparation theorem, $B[[T_{d-1}]]/(g_{\mathbf{a}})$ is finite over B. This implies that $A'/\mathbf{a}$ is also finite over B and, by [ILO, Exp. V, 2.2.6], $\mathbf{a}$ comes from an ideal of C'.

Recall now that the henselisation $B\{T_{d-1}\}$ is a filtered inductive limit of finitely generated étale $B[T_{d-1}]$-algebras D with a unique maximal ideal $\mathbf{m}$ over the zero ideal of $B[T_{d-1}]$ and $B[[T_{d-1}]] \widetilde{\to} \widehat{D}_{\mathbf{m}}$. Notice that the dimension of every such D is equal to d. By [EGAIV, Theorem 8.8.2], we can find such D and a finite D-algebra C for which there is an isomorphism $C \otimes_D B\{T_{d-1}\} \widetilde{\to} C'$ and all of the ideals $\mathbf{a}'_i$ are induced by ideals of C. It follows that C is of dimension d, and the preimage of $\mathbf{m}_{A'}$ with respect to induced homomorphism $C \to A'$ is a maximal ideal $\mathbf{n}$ with $\widehat{C}_{\mathbf{n}} \widetilde{\to} A'$. ∎

Proof of Theorem 2.1.3. STEP 1. Suppose we are in the situation of Theorem 2.1.3. Since $\mathcal{X}$ is quasi-compact, it suffices to show that every closed point $x \in \mathcal{X}$ has an open neighborhood $\mathcal{U}$ which admits a finite covering $\{\mathcal{X}_i \to \mathcal{U}\}_{i \in I}$ in the alteration topology of $\mathcal{U}$ with the properties stated in the theorem. If $x \notin \widetilde{\mathcal{X}}$, this follows from Gabber's theorem [ILO, Exp. VII, Theorem 1.1]. Suppose therefore that $x \in \widetilde{\mathcal{X}}$. We may assume that $\mathcal{X}$ is of dimension $d \geq 1$ and that the statement is true for schemes of dimension at most $d-1$. By §2.2, one may assume that $\mathcal{X} = \mathrm{Spec}(A)$, where A is a local noetherian normal complete flat R-algebra of dimension d. If $d = 1$, the required statement follows from Epp's theorem [Epp 1.9]. Suppose therefore that $d \geq 2$.

STEP 2. Consider a morphism of schemes $f : \mathcal{X}' \to \mathcal{X}$ that possesses the properties of Proposition 2.4.1. Since it is an alteration, we can replace $\mathcal{X}$ by $\mathcal{X}'$, $\mathcal{Z}$ by $f^{-1}(\mathcal{Z})_{\mathrm{red}}$ and x by the closed point of $\mathcal{X}'$, and so we may assume that there are morphisms of schemes $\mathcal{X} \xrightarrow{g} \mathcal{Y} = \mathrm{Spec}(C) \xrightarrow{\varphi} \mathcal{S} = \mathrm{Spec}(B)$ with the properties of that proposition for the closed subscheme $\mathcal{Z}$. In particular, g induces an isomorphism $\widehat{\mathcal{X}} = \mathrm{Spf}(A) \widetilde{\to} \widehat{\mathcal{Y}}_{/y}$ for a closed point $y \in \widetilde{\mathcal{Y}}$ and $\mathcal{Z} = g^{-1}(\mathcal{W})$ for a closed subset $(\widetilde{\mathcal{Y}})_{\mathrm{red}} \subset \mathcal{W} \subset \mathcal{Y}$.

Suppose there is a finite covering $\{\mathcal{Y}_i \xrightarrow{\psi_i} \mathcal{Y}\}_{i \in I}$ in the alteration topology of $\mathcal{Y}$ such that, for every $i \in I$, $(\mathcal{Y}_i, \mathcal{W}_i)$ with $\mathcal{W}_i = \psi_i^{-1}(\mathcal{W})_{\mathrm{red}}$ is an R_i-snc pair, where R_i is the ring of integers of a finite extension of the fraction field of R. Since the morphism g is flat, it follows that $\{\mathcal{X}_i \to \mathcal{X}\}_{i \in I}$ with $\mathcal{X}_i = \mathcal{Y}_i \times_{\mathcal{Y}} \mathcal{X}$ is a finite covering in the alteration topology of $\mathcal{X}$ (see [ILO, Exp. II, 2.3.4]). Let $\mathcal{Z}_i$ be the preimage of $\mathcal{W}_i$ in $\mathcal{X}_i$ (it coincides with the preimage of $\mathcal{Z}$ in $\mathcal{X}_i$). We claim that each $(\mathcal{X}_i, \mathcal{Z}_i)$ is an R_i-snc pair. Indeed, since $\mathcal{Y}$ is an excellent scheme, the morphism g is regular, and from [EGAIV, 6.8.3] it follows that the morphisms $\mathcal{X}_i \to \mathcal{Y}_i$ are regular. Hence, by [EGAIV, 6.5.2(ii)], regularity of $\mathcal{Y}_i$ implies regularity of $\mathcal{X}_i$. For the same reason, the preimage of any regular subscheme of $\mathcal{Y}_i$ in $\mathcal{X}_i$ is regular. This implies that the support of $\mathcal{Z}_i$ is a divisor with strict normal crossings. Since the R_i-special fiber $\widetilde{\mathcal{X}}_i$ of $\mathcal{X}_i$ is the preimage of the R_i-special fiber $\widetilde{\mathcal{Y}}_i$ of $\mathcal{Y}_i$, it follows also that the support of $\widetilde{\mathcal{X}}_i$ of $\mathcal{X}_i$ is a divisor with strict normal crossings, and it is regular at the generic points of all of its irreducible components. Then $\widetilde{\mathcal{X}}_i$ is reduced, by [ILO, Exp. IV, 4.2.4]. This implies that $\widetilde{\mathcal{X}}_i$ is a divisor with strict normal crossings, and the claim follows.

Thus, replacing $\mathcal{X}$ by $\mathcal{Y}$, we may assume that $\mathcal{X}$ is an integral scheme of dimension d provided with a dominant morphism of finite type of relative dimension one $\varphi : \mathcal{X} \to \mathcal{S} = \mathrm{Spec}(B)$ and x is a closed point in $\widetilde{\mathcal{X}}$. We can compactify the morphism φ and assume that it is proper. Finally, replacing $\mathcal{X}$ by a blow-up, we may assume that $\mathcal{Z}$ is a divisor.

Step 3. By de Jong's theorem [deJ2, 2.4], there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{S} \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ \mathcal{X}' & \xrightarrow{\varphi'} & \mathcal{S}' \end{array}$$

where f and f' are alterations, φ' is a projective semi-stable family of curves with smooth irreducible generic fiber, and $f'^{-1}(\mathcal{Z})_{\mathrm{red}} \subset \mathcal{D} \cup \varphi'^{-1}(\mathcal{T})_{\mathrm{red}}$, where $\mathcal{D}$ is a divisor of $\mathcal{X}'$ étale over $\mathcal{S}'$ which lies in the smoothness locus of φ', and $\mathcal{T}$ is a nowhere dense closed subset of $\mathcal{S}'$ that contains $(\widetilde{\mathcal{S}'})_{\mathrm{red}}$. We can therefore replace $\mathcal{S}$ and $\mathcal{X}$ by $\mathcal{S}'$ and $\mathcal{X}'$, respectively, and assume that φ is a projective semi-stable family of curves with smooth irreducible generic fiber. Of course, the scheme $\mathcal{S}$ is not necessarily regular. Notice also that the above property of φ is preserved by any base change. By the induction hypothesis, there exists a finite covering $\{\mathcal{S}_i \stackrel{\psi_i}{\to} \mathcal{S}\}_{i\in I}$ in the alteration topology of $\mathcal{S}$ such that each $(\mathcal{S}_i, \psi_i^{-1}(\mathcal{T})_{\mathrm{red}})$ is an R_i-snc pair for the ring of integers R_i of a finite extension of the fraction field of R. We can therefore replace R by R_i, $\mathcal{S}$ by $\mathcal{S}_i$, $\mathcal{X}$ by $\mathcal{X} \otimes_{\mathcal{S}} \mathcal{S}_i$, $\mathcal{Z}$ by its preimage in the latter, and x by a closed point of the latter over it, and assume that $(\mathcal{S}, \mathcal{T})$ is an R-snc pair. It follows that, if the point x lies in the smooth locus of the morphism φ, it has an open neighborhood $\mathcal{U}$ for which $(\mathcal{U}, \mathcal{Z} \cap \mathcal{U})$ is an R-snc pair. Assume therefore that x does not lie in the smoothness locus of φ. Since the divisor $\mathcal{D}$ lies in the latter set, we can shrink $\mathcal{X}$ and assume that $\mathcal{D} = \emptyset$. (In particular, φ is already not projective.) Since $\mathcal{Z}$ is a divisor, we may increase it so that $\mathcal{Z} = \varphi^{-1}(\mathcal{T})$.

Step 4. Shrinking $\mathcal{S}$, we may assume that $\mathcal{S} = \mathrm{Spec}(C)$ is affine, $\mathcal{T}$ has n irreducible components for $1 \le n \le d-1$, each of them is generated by a function from a regular system of parameters of C, and the point $\varphi(x)$ lies in the intersection of all of them. Shrinking $\mathcal{S}$ again and replacing $\mathcal{X}$ by an étale neighborhood of the point x, we may assume that $\mathcal{X} = \text{Spec}(A)$ is affine and that it admits an étale morphism to an affine scheme of the form $\text{Spec}(C[u,v]/(uv-c))$ for an element $c \in C$ invertible at $\mathcal{S}\backslash\mathcal{T}$. Since x does not lie in the smoothness locus of φ, we have $u(x) = v(x) = 0$. To finish the proof, we apply Kato's results [Kato] on resolution of toric singularities which are briefly recalled below.

The schemes $\mathcal{S}$ and $\mathcal{X}$ are provided with fine saturated logarithmic structures $(\mathcal{S}, N_{\mathcal{S}})$ and $(\mathcal{X}, M_{\mathcal{X}})$ which are log-regular in the sense of [Kato, (2.1)]. Recall (see [ILO, Exp. VI, §1]) that $N_{\mathcal{S}}$ and $M_{\mathcal{X}}$ are the subsheaves (in the Zariski topology) of the multiplicative submonoids of $\mathcal{O}_{\mathcal{S}}$ and $\mathcal{O}_{\mathcal{X}}$ consisting of functions invertible outside $\mathcal{T}$ and $\mathcal{Z}$, respectively. In our case, the support of $\mathcal{Z}$ coincides with the set $F(\mathcal{X})$ consisting of the points $y \in \mathcal{X}$ such that the maximal ideal $\mathbf{m}_y$ of $\mathcal{O}_{\mathcal{X},y}$ is generated by non-invertible elements of $M_{\mathcal{X},y}$. The set $F(\mathcal{X})$ provided with the induced topology and the inverse image of the sheaf $M_{\mathcal{X}}/\mathcal{O}^*_{\mathcal{X}}$ is monoidal space which is an affine fan in the sense of [Kato, (9.3) and (10.1)].

If $F' \to F(\mathcal{X})$ is a proper subdivision [Kato, (9.6) and (9.7)], then the logarithmic scheme $(\mathcal{X}', M_{\mathcal{X}'}) = (\mathcal{X}, M_{\mathcal{X}}) \times_{F(\mathcal{X})} F'$ is also log-regular, F' is identified with the fan $F(\mathcal{X}')$ of $(\mathcal{X}', M_{\mathcal{X}'})$ (and with the support $\mathcal{Z}'$ of the preimage of $\mathcal{Z}$ in $\mathcal{X}'$), and the morphism $\mathcal{X}' \to \mathcal{X}$ is proper birational [Kato, (10.3)]. Moreover, the scheme $\mathcal{X}'$ is regular at a point $y' \in \mathcal{Z}'$ if and only if the quotient $M_{\mathcal{X}',y'}/\mathcal{O}^*_{\mathcal{X}',y'}$ is isomorphic to a power of $\mathbf{N}$, the additive monoid of non-negative integers $\{0,1,2,\ldots\}$. In this case, the intersection of $\mathcal{Z}'$ with an open neighborhood of y' is a divisor with strict normal crossings, and representatives of the free generators of the stalk in $M_{\mathcal{X}',y'}$ define the irreducible components of $\mathcal{Z}'$ that pass through y'. If, in addition, $y' \in \widetilde{\mathcal{X}}'$ and the product of those of the representatives that define the irreducible components of $\widetilde{\mathcal{X}}'$ that pass through y' is equal to a uniformizing element of R, then $\widetilde{\mathcal{X}}'$ is a divisor with strict normal crossings in an open neighborhood of y' in $\mathcal{X}'$, i.e., $(\mathcal{X}', \mathcal{Z}')$ is an R-snc pair at y'. We are going to construct a required finite covering in the alteration topology of $\mathcal{X}$, which is a composition of such proper birational morphisms and open coverings and which consists of log regular schemes with the above properties.

Step 5. We fix elements $c_1, \ldots, c_n \in C$ that define the irreducible components of $\mathcal{T}$. Multiplying one of them by an invertible element of C, we may assume that the product of c_i's that define the irreducible components of $\widetilde{\mathcal{S}}$ is a uniformizing element π of R. Multiplying the element c by an invertible element of C, we may assume that c lies in the submonoid $P \subset C$ generated by $c_1, \dots, c_n$. Since the latter are a part of a regular system of parameters of C, the monoid P is isomorphic to $\mathbf{N}^n$. Let Q be the submonoid of A generated by P and the elements u and v. Then P and Q are finitely generated saturated monoids, and the canonical homomorphisms $P \to C$ and $Q \to A$ are charts of the logarithmic structures $N_{\mathcal{S}}$ and $M_{\mathcal{X}}$.

Step 6. Let $c = c_1^{r_1} \cdots c_m^{r_m}$ with $1 \le m \le n$ and $r_i \ge 1$ for $1 \le i \le m$. Suppose the elements $c_1, \dots, c_k$ and $c_{m+1}, \dots, c_{m+l}$ for $0 \le k \le m$ and $0 \le l \le n-m$ define the irreducible components of $\widetilde{\mathcal{S}}$. By Step 5, one has

$$\pi = c_1 \cdots c_k \cdot c_{m+1} \cdots c_{m+l}.$$

(The scheme $\mathcal{S}$ is not used anymore in what follows.) The elements

$$c_1, \dots, c_n, c_{n+1} = u$$

are free generators of the abelian group Q^{gr}, and the submonoid $Q \subset Q^{gr}$ is generated by the elements $c_1, \dots, c_n, c_{n+1}, c \cdot c_{n+1}^{-1}$. Let L be the dual abelian group $\operatorname{Hom}(Q^{gr}, \mathbf{Z})$ whose operation is written additively. Its basis is formed by the elements $e_1, \dots, e_n, e_{n+1}$ for which $\langle e_i, c_j \rangle = \delta_{ij}$. If σ is a strongly convex rational cone in $L_{\mathbf{R}} = L \otimes_{\mathbf{Z}} \mathbf{R}$, the intersection of each one-dimensional face of σ with L is isomorphic to $\mathbf{N}$, i.e., it is generated by a unique element, and we denote by $\mathcal{E}(\sigma)$ the set of such generators. Notice that the monoid $\sigma^{\perp} = \{y \in Q^{gr} \big| \langle \ell, y \rangle \ge 0 \text{ for all } \ell \in \sigma\}$ is isomorphic to $\mathbf{N}^{n+1}$ if and only if elements of $\mathcal{E}(\sigma)$ are free generators of the abelian group L.

Step 7. Let σ be the strongly convex rational cone in $L_{\mathbf{R}}$ dual to Q, i.e.,

$$\sigma = Q^{\perp} = \{\ell \in L_{\mathbf{R}} \big| \langle \ell, y \rangle \ge 0 \text{ for all } y \in Q\}.$$

Then

$$\mathcal{E}(\sigma) = \{e_1, \dots, e_n, e_1 + r_1 e_{n+1}, \dots, e_m + r_m e_{n+1}\}.$$

The cone σ can be represented as a union $\sigma_1 \cup \cdots \cup \sigma_m$ with

$$\mathcal{E}(\sigma_i) = \{e_1 + r_1 e_{n+1}, \dots, e_i + r_i e_{n+1}, e_i, \dots, e_n\}.$$

We claim that, for $1 \le i, j \le m$, the intersection $\sigma_i \cap \sigma_j$ is a face in both σ_i and σ_j. Indeed, suppose that $i < j$, and set $d = c_1^{2r_1} \cdots c_i^{2r_i} c_{i+1} \cdots c_{j-1} c_{n+1}^{-2} \in Q^{gr}$. Then the number $\langle \ell, d \rangle$ is zero for $\ell \in \mathcal{E}(\sigma_i) \cap \mathcal{E}(\sigma_j)$, positive for $\ell \in \mathcal{E}(\sigma_i) \backslash \mathcal{E}(\sigma_j)$, and negative for $\ell \in \mathcal{E}(\sigma_j)\backslash\mathcal{E}(\sigma_i)$. This implies that every element from $\sigma_i \cap \sigma_j$ is a linear combination of elements from $\mathcal{E}(\sigma_i) \cap \mathcal{E}(\sigma_j)$ and, therefore,

$$\mathcal{E}(\sigma_i \cap \sigma_j) = \mathcal{E}(\sigma_i) \cap \mathcal{E}(\sigma_j).$$

Since any subset of $\mathcal{E}(\sigma_i)$ (resp. $\mathcal{E}(\sigma_j)$) generates a face of σ_i (resp. σ_j), the claim follows.

The claim implies that $\sigma_1, \ldots, \sigma_m$ are the cones of maximal dimension $n+1$ in the subdivision of σ formed by their joint intersections. Let F' be the corresponding proper subdivision of $F(\mathcal{X})$, and let $(\mathcal{X}', M_{\mathcal{X}'}) = (\mathcal{X}, M_{\mathcal{X}}) \times_{F(\mathcal{X})} F'$. The scheme $\mathcal{X}'$ admits an open covering by the affine schemes $\mathcal{X}_i = \mathrm{Spec}(A_i)$ for $A_i = A \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[Q_i]$, where Q_i is the submonoid of Q^{gr} dual to σ_i. It suffices therefore to construct a proper subdivision of each cone σ_i so that the corresponding proper birational morphism $\mathcal{X}'_i \to \mathcal{X}_i$ possesses the required properties.

Step 8. For $1 \le i \le m$, the cone σ_i can be represented as a union $\sigma_{i1} \cup \cdots \cup \sigma_{ir_i}$ with

$$\mathcal{E}(\sigma_{ij}) = \{e_1 + r_1 e_{n+1}, \ldots, e_{i-1} + r_{i-1} e_{n+1}, e_i + (j-1) e_{n+1}, e_i + j e_{n+1}, e_{i+1}, \ldots, e_n\}.$$

Notice that the latter elements form a basis of L. The reasoning from Step 7 shows that the intersection $\sigma_{ij} \cap \sigma_{il}$ is a face in both σ_{ij} and σ_{il}. Namely, suppose that $1 \le j < l \le r_i$. If $l = j+1$ (resp. $l > j+1$), we set $d = c_1^{r_1} \cdots c_{i-1}^{r_{i-1}} c_i^j c_{n+1}^{-1}$ (resp. $c_1^{2r_1} \cdots c_{i-1}^{2r_{i-1}} c_i^{2j+1} c_{n+1}^{-2}$). Then the number $\langle \ell, d \rangle$ is zero for $\ell \in \mathcal{E}(\sigma_{ij}) \cap \mathcal{E}(\sigma_{il})$, positive for $\ell \in \mathcal{E}(\sigma_{ij})\backslash\mathcal{E}(\sigma_{il})$, and negative for $\ell \in \mathcal{E}(\sigma_{il})\backslash\mathcal{E}(\sigma_{ij})$. It follows that $\sigma_{i1}, \ldots, \sigma_{ir_i}$ are the cones of maximal dimension $n+1$ in the subdivision of σ_i formed by their joint intersections. Let F'_i be the corresponding proper subdivision of $F(\mathcal{X}_i)$. We set $(\mathcal{X}'_i, M_{\mathcal{X}'_i}) = (\mathcal{X}_i, M_{\mathcal{X}_i}) \times_{F(\mathcal{X}_i)} F'_i$ and denote by $\mathcal{Z}'_i$ the preimage of $\mathcal{Z}$ in $\mathcal{X}'_i$. We claim that $(\mathcal{X}'_i, \mathcal{Z}'_i)$ is an R-snc pair.

Indeed, the monoid $Q_{ij} = \sigma_{ij}^{\perp}$ is generated by the $n+1$ elements

$$c_1, \ldots, c_{i-1}, c_1^{r_1} \cdots c_{i-1}^{r_{i-1}} c_i^j c_{n+1}^{-1}, c_1^{-r_1} \cdots c_{i-1}^{-r_{i-1}} c_i^{1-j} c_{n+1}, c_{i+1}, \ldots, c_n$$

which form the basis of the abelian group Q^{gr} dual to the above basis of L. Let $\mathcal{X}_{ij} = \mathrm{Spec}(A_{ij})$ be the open subscheme of $\mathcal{X}'_i$ that corresponds to the cone σ_{ij}. (One has $A_{ij} = A_i \otimes_{\mathbf{Z}[Q_i]} \mathbf{Z}[Q_{ij}]$.) It is a regular affine scheme, and the canonical homomorphism $Q_{ij} \to A_{ij}$ is a chart of the logarithmic structure $M_{\mathcal{X}'_i}$ on $\mathcal{X}_{ij}$. If $i \ge k+1$, the irreducible components of $\widetilde{\mathcal{X}}_{ij}$ are defined by the elements $c_1, \dots, c_k, c_{m+1}, \dots, c_{m+l}$. If $i \le k$, the irreducible components of $\widetilde{\mathcal{X}}_{ij}$ are defined, in addition, by the elements $c_1^{r_1} \cdots c_{i-1}^{r_{i-1}} c_i^j c_{n+1}^{-1}$ and $c_1^{-r_1} \cdots c_{i-1}^{-r_{i-1}} c_i^{1-j} c_{n+1}$. In both cases, π is the product of all of these elements. This implies that the pair $(\mathcal{X}'_i, \mathcal{Z}'_i)$ possesses the required properties (from Step 4), i.e., it is an R-snc pair. ∎

3. Finiteness theorem for special formal schemes

3.1. Formulation of results. Let k be a non-Archimedean field with nontrivial discrete valuation. The nearby cycles and vanishing cycles functors Θ and Ψ_η for a special formal scheme $\mathfrak{X}$ are defined in the same way as for locally finitely presented formal schemes (see [Ber6, §2]). There is also a canonical action of the Galois group $G = \mathrm{Gal}(k^s/k)$ on the vanishing cycles sheaves compatible with the action of G on $\mathfrak{X}_{\overline{s}}$, but this action is not necessarily continuous (see [Ber6, Remark 2.6(i)] and Corollary 3.1.2 below).

We say that an étale sheaf F on the generic fiber $\mathfrak{X}_\eta$ of a special formal scheme $\mathfrak{X}$ over k° is **$\mathfrak{X}$-constructible** if every point of $\mathfrak{X}$ has an open affine neighborhood $\mathfrak{X}' = \mathrm{Spf}(A)$ such that the restriction of F to $\mathfrak{X}'_\eta$ is the pullback of an étale constructible sheaf on $\mathrm{Spec}(A \otimes_{k^\circ} k)$. For example, any locally constant sheaf on $\mathfrak{X}_\eta$, which is induced by a finite discrete G-module, is $\mathfrak{X}$-constructible. The category of abelian $\mathfrak{X}$-constructible sheaves on $\mathfrak{X}_\eta$ is abelian, and the inverse image $F\big|_{\mathfrak{Y}_\eta} = \varphi_\eta^*(F)$ of an $\mathfrak{X}$-constructible sheaf F on $\mathfrak{X}_\eta$ with respect to a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is an $\mathfrak{Y}$-constructible sheaf on $\mathfrak{Y}_\eta$.

Theorem 3.1.1: *Let $\mathfrak{X}$ be a special formal scheme over k°, and F an abelian $\mathfrak{X}$-constructible sheaf on $\mathfrak{X}_\eta$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *Then*

(i) *the sheaves $R^q\Psi_\eta(F)$ are constructible;*
(ii) *if the residue field $\widetilde{k}$ is perfect, the sheaves $R^q\Theta(F)$ are constructible.*

Corollary 3.1.2: *In the situation of Theorem 3.1.1, if $\mathfrak{X}$ is quasicompact, the étale cohomology groups $H^q(\mathfrak{X}_{\overline{\eta}}, F)$ are finite discrete G-modules. In particular, the action of G on the sheaves $R^q\Psi_\eta(F)$ is discrete.*

Proof. By [Ber6, Corollary 2.3(ii)], there is a spectral sequence

$$E_2^{pq} = H^p(\mathfrak{X}_{\overline{s}}, R^q\Psi_\eta(F)) \Longrightarrow H^{p+q}(\mathfrak{X}_{\overline{\eta}}, F)$$

and, therefore, the finiteness statement follows from constructibility of the sheaves $R^q\Psi_\eta(F)$ and Deligne's finiteness theorem [SGA4$\frac{1}{2}$, Th. finitude, 1.1]. To prove the discreteness statement, we notice that there is an increasing sequence $X_1 \subset X_2 \subset \cdots$ of compact analytic subdomains of $\mathfrak{X}_\eta$ such that each X_i lies in the topological interior of X_{i+1} of $\mathfrak{X}_\eta$ and $\mathfrak{X}_\eta = \bigcup_{i=1}^\infty X_i$. By [Ber2, 6.3.12] (see also [Ber8, 4.2]), there is a canonical isomorphism

$$H^q(\mathfrak{X}_{\overline{\eta}}, F) \widetilde{\to} \varprojlim H^q(\overline{X}_i, F).$$

By the above, the group on the left-hand side is finite and, by Theorem 3.1.1, the groups on the right-hand side are finite. It follows that there exists a sufficiently large $i \geq 1$ such that the homomorphism $H^q(\mathfrak{X}_{\overline{\eta}}, F) \to H^q(\overline{X}_i, F)$ is injective. Since X_i is compact, $H^q(\overline{X}_i, F)$ is a discrete G-module and, therefore, the same is true for $H^q(\mathfrak{X}_{\overline{\eta}}, F)$. ∎

The following two statements were proved in [Ber6, 4.1 and 4.5] under the assumption that all of the formal schemes considered are formal completions of schemes of finite type over k° along subschemes of their closed fibers. This assumption was necessary because constructibility of the nearby and vanishing cycles sheaves and finiteness of the étale cohomology groups of compact analytic spaces were not available in general. The same proof works in the general case due to Theorems 3.1.1 and 1.1.1.

Let $\mathfrak{T}$ be a fixed quasicompact special formal scheme over k°, and let F be an abelian étale sheaf on $\mathfrak{T}_\eta$. As in §1.1, given special formal schemes $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{T}$, any $\mathfrak{T}$-morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ gives rise to homomorphisms $\theta^q(\varphi, F)$ between sheaves of nearby cycles on $\mathfrak{Y}_s$ and $\theta^q_\eta(\varphi, F)$ between sheaves of vanishing cycles on $\mathfrak{Y}_{\overline{s}}$, respectively.

Corollary 3.1.3: *Given quasicompact special formal schemes $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{T}$ and an abelian $\mathfrak{T}$-constructible sheaf F on $\mathfrak{T}_\eta$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$*, there exists an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ such that, for any pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ that coincide modulo $\mathcal{J}$, one has $\theta^q_\eta(\varphi, F) = \theta^q_\eta(\psi, F)$ for all $q \geq 0$. If the field $\widetilde{k}$ is perfect, the same fact holds for homomorphisms between sheaves of nearby cycles.* ∎

For a prime l, a **$\mathfrak{T}$-constructible $\mathbf{Z}_l$-sheaf** on $\mathfrak{T}_\eta$ is a projective system

$$F = (F_m)_{m \geq 0}$$

of $\mathfrak{T}$-constructible $\mathbf{Z}/l^{m+1}\mathbf{Z}$-modules F_m such that, for each $m \geq 1$, the canonical homomorphism induces an isomorphism

$$F_m \otimes_{\mathbf{Z}/l^{m+1}\mathbf{Z}} \mathbf{Z}/l^m\mathbf{Z} \widetilde{\to} F_{m-1}.$$

Furthermore, for a special formal scheme $\mathfrak{X}$ over $\mathfrak{T}$ and an ideal of definition $\mathcal{I}$ of $\mathfrak{X}$, we denote by $\mathcal{G}_{\mathcal{I}}(\mathfrak{X}/\mathfrak{T})$ the subgroup of the group $\mathcal{G}(\mathfrak{X}/\mathfrak{T})$ of $\mathfrak{T}$-automorphisms of $\mathfrak{X}$ consisting of the automorphisms trivial modulo $\mathcal{I}$.

COROLLARY 3.1.4: *Let F be an $\mathfrak{T}$-constructible $\mathbf{Z}_l$-sheaf on $\mathfrak{T}_\eta$ with $l \neq \mathrm{char}(\widetilde{k})$. Given a quasicompact formal scheme $\mathfrak{X}$ over $\mathfrak{T}$, there exists an ideal of definition $\mathcal{I}$ of $\mathfrak{X}$ such that the group $\mathcal{G}_{\mathcal{I}}(\mathfrak{X}/\mathfrak{T})$ acts trivially on all of the sheaves $R^q\Psi_\eta(F_m|_{\mathfrak{X}_\eta})$. If the field $\widetilde{k}$ is perfect, the same statement holds for the nearby cycles sheaves.* ∎

The following theorem is an extension to special formal schemes and $\mathfrak{X}$-constructible sheaves of the property of formal schemes locally finitely presented over k°, established in [Ber3, Theorem 4.9] (see also [Ber6, Remark 2.6(i)]).

Let K be a discretely valued non-Archimedean field over k. For a special formal scheme $\mathfrak{X}$ over k°, we denote by $\mathfrak{X}_{\eta_K}$ and $\mathfrak{X}_{s_K}$ the generic and closed fibers of the special formal scheme $\mathfrak{X}_K = \mathfrak{X}\widehat{\otimes}_{k^\circ} K^\circ$ over K°, and denote by Ψ_{η_K} the vanishing cycles functor for $\mathfrak{X}_K$. We fix an embedding of fields $k^s \hookrightarrow K^s$ and denote by g the induced morphism $\mathfrak{X}_{\overline{s}_K} \to \mathfrak{X}_{\overline{s}}$.

THEOREM 3.1.5: *In the above situation, let F be an abelian $\mathfrak{X}$-constructible sheaf on $\mathfrak{X}_\eta$ with torsion orders prime to $\mathrm{char}(\widetilde{k})$. Then there is a canonical isomorphism $g^*(R\Psi_\eta(F)) \widetilde{\to} R\Psi_{\eta_K}(F_K)$, where F_K is the pullback of F to $\mathfrak{X}_{\eta_K}$.*

The last Theorem 3.1.6 is a refinement of Corollary 3.1.2 in the case when the field $\widetilde{k}$ is perfect. For a finite extension K of k in k^{a}, let Θ^K denote the nearby cycles functor for the formal scheme $\mathfrak{X}_K = \mathfrak{X}\widehat{\otimes}_{k^\circ} K^\circ$ and, for an étale sheaf on $\mathfrak{X}_\eta$, we set $\Psi'_\eta(F) = \varinjlim \overline{i}_K^*(\Theta^K(F_K))$, where K runs through finite extensions of k in k^{a}, $\overline{i}_K$ is the morphism $\mathfrak{X}_{\overline{s}} \to \mathfrak{X}_{s_K}$, and F_K is the pullback of F on $\mathfrak{X}_{\eta_K}$. The correspondence $F \mapsto \Psi'_\eta(F)$ is a functor to the category of étale sheaves on $\mathfrak{X}_{\overline{s}}$ provided with a continuous action of the Galois group G compatible with its action on $\mathfrak{X}_{\overline{s}}$. There is a canonical morphism of functors $\Psi'_\eta \to \Psi_\eta$, which is an isomorphism if $\mathfrak{X}$ is of locally finite type over k° (see [Ber3, Lemma 4.3]).

Theorem 3.1.6: *Suppose that the residue field* $\widetilde{k}$ *is perfect, and let* F *be an abelian* $\mathfrak{X}$*-constructible sheaf on* $\mathfrak{X}_\eta$ *with torsion orders prime to* $\mathrm{char}(\widetilde{k})$*. Then*

(i) *if* $\mathfrak{X}$ *is quasicompact, then* $\varinjlim H^q(\mathfrak{X}_{\eta_K}, F) \widetilde{\to} H^q(\mathfrak{X}_{\overline{\eta}}, F)$ *for all* $q \geq 0$*;*
(ii) *the homomorphism* $R\Psi'_\eta(F) \to R\Psi_\eta(F)$ *is an isomorphism.*

Remark 3.1.7: Let Q be an étale sheaf on $\mathfrak{X}_{\overline{s}}$ provided with a continuous action of the group G which is compatible with the action of G on $\mathfrak{X}_{\overline{s}}$. If I is the inertia subgroup of G, then Q^I is an étale sheaf on $\mathfrak{X}_{\overline{s}}$ provided with an action of Galois group of $\widetilde{k}$ and, therefore, it is the pullback of an étale sheaf on $\mathfrak{X}_s$. Let $\mathcal{I}^G(Q)$ denote the latter sheaf. By the definition of the functor Ψ'_η, there is a canonical isomorphism of functors $\Theta \widetilde{\to} \mathcal{I}^G \circ \Psi'_\eta$. It follows that, for any abelian étale sheaf F on $\mathfrak{X}_\eta$, there is a canonical isomorphism $R\Theta(F) \widetilde{\to} R\mathcal{I}^G(R\Psi'_\eta(F))$. Thus, in the situation of Theorem 3.1.6, there is a canonical isomorphism $R\Theta(F) \widetilde{\to} R\mathcal{I}^G(R\Psi_\eta(F))$ and, in particular, there is a Hochschild–Serre spectral sequence $E_2^{p,q} = \mathcal{H}^p(G, R^q\Psi_\eta(F)) \Longrightarrow R^{p+q}\Theta(F)$.

3.2. A uniformization theorem for special formal schemes. A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ between special formal schemes over k° is said to be of locally finite type if the preimage of every open affine subscheme $\mathrm{Spf}(A)$ of $\mathfrak{X}$ is a union of open affine subschemes $\mathrm{Spf}(B)$ with B isomorphic to a quotient of the adic A-algebra $A\{T_1, \ldots, T_m\}$, $m \geq 0$. It is equivalent to the property that the subsheaf of ideals of $\mathcal{O}_{\mathfrak{Y}}$ generated by an ideal of definition of $\mathfrak{X}$ is an ideal of definition of $\mathfrak{Y}$. If such a morphism is quasi-compact, it is said to be of finite type.

Theorem 3.2.1: *Let* $\mathfrak{X}$ *be a special formal scheme over* k°*, and suppose that the residue field* $\widetilde{k}$ *is perfect. Then there exists a morphism of finite type* $\coprod_{i\in I} \mathfrak{Y}_i \to \mathfrak{X}$ *with induced surjective map* $\coprod_{i\in I} \mathfrak{Y}_{i,\eta} \to \mathfrak{X}_\eta$ *and each* $\mathfrak{Y}_i$ *of the form* $\widehat{\mathcal{X}}_{/\mathcal{Y}}$*, where* $\mathcal{X}$ *is a strictly semi-stable affine scheme over* k'° *for a finite extension of* k *and* $\mathcal{Y}$ *is a union of irreducible components of* $\mathcal{X}_s$*.*

Let k be an arbitrary non-Archimedean field with nontrivial discrete valuation, and A be a special k°-algebra. We set $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathcal{X} = \mathrm{Spec}(A)$, and denote by $\mathcal{J}$ a subsheaf of ideals of $\mathcal{O}_{\mathcal{X}}$ which corresponds to a fixed ideal of definition of $\mathfrak{X}$. Recall that A is an excellent ring by results of P. Valabrega [Val1, Proposition 7] (in the equicharacteristic case) and [Val2, Theorem 9] (in the mixed characteristic case). It follows that any scheme of locally finite type over A is also excellent. It follows also that A is regular if and only if the completion $\widehat{\mathcal{O}}_{\mathfrak{X},x}$ of $\mathcal{O}_{\mathfrak{X},x}$ (which coincides with the completion $\widehat{\mathcal{O}}_{\mathcal{X},x}$ of $\mathcal{O}_{\mathcal{X},x}$) is regular for each closed point $x \in \mathfrak{X}_s$. (This allows one to define in the evident way the notion of a regular special formal scheme over k°.)

A scheme of **locally finite type over** $\mathcal{X}$ is a scheme $\mathcal{Y}$ over $\mathcal{X}$ which is a locally finite union of affine schemes of finite type over $\mathcal{X}$. The completion $\widehat{\mathcal{Y}}$ of $\mathcal{Y}$ with respect to the ideal $\mathcal{J}\mathcal{O}_\mathcal{Y}$ is a special formal scheme over k°, and the correspondence $\mathcal{Y} \mapsto \widehat{\mathcal{Y}}$ is a well defined functor. For example, $\widehat{\mathcal{X}} = \mathfrak{X}$. The generic fiber $\widehat{\mathcal{Y}}_\eta$ of $\widehat{\mathcal{Y}}$ is a paracompact (and, in particular, Hausdorff) strictly k-analytic space, and the closed fiber $\widehat{\mathcal{Y}}_s$ of $\widehat{\mathcal{Y}}$ is a scheme of locally finite type over $\widetilde{k}$ which coincides with $\mathcal{Y}_s = \mathcal{Y} \times_\mathcal{X} \mathfrak{X}_s$. There is a canonical morphism of locally ringed spaces $\widehat{\mathcal{Y}} \to \mathcal{Y}$ and, for any closed point $\mathbf{y} \in \widehat{\mathcal{Y}}_s = \mathcal{Y}_s$, this morphism induces an isomorphism of completions of local rings $\widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}} \widetilde{\to} \widehat{\mathcal{O}}_{\widehat{\mathcal{Y}},\mathbf{y}}$. In particular, if the scheme $\mathcal{Y}$ is regular, then so is the formal scheme $\widehat{\mathcal{Y}}$. To establish properties of the functor $\mathcal{Y} \mapsto \widehat{\mathcal{Y}}_\eta$, we describe $\widehat{\mathcal{Y}}_\eta$ in terms of another k-analytic space closely related to it.

Recall (see [Ber6, §1]) that the k-analytic space $\mathfrak{X}_\eta$ is a union of an increasing sequence of affinoid subdomains $V_1 \subset V_2 \subset \cdots$ such that $V_i = \mathcal{M}(\mathcal{A}_{V_i})$ is a Weierstrass subdomain of V_{i+1} and lies in the topological interior of V_{i+1} in $\mathfrak{X}_\eta$. Any scheme $\mathcal{Y}$ of locally finite type over $\mathcal{X}_\eta = \mathrm{Spec}(A \otimes_{k^\circ} k)$ defines a sequence of schemes $\mathcal{Y}_i = \mathcal{Y} \otimes_{\mathcal{X}_\eta} \mathcal{V}_i$, where $\mathcal{V}_i = \mathrm{Spec}(\mathcal{A}_{V_i})$. By [Ber2, §2.6], one can associate with each $\mathcal{Y}_i$ a k-analytic space $\mathcal{Y}_i^{\mathrm{an}}$. It is easy to see that the canonical morphism $\mathcal{Y}_i^{\mathrm{an}} \to \mathcal{Y}_{i+1}^{\mathrm{an}}$ identifies $\mathcal{Y}_i^{\mathrm{an}}$ with a closed analytic domain in $\mathcal{Y}_{i+1}^{\mathrm{an}}$. We can therefore glue all of them and get a k-analytic space $\mathcal{Y}^{\mathrm{an}} = \bigcup_{i=1}^\infty \mathcal{Y}_i^{\mathrm{an}}$. For example, $\mathcal{X}_\eta^{\mathrm{an}} = \mathfrak{X}_\eta$.

The properties of the functor $\mathcal{Y} \mapsto \mathcal{Y}^{\mathrm{an}}$, established in [Ber2, §2.6] in the particular case when $\mathfrak{X}$ is locally finitely presented (and the valuation on k is not necessarily discrete) are easily extended to our case. We only mention that there is a canonical surjective and flat morphism of locally ringed spaces $\mathcal{Y}^{\mathrm{an}} \to \mathcal{Y}$, and that a morphism $\varphi : \mathcal{Z} \to \mathcal{Y}$ between schemes of locally finite type over $\mathcal{X}_\eta$ is surjective (resp. separated; resp. proper; resp. finite; resp. a closed immersion; resp. étale; resp. smooth) if and only if the morphism of k-analytic spaces $\varphi^{\mathrm{an}} : \mathcal{Z}^{\mathrm{an}} \to \mathcal{Y}^{\mathrm{an}}$ possesses the corresponding property. In particular, the above morphism $\mathcal{Y}^{\mathrm{an}} \to \mathcal{Y}$ gives rise to a morphism between the corresponding étale sites. (The particular case of this morphism for $\mathcal{Y} = \mathrm{Spec}(A \otimes_{k^\circ} k)$ was used in the definition of an $\mathfrak{X}$-constructible sheaf.)

Suppose now that $\mathcal{Y}$ is a scheme of locally finite type over $\mathcal{X}$. If $\mathcal{Y} = \mathrm{Spec}(B)$ is affine, then $\mathcal{Y}^{\mathrm{an}}_\eta$ coincides with the set of all multiplicative semi-norms $|\ | : B \to \mathbf{R}_+$ which extend the valuation on k, take values at most one at all elements of A, and are strictly less than one at all elements of an ideal of definition of $\mathfrak{X}$. The k-analytic space $\widehat{\mathcal{Y}}_\eta$ coincides with the subset of the latter which consists of the multiplicative seminorms that take values at most one at all elements of B. In general, there is a canonical morphism $\widehat{\mathcal{Y}}_\eta \to \mathcal{Y}^{\mathrm{an}}_\eta$ which, in the case when $\mathcal{Y}$ is separated, identifies $\widehat{\mathcal{Y}}_\eta$ with a closed analytic subdomain of $\mathcal{Y}^{\mathrm{an}}_\eta$. If $\mathcal{Y}$ is proper over $\mathcal{X}$, then $\widehat{\mathcal{Y}}_\eta \widetilde{\to} \mathcal{Y}^{\mathrm{an}}_\eta$ (it is a particular case of (ii) below).

Lemma 3.2.2: *Let $\varphi : \mathcal{Z} \to \mathcal{Y}$ be a morphism between schemes of locally finite type over $\mathcal{X}$, and let $\varphi^{\mathrm{an}}_\eta$ and $\widehat{\varphi}_\eta$ be the induced morphisms $\mathcal{Z}^{\mathrm{an}}_\eta \to \mathcal{Y}^{\mathrm{an}}_\eta$ and $\widehat{\mathcal{Z}}_\eta \to \widehat{\mathcal{Y}}_\eta$, respectively. Then*

(i) *if φ is of finite type, $\widehat{\varphi}_\eta$ is a compact map;*

(ii) *if φ is proper, the following commutative diagram is cartesian*

$$\begin{array}{ccc} \mathcal{Z}^{\mathrm{an}}_\eta & \xrightarrow{\varphi^{\mathrm{an}}_\eta} & \mathcal{Y}^{\mathrm{an}}_\eta \\ \uparrow & & \uparrow \\ \widehat{\mathcal{Z}}_\eta & \xrightarrow{\widehat{\varphi}_\eta} & \widehat{\mathcal{Y}}_\eta \end{array}$$

(iii) *if φ is proper surjective, $\widehat{\varphi}_\eta$ is a compact surjective map.*

Proof. (i) It suffices to consider the case when $\mathcal{Y} = \mathrm{Spec}(B)$ and $\mathcal{Z} = \mathrm{Spec}(C)$ are affine. In this case B is isomorphic to a quotient of the ring of polynomials $A[T_1, \ldots, T_n]$ by an ideal. It follows that $\widehat{\mathcal{Z}}$ is a Zariski closed formal subscheme of the direct product of $\widehat{\mathcal{Y}}$ and the n-dimensional formal affine space $\mathfrak{A}^n = \mathrm{Spf}(k^\circ\{T_1, \ldots, T_n\})$ and, therefore, $\widehat{\mathcal{Z}}_\eta$ is a closed analytic subspace of the direct product of $\widehat{\mathcal{Y}}_\eta$ and the n-dimensional closed unit polydisc. This implies that the map $\widehat{\varphi}_\eta$ is compact.

(ii) and (iii) follow from the valuative criterion of properness and the above description of the spaces $\widehat{\mathcal{Y}}_\eta$ and $\widehat{\mathcal{Z}}_\eta$. ∎

Corollary 3.2.3: *Let $\mathcal{Y}$ be a scheme of finite type over $\mathcal{X}$. If $\{\mathcal{Y}_i \stackrel{\varphi_i}{\to} \mathcal{Y}\}_{i \in I}$ is a covering in the alteration topology of $\mathcal{Y}$, then $\widehat{\mathcal{Y}}_\eta = \bigcup_{i \in I} \widehat{\varphi}_{i,\eta}(\widehat{\mathcal{Y}}_{i,\eta})$.*

Proof. If it is a proper surjective and maximally finitely dominant morphism $\mathcal{Z} \to \mathcal{Y}$, the required property follows from Lemma 3.2.2(iii). If it is a covering in the Zariski topology of $\mathcal{Y}$, it gives rise to a Zariski topology covering $\{\widehat{\mathcal{Y}}_i \stackrel{\widehat{\varphi}_i}{\to} \widehat{\mathcal{Y}}\}_{i\in I}$ of the special formal scheme $\widehat{\mathcal{Y}}$, and the required property follows from [Ber6, 2.1(ii)]. ∎

THEOREM 3.2.4: *Let A be a special k°-algebra, and let $\mathcal{X}$ be a scheme of finite type over A flat over k°. Suppose that the residue field $\widetilde{k}$ is perfect. Then there exists a finite covering $\{\mathcal{X}_i \stackrel{\varphi_i}{\to} \mathcal{X}\}_{i\in I}$ in the alteration topology of $\mathcal{X}$ such that, for every $i \in I$, $\widehat{\mathcal{X}}_i$ is isomorphic to a formal scheme of the form $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$, where $\mathcal{Y}$ is a strictly semi-stable scheme over k'° for a finite extension k' of k and $\mathcal{Z}$ is a union of irreducible components of $\mathcal{Y}_s$.*

First of all, notice that Theorem 3.2.4 implies Theorem 3.2.1. Indeed, in order to prove the latter we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine. Replacing A by the quotient by its k°-torsion (which does not change $\mathfrak{X}_\eta$), we may assume that $\mathfrak{X}$ is flat over k°. Then the required fact follows from Theorem 3.2.4 and Corollary 3.2.3 applied to the affine scheme $\mathcal{X} = \mathrm{Spec}(A)$.

Proof. STEP 1. Since the statement is local in the alteration topology of the affine scheme $\mathcal{X}$, we can replace $\mathcal{X}$ by an open affine subscheme of a blow-up of $\mathcal{X}$, and so we may assume that the closed subscheme $\mathcal{X}_s$ is a Cartier divisor in $\mathcal{X}$. We now apply Theorem 2.1.3 to the pair $(\mathcal{X}, \widetilde{\mathcal{X}})$. It follows that there is a finite covering $\{\mathcal{X}_i \stackrel{\varphi_i}{\to} \mathcal{X}\}_{i\in I}$ in the alteration topology of $\mathcal{X}$ such that, for every $i \in I$, $\mathcal{X}_i$ is an affine k_i°-snc scheme, where k_i is a finite extension of k. Replacing $\mathcal{X}$ by any of the affine schemes $\mathcal{X}_i$ and the field k by k_i, we may assume that $\mathcal{X} = \mathrm{Spec}(B)$ is an k°-snc scheme. Replacing $\mathcal{X}_s$ by its support, we may assume that it is reduced and a union of irreducible components of $\widetilde{\mathcal{X}}$. Notice that if $\mathcal{U}$ is an open subset of $\mathcal{X}$ which has empty intersection with $\mathcal{X}_s$, then $\widehat{\mathcal{U}}$ is empty, and so it trivially possesses the required property. Thus, it suffices to find a required covering in the alteration topology of an open neighborhood in $\mathcal{X}$ of every closed point $\mathbf{x} \in \mathcal{X}_s$. Shrinking $\mathcal{X}$, we may assume that $\mathbf{x}$ lies in the intersection of all irreducible components $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ of $\widetilde{\mathcal{X}}$.

STEP 2. Let $t_1, \ldots, t_d$ be a regular system of parameters of $\mathcal{O}_{\mathcal{X},\mathbf{x}}$ such that each t_i for $1 \le i \le n \le d$ defines $\mathcal{Z}_i$ in an open neighborhood of $\mathbf{x}$. Shrinking $\mathcal{X}$, we may assume that $t_1, \ldots, t_d \in B$ and, multiplying t_1 by an invertible element of B, we may assume that $\pi = t_1 \cdots t_n$ is a uniformizing element of k°. Suppose also that the irreducible components of $\mathcal{X}_s$ are $\mathcal{Z}_1, \dots, \mathcal{Z}_m$ with $1 \le m \le n$. Then the elements $t_{m+1}, \dots, t_n$ define the other irreducible components of $\widetilde{\mathcal{X}}$, and the element $t_1 \cdots t_m$ generates the ideal $\mathbf{b}$ that defines the reduced closed subscheme $\mathcal{X}_s$. The formal completion of $\mathcal{X}$ is the affine formal scheme $\widehat{\mathcal{X}} = \text{Spf}(\widehat{B})$, where $\widehat{B}$ is the $\mathbf{b}$-adic completion of B. We set $\mathcal{X}' = \text{Spec}(B')$ for $B' = k^\circ[T_1, \dots, T_d]/(T_1 \cdots T_n - \pi)$ and denote by $\mathbf{b}'$ the ideal of B' generated by the element $T_1 \cdots T_m$ and by $\widehat{B}'$ the $\mathbf{b}'$-adic completion of B'. We claim that one can shrink $\mathcal{X}$ so that the morphism of special formal schemes $\widehat{\mathcal{X}} \to \widehat{\mathcal{X}}'$ that corresponds to the homomorphism $B' \to B : T_i \mapsto t_i$ is étale.

Lemma 3.2.5: *Let $\mathcal{X} = \mathrm{Spec}(A)$ be a reduced affine scheme of finite type over a perfect field K, and x a closed point of $\mathcal{X}$. Suppose there are elements $t_1, \dots, t_d \in A$ such that $t_1 \cdots t_m = 0$ with $1 \le m \le d$ and, for every $1 \le i \le m$, the elements $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_d$ form a regular system of parameters of the closed subscheme $\mathcal{X}_i = \mathrm{Spec}(A/(t_i))$ at x. Then the morphism $\mathcal{X} \to \mathcal{X}' = \mathrm{Spec}(A')$ induced by the homomorphism*

$$A' = K[T_1, \dots, T_d]/(T_1 \cdots T_m) \to A : T_i \mapsto t_i$$

is étale at x.

Proof. Let x' be the image of x in $\mathcal{X}'$. Since the morphism considered is of finite type, it suffices to show that the induced homomorphism between completions of local rings $\widehat{\mathcal{O}}_{\mathcal{X}',x'} \to \widehat{\mathcal{O}}_{\mathcal{X},x}$ is étale. The assumptions (that include perfectness of K) imply that the closed subscheme $\mathcal{Y} = \text{Spec}(A/(t_1, \dots, t_m))$ is smooth at x and, therefore, the complete local ring $B = \widehat{\mathcal{O}}_{\mathcal{Y},x} = \widehat{\mathcal{O}}_{\mathcal{X},x}/(t_1, \dots, t_m)$ is formally smooth over K. It follows that the canonical surjection $\widehat{\mathcal{O}}_{\mathcal{X},x} \to B$ has a section $B \to \widehat{\mathcal{O}}_{\mathcal{X},x}$. We claim that the induced homomorphism

$$\alpha : C = B[[T_1, \dots, T_m]]/(T_1 \cdots T_m) \to \widehat{\mathcal{O}}_{\mathcal{X},x} : T_i \mapsto t_i$$

is a bijection. Indeed, since the schemes $\mathcal{X}_i$ are smooth at the point x, α induces bijections $C/(T_i) \widetilde{\to} \widehat{\mathcal{O}}_{\mathcal{X},x}/(t_i)$ and, since C embeds in the direct product of C_i's, α is injective. That it is surjective follows from [Bou, Ch. III, §2, n° 11, Prop. 14]. Since $\mathcal{Y}$ is smooth at x, the canonical homomorphism $K[[T_{m+1}, \dots, T_d]] \to B$ is étale. This implies étaleness of the homomorphism $\widehat{\mathcal{O}}_{\mathcal{X}',x'} \to \widehat{\mathcal{O}}_{\mathcal{X},x}$. ∎

By Lemma 3.2.5, we can shrink $\mathcal{X}$ and assume that the induced morphism $\mathcal{X}_s \to \mathcal{X}'_s = \mathrm{Spec}(\widetilde{k}[T_1, \dots, T_d]/(T_1 \cdots T_m))$ is étale. By [Ber6, 2.1(i)], there exists an étale morphism $\mathfrak{Y} = \mathrm{Spf}(C) \to \widehat{\mathcal{X}}'$ with $\mathfrak{Y}_s \widetilde{\to} \mathcal{X}_s$ over $\mathcal{X}'_s$. Since C is formally étale over $\widehat{B}'$, the latter isomorphism is induced by a unique homomorphism $C \to \widehat{B}$ over $\widehat{B}'$ ([EGAIV, Ch. 0, 19.3.10]). From [Bou, Ch. III, §2, n° 11, Prop. 14] it follows that the homomorphism $C \to \widehat{B}$ is surjective. Since both rings are regular of the same dimension, we get $C \widetilde{\to} \widehat{B}$, and the claim follows.

Step 3. For a special k°-algebra D, let D_s denote its quotient by the Jacobson radical (which coincides with the maximal ideal of definition of D, see [Ber6, §1]) and, for an element $f \in D$ and a polynomial $P \in D[S]$, let f_s and P_s denote their images in D_s and $D_s[S]$, respectively. By the local description of étale morphisms of schemes, we can shrink $\mathcal{X}$ and assume that $\widehat{B}_s = (\widehat{C}_s)_{f_s}$, where $C = B'[S]/(P)$ for a monic polynomial $P \in B'[S]$, $\widehat{C}$ is the $\mathbf{b}'C$-adic completion of C, and an element $f \in C$ is such that the image of the derivative P' is invertible in $(\widehat{C}_s)_{f_s}$. Then $\widehat{B} = \widehat{C}_{\{f\}}$, i.e., $\widehat{B}$ coincides with the $\mathbf{b}'C_f$-adic completion of C_f. By the construction, the morphism $\mathcal{Y} = \mathrm{Spec}(C_f) \to \mathcal{X}' = \mathrm{Spec}(B')$ is étale at the point $\mathbf{x}$. We can therefore shrink $\mathcal{X}$ and $\mathcal{Y}$ so that $\mathcal{X}_s = \mathrm{Spec}(C_f/\mathbf{b}'C_f)$ and $\mathcal{Y}$ is étale over $\mathcal{X}'$. In particular, $\mathcal{Y}$ is strictly semi-stable over k°. Thus, if $\mathcal{Z}$ is the union of the irreducible components of $\mathcal{Y}_s = \mathrm{Spec}(C_f/(\pi))$ which are the preimages of the irreducible components of $\mathcal{X}'_s = \mathrm{Spec}(B'/(\pi))$ defined by the elements $T_1, \dots, T_m$, we get $\widehat{\mathcal{X}} = \widehat{\mathcal{Y}}_{/\mathcal{Z}}$. ∎

Remark 3.2.6: In what follows, we use a consequence of Theorem 3.2.1 which tells that in its situation there exists a morphism of finite type $\mathfrak{Y} \to \mathfrak{X}$ such that the map $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is surjective and $\mathfrak{Y}$ is **locally algebraic** in the sense that locally in the étale topology it is isomorphic to a formal scheme of the form $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, where $\mathcal{X}$ is a scheme of finite type over k° and $\mathcal{Y}$ is a subscheme of $\mathcal{X}_s$.

3.3. Simplicial formal schemes. Let k be a non-Archimedean field with nontrivial discrete valuation, and let $\mathfrak{X}_\bullet$ be a simplicial object in the category of special formal schemes over k° (for brevity, it will be called a simplicial formal scheme over k°). Its generic fiber is the simplicial object $\mathfrak{X}_{\bullet,\eta}$ of the category of paracompact strictly k-analytic spaces, and its closed fiber is the simplicial object $\mathfrak{X}_{\bullet,s}$ of the category of schemes of locally finite type over $\widetilde{k}$.

The nearby cycles and vanishing cycles functors $\Theta : (\mathfrak{X}_{\bullet,\eta})_{\widetilde{\text{ét}}} \to (\mathfrak{X}_{\bullet,s})_{\widetilde{\text{ét}}}$ and $\Psi_\eta : (\mathfrak{X}_{\bullet,\eta})_{\widetilde{\text{ét}}} \to (\mathfrak{X}_{\bullet,\overline{s}})_{\widetilde{\text{ét}}}$ extend the corresponding functors from [Ber6] componentwise. Namely, if $F^\bullet = (F^n)_{n\geq 0}$ is an étale sheaf on $\mathfrak{X}_{\bullet,\eta}$, then

$$\Theta(F^\bullet)^n = \Theta(F^n) \quad \text{and} \quad \Psi_\eta(F^\bullet)^n = \Psi_\eta(F^n).$$

If $F^{\cdot,\bullet}$ is a complex of étale abelian sheaves on $\mathfrak{X}_{\bullet,\eta}$ and $F^{\cdot,\bullet} \to G^{\cdot,\bullet}$ is a resolution by flabby sheaves, then $R\Theta(F^{\cdot,\bullet}) = \Theta(G^{\cdot,\bullet})$ and $R\Psi_\eta(F^{\cdot,\bullet}) = \Psi_\eta(G^{\cdot,\bullet})$ and, in particular, $R\Theta(F^{\cdot,\bullet})^n = R\Theta(F^{\cdot,n})$ and $R\Psi_\eta(F^{\cdot,\bullet})^n = R\Psi_\eta(F^{\cdot,n})$. The following statement follows straightforwardly from [Ber6, 2.3(ii)].

Proposition 3.3.1: *Given a morphism $\varphi : \mathfrak{Y}_\bullet \to \mathfrak{X}_\bullet$ of simplicial formal schemes, one has*

$$R\Theta(R\varphi_{\eta*}F^{\cdot,\bullet})\widetilde{\to}R\varphi_{s*}(R\Theta F^{\cdot,\bullet}) \quad \textit{and} \quad R\Psi_\eta(R\varphi_{\eta*}F^{\cdot,\bullet})\widetilde{\to}R\varphi_{\overline{s}*}(R\Psi_\eta F^{\cdot,\bullet})$$

for all $F^{\cdot,\bullet} \in D^+(\mathfrak{Y}_{\bullet,\eta}, \mathbf{Z})$. ∎

Proposition 3.3.2: *Let $a : \mathfrak{X}_\bullet \to \mathfrak{S}$ be an augmented simplicial formal scheme over k° such that $\mathfrak{X}_{\bullet,\eta} \to \mathfrak{S}_\eta$ is a hypercovering in the étale topology of universal $\mathbf{Z}/d\mathbf{Z}$-cohomological descent for $d \geq 1$ prime to $\text{char}(\widetilde{k})$. Then for any étale $\mathbf{Z}/d\mathbf{Z}$-module F on $\mathfrak{S}_\eta$, there are canonical isomorphisms*

$$R\Theta(F)\widetilde{\to}Ra_{s*}(R\Theta(a_\eta^*F)) \quad \textit{and} \quad R\Psi_\eta(F)\widetilde{\to}Ra_{\overline{s}*}(R\Psi_\eta(a_\eta^*F)).$$

Proof. The assumption implies that $F\widetilde{\to}Ra_{\eta*}(a_\eta^*F)$. By Example 1.2.1(vi), one has $Ra_{\eta*} = \mathrm{s}Ra_{\bullet\eta*}$ and $Ra_{\overline{s}*} = \mathrm{s}Ra_{\bullet\overline{s}*}$. Since the functor $R\Psi_\eta$ on simplicial formal schemes is calculated componentwise and, by Example 1.2.1(v), the functor preserves quasi-isomorphisms, one has $R\Psi_\eta(\mathrm{s}G^{\cdot,\bullet})\widetilde{\to}\mathrm{s}R\Psi_\eta(G^{\cdot,\bullet})$ for all complexes of $\mathbf{Z}/d\mathbf{Z}$-modules $G^{\cdot,\bullet}$ on the constant simplicial formal scheme $\mathfrak{S}_\bullet$; Proposition 3.3.1 applied to the morphism $a_\bullet : \mathfrak{X}_\bullet \to \mathfrak{S}_\bullet$ implies that

$$R\Psi_\eta(F)\widetilde{\to}\mathrm{s}R\Psi_\eta(Ra_{\bullet\eta*}(a_\eta^*F))\widetilde{\to}\mathrm{s}Ra_{\bullet\overline{s}*}(R\Psi_\eta(a_\eta^*F))\widetilde{\to}Ra_{\overline{s}*}(R\Psi_\eta(a_\eta^*F)).$$

The same reasoning is applicable to the nearby cycles functor Θ. ∎

We say that an augmented simplicial formal scheme $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ over k° is **locally algebraic** (resp. **strictly semi-stable**) if each $\mathfrak{Y}_n$ is locally algebraic (resp. is a disjoint union of special formal schemes of the form $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, where $\mathcal{X}$ is a strictly semi-stable scheme over k'° for a finite extension k' of k and $\mathcal{Y}$ is a union of irreducible components of $\mathcal{X}_s$). Furthermore, we say that an augmented simplicial formal scheme $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ over k° is a **compact hypercovering of** $\mathfrak{X}$ if all of the morphisms $\mathfrak{Y}_n \to \mathfrak{X}$ are of finite type and the augmented k-analytic space $\mathfrak{Y}_{\bullet,\eta} \to \mathfrak{X}_\eta$ is a compact hypercovering of $\mathfrak{X}_\eta$. The uniformization Theorem 3.2.1 implies that, if the field $\widetilde{k}$ is perfect, then every special formal scheme over k° admits a strictly semi-stable (and, in particular, locally algebraic) compact hypercovering.

3.4. Proof of Theorems 3.1.1 and 3.1.5. First of all, we notice that Theorem 3.1.5 holds if the residue field $\widetilde{K}$ is purely inseparable over $\widetilde{k}$. Indeed, this follows from the facts that the categories of schemes étale over $\mathfrak{X}_{\overline{s}}$ and over $\mathfrak{X}_{\overline{s}_K}$ are equivalent, the sheaf $R^q\Psi_\eta(F)$ is associated with the presheaf $\mathfrak{Y}_s \mapsto H^q(\mathfrak{Y}_{\overline{\eta}}, F)$ ([Ber6, 2.2(ii)]), and the étale cohomology groups are invariant under algebraically closed extensions of the ground field ([Ber2, 7.6.1]).

Furthermore, if the field $\widetilde{k}$ is not perfect, let k' be a subfield of k^{s} which is maximal with respect to the properties that $\widetilde{k}'$ is purely inseparable over $\widetilde{k}$ and $|k'| = |k|$. Then the field $\widetilde{k}'$ is perfect. (Indeed, if this is not so, then for an element $\alpha \in k'^\circ$ whose residue is not contained in $(\widetilde{k}')^p$ the extension k'' of k' generated by a root of the polynomial $T^p + \pi T - \alpha$, where $p = \mathrm{char}(\widetilde{k})$ and π is a nonzero element of $k'^{\circ\circ}$, is a separable extension of k' for which $\widetilde{k}''$ is purely inseparable over $\widetilde{k}'$ and $|k''| = |k'|$.) By the above remark, in order to prove Theorem 3.1.1(i) and Theorem 3.1.5, we can replace the field k by the completion of k', and so we may assume that the residue field $\widetilde{k}$ is perfect.

Since the statements are local with respect to $\mathfrak{X}$, we may always assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine. We set $\mathcal{X} = \mathrm{Spec}(A)$ and $\mathcal{X}_\eta = \mathrm{Spec}(\mathcal{A})$, where $\mathcal{A} = A \otimes_{k^\circ} k$.

Case 1: The sheaf F is constant, i.e., $F = \Lambda_{\mathfrak{X}_\eta}$ for a finite abelian group Λ of order prime to $\mathrm{char}(\widetilde{k})$. Let $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ be a locally algebraic compact hypercovering.

Theorem 3.1.1: By [Ber6, 3.1] and [SGA4$\frac{1}{2}$, Th. finitude, 3.2 (resp. 3.11)], the cohomology sheaves of the complexes $R\Psi_\eta(\Lambda_{\mathfrak{Y}_{n,\eta}})$ (resp. $R\Theta(\Lambda_{\mathfrak{Y}_{n,\eta}})$) for all $n \ge 0$ are constructible. Deligne's finiteness theorem [SGA4$\frac{1}{2}$, Th. finitude, 1.1] then implies that the cohomology sheaves of the complexes $Ra_{n\overline{s}*}(R\Psi_\eta(\Lambda_{\mathfrak{Y}_{n,\eta}}))$ (resp. $Ra_{n\overline{s}*}(R\Theta(\Lambda_{\mathfrak{Y}_{n,\eta}}))$) for all $n \ge 0$ are constructible, and constructibility of the sheaves $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ (resp. $R^q\Theta(\Lambda_{\mathfrak{X}_\eta})$) follows from Proposition 3.3.2.

Theorem 3.1.5: By Proposition 3.3.2, one has

$$R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta}) = Ra_{\overline{s}*}(R\Psi_\eta(\Lambda_{\mathfrak{Y}_{\bullet,\eta}}))$$

and

$$R\Psi_{\eta_K}(\Lambda_{\mathfrak{X}_{\eta_K}}) = Ra_{\overline{s}_K*}(R\Psi_{\eta_K}(\Lambda_{\mathfrak{Y}_{\bullet,\eta_K}})).$$

Since the formation of vanishing cycles sheaves of schemes is compatible with the changes of henselian discrete valuation rings [SGA4$\frac{1}{2}$, Th. finitude, 3.7], we have

$$g_\bullet^*(R\Psi_\eta(\Lambda_{\mathfrak{Y}_{\bullet,\eta}}))\widetilde{\to}R\Psi_{\eta_K}(\Lambda_{\mathfrak{Y}_{\bullet,\eta_K}}),$$

and the required statement follows from the fact that the morphism of schemes $\mathrm{Spec}(\widetilde{K}^{\mathrm{a}}) \to \mathrm{Spec}(\widetilde{k}^{\mathrm{a}})$ is universally acyclic.

Case 2: F is the pullback of a sheaf of the form $f_*(\Lambda_{\mathcal{Y}})$, where f is a finite morphism $\mathcal{Y} = \mathrm{Spec}(\mathcal{B}) \to \mathcal{X}_\eta$ with integral $\mathcal{B}$. We claim that the homomorphism $\mathcal{A} \to \mathcal{B}$ is induced by a homomorphism of special k°-algebras $A \to B$ with $\mathcal{B} = B \otimes_{k^\circ} k$ for a special k°-algebra B. Indeed, the image $f(\mathcal{Y})$ is a Zariski closed subset of $\mathcal{X}_\eta$. Let $\mathcal{Z}$ be the schematic closure of $f(\mathcal{Y})$ in $\mathcal{X}$. Then $\mathcal{Z} = \mathrm{Spec}(C)$ for a quotient C of A by a prime ideal. The canonical homomorphism $\mathcal{C} = C \otimes_{k^\circ} k \to \mathcal{B}$ is injective and finite. Since the special k°-algebra C is excellent, it is a Japanese ring. It follows that the integral closure B of C in $\mathcal{B}$ is finite over C and, in particular, B is a special k°-algebra, and the claim follows. Thus, the morphism f is induced by the finite morphism of special formal schemes $h : \mathfrak{Y} = \mathrm{Spf}(B) \to \mathfrak{X}$, and one has $F = h_{\eta*}(\Lambda_{\mathfrak{Y}_\eta})$. By [Ber6, Corollary 2.3(ii)], it follows that $R\Theta(F) = h_{s*}(R\Theta(\Lambda_{\mathfrak{Y}_\eta}))$ and $R\Psi_\eta(F) = h_{\overline{s}*}(R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta}))$ and, by the previous case, the sheaves $R^q\Theta(F)$ and $R^q\Psi_\eta(F)$ are constructible (Theorem 3.1.1). For the same reason, one has $R\Psi_{\eta_K}(F_K) = h_{\overline{s}_K*}(R\Psi_{\eta_K}(\Lambda_{\mathfrak{Y}_{\eta_K}}))$ and, by the previous case, $g'^*(R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta}))\widetilde{\to}R\Psi_{\eta_K}(\Lambda_{\mathfrak{Y}_{\eta_K}})$. Again, since the morphism $\mathrm{Spec}(\widetilde{K}^{\mathrm{a}}) \to \mathrm{Spec}(\widetilde{k}^{\mathrm{a}})$ is universally acyclic, we get $g^*(R\Psi_\eta(F))\widetilde{\to}R\Psi_{\eta_K}(F_K)$, where F_K is the pullback of F to $\mathfrak{X}_{\eta_K}$ (Theorem 3.1.5).

Case 3: F is arbitrary. Shrinking $\mathfrak{X}$, we may assume that F is the pullback of an abelian constructible sheaf $\mathcal{F}$ on $\mathcal{X}_\eta$. By [SGA4, Exp. IX, 2.14(ii)], one can find a finite family of finite morphisms $\{\mathcal{Y}_i \stackrel{p_i}{\to} \mathcal{X}_\eta\}_{i\in I}$ and a monomorphism $\mathcal{F} \to \bigoplus_{i\in I} p_{i*}(\mathcal{G}_i)$, where $\mathcal{G}_i$ is an abelian constant sheaf on $\mathcal{Y}_i = \mathrm{Spec}(\mathcal{B}_i)$. Replacing each $\mathcal{Y}_i$ by its family of irreducible components provided with the reduced structure, we may assume that all $\mathcal{Y}_i$'s are integral (in order to use Case 2). In this way we get a long exact sequence $0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots$ of constructible sheaves on $\mathcal{X}_\eta$ such that, for every $n \geq 0$, the nearby cycles sheaves $R^q\Theta(F_n)$ and vanishing cycles sheaves $R^q\Psi_\eta(F_n)$ of the pullback F_n of $\mathcal{F}_n$ to $\mathfrak{X}_\eta$ are constructible, and the latter are preserved by extensions of the ground field. The statements of Theorems 3.1.1 and 3.1.5 are now easily obtained from the spectral sequence that relates the nearby cycles and vanishing cycles sheaves of F and F_K with those of F_n's and $(F_n)_K$'s, respectively. ∎

3.5. Proof of Theorem 3.1.6. As in [Ber6, §2], we fix a functor $\mathfrak{Y}_s \mapsto \mathfrak{Y}$ from the category of schemes étale over $\mathfrak{X}_{\overline{s}}$ to the category of formal schemes étale over $\mathfrak{X}\widehat{\otimes}_{k^\circ}(k^{\mathrm{a}})^\circ$. Suppose $\mathfrak{Y}_s$ is quasicompact. Then $\mathfrak{Y} = \mathfrak{Y}'\widehat{\otimes}_{K^\circ}(k^{\mathrm{a}})^\circ$ for a finite (unramified) extension K of k in k^{a} and a special formal scheme $\mathfrak{Y}'$ over K° étale over $\mathfrak{X}_K$.

Lemma 3.5.1: *In the above situation, let F be an étale sheaf on $\mathfrak{X}_\eta$. Then*

(i) *there is a canonical bijection $\varinjlim F(\mathfrak{Y}'_{\eta_{K'}}) \tilde{\to} \Psi'_\eta(F)(\mathfrak{Y}_s)$, where the inductive limit is taken over finite extensions K' of K in k^{a};*

(ii) *if F is abelian soft, then the sheaf $\Psi'_\eta(F)$ is flabby;*

(iii) *if F is abelian, $R^q\Psi'_\eta(F)$ is associated to the presheaf*

$$\mathfrak{Y}_s \mapsto \varinjlim H^q(\mathfrak{Y}'_{\eta_{K'}}, F)$$

on the family of quasicompact $\mathfrak{Y}_s$.

Proof. (i) Let $\widetilde{F}(\mathfrak{Y}_s)$ denote the set on the left-hand side. By the construction, $\Psi'_\eta(F)$ is the sheaf associated to the presheaf $\mathfrak{Y}_s \mapsto \widetilde{F}(\mathfrak{Y}_s)$. Hence it suffices to show that the latter is a sheaf on the family of quasicompact $\mathfrak{Y}_s$. Given an étale covering $\{\mathfrak{Y}_{i,s} \to \mathfrak{Y}_s\}_{i\in I}$, we have to verify that the following sequence of sets is exact,

$$\widetilde{F}(\mathfrak{Y}_s) \longrightarrow \prod_i \widetilde{F}(\mathfrak{Y}_{i,s}) \rightrightarrows \prod_{i,j} \widetilde{F}(\mathfrak{Y}_{i,s} \times_{\mathfrak{Y}_s} \mathfrak{Y}_{j,s}).$$

Since $\mathfrak{Y}_s$ is quasicompact, we may assume that the covering is finite and all $\mathfrak{Y}_{i,s}$ are also quasicompact. We can therefore find a finite extension K of k in k^{a} such that $\mathfrak{Y}$ and all $\mathfrak{Y}_i$ come from special formal schemes $\mathfrak{Y}'$ and $\mathfrak{Y}'_i$ over K°. By [Ber3, Theorem 3.3(i)], for every finite extension K' of K in k^{a} there is an exact sequence

$$F(\mathfrak{Y}'_{\eta_{K'}}) \longrightarrow \prod_i F(\mathfrak{Y}'_{i,\eta_{K'}}) \rightrightarrows \prod_{i,j} F((\mathfrak{Y}'_{i}\times_{\mathfrak{Y}'}\mathfrak{Y}'_j)_{\eta_{K'}}).$$

Since the set I is finite, exactness of the required sequence follows.

(ii) It suffices to verify that the Čech cohomology groups of the sheaf $\Psi'_\eta(F)$ associated to an étale covering of $\mathfrak{Y}_s$ (which is étale over $\mathfrak{X}_s$) are trivial. For this we may assume that all of the schemes from the covering are quasicompact. We then can use (i) and the reasoning from the proof of the similar statement [Ber3, 4.1(iii)].

The statement (iii) is now easy. ∎

Corollary 3.5.2: (i) *For an étale morphism $\mathfrak{Y} \to \mathfrak{X}$ and an étale abelian sheaf F on $\mathfrak{X}_\eta$, the natural arrow $R\Psi'_\eta(F)\big|_{\mathfrak{Y}_{\overline{s}}} \to R\Psi'_\eta(F\big|_{\mathfrak{Y}_\eta})$ is an isomorphism.*

(ii) *For a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ and an étale abelian sheaf $F^\cdot \in D^+(\mathfrak{Y}_\eta, \mathbf{Z})$, there is a canonical isomorphism $R\Psi'_\eta(R\varphi_{\eta *}(F^\cdot)) \widetilde{\to} R\varphi_{\overline{s}*}(R\Psi'_\eta(F^\cdot))$.* ∎

For an étale abelian sheaf F on a k-analytic space X, we set

$$'H^q(\overline{X}, F) = \varinjlim H^q(X\widehat{\otimes}_k K, F),$$

where the inductive limit is taken over finite extensions of k in k^{a}. It is a discrete G-module which corresponds to the high direct image $R^q f_*(F)$ of F with respect to the canonical morphism $f : X \to \mathcal{M}(k)$. Applying Corollary 3.5.2(ii) and Lemma 3.5.1(iii) to the canonical morphism $\mathfrak{X} \to \mathrm{Spf}(k^\circ)$ for quasicompact $\mathfrak{X}$, we get for an étale abelian sheaf F on $\mathfrak{X}_\eta$ a spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{X}_{\overline{s}}, R^q\Psi'_\eta(F)) \Longrightarrow {}'H^{p+q}(\mathfrak{X}_{\overline{\eta}}, F).$$

By [Ber6, 2.3(ii)], there is a similar spectral sequence for the functor Ψ_η that converges to the groups $H^{p+q}(\mathfrak{X}_{\overline{\eta}}, F)$. Thus, the statement (ii) of Theorem 3.1.6 implies (i).

We also notice that the functor Ψ'_η extends to simplicial formal schemes as in §3.3 and, due to the above results, the analogs of Propositions 3.3.1 and 3.3.2 are valid for it as well. Therefore the reasoning from the proof of Theorems 3.1.1 and 3.1.4 can be used here if we verify the required statement (ii) in the particular case when the sheaf F is constant and $\mathfrak{X}$ is of the form $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, where $\mathcal{X}$ is a scheme of finite type over k° and $\mathcal{Y}$ is a subscheme of $\mathcal{X}_s$. We may even assume that $F = \mathcal{F}^{\mathrm{an}}$, where $\mathcal{F}$ is an abelian constructible sheaf on $\mathcal{X}_\eta$ with torsion orders prime to $\mathrm{char}(\widetilde{k})$. By the comparison theorem [Ber6, 3.1], there are canonical isomorphisms $(R\Theta^K(\mathcal{F}_K))\big|_{\mathcal{Y}_{\widetilde{K}}} \widetilde{\to} R\Theta^K(F_K)$ for all finite extensions K of k in k^{a}, where $\mathcal{F}_K$ is the pullback of $\mathcal{F}$ on $\mathcal{X}_{\eta_K}$. Since $\varinjlim \bar{i}^*_K(R\Theta^K(\mathcal{F}_K))\widetilde{\to}R\Psi_\eta(\mathcal{F})$ and, by the same comparison theorem, $(R\Psi_\eta(\mathcal{F}))\big|_{\mathcal{Y}_{\bar{s}}}\widetilde{\to}R\Psi_\eta(F)$, we get the required isomorphism $R\Psi'_\eta(F)\widetilde{\to}R\Psi_\eta(F)$. ∎

Remark 3.5.3: The assumption on perfectness of the residue field $\widetilde{k}$ is used in order to apply the uniformization Theorem 3.2.1 which reduces the situation to the case when $\mathfrak{X}$ is locally algebraic and F comes from constructible sheaves on the corresponding schemes (e.g., $F = \Lambda_{\mathfrak{X}_\eta}$, where Λ is a finite G-module of order prime to char$(\widetilde{k})$). In that case, perfectness of $\widetilde{k}$ is not necessary, and it is very likely this assumption is not necessary at all.

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