

Finsler metrics of scalar flag curvature and projective invariants

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Abstract. In this paper, we define a new projective invariant and call it \widetilde{W} -curvature. We prove that a Finsler manifold with dimension $n \geq 3$ is of constant flag curvature if and only if its \widetilde{W} -curvature vanishes. Various kinds of projectively flatness of Finsler metrics and their equivalency on Riemannian metrics are also studied.

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Key words: Projective transformation; scalar flag curvature.

1 Introduction

One of the fundamental problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature. The best well-known result towards this question is due to Akbar-Zadeh, which classified compact Finsler manifolds with non-positive constant flag curvature [3]. In a 25 year research, initiated by famous Yasuda-Shimada's theorem [16] and finished by Bao-Robless-Shen's theorem [7], Randers metrics of constant flag curvature have been classified.

On the other hand, there are some well-known projective invariants of Finsler metrics namely, Douglas curvature [5][6][10], Weyl curvature, generalized Douglas - Weyl curvature [4][13] and another projective invariant which is due to Akbar-Zadeh [1]. In [21], Weyl introduces a projective invariant for Riemannian metrics. Then Douglas extends Weyl's projective invariant to Finsler metrics [10]. Finsler metrics with vanishing projective Weyl curvature are called *Weyl metrics*. In [18], Z. Szabó proves that Weyl metrics are exactly Finsler metrics of scalar flag curvature.

In [3], Akbar-Zadeh introduces the non-Riemannian quantity \mathbf{H} which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle, and recently has been studied [11][12]. Akbar-Zadeh proves that for a Weyl manifold of dimension $n \geq 3$, the flag curvature is constant if and only if $\mathbf{H} = 0$. The natural question is: Is there any projectively invariant quantity which characterizes Finsler metrics of constant flag curvature?

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In this paper, using Akbar-Zadeh's method in [1], we define a new projective invariant and call it \widetilde{W} -curvature (see the equation (3.14)). We show that the \widetilde{W} -curvature is another candidate for characterizing Finsler metrics of constant flag curvature. More precisely, we prove the following

Theorem 1.1. *Let (M, F) be a Finsler manifold with dimension $n \geq 3$. Then F is of constant flag curvature if and only if $\widetilde{W} = 0$.*

By Akbar-Zadeh's theorem and Theorem 1.1, we have the following

Corollary 1.1. *Let (M, F) be a Finsler manifold with dimension $n \geq 3$. Suppose that F is of scalar flag curvature. Then $\mathbf{H} = 0$ if and only if $\widetilde{W} = 0$.*

Throughout this paper, we use the Berwald connection on Finsler manifolds [19][20]. The h - and v -covariant derivatives of a Finsler tensor field are denoted by “ $|$ ” and “ $\dot{}$,” respectively.

2 Preliminaries

Let M be an n -dimensional C^∞ manifold. Denote by $T_x M$ the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_x M$ the tangent space of M . A *Finsler metric* on M is a function $F : TM \rightarrow [0, \infty)$ which has the following properties: (i) F is C^∞ on TM_0 ; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM , and (iii) for each $y \in T_x M$, the following quadratic form g_y on $T_x M$ is positive definite,

$$g_y(u, v) := \frac{1}{2} [F^2(y + su + tv)]|_{s,t=0}, \quad u, v \in T_x M.$$

Given a Finsler manifold (M, F) , then a global vector field \mathbf{G} is induced by F on TM_0 , which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, for all $\lambda > 0$. Functions G^i are given by

$$(2.1) \quad G^i := \frac{1}{4} g^{il} \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k,$$

where g_{ij} is the vertical Hessian of $F^2/2$ and g^{ij} denotes its inverse. \mathbf{G} is called the associated *spray* to (M, F) . The projection of an integral curve of \mathbf{G} is called a *geodesic* in M . In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$ [14].

For a vector v^i vertical and horizontal covariant derivative with respect to Berwald connection are given by

$$v^i_{,k} = \dot{\partial}_k v^i, \quad v^i|_k = d_k v^i + G^i_{jk} v^j,$$

where $d_k = \partial_k - G^m_k \dot{\partial}_m$, $\partial_k = \frac{\partial}{\partial x^k}$, $\dot{\partial}_k = \frac{\partial}{\partial y^k}$, $G^i_k = \dot{\partial}_k G^i$ and $G^i_{jk} = \dot{\partial}_j G^i_k$.

In [1], Akbar-Zadeh considers a non-Riemannian quantity \mathbf{H} which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. The quantity $\mathbf{H} = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of \mathbf{E} along geodesics, where $E_{ij} = \frac{1}{2} \partial_m G_{ij}^m$ [12]. More precisely $H_{ij} := E_{ij|m} y^m$. In local coordinates, we have

$$2H_{ij} = y^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial x^m} - 2G^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^i} \frac{\partial^3 G^k}{\partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^j} \frac{\partial^4 G^k}{\partial y^i \partial y^k \partial y^m}.$$

The Riemannian curvature tensor of Berwald connection are given by

$$K_{hjk}^i = d_j G_{hk}^i + G_{hk}^m G_{mj}^i - d_k G_{hj}^i + G_{hj}^m G_{mk}^i.$$

Let $K_{jk}^i = K_{0jk}^i$ and $K_k^i = K_{0k}^i$. Then we have

$$K_{jk}^i = \frac{1}{3} \{ \dot{\partial}_j K_k^i - \dot{\partial}_k K_j^i \}.$$

Then, the Riemann curvature operator of Berwald connection at $y \in T_x M$ is defined by $\mathbf{K}_y = K_k^i dx^k \otimes \frac{\partial}{\partial x^i} |_x : T_x M \rightarrow T_x M$, which is a family of linear maps on tangent spaces. The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald [8]. For a flag $P = \text{span}\{y, u\} \subset T_x M$ with flagpole y , the *flag curvature* $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{K}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, which is just the sectional curvature of P in Riemannian geometry. We say that a Finsler metric F is of *scalar curvature* if for any $y \in T_x M$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent space TM_0 . If $\mathbf{K} = \text{constant}$, then F is said to be of *constant flag curvature*.

The projective Weyl curvature is defined as follows

$$W_{jkl}^i := K_{jkl}^i - \frac{1}{1-n^2} \left\{ -\delta_j^i (\tilde{K}_{kl} - \tilde{K}_{lk}) - \delta_k^i \tilde{K}_{jl} + \delta_l^i \tilde{K}_{jk} - y^r \dot{\partial}_j (\tilde{K}_{kl} - \tilde{K}_{lk}) \right\}$$

where $\tilde{K}_{jk} := nK_{jk} + K_{kj} + y^r \dot{\partial}_j K_{kr}$. As it is well known, a Finsler metric is of scalar flag curvature if and only if $W_{jkl}^i = 0$.

3 C-projective Weyl curvature

Let $\phi : F^n \rightarrow \bar{F}^n$ be a diffeomorphism. We call ϕ a projective mapping if there exists a positive homogeneous scalar function $P(x, y)$ of degree one satisfying

$$\bar{G}^i = G^i + Py^i.$$

In this case, P is called the projective factor ([17]). Under a projective transformation with projective factor P , the Riemannian curvature tensor of Berwald connection change as follows

$$(3.1) \quad \bar{K}^i_{hjk} = K^i_{hjk} + y^i \dot{\partial}_h Q_{jk} + \delta_h^i Q_{jk} + \delta_j^i \dot{\partial}_h Q_k - \delta_k^i \dot{\partial}_h Q_j,$$

where $Q_i = d_i P - P P_i$ and $Q_{ij} = \dot{\partial}_i Q_j - \dot{\partial}_j Q_i$. A projective transformation with projective factor P is said to be C -projective if $Q_{ij} = 0$.

Let X be a projective vector field on a Finsler manifold (M, F) . Let the vector field X in a local coordinate (x^i) on M be written in the form $X = X^i(x) \partial_i$. Then the complete lift of X is denoted by \hat{X} and locally defined by $\hat{X} = X^i \partial_i + y^j \partial_j X^i \dot{\partial}_i$. Suppose that $\mathcal{L}_{\hat{X}}$ stands for Lie derivative with respect to the complete lift of X . Then we have

$$(3.2) \quad \begin{aligned} \mathcal{L}_{\hat{X}} G^i &= P y^i, \\ \mathcal{L}_{\hat{X}} G^i_k &= \delta_k^i P + y^i P_k, \\ \mathcal{L}_{\hat{X}} G^i_{jk} &= \delta_j^i P_k + \delta_k^i P_j + y^i P_{jk}, \\ \mathcal{L}_{\hat{X}} G^i_{jkl} &= \delta_j^i P_{kl} + \delta_k^i P_{jl} + \delta_l^i P_{kj} + y^i P_{jkl}, \end{aligned}$$

$$(3.3) \quad \mathcal{L}_{\hat{X}} K^i_{jkl} = \delta_j^i (P_{l|k} - P_{k|l}) + \delta_l^i P_{j|k} - \delta_k^i P_{j|l} + y^i \dot{\partial}_j (P_{l|k} - P_{k|l}).$$

Since $Q_{ij} = P_{i|j} - P_{j|i}$, we have

$$(3.4) \quad \mathcal{L}_{\hat{X}} K^i_{jkl} = \delta_j^i Q_{lk} + \delta_l^i P_{j|k} - \delta_k^i P_{j|l} + y^i \dot{\partial}_j Q_{lk}.$$

We have

$$(3.5) \quad \dot{\partial}_j P_{k|l} = P_{jk|l} - P_r G^r_{jkl}.$$

Contracting i and k in (3.4), we get

$$(3.6) \quad \mathcal{L}_{\hat{X}} K_{jl} = P_{l|j} - n P_{j|l} + P_{j|l} y^s.$$

Consequently

$$(3.7) \quad \mathcal{L}_{\hat{X}} (y^r \dot{\partial}_l K_{jr}) = -(n+1) P_{j|l} y^s.$$

Hence

$$(3.8) \quad P_{j|l} y^s = -\frac{1}{n+1} L(\hat{X})(y^r \dot{\partial}_l K_{jr}),$$

and

$$(3.9) \quad \mathcal{L}_{\hat{X}} (K_{jl} + \frac{1}{n+1} y^r \dot{\partial}_l K_{jr}) = P_{l|j} - n P_{j|l},$$

$$(3.10) \quad \mathcal{L}_{\hat{X}} (K_{lj} + \frac{1}{n+1} y^r \dot{\partial}_j K_{lr}) = P_{j|l} - n P_{l|j}.$$

Using (3.9) and (3.10), one can obtain

$$(3.11) \quad P_{j|l} = \frac{1}{1-n^2} \mathcal{L}_{\hat{X}} \left\{ K_{lj} + \frac{1}{n+1} y^r \dot{\partial}_j K_{lr} + n K_{jl} + \frac{n}{n+1} y^r \dot{\partial}_l K_{jr} \right\}.$$

If $Q_{ij} = 0$, then (3.4) reduces to the following

$$(3.12) \quad \mathcal{L}_{\hat{X}} K_{jkl}^i = \delta_l^i P_{j|k} - \delta_k^i P_{j|l}.$$

Using (3.11) and eliminating $P_{j|l}$ from (3.12), we are led to the following tensor

$$(3.13) \quad \begin{aligned} \widetilde{W}_{jkl}^i := & K_{jkl}^i - \frac{1}{1-n^2} \delta_l^i \left\{ \widetilde{K}_{jk} + \frac{n}{n+1} y^r (\dot{\partial}_k K_{jr} - \dot{\partial}_j K_{kr}) \right\} \\ & + \frac{1}{1-n^2} \delta_k^i \left\{ \widetilde{K}_{jl} + \frac{n}{n+1} y^r (\dot{\partial}_l K_{jr} - \dot{\partial}_j K_{lr}) \right\}. \end{aligned}$$

Since $y^j y^r \dot{\partial}_k K_{jr} = 0$, if we put $\widetilde{W}_k^i := \widetilde{W}_{jkl}^i y^j y^l$, then we have

$$(3.14) \quad \widetilde{W}_k^i = K_k^i - \frac{1}{1-n^2} \left\{ y^i \widetilde{K}_{0k} - \delta_k^i \widetilde{K}_{00} \right\}.$$

The tensor \widetilde{W}_k^i is said to be *C-projective Weyl curvature* or \widetilde{W} -curvature. According to the way we construct \widetilde{W} , it is easy to see that \widetilde{W} is *C-projective invariant tensor*. A Finsler metric F is called *C-projective Weyl metric* if its *C-projective Weyl-curvature* vanishes. First, we prove that the class of Weyl metrics contains the class of *C-projective Weyl metrics*.

Theorem 3.1. *Let F be a C-projective Weyl metric. Then F is a Weyl metric.*

Proof. By assumption, we have the following

$$(3.15) \quad K_k^i - \frac{1}{1-n^2} \left\{ y^i \widetilde{K}_{0k} - \delta_k^i \widetilde{K}_{00} \right\} = 0.$$

Contracting (3.15) with y_i implies that

$$(3.16) \quad F^2 \widetilde{K}_{0k} - y_k \widetilde{K}_{00} = 0.$$

Hence

$$(3.17) \quad \widetilde{K}_{0k} = F^{-2} y_k \widetilde{K}_{00}.$$

Plugging (3.17) into (3.15), we get

$$(3.18) \quad K_k^i = \frac{1}{1-n^2} \widetilde{K}_{00} h_k^i,$$

which means that F is of scalar flag curvature. Hence, F is a Weyl metric. \square

4 Proof of Theorem 1.1

To prove Theorem 1.1, we need to find the \widetilde{W} -curvature of Weyl metrics.

Proposition 4.1. *Let F be a Finsler metric of scalar flag curvature λ . Then \widetilde{W} -curvature is given by*

$$(4.1) \quad \widetilde{W}_k^i = \frac{1}{3}F^2 y^i \lambda_k,$$

where $\lambda_k := \dot{\partial}_k \lambda$.

Proof. By assumption, the Riemannian curvature of Berwald connection is in the following form.

$$(4.2) \quad \begin{aligned} K_{jkl}^i &= \lambda(\delta_k^i g_{jl} - \delta_l^i g_{jk}) + \lambda_j F(\delta_k^i F_l - \delta_l^i F_k) + \frac{1}{3}F^2(h_k^i \lambda_{jl} - h_l^i \lambda_{jk}) \\ &\quad + \frac{1}{3}\lambda_l F(2\delta_k^i F_j - 2\delta_j^i F_k - g_{jk} \ell^i) \\ &\quad - \frac{1}{3}F\lambda_k(2\delta_l^i F_j - 2\delta_j^i F_l - g_{jl} \ell^i). \end{aligned}$$

where $\lambda_{ij} = \dot{\partial}_j \lambda_i$. Hence, we have

$$(4.3) \quad K_k^i = \lambda F^2 h_k^i.$$

Then, we get the following relations.

$$(4.4) \quad \begin{aligned} K_{jl} &= (n-1)(\lambda g_{jl} + FF_l \lambda_j) + \frac{n-2}{3}(F^2 \lambda_{jl} + 2FF_j \lambda_l), \\ K_{00} &= \lambda(n-1)F^2, \quad \widetilde{K}_{00} = \lambda(n^2-1)F^2, \\ K_{k0} &= \lambda(n-1)FF_k + \frac{2n-1}{3}F^2 \lambda_k, \\ K_{0k} &= \lambda(n-1)FF_k + \frac{n-2}{3}F^2 \lambda_k, \\ \widetilde{K}_{0k} &= (n^2-1)(\lambda FF_k + \frac{1}{3}F^2 \lambda_k). \end{aligned}$$

Plugging (4.3) and (4.4) into (3.14), we get the result. \square

Lemma 4.1. *Let (M, F) be a C -projective Weyl manifold with dimension $n \geq 3$. Then F is of constant flag curvature.*

Proof. By Theorem 3.1 and Proposition 4.1, we have

$$\widetilde{W}_k^i = \frac{1}{3}F^2 y^i \lambda_k.$$

From assumption, we get $\lambda_k = 0$. It means that F is of isotropic flag curvature. The result follows by Schur's Lemma. \square

Now, let us consider the case F being of constant flag curvature.

Lemma 4.2. *Let F be a Finsler metric of constant flag curvature $\mathbf{K} = \lambda$. Then F is C -projective Weyl metric.*

Proof. If F is of constant flag curvature λ , then (4.2) reduces to the following

$$(4.5) \quad K_{jkl}^i = \lambda(g_{jl}\delta_k^i - g_{jk}\delta_l^i).$$

Hence

$$(4.6) \quad K_{jl} = \lambda(1-n)g_{jl}, \quad \tilde{K}_{jk} = \lambda(1-n^2)g_{jl}.$$

Plugging (4.6) into (3.13), we obtain $\widetilde{W}_{jkl}^i = 0$ and consequently $\widetilde{W}_k^i = 0$. \square

5 Reduction in Riemannian manifolds

As mentioned before, in Finsler metrics F^n of scalar flag curvature with $(n \geq 3)$, we have this equivalence $\widetilde{W} = 0$ if and only if $\mathbf{H} = 0$. Observing C -projective invariance of \widetilde{W} -curvature, one can conjecture that \mathbf{H} -curvature must be C -projective invariant too. Here, we prove that this is true. By definition, $H_{ij} = E_{ij|s}y^s$. Under a projective transformation with the projective factor P , we have the following relations:

$$\begin{aligned} \bar{E}_{ij} &= E_{ij} + \frac{n+1}{2}P_{ij}, \\ y^l \bar{d}_l &= y^l d_l - 2Py^m \dot{\partial}_m, \\ \bar{E}_{mj} \bar{G}_i^m &= E_{mj} G_i^m + PE_{ij} + \frac{n+1}{2}(P_{mj} G_i^m + PP_{ij}). \end{aligned}$$

Now, we can prove the following

Proposition 5.1. *\mathbf{H} -curvature is C -projective invariant.*

Proof. Under a projective transformation, we have

$$\begin{aligned} \bar{H}_{ij} &= \bar{E}_{ij|l}y^l \\ &= y^l \bar{d}_l \bar{E}_{ij} - \bar{E}_{mj} \bar{G}_i^m - \bar{E}_{im} \bar{G}_j^m \\ &= (y^l d_l \bar{E}_{ij} - 2Py^m \dot{\partial}_m \bar{E}_{ij}) - \bar{E}_{mj} \bar{G}_i^m - \bar{E}_{im} \bar{G}_j^m \\ &= y^l d_l E_{ij} + \frac{n+1}{2}y^l d_l P_{ij} + 2PE_{ij} + (n+1)PP_{ij} - \bar{E}_{mj} \bar{G}_i^m - \bar{E}_{im} \bar{G}_j^m \\ &= y^l d_l E_{ij} - E_{mj} G_i^m - E_{im} G_j^m + \frac{n+1}{2}(y^l d_l P_{ij} - P_{mj} G_i^m - P_{im} G_j^m) \\ (5.1) \quad &= H_{ij} + \frac{n+1}{2}(y^l d_l P_{ij} - P_{mj} G_i^m - P_{im} G_j^m). \end{aligned}$$

On the other hand, we have

$$(5.2) \quad \begin{aligned} y^l \dot{\partial}_i Q_{jl} &= y^l d_l P_{ij} - P_{mj} G_i^m - y^l d_j P_{il} \\ &= y^l d_l P_{ij} - P_{mj} G_i^m - P_{mi} G_j^m \end{aligned}$$

Plugging (5.2) into (5.1) yields

$$(5.3) \quad \bar{H}_{ij} = H_{ij} + \frac{n+1}{2} y^l \dot{\partial}_i Q_{jl}.$$

We deal with C -projective mapping, i.e., $Q_{ij} = 0$. Hence $\bar{H}_{ij} = H_{ij}$. This completes the proof. \square

A locally projectively flat Finsler manifold (M, F) with the projective factor P is said to be locally C -projectively flat if P satisfies $Q_{ij} = 0$, this means F is locally C -projectively related to a locally Minkowskian metric.

Example. Let Θ be the Funk metric on the Euclidean unit ball $B^n(1)$, i.e.,

$$\Theta(x, y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)} + \langle x, y \rangle}{1 - |x|^2}, \quad y \in T_x B^n(1) \simeq \mathbb{R}^n,$$

where \langle, \rangle and $|\cdot|$ denotes the Euclidean inner product and norm on \mathbb{R}^n , respectively. For a constant vector $a \in \mathbb{R}^n$, let F be the Finsler metric given by

$$(5.4) \quad F := \left\{ 1 + \langle a, x \rangle + \frac{\langle a, y \rangle}{\Theta} \right\} \{ \Theta + \Theta_{x^k} x^k \}.$$

In [15], Shen proves that F is projectively flat with projective factor $P = \Theta$. A direct computation shows that $Q_{ij} = 0$. Hence, F is locally C -projectively flat. Moreover, Shen proves that F is of constant flag curvature $\mathbf{K} = 0$.

Every locally Minkowskian metric has vanishing \mathbf{H} -curvature. It is well known that every locally projectively flat Finsler metric is of scalar flag curvature. In the case of locally C -projectively flat Finsler metrics we have the following

Corollary 5.1. *Let F be a locally C -projectively flat Finsler metric. Then F is of constant flag curvature.*

In studying the subgroups of the group of projective transformations, Akbar-Zadeh considers projective vector fields satisfying $P_{ij} = 0$ and calls this kind of vector fields, restricted projective vector field [1]. The condition $P_{ij} = 0$ means that the projective factor P is linear, which is always true in Riemannian manifolds. Hence, in Riemannian manifolds, every projective transformation is restricted.

Let us define locally restricted projectively flatness similar to C -projectively flatness. Note that Finsler metric given in Example 1 is not locally restricted projectively flat. In fact, a restricted projective vector field with $P = a_i(x)y^i$ is C -projective vector field, if $a_i(x)$ is gradient, that is $P = d\sigma$ for some scalar function on the underlying manifold.

Using (3.11) and eliminating $P_{j|l}$ from (3.4), Akbar-Zadeh introduces the following tensor

$$(5.5) \quad \begin{aligned} {}^*W_{jkl}^i &:= K_{jkl}^i - \frac{1}{n^2 - 1} \left\{ \delta_k^i (nK_{jl} + K_{lj}) - \delta_l^i (nK_{jk} + K_{kj}) \right\} \\ &\quad - \frac{1}{n + 1} \delta_j^i (K_{kl} - K_{lk}). \end{aligned}$$

Under a C -projective mapping, we have

$$(5.6) \quad {}^* \overline{W}^i_{jkl} = {}^* W^i_{jkl} + 2\delta_k^i \dot{\partial}_l Q_j - 2\delta_l^i \dot{\partial}_j Q_k.$$

This means that ${}^* W^i_{jkl}$ is not a C -projective invariant. In fact, ${}^* W^i_{jkl}$ is a restricted projective invariant. We call ${}^* W^i_{jkl}$ *restricted projective Weyl-curvature*. The geometric importance of the restricted projective Weyl-curvature is to characterize Finsler metrics of constant flag curvature, i.e., a Finsler metric F^n with ($n \geq 3$) is of constant flag curvature if and only if F has vanishing restricted projective Weyl-curvature ([2] page 209).

Now let F be a Riemannian metric. By Beltrami's well-known theorem, locally projectively flat Riemannian manifolds are exactly Riemannian manifolds of constant sectional curvature. Summarizing up, we get the following reduction theorem in Riemannian manifolds.

Theorem 5.1. *Let (M, F) be Riemannian manifold with dimension $n \geq 3$. Then the following are equivalent.*

1. F is locally projectively flat.
2. F is locally restricted projectively flat.
3. F is locally C -projectively flat.

This is not true in generic Finslerian manifolds. The non-equivalence between these kind of projective mappings in Finsler manifolds reveals the complexity of Finsler spaces.

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