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FINSLER STRUCTURES FOR THE PART METRIC AND HILBERT'S PROJECTIVE METRIC AND APPLICATIONS TO ORDINARY DIFFERENTIAL EQUATIONS

Roger D. Nussbaum*

Mathematics Department, Rutgers University, New Brunswick, NJ 08903

Dedicated to the memory of Peter Hess

0. Introduction. Over the past thirty years, a powerful theory of monotone dynamical systems has been developed by many authors. A *partial* list of contributors would include N. Alikakos, E. N. Dancer, M. Hirsch, P. Hess, M.A. Krasnoselskii, U. Krause, H. Matano, P. Polacik, H.L. Smith, P. Takǎč and H. Thieme. If one understands the subject more generally as a chapter in the study of linear and nonlinear operators which map a subset of a "cone" C_1, into a cone C_2, then the relevant literature, encompassing as it does the beautiful classical theory of positive linear operators, is enormous. Usually, in the study of monotone dynamical systems, it has been assumed that the map or flows in question are "strongly monotone." In this paper we shall try to show that a significant part of this theory does not depend on monotonicity, and is a special case of results about maps T which take a metric space (M, ρ) into itself and satisfy

$$\rho(T(x), T(y)) < \rho(x, y) \quad \text{for all } x \neq y \quad \text{or} \tag{0.1}$$

$$\rho(T(x), T(y)) \leq \rho(x, y) \quad \text{for all } x, y, \tag{0.2}$$

or sometimes are just assumed Lipschitzian. We will usually assume that M is a subset of a cone C in a vector space and ρ will be either the "part metric" or "Hilbert's projective metric" (see Section 1 below). We specifically emphasize that we allow equation (0.2) (so T is "nonexpansive") and that in this case there are many intriguing open and apparently difficult questions concerning the behaviour of iterates of T: see Section 3 below.

As we have already remarked, the assumption of strong monotonicity has usually been made; and in many applications this is a natural assumption. It seems less widely known that there are important applications where strong monotonicity fails and where, in addition, equation (0.2), but not equation (0.1), is satisfied. To illustrate this point, we mention a class of examples which arises in statistical mechanics [14, 15, 21, 22], in machine scheduling problems [4, 16, 17] and elsewhere. Let S denote a compact Hausdorff space, $C(S) = X$, the Banach space of continuous, real-valued functions on S (in the sup norm), K the cone of nonnegative functions in $C(S)$ and $\overset{\circ}{K}$ the interior of K. If S is the set of positive integers i, $1 \leq i \leq n$, $C(S) = \mathbb{R}^n$ and

$$K := K^n := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n\}.$$

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Let $a(s,t)$ be a given continuous, real-valued function on $S \times S$ (write $a(s,t) = a_{ij}$ if $S = \{i : 1 \le i \le n,\ i \in \mathbb{Z}\}$) and define $F : X \to X$ and $G : X \to X$ by

$$(F(x))(s) = \max_{t \in S}(a(s,t) + X(t)), \quad (G(x))(s) = \min_{t \in S}(a(s,t) + x(t)). \tag{0.3}$$

One is interested in behaviour of iterates of F or of G. By making the change of variables $x(t) = \log(y(t))$ for $y \in \overset{\circ}{K}$ and $a(s,t) = \log(c(s,t))$, one can equivalently study the maps $\tilde{F} : \overset{\circ}{K} \to \overset{\circ}{K}$, and $\tilde{G} : \overset{\circ}{K} \to \overset{\circ}{K}$ given by

$$(\tilde{F}(y))(s) = \max_{t \in S}(c(s,t)y(t)), \quad (\tilde{G}(y))(s) = \min_{t \in S}(c(s,t)y(t)). \tag{0.4}$$

The maps F and G are monotone ($u \le v$ implies $F(u) \le F(v)$), compact and nonexpansive with respect to the sup norm on X; and $\tilde{F}$ and $\tilde{G}$ are monotone, nonexpansive with respect to the part metric and Hilbert's projective metric on $\overset{\circ}{K}$, and homogeneous of degree one. The maps are not, in general, strongly monotone, nor do they satisfy equation (0.1).

In this generality, not too much is rigorously known about behaviour of iterates of F or G. If $X = \mathbb{R}^n$, the maps F and G are given coordinate-wise by

$$F_i(x) = \max\{a_{ij} + x_j : 1 \le j \le n\} \text{ and } G_i(x) = \min\{a_{ij} + x_j : 1 \le j \le n\}. \tag{0.5}$$

Essentially complete analyses of the behaviour of iterates of F and G were obtained independently in [4] and [41], which contain further references to the extensive literature on equation (0.5).

For $\delta = \pm 1$ and S any set of reals, define $\mu_\delta(S) = \max(\{s : s \in S\})$ if $\delta = +1$ and $\mu_\delta(S) = \min(\{s : s \in S\})$ if $\delta = -1$. Let ε_{ij}, $1 \le i, j \le n$, and δ_i, $1 \le i \le n$, be given sets of real numbers with $|\varepsilon_{ij}| = 1 = |\delta_i|$ for all i, j. Define a map $H : \mathbb{R}^n \to R^n$ coordinate-wise by

$$H_i(x) = \mu_{\delta_i}(\{a_{ij} + \varepsilon_{ij}x_j : 1 \le j \le n\}), \tag{0.6}$$

so H generalizes F and G in equation (0.5). It is easy to show (use Proposition 1.2 below) that H is nonexpansive with respect to the l_∞-norm on $\mathbb{R}^n$ and H is monotone (but not strongly monotone) if $\varepsilon_{ij} = 1$ for all i, j. However, it is interesting to note that even if one assumes that H has a fixed point and is monotone, the detailed analysis in [4] and [41] concerning iterates of H fails completely; and the same is certainly true for the general map in equation (0.6), although results described in Section 3 (see Theorem 3.1) provide some information.

A first step in applying results about maps which are Lipschitz in the part metric or Hilbert's projective metric is to determine useful criteria for computing the Lipschitz constant of a map in these metrics. For maps which are monotone, some results in this direction can be found in the literature: see [11], [19], [29], [35], [36], [37], [42], [52], [56]. We especially mention beautiful, classical results concerning the Lipschitz constant of positive linear operators with respect to Hilbert's projective metric: see [7], [8], [11], [12], [13], [19], [20], [27] and the discussion on pp. 42–45 of [36]. However, very little has been done on the problem of computing Lipschitz constants for maps which may not be monotone (although the reader should note Theorem 4.1 in [29]). For this reason Sections 1 and 2 of this paper are devoted to establishing geometric facts about the part metric p and Hilbert's projective metric d, these metrics being considered on appropriate subsets S of a general cone K in a normed linear space X. The key step is to determine a class of minimal geodesics for p and d and to describe "Finsler structures" for (S, p) and (S, d). Once a Finsler structure has been given, it is possible to give useful formulas for the Lipschitz constants of maps f defined on S. It may be interesting to note that our results provide new information even in the case of linear maps: see Corollary 2.2, Remark 2.2 and Remark 2.3 in Section 2.

In the important special case that $K = K^n \subseteq \mathbb{R}^n$ and $S = \overset{\circ}{K^n}$ or $S = S_\psi = \{x \in \overset{\circ}{K^n} : \psi(x) = 1\}$, where ψ is a given linear functional which is positive on $\overset{\circ}{K^n}$, these results are not new. In fact define $\Phi : \overset{\circ}{K^n} \to \mathbb{R}^n$ by

$$\Phi(x) = \log(x) = (\log(x_1), \log(x_2), \dots, \log(x_n)).$$

It is observed in [35] and in Proposition 1.6 on p. 20 of [36] that $\Phi : (\overset{\circ}{K^n}, p) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ is an isometry onto, so $(\overset{\circ}{K^n}, p)$ obtains a Finsler structure from $(\mathbb{R}^n, \|\cdot\|_\infty)$. Similarly, define $V = \{y \in \mathbb{R}^n : y_n = 0\}$ and define a norm ω on V by

$$\omega(y) = (\max_i y_i) - (\min_i y_i).$$

It is observed in [35] and in Proposition 1.7 on p. 22 of [36] that (S_ψ, d) is isometric to (V, ω), with the isometry given by Φ if $S_\psi = \{x : x_n = 1\}$. Thus (S_ψ, d) has a Finsler structure or, equivalently, the space of rays in $\overset{\circ}{K^n}$ with metric d has a Finsler structure. Wojtkowski [55] has independently observed the existence of a Finsler structure on $(\overset{\circ}{K^n}, d)$.

Section 3 of this paper is devoted to some applications. For reasons of length, we restrict ourselves to the case of the part metric and ordinary differential equations in finite dimensional Banach spaces. The first part of the section basically describes known results but, with an eye to later applications, gives the results in greater generality than in the literature. If K is a cone with nonempty interior in a finite dimensional Banach space X, $B \subset B_1 \subset \overset{\circ}{K}$ and $f : \mathbb{R} \times B_1 \to X$ is a locally Lipschitzian map, the remainder of the section is concerned with

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0 \in B, \tag{0.7}$$

which has a solution $x(t) = x(t; t_0, x_0)$. Usually, it is assumed that $f(t+1, u) = f(t, u)$ for all $t \in \mathbb{R}$, $u \in B_1$. Conditions are given which insure that $x(t; t_0, x_0) \in B$ for all $x_0 \in B$ and $t \geq t_0$. For $t \geq t_0$, estimates are given for the Lipschitz constant with respect to the part metric of the map $U(t, t_0) : B \to B$ given by

$$U(t, t_0)(x_0) = x(t; t_0, x_0).$$

These results are applied to the special case that $K = K^n$, where relatively simple explicit formulas are possible. Our theorems give generalizations, without monotonicity assumptions, of results concerning cooperative systems of differential equations. As an example, we consider variants of equations studied by Aronsson and Mellander [2] and Lajmanovich and Yorke [31]. Even in the well-studied monotone case, our theorems sometimes provide new information, especially in the absence of irreducibility assumptions.

1. Finsler structure and Lipschitz maps for the part metric. In this section we develop some general ideas which we will need for later applications to differential equations and iterated nonlinear maps.

If V is a real vector and C is a subset of V, we shall call C a "cone" (with vertex at 0) if (a) C is convex, (b) $tC = \{tx : x \in C\} \subset C$ for all $t \geq 0$ and (c) $C \cap (-C) = \{0\}$. If C satisfies (a) and (b) but not necessarily (c), we shall call C a wedge. In contrast to much of the literature, we do not necessarily assume that V is a Hausdorff topological vector space or that C is closed in V. If V is a Hausdorff t. v. s. and C is a cone in V and C is closed, we shall call C a "closed cone." If C is a cone in a real vector space V, C induces a partial ordering $\leq_C$ on V by

$$x \leq_C y \text{ if and only if } y - x \in C. \tag{1.1}$$

If C is obvious we write $\leq$ instead of $\leq_C$. If $x \in C$ and $y \in V$, we say that "x dominates y" if there exist real numbers α and β with

$$\alpha x \leq_C y \leq_C \beta x. \tag{1.2}$$

If, also, $x \neq 0$, we use Bushell's notation[11] and define

$$\begin{aligned} M(y/x; C) &= \inf\{\beta \in \mathbb{R} : y \leq_C \beta x\} \\ m(y/x; C) &= \sup\{\alpha \in \mathbb{R} : \alpha x \leq_C y\} \\ \omega(y/x; C) &= M(y/x; C) - m(y/x; C). \end{aligned} \tag{1.3}$$

We make the convention that $\omega(0/0; C) = 0$. If C is obvious, we shall write $M(y/x)$ instead of $M(y/x; C)$, etc. The quantity $\omega(y/x; C)$ is called the "oscillation of y over x." If $V = C(S)$, the space of continuous real-valued functions on a compact space S, and C is the cone of nonnegative functions on S, then $M(y/x; C)$ is the usual maximum of $z(s) = y(s)x(s)^{-1}$ for $s \in S$, $m(y/x; C)$ is the minimum of z on S and $\omega(y/x; C)$ is the usual oscillation of z, namely, $\max_{s \in S} z(s) - \min_{s \in S} z(s)$.

If $x, y \in C - \{0\}$ we shall say that "x is comparable to y in C" and write $x \sim_C y$ (or $x \sim y$ if there is no chance of confusion) if there exist positive reals α and β with

$$\alpha x \leq y \leq \beta x.$$

It is easy to see that $\sim_C$ defines an equivalence relation on $C - \{0\}$. If $u \in C - \{0\}$ we shall always define

$$P(u) = \{x \in C : x \sim_C u\}. \tag{1.4}$$

$P(u)$ is called the part of C equivalent to u. On $P(u) \times P(u)$, we can define two important functions, the "part metric" or "Thompson's metric" p and "Hilbert's projective metric" d:

$$\begin{aligned} p(x, y; C) &= \log(\max(M(y/x; C), (m(y/x; C))^{-1})) \\ d(x, y; C) &= \log(M(y/x; C)(m(y/x; C))^{-1}). \end{aligned} \tag{1.5}$$

As usual, we shall write $p(x, y)$ and $d(x, y)$ when there is no danger of confusion. It is useful to note that

$$p(x, y) = \log(\inf\{R \geq 1 : R^{-1}x \leq y \leq Rx\}).$$

We make the convention that $p(0, 0) = 0$ and $d(0, 0) = 0$.

The following lemma lists the basic properties of p and d; proofs (in slightly less general settings) are given in [11] and [52] or can be supplied by the reader. See also Chapter I of [36].

Lemma 1.1. *Let C be a cone in a real vector space V. If x, y and z are comparable elements of $C - \{0\}$ and λ and μ are positive reals, it follows that*

$$d(x, y) = d(y, x), d(x, z) \leq d(x, y) + d(y, z)$$

and

$$d(\lambda x, \mu y) = d(x, y) \quad \text{and} \quad d(x, \lambda x) = 0.$$

Similarly, it is true that

$$p(x, y) = p(y, x) \quad \text{and} \quad p(x, z) \leq p(x, y) + p(y, z).$$

In our generality a technical difficulty arises: it may happen that $p(x, y) = 0$ for $y \neq x$ or $d(x, y) = 0$ for $y \neq \lambda x$, $\lambda > 0$. Both phenomena occur if $V = \mathbb{R}^2$ and $C = \{(0, 0)\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$. See [19] for further details. For simplicity we shall make further restrictions on C to eliminate the possibilities described above.

If V is a real vector space and $x, y \in V$, we shall always define

$$V(x, y) = \{ax + by : a, b \in \mathbb{R}\}, \tag{1.7}$$

so $V(x, y)$ is a finite dimensional real vector space with $\dim(V(x, y)) \leq 2$. As is well-known (see [44], Chapter 1), there is a unique topology on $V(x, y)$ which makes $V(x, y)$ a Hausdorff topological vector space, and we shall always assume that $V(x, y)$ is given this topology.

Definition 1.1. Let C be a cone in a real vector space V. We shall say that C is "almost Archimedean" if, for all $x, y \in V$, the closure of $C \cap V(x, y)$ in $V(x, y)$ is a cone.

The concept of "almost Archimedean" was apparently introduced by F. F. Bonsall [10], who used an ostensibly different definition.

Definition 1.2. (See [19]). Let C be a cone in a real vector space V. We shall say that C is "almost Archimedean" if, whenever y and z are elements of V with $-\varepsilon y \leq_C z \leq_C \varepsilon y$ for all $\varepsilon > 0$, it follows that $z = 0$.

It is not hard to prove that Definitions 1.1 and 1.2 are equivalent. We shall leave the verification of this equivalence to the reader. The following lemma is the motivation for introducing the definition of almost Archimedean.

Lemma 1.2. *Let C be an almost Archimedean cone in a real vector space V. If $x \sim_C y$ and $p(x, y; C) = 0$, then $x = y$; and if $x \sim_C y$ and $d(x, y; C) = 0$, then there exists $\beta > 0$ with $y = \beta x$.*

Proof. First, suppose that $x \sim_C y$ and let $D = C \cap V(x, y)$ and $\overline{D}$ = the closure of D in $V(x, y)$, so $\overline{D}$ is a cone. If $p(x, y; C) = 0$, we must have

$$\inf\{R \geq 1 : R^{-1}x \leq_C y \leq_C Rx\} = 1.$$

If $u, v \in D$, it is easy to check that $u \leq_D v$ if and only if $u \leq_C v$. Thus we obtain that for all $R > 1$,

$$R^{-1}x \leq_D y \leq_D Rx.$$

It follows that $Rx - y \in \overline{D}$ and $y - R^{-1}x \in \overline{D}$ for all $R > 1$, and taking limits we see that $x - y \in \overline{D}$ and $y - x \in \overline{D}$. Since $\overline{D}$ is assumed a cone, $y = x$.

If $x \sim_C y$ and $d(x, y; C) = 0$, we have that $m(y/x; C) = M(y/x; C)$. It is easy to check that $m(y/x; D) = m(y/x; C)$ and $M(y/x; C) = M(y/x; D) = \beta > 0$. It follows that there exist sequences (α_n) and (β_n) of positive reals with

$$\alpha_n x \leq_D y \leq_D \beta_n x \quad \text{and} \quad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \beta.$$

Thus we have that $y - \alpha_n x \in \overline{D}$ and $\beta_n x - y \in \overline{D}$. Taking limits we conclude that $\beta x - y \in \overline{D}$ and $y - \beta x \in \overline{D}$, which implies (since $\overline{D}$ is a cone) that $y = \beta x$. $\square$

If C is an almost Archimedean cone in a real vector space V and $u \in C - \{0\}$, we define a set V_u by

$$V_u = \{y \in V : \exists \alpha > 0 \quad \text{with} \quad -\alpha u \leq y \leq \alpha u\}. \tag{1.8}$$

We define a norm $|\cdot|_u$ on V_u by

$$|y|_u = \inf\{\alpha > 0 : -\alpha u \leq y \leq \alpha u\}. \tag{1.9}$$

Definition 1.2 implies that if $|y|_u = 0$, then $y = 0$; the verification that $|\cdot|_u$ is a norm is left to the reader. See [36], p. 14, for further references. If $v \sim_C u$, one can easily verify that $|\cdot|_v$ and $|\cdot|_u$ are equivalent norms. (Recall that norms $|\cdot|$ and $\|\cdot\|$ on a vector space W are "equivalent" if there exist positive constants A and B with $A|w| \leq \|w\| \leq B|w|$ for all $w \in W$). We shall always consider V_u a normed linear space with norm $|\cdot|_u$.

For V_u as above, we define $C_u = C \cap V_u$ and note that C_u is a cone in V_u and the $P(u)$ (see equation (1.4)) is the interior of C_u in V_u. If $D := C_u$ and $x, y \in V_u$, one can also see that $x \leq_D y$ if and only if $x \leq_C y$ and that $|x|_u \leq |y|_u$ if $0 \leq_C x \leq_C y$.

For the reader's convenience we collect in the next proposition some results concerning the connection between the topologies induces by p, d and the norm on a normed linear space. The results given are refinements of those in Chapter I of [36] and are related to theorems in [52], [11] and in Chapter 3 of [18].

Proposition 1.1. *Let C be a cone in a normed linear space $(V, \|\cdot\|)$. Assume that there exists a constant $A \geq 1$ such that $\|x\| \leq A\|y\|$ for all $x, y \in C$ with $0 \leq x \leq y$. Then C is almost Archimedean. If $u \in C - \{0\}$ and $P(u)$, V_u and $|\cdot|_u$ are defined by equations* (1.4), (1.8) *and* (1.9) *respectively the topology induced on $P(u)$ by the part metric p is the same as the topology induced on $P(u)$ by $|\cdot|_u$. For all $x, y \in P(u)$ we have*

$$\|x - y\| \leq A[\exp(p(x, y)) - 1][\|x\| + \|y\|] \qquad \textit{and} \tag{1.10}$$

$$|x - y|_u \leq [\exp(p(x, y)) - 1][|x|_u + |y|_u]. \tag{1.11}$$

If $y \in P(u)$, then there exists $r = r(y) > 0$ such that $\{z \in V_u : |z - y|_u < r\} \subset P(u)$ and for all $x \in V_u$ with $|x - y|_u < r$ we have

$$p(x, y) \leq \max\left(\log\left(\tfrac{r}{r - |x-y|_u}\right), \log\left(\tfrac{r + |y-x|_u}{r}\right)\right). \tag{1.12}$$

If there exists $\rho = \rho(y) > 0$ such that $\{z \in V_u : \|z - y\| < \rho\} \subset P(u)$, then for all $x \in V_u$ with $\|x - y\| < \rho$ we have

$$p(x, y) \leq \max\left(\log\left(\tfrac{\rho}{\rho - \|x-y\|}\right), \log\left(\tfrac{\rho + \|x-y\|}{\rho}\right)\right). \tag{1.13}$$

If $\Sigma = \{x \in P(u) : \|x\| = 1\}$ (respectively, $S = \{x \in P(u) : |x|_u = 1\}$) the topologies induced on Σ (respectively, S) by the norm $|\cdot|_u$ and by p are the same; and the topologies induced on S by $|\cdot|_u$, p and Hilbert's projective metric d are the same. Furthermore, for all $x, y \in \Sigma$ we have

$$\|x-y\| \leq 2A\kappa[\exp(d(x,y))-1], \quad \textit{where } \kappa := \min(m(x/y; C), m(y/x; C)) \leq 1. \tag{1.14}$$

For all $x, y \in S$ and for κ as in equation (1.14) *we have*

$$|x-y|_u \leq 2\kappa[\exp(d(x,y))-1]. \tag{1.15}$$

If $y \in P(u)$, r is defined as in equation (1.12), *$x \in P(u)$ and $|x-y|_u < r$, we have*

$$d(x,y) \leq \log\left[\frac{r+|x-y|_u}{r-|x-y|_u}\right] \tag{1.16}$$

If there exists $\rho = \rho(y) > 0$ as defined in equation (1.13), *then for all $x \in P(u)$ with $\|x-y\| < \rho$ we have*

$$d(x,y) \leq \log\left[\frac{\rho+\|x-y\|}{\rho-\|x-y\|}\right]. \tag{1.17}$$

If the interior of C is nonempty in $(V, \|\cdot\|)$ and $u \in \overset{\circ}{C}$ (so $P(u) = \overset{\circ}{C}$) then $|\cdot|_u$ and $\|\cdot\|$ are equivalent norms on V and give the same topology on $\overset{\circ}{C}$, Σ and S.

Proof. We use Definition 1.2 to prove that C is almost Archimedean. Suppose that $y, z \in V$ and $-\varepsilon y \leq z \leq \varepsilon y$ for all $\varepsilon > 0$, so $0 \leq z + \varepsilon y \leq 2\varepsilon y$ for all $\varepsilon > 0$. It follows that

$$\|z+\varepsilon y\| \leq 2\varepsilon A\|y\| \quad \text{and} \quad \|z\| \leq \varepsilon\|y\| + \|z+\varepsilon y\| \leq \varepsilon(2A+1)\|y\|.$$

Letting $\varepsilon \to 0^+$, we conclude that $z = 0$ and C is almost Archimedean. Lemmas 1.1 and 1.2 now imply that p gives a metric on $P(u)$ and d a metric on S or Σ.

The argument above also shows that for all $z \in V_u$,

$$\|z\| \leq (2A+1)\|u\|\,|z|_u.$$

It suffices to show that if $|z|_u = 1$, then $\|z\| \leq (2A+1)\|u\|$. However, if $|z|_u = 1$, we find that $0 \leq z + u \leq 2u$, so

$$\|z\| \leq \|u\| + \|z+u\| \leq 2A\|u\| + \|u\| = (2A+1)\|u\|.$$

If d, and d_2 are any two metrics defined on a set $\Gamma \times \Gamma$, they determine topologies on Γ. These topologies are the same if and only if for every $r > 0$ and every $x \in \Gamma$, there exists $\sigma = \sigma(r,x) > 0$ so that $B_\sigma^2(x) \subset B_r^1(x)$ and $B_\sigma^1(x) \subset B_r^2(x)$, where $B_s^j(x) := \{y \in \Gamma : d_j(y,x) < s\}$. Using this characterization, the reader can verify that the statements in Proposition 1.1 asserting that topologies given by various metrics are the same follow easily from equation (1.10) – equation (1.17).

To prove equation (1.10) and (1.11), suppose that $x, y \in P(u)$. If $R = \exp(p(x,y))$ and $R_1 > R$ we know that

$$R_1^{-1}y \leq x \leq R_1 y \quad \text{and} \quad 0 \leq x - R_1^{-1}y \leq (R_1 - R_1^{-1})y.$$

It follows from these inequalities and by letting $R_1 \to R$ that

$$\|x - R^{-1}y\| \leq A(R - R^{-1})\|y\| \quad \text{and} \quad |x - R^{-1}y|_u \leq (R - R^{-1})|y|_u.$$

These arguments are symmetric in the roles of x and y, so we obtain by interchanging x and y that

$$\|y - R^{-1}x\| \leq A(R - R^{-1})\|x\| \quad \text{and} \quad |y - R^{-1}x|_u \leq (R - R^{-1})|x|_u.$$

Adding these inequalities and using the triangle inequality yields $(R^{-1} + 1)\|x - y\| \leq \|x - R^{-1}y\| + \|R^{-1}x - y\| \leq A(R - R^{-1})(\|x\| + \|y\|)$ and $(R^{-1} + 1)|x - y|_u \leq |x - R^{-1}y|_u + |R^{-1}x - y|_u \leq (R - R^{-1})(|x|_u + |y|_u)$. After dividing by $R^{-1} + 1$, these inequalities give equation (1.10) and (1.11).

If $y \in P(u)$, we leave to the reader the easy verification that there exists $r(y) > 0$ such that if $x \in V_u$ and $|x - y|_u < r(y) = r$, then $x \in P(u)$. It *may* also happen that there exists $\rho = \rho(y) > 0$ with $\{x \in V_u : \|x - y\| < \rho\} \subset P(u)$. Essentially the same argument used in Remark 1.4, p. 16 in [36], shows that if $|x - y| < r$, then

$$r(r + |x - y|_u)^{-1} \leq m(y/x) \quad \text{and} \quad r(r - |x - y|_u)^{-1} \geq M(y/x) \tag{1.18}$$

Similarly, if $\rho = \rho(y) > 0$ exists,

$$\rho(\rho + \|x - y\|)^{-1} \leq m(y/x) \quad \text{and} \quad \rho(\rho - \|x - y\|)^{-1} \geq M(y/x) \tag{1.19}$$

These inequalities imply equation (1.12) and equation (1.13) and also yield equation (1.16) and equation (1.17).

In order to prove equation (1.14) and equation (1.15), suppose either that $x, y \in \Sigma$ or that $x, y \in S$. We can assume that $x \neq y$ and define $\alpha = m(y/x; C)$ and $\beta = M(y/x; C)$, $\beta > \alpha$. It is easy to verify that

$$m(y/x) = (M(x/y))^{-1} \quad \text{and} \quad M(y/x) = (m(x/y))^{-1}.$$

Thus, if $\alpha \geq 1$, we must have that $\beta^{-1} = m(x, y) < 1$. It follows that, possibly by interchanging the roles of x and y, we can assume that

$$\alpha = \kappa = \min(m(y/x), m(x/y)) < 1.$$

Select an increasing sequence $\alpha_n < \alpha$ with $\lim_{n\to\infty} \alpha_n = \alpha$ and a decreasing sequence $\beta_n > \beta$ with $\lim_{n\to\infty} \beta_n = \beta$. By definition of α and β we have that

$$0 \leq y - \alpha_n x \leq (\beta_n - \alpha_n)x.$$

If $x, y \in \Sigma$, this implies that

$$\|y - \alpha_n x\| \leq A(\beta_n - \alpha_n)\|x\| = A\alpha_n\left[\left(\tfrac{\beta_n}{\alpha_n}\right) - 1\right];$$

while if $x, y \in S$ we obtain

$$|y - \alpha_n x|_u \leq (\beta_n - \alpha_n)|x|_u = \alpha_n\left[\left(\tfrac{\beta_n}{\alpha_n}\right) - 1\right].$$

Letting n approach ∞ we see that for $x, y \in \Sigma$ we have

$$\|y - \alpha x\| \leq A\kappa\big[\exp(d(x, y)) - 1\big];$$

while for $x, y \in S$, we obtain

$$|y - \alpha x|_u \leq \kappa[\exp(d(x, y)) - 1].$$

Note that if $x, y \in \Sigma$ we have

$$\|y - \alpha x\| \geq \|y\| - \alpha\|x\| = (1 - \alpha),$$

and if $x, y \in S$ we have

$$|y - \alpha x|_u \geq (1 - \alpha).$$

Thus, for $x, y \in \Sigma$ we see that

$$\|y-x\| \leq \|y-\alpha x\|+\|\alpha x-x\| \leq \|y-\alpha x\|+(1-\alpha) \leq 2\|y-\alpha x\| \leq 2A\kappa[\exp(d(x, y))-1],$$

and a similar argument yields equation (1.15).

If $u \in \overset{\circ}{C}$, we know that $V_u = V$, and we have already seen that there exists B with $\|z\| \leq B|z|_u$ for all $z \in V$. Conversely, select $\rho > 0$ so that $\{x : \|x - u\| \leq \rho\} \subset C$. It follows that for all z with $\|z\| \leq 1$,

$$u \pm \rho\Big(\frac{z}{\|z\|}\Big) \geq 0,$$

which implies that for $\|z\| \leq 1$

$$\Big(\frac{\|z\|}{\rho}\Big)u \geq z \geq -\Big(\frac{\|z\|}{\rho}\Big)u.$$

The latter inequality implies that for all $z \in V$,

$$|z|_u \leq \rho^{-1}\|z\|. \qquad \square$$

Remark 1.1. A major motivation for studying the part metric and Hilbert's projective metric is that, in the notation of Proposition 1.1, (S, d), its isometric image (Σ, d), and $(P(u), p)$ are often complete metric spaces. Specifically, suppose that $(V, \|\cdot\|)$ is a Banach space and C is a closed, normal cone in V. (A cone C in a normed linear space V is "normal" if there exists a constant A with $\|x\| \leq A\|y\|$ for all $x, y \in C$ with $0 \leq x \leq y$). If $u \in C - \{0\}$ and $P(u)$ is as in equation (1.4), A.C. Thompson [51] has proved that $P(u)$ is a complete metric space with respect to the part metric p. Other authors have proved that if $\Sigma = \{x \in P(u) : \|x\| = 1\}$ and d denotes Hilbert's projective metric, (Σ, d) is a complete metric space. Furthermore, if

$$V_u = \{x \in V : \exists \alpha > 0 \text{ with } -\alpha u \leq x \leq \alpha u\} \text{ and } |x|_u = \inf\{\alpha > 0 : -\alpha u \leq x \leq \alpha u\},$$

$(V_u, |\cdot|_u)$ is a Banach space and $C_u := C \cap V_u$ is a closed, normal cone in $(V_u, \|\cdot\|_u)$ with nonempty interior $P(u)$ in V_u. Further details and references to the literature are given in [36], pp. 12–18.

To simplify our further work we make another definition.

Definition 1.3. If C is a cone in a real vector space V, we shall say that C is "Archimedean" if $C \cap V(x, y)$ is closed in $V(x, y)$ (see equation (1.7)) for all $x, y \in V$.

It is easy to check that if C is an Archimedean cone, $x \in C - \{0\}$ dominates $y \in V$, and $\alpha = m(y/x; C)$ and $\beta = M(y/x; C)$, then

$$\alpha x \leq_C y \leq_C \beta x.$$

For almost Archimedean cones this equation may fail.

We also need the idea of a "minimal geodesic."

Definition 1.4. If (S, ρ) is a metric space, a map $\varphi : [0, 1] \to S$ will be called a minimal geodesic (with respect to ρ) from $x_0 = \varphi(0)$ to $x_1 = \varphi(1)$ if, whenever $0 \leq t_1 < t_2 \leq 1$ we have

$$\rho(\varphi(t_1), \varphi(t_2)) = (t_2 - t_1)\rho(x_0, x_1). \tag{1.20}$$

We shall say that (S, ρ) is "geodesically convex" if for all $x, y \in S$, there exists a minimal geodesic $\varphi : [0, 1] \to S$ with $\varphi(0) = x$ and $\varphi(1) = y$.

If (E, d) and (F, ρ) are general metric spaces and $f : D \subset E \to F$ is a map, our main interest is in finding conditions which insure that f is Lipschitz with Lipschitz constant c. However, our next proposition and subsequent work will show that this question is closely related to the existence of minimal geodesics, so we shall have to study minimal geodesics for the part metric and Hilbert's projective metric.

Proposition 1.2. *Let (E, d) and (F, ρ) be metric spaces and suppose that $f : D \subset E \to F$ is a continuous map. Suppose that $D = \bigcup_{\alpha \in A} D_\alpha$, where each D_α is a closed subset of D, and suppose that $f \mid D_\alpha$ is a Lipschitz map with constant c for all $\alpha \in A$ (so $\rho(f(u), f(v)) \leq cd(u, v)$ for all $u, v \in D_\alpha$). Assume that $x, y \in D$ and that there exists a minimal geodesic $\psi : [0, 1] \to D$ with $\psi(0) = x$ and $\psi(1) = x$. Assume also that there exist $n < \infty$ and D_{α_j}, $1 \leq j \leq n$, with $\{\psi(t) : 0 \leq t \leq 1\} \subset \bigcup_{j=1}^n D_{\alpha_j}$. Then it follows that $\rho(f(x), f(y)) \leq cd(x, y)$.*

Proof. The proof is by induction on n. If $n = 1$, the result is obvious, so we assume $n > 1$ and suppose the proposition is true for all $m < n$.

Define $J = \{i : 1 \leq i \leq n \text{ and } x \in D_{\alpha_i}\}$ and $t_* = \sup\{s \geq 0 : \psi(s) \in D_{\alpha_i} \text{ for some } i \in J\}$.

Because each D_α is closed, there exists $k \in J$ with $x = \psi(0) \in D_{\alpha_k}$ and $\psi(t_*) = x_* \in D_{\alpha_k}$. If $t_* = 1$, we are done, because $f \mid D_{\alpha_k}$ is Lipschitz with constant c. Thus we assume that $t_* < 1$. If $J_* = \{i \mid i \in J, 1 \leq i \leq n\}$ we know that

$$\{\psi(s) : t_* < s \leq 1\} \subset \bigcup_{i \in J_*} D_{\alpha_i}.$$

For fixed $s > t_*$, $\psi(t)$, $s \leq t \leq 1$, gives (after reparametrization) a minimal geodesic from $\psi(s)$ to $\psi(1) = y$. Because $|J_*| < n$, the inductive hypothesis gives

$$\rho(f(\psi(s)), f(y)) \leq cd(\psi(s), y) = (1 - s)cd(x, y).$$

Taking the limit as $s \to t_*^+$ gives

$$\rho(f(x_*), f(y)) \leq (1 - t_*)cd(x, y).$$

Because x_* and x both lie in D_{α_k} we also obtain

$$\rho(f(x_*), f(x)) \leq cd(x_*, x) = ct_* d(x, y).$$

The triangle inequality finally gives

$$\rho(f(x), f(y)) \leq cd(x, y). \qquad \square$$

In practice it may happen that (E, d) is geodesically convex and that there exists a non-expansive retraction r of E onto D (so $d(r(u), r(v)) \leq d(u, v)$ for all $u, v \in E$). If $\psi : [0, 1] \to E$ is a minimal geodesic then $r \circ \psi : [0, 1] \to D$ is a minimal so (D, d) is also geodesically convex.

In our next lemma, which refines Proposition 1.12 on p. 34 in [36], we give explicit formulas for minimal geodesics with respect to the part metric in Archimedean cones. In general, Lemma 1.3 implies that there are infinitely many minimal geodesics connecting x_0 to x_1 in $P(u)$.

Lemma 1.3. *Let C be an Archimedean cone in a real vector space V. If x and y are any two comparable elements of C and α and β are positive reals with $\alpha \leq m(y/x; C)$ and $\beta \geq M(y/x; C)$, define a function $\varphi(t; x, y, \alpha, \beta)$ for $0 \leq t \leq 1$ by*

$$\varphi(t; x, y, \alpha, \beta) = \begin{cases} \left(\frac{\beta^t - \alpha^t}{\beta - \alpha}\right)y + \left(\frac{\beta\alpha^t - \alpha\beta^t}{\beta - \alpha}\right)x, & \text{for } \beta > \alpha \\ \alpha^t x, & \text{for } \beta = \alpha. \end{cases} \tag{1.21}$$

Then it follows that

$$\varphi(t; x, y, \alpha, \beta) = \varphi(1 - t; y, x, \beta^{-1}, \alpha^{-1}), \tag{1.22}$$

$$\alpha^t x \leq \varphi(t; x, y, \alpha, \beta) \leq \beta^t x \quad \text{and} \tag{1.23}$$

$$(\beta^{-1})^{(1-t)} y \leq \varphi(t; x, y, \alpha, \beta) \leq (\alpha^{-1})^{(1-t)} y. \tag{1.24}$$

If $0 \leq t_1 < t_2 \leq 1$ and $t_1 = st_2$ and $w = \varphi(t_2; x, y, \alpha, \beta)$, then

$$\varphi(s; x, w, \alpha^{t_2}, \beta^{t_2}) = \varphi(t_1; x, y, \alpha, \beta) \quad \text{and} \tag{1.25}$$

$$p(\varphi(t_1; x, y, \alpha, \beta), \varphi(t_2; x, y, \alpha, \beta)) \leq (t_2 - t_1) \max(\log(\alpha^{-1}), \log(\beta)), \tag{1.26}$$

where p denotes the part metric on C. If

(1) $\alpha = m(y/x; C)$ *and* $\alpha^{-1} \geq \beta \geq M(y/x; C)$ *or*
(2) $\beta = M(y/x; C)$ *and* $\beta^{-1} \leq \alpha \leq m(y/x; C)$,

then $t \to \varphi(t; x, y, \alpha, \beta)$ is a minimal geodesic with respect to p from x to y.

Proof. We leave the case that $\alpha = \beta$ (so $y = \beta x$) to the reader, and we assume that $\alpha \leq m(y/x)$, $\beta \geq M(y/x)$ and $\alpha < \beta$. The same function $\varphi(t; x, y, \alpha, \beta)$ was considered in Proposition 1.12 of [36], but under the assumption that $\alpha = m(y/x)$ and $\beta = M(y/x)$. However the same argument as in [36] shows that

$$\alpha^t x \leq \varphi(t; x, y, \alpha, \beta) \leq \beta^t x.$$

A simple calculation (left to the reader) also gives equation (1.22), and equation (1.22) and equation (1.23) imply equation (1.24). Equations (1.23) and (1.24) and the definition of p imply

$$\begin{aligned} p(\varphi(t; x, y, \alpha, \beta), x) &\leq t \ \max(\log(\alpha^{-1}), \log(\beta)) \\ p(\varphi(t; x, y, \alpha, \beta), y) &\leq (1-t) \max(\log(\alpha^{-1}), \log(\beta)). \end{aligned} \tag{1.27}$$

It remains to prove equation (1.25). Notice that equation (1.23) implies that $\alpha^{t_2} x \leq w \leq \beta^{t_2} x$, so $\varphi(s; x, w, \alpha^{t_2}, \beta^{t_2})$ is defined. Another calculation (left to the reader) proves equation (1.25). If we use equation (1.25) and equation (1.27) we find (for s and w as in the statement of the Lemma)

$$\begin{aligned} p(\varphi(t_1; x, y, \alpha, \beta), \varphi(t_2; x, y, \alpha, \beta)) &= p(\varphi(s; x, w, \alpha^{t_2}, \beta^{t_2}), w) \\ &\leq s \max(\log(\alpha^{-t_2}), \log(\beta^{t_2})) \\ &= (t_2 - t_1) \max(\log(\alpha^{-1}), \log(\beta)). \end{aligned} \tag{1.28}$$

The final assumptions of the lemma imply

$$p(x, y) = \max(\log(\alpha^{-1}), \log(\beta)),$$

so equation (1.28) gives, for $0 \leq t_1 < t_2 \leq 1$,

$$p(\varphi(t_1; x, y, \alpha, \beta), \varphi(t_2; x, y, \alpha, \beta)) \leq (t_2 - t_1) p(x, y). \tag{1.29}$$

If we write $\varphi(t) = \varphi(t; x, y, \alpha, \beta)$ and if strict inequality holds in (1.29) for some $t_1 < t_2$, then we obtain

$$\begin{aligned} p(x, y) &\leq p(x, \varphi(t_1)) + p(\varphi(t_1), \varphi(t_2)) + p(\varphi(t_2), y) \\ &< t_1 p(x, y) + (t_2 - t_1) p(x, y) + (1 - t_2) p(x, y) = p(x, y), \end{aligned}$$

which gives a contradiction. Thus equality holds in equation (1.29) and $t \to \varphi(t)$ is a minimal geodesic.

Remark 1.2. If C is only almost Archimedean, one can give an analogue of Lemma 1.3 by working with maps φ which are "almost" minimal geodesics, namely $\varphi(t; x, y, \alpha, \beta)$ for appropriate $\alpha < m(y/x; C)$ and $\beta > M(y/x; C)$. This leads to slight technical complications which we have chosen to avoid. However, a version of Theorem 1.1 below can be given for almost Archimedean cones.

We are now in a position to state our first theorem. Recall that if $u \in C - \{0\}$ and V_u and $P(u)$ are given by equation (1.8) and equation (1.4) respectively, then for any $x \in P(u)$ one has a norm $\|\cdot\|_x$ on V_u defined by

$$|v|_x = \inf\{\alpha > 0 : -\alpha x \leq v \leq \alpha x\} \tag{1.30}$$

and any two such norms are equivalent. If $\varphi : [0, 1] \to V_u$, we shall say that φ is piecewise C^1 if it is piecewise C^1 with respect to the norm topology on V_u given by $|\cdot|_u$ (or any $|\cdot|_x$, $x \in P(u)$).

Theorem 1.1. *Let C be an Archimedean cone in a real vector space V. For $u \in C - \{0\}$, let $P(u)$ be given by equation (1.4) and V_u by equation (1.8), so V_u is a normed linear space with respect to $|\cdot|_u$ (equation (1.9)). Let $C_u = C \cap V_u$ and let p denote the part metric (equation (1.5)) on C. Let $G \subset H$ be subsets of $P(u)$ and assume that for any two points $x_0, x_1 \in G$ there exists a piecewise C', minimal geodesic φ (with respect to p) with $\varphi(0) = x_0$, $\varphi(1) = x_1$ and $\varphi(t) \in H$ for $0 \leq t \leq 1$. Let S denote the set of piecewise C^1 maps $\psi : [0, 1] \to H$. Then for any $x_0, x_1 \in G$,*

$$p(x_0, x_1; C) = \inf\left\{\int_0^1 |\psi'(t)|_{\psi(t)} dt : \psi \in S, \psi(0) = x_0 \text{ and } \psi(1) = x_1\right\}. \tag{1.31}$$

Proof. If $\psi : [a, b] \to P(u)$ is a C^1 map, one can check that $t \to |\psi'(t)|_{\psi(t)}$ is continuous on $[a, b]$. Thus the integrals in equation (1.31) are defined. By assumption there exists a piecewise C^1, minimal geodesic $\varphi : [0, 1] \to H$ with $\varphi(0) = x_0$ and $\varphi(1) = x_1$. If $0 \leq s < t \leq 1$, we know that

$$p(\varphi(s), \varphi(t)) = (t - s)p(x_0, x_1),$$

which implies that

$$\varphi(s) \exp(-(t - s)p(x_0, x_1)) \leq \varphi(t) \leq \varphi(s) \exp((t - s)p(x_0, x_1)).$$

Subtracting $\varphi(s)$ from this equation, dividing by $t - s$ and letting t approach s from above gives

$$-p(x_0, x_1)\varphi(s) \leq \varphi'_+(s) \leq p(x_0, x_1)\varphi(s), \tag{1.32}$$

where $\varphi'_+(s)$ denotes the right hand derivative of φ at s. By definition of $|\cdot|_{\varphi(s)}$, equation (1.32) implies that

$$|\varphi'_+(s)|_{\varphi(s)} \leq p(x_0, x_1), \quad 0 \leq s < 1. \tag{1.33}$$

A similar argument shows that

$$|\varphi'_-(s)|_{\varphi(s)} \leq p(x_0, x_1), \quad 0 < s \leq 1, \tag{1.34}$$

where $\varphi'_-(s)$ denotes the left hand derivative of φ. Since $\varphi'(s)$ is assumed to exist except at finitely many points in (0, 1) we conclude that

$$p(x_0, x_1) = \int_0^1 p(x_0, x_1) ds \geq \int_0^1 |\varphi'(s)|_{\varphi(s)} ds,$$

which implies that $p(x_0, x_1)$ is greater than or equal to the right hand side in equation (1.31).

Conversely, suppose that $\psi \in S$, $\psi(0) = x_0$ and $\psi(1) = x_1$. We can assume for definiteness that $\beta = M(x_1/x_0; C) \geq m(x_1/x_0; C)^{-1}$. We know that $P(u) = \overset{\circ}{C}_u$ is an open convex set in the normed linear space $(V_u, |\cdot|_u)$, that $\beta x_0 - x_1 \in C_u$ and that $\beta x_0 - x_1 \notin P(u)$ (otherwise, there would exist $\beta' < \beta$ with $\beta' x_0 - x_1 \in C$). Thus we are in the situation of the Hahn-Banach theorem: there exists a continuous (with respect to $|\cdot|_u$) nonzero linear functional $f : V_u \to \mathbb{R}$ and a number γ with $f(y) \geq \gamma$ for all $y \in P(u)$ and $f(\beta x_0 - x_1) \leq \gamma$. Because C_u is in the closure of $P(u)$ in V_u, $f(y) \geq \gamma$ for all $y \in C_u$; in particular, $\beta x_0 - x_1 \in C_u$ and we must have $f(\beta x_0 - x_1) = \gamma$. Because $ty \in P(u)$ if $y \in P(u)$ and

$t > 0$, $\gamma \leq tf(y)$ for all $t > 0$ and $y \in P(u)$. This implies that $\gamma \leq 0$ and $f(y) \geq 0$ for all $y \in P(u)$. Because $(\beta x_0 - x_1) + \varepsilon y \in P(u)$ if $\varepsilon > 0$ and $y \in P(u)$, we conclude that

$$0 \leq f((\beta x_0 - x_1) + \varepsilon y) = \gamma + \varepsilon f(y),$$

and it follows that $\gamma \geq 0$ and hence that $\gamma = 0$. Thus we see that $f(y) \geq 0$ for all $y \in C_u$ and $\beta f(x_0) = f(x_1)$. Because f is not the zero functional and $f(y) \geq 0$ for $y \in C_u$, it is easy to see that $f(y) > 0$ for all $y \in P(u)$.

If $\gamma(t) := |\psi'(t)|_{\psi(t)}$, we have

$$-\gamma(t)\psi(t) \leq \psi'(t) \leq \gamma(t)\psi(t).$$

Applying f to this equation and recalling that $f(\psi(t)) > 0$, we conclude that

$$\frac{d}{dt}\log(f(\psi(t))) = \frac{f(\psi'(t))}{f(\psi(t))} \leq \gamma(t) = |\psi'(t)|_{\psi(t)}.$$

Integrating this inequality from 0 to 1 gives

$$\log(f(x_1)) - \log(f(x_0)) = \log(\beta) \leq \int_0^1 |\psi'(t)|_{\psi(t)} dt.$$

We chose $\log(\beta) = p(x_0, x_1)$, so this completes the proof of equation (1.31).

Remark 1.3. Our proof actually shows that if $\varphi : [0, 1] \to P(u)$ is any piecewise C^1 minimal geodesic (with respect to p) from x_0 to x_1, then

$$p(x_0, x_1; C) = \int_0^1 |\varphi'(t)|_{\varphi(t)} dt.$$

Thus "inf" in equation (1.31) can be replaced by "min."

It is useful, in the context of Theorem 1.1, to allow maps $\varphi : [0, 1] \to P(u)$ which are only Lipschitz (with respect to $|\cdot|_u$). In the general infinite dimensional setting this leads to technical complications. For example, if $V = C[0, 1]$, C is the cone of nonnegative functions on $[0, 1]$, and $\varphi : [0, 1] \to \overset{\circ}{C}$ is defined by $\varphi(t)(x) = \exp(|x - t|)$ for $0 \leq x \leq 1$, one can prove that φ is Lipschitz and a minimal geodesic with respect to the part metric but that φ is nowhere Fréchet differentiable. Thus, in our next theorem, we allow Lipschitz φ, but we restrict V to be finite dimensional. The main technical difficulty is to prove that $t \to |\varphi'(t)|_{\varphi(t)}$ is bounded and Lebesgue measurable.

Theorem 1.2. *Let C be a closed cone with nonempty interior $\overset{\circ}{C}$ in a finite dimensional Banach space $(V, \|\cdot\|)$. Let $G \subset H$ be subsets of $\overset{\circ}{C}$ and assume that for any two points $x_0, x_1 \in G$ there exists a minimal geodesic φ (with respect to the part metric p on C) with $\varphi(0) = x_0$, $\varphi(1) = x_1$, and $\varphi(t) \in H$ for $0 \leq t \leq 1$. Let S denote the set of Lipschitz (with respect to $\|\cdot\|$) maps $\psi : [0, 1] \to H$. For any $\psi \in S$, $\psi'(t)$ is defined almost everywhere and $t \to |\psi'(t)|_{\psi(t)}$ is a bounded, Lebesgue measurable map. For any $x_0, x_1 \in G$,*

$$p(x_0, x_1; C) = \inf\left\{\int_0^1 |\psi'(t)|_{\psi(t)} dt : \psi \in S,\ \psi(0) = x_0,\ \psi(1) = x_1\right\}.$$

Proof. Select $u \in \overset{\circ}{C}$. It is well-known that any closed cone C in a finite dimensional Banach space V is normal. Thus we know (see Remark 1.1) that $|\cdot|_u$ and $\|\cdot\|$ are equivalent norms on $V = V_u$. By using Proposition 1.1 one can see that any map $\theta : [0, 1] \to \overset{\circ}{C}$ which is Lipschitz with respect to $\|\cdot\|$ is Lipschitz with respect to the part metric p and conversely. Since a minimal geodesic $\varphi : [0, 1] \to H$ (with respect to p) is Lipschitz with respect to p, it is also Lipschitz with respect to $\|\cdot\|$ and hence an element of S. Conversely, every element of S is a Lipschitz map with respect to p.

Standard real variables implies that every Lipschitz map $\theta : [0, 1] \to (V, \|\cdot\|)$ is Fréchet differentiable almost everywhere, $t \to \theta'(t)$ is Lebesgue measurable and $\|\theta'(t)\|$ is uniformly bounded. Thus, if $\psi \in S$, $t \to \psi'(t)$ is a bounded, Lebesgue measurable function.

It remains to prove that $t \to |\psi'(t)|_{\psi(t)}$ is a bounded, Lebesgue measurable map. We prove a slightly more general fact: If $\psi_1 : [0, 1] \to V$ is a bounded, Lebesgue measurable map and $\psi_2 : [0, 1] \to \overset{\circ}{C}$ is a continuous map, we claim that $t \to |\psi_1(t)|_{\psi_2(t)}$ is a bounded, Lebesgue measurable map. To prove this, first recall the well-known fact that there exists a countable family f_i, $i \geq 1$ of continuous linear functionals such that $\|f_i\| = 1$, for all $z \in V$, and $z \in C$ if and only if $f_i(z) \geq 0$ for all i. The proof is an application of the Hahn-Banach theorem. If $x \in \overset{\circ}{C}$ and $v \in V$, the reader can verify that

$$|v|_x = \sup_i \left(\frac{|f_i(v)|}{f_i(x)}\right).$$

(Note that necessarily $f_i(x) > 0$ for all $x \in \overset{\circ}{C}$, or we would have $f_i = 0$). The set $\{\psi_2(t) : 0 \leq t \leq 1\}$ is a compact subset of $\overset{\circ}{C}$, so a simple compactness argument implies that there exists $r > 0$ with

$$B_r(\psi_2(t)) := \{z : \|z - \psi_2(t)\| < r\} \subset \overset{\circ}{C}$$

for $0 \leq t \leq 1$. We know that $f_i(\psi_2(t) + v) > 0$ for all v with $\|v\| < r$, and since $\|f_i\| = 1$, we must have

$$f_i(\psi_2(t)) \geq r$$

for $i \geq 1$, $0 \leq t \leq 1$. By assumption, there exists A so $\|\psi_1(t)\| \leq A$ almost everywhere, so

$$|f_i(\psi_1(t))| \leq A \ \text{ a.e.}$$

Basic measure theory also implies that $t \to f_i(\psi_1(t))$ and $t \to f_i(\psi_2(t))^{-1}$ are Lebesgue measurable, so $t \to f_i(\psi_1(t)) f_i(\psi_2(t))^{-1}$ is Lebesgue measurable and

$$|\psi_1(t)|_{\psi_2(t)} = \sup_{i \geq 1} \left(\frac{|f_i(\psi_1(t))|}{f_i(\psi_2(t))}\right) \leq \frac{A}{r}$$

gives a Lebesgue measurable function bounded by $\frac{A}{r}$.

If $\varphi : [0, 1] \to H$ is a minimal geodesic (with respect to p) with $\varphi(0) = x_0$ and $\varphi(1) = x_1$, the same argument used in the first part of the proof of Theorem 1.1 shows that

$$|\varphi'(s)|_{\varphi(s)} \leq p(x_0, x_1)$$

for all s such that $\varphi'(s)$ exists. This proves that

$$p(x_0, x_1; C) \geq \int_0^1 |\varphi'(s)|_{\varphi(s)} ds.$$

If $\psi \in S$, essentially the same argument used in Theorem 1.1 shows that

$$p(x_0, x_1; C) \leq \int_0^1 |\psi'(s)|_{\psi(s)} ds,$$

which completes the proof.

Remark 1.4. In Theorem 1.1 we actually work in the normed linear space V_u, and $C \cap V_u$ is a normal, Archimedean cone in V_u with nonempty interior $P(u)$. Thus, in Theorem 1.1, we might as well assume that C is a normal, Archimedean cone with nonempty interior $\overset{\circ}{C}$ in a normed linear space V and $u \in \overset{\circ}{C}$, which is the framework of Theorem 1.2.

Remark 1.5. We have not formally defined "Finsler structure." The set $P(u)$ is an open subset of the normed linear space $(V_u, \|\cdot\|_u)$ and can be considered a manifold modeled on V_u. At each point $x \in P(u)$, the tangent space to $P(u)$ at x is V_u, but we equip the tangent space with the norm $|\cdot|_x$ (equation (1.30), which depends continuously on x in a natural sense. This gives a Finsler structure on $P(u)$. If $\varphi : [a, b] \to P(u)$ is a C^1 map, the length of the curve φ with respect to the Finsler structure is, by definition,

$$l(\varphi) = \int_a^b |\varphi'(t)|_{\varphi(t)} dt.$$

Define S_1 to be the set of C^1 maps $\varphi : [0, 1] \to P(u)$ and define a function q on $P(u) \times P(u)$ by

$$q(x_0, x_1) = \inf\{l(\varphi) : \varphi \in S_1, \varphi(0) = x_0 \text{ and } \varphi(1) = x_1\}.$$

It is easy to show that q is a metric on $P(u)$, and it follows from Lemma 1.3 and Theorem 1.1 (with $H = P(u)$) that

$$p(x_0, x_1; C) = q(x_0, x_1)$$

for all $x_0, x_1 \in P(u)$.

For applications of Theorem 1.1, it is useful to choose $H \supset G$ as small as possible. The following Corollary illustrates this point.

Corollary 1.1. *Let C be an Archimedean cone in a real vector space V. For $u \in C - \{0\}$, let $(V_u, |\cdot|_u)$ be the normed linear space given by equation* (1.8) *and let $P(u)$ be given by equation* (1.4), *so $P(u)$ is the interior of $C \cap V_u$ in V_u. Assume that G and H are subsets of $P(u)$ with $G \subset H$ and that G and H satisfy one of the following conditions:*

(a) *$G \cup \{0\}$ is convex.*

(b) *There are positive numbers $a < b$ and $v \in P(u)$ such that $G = \{x \in C : av \leq x \leq bv\}$ and $H \supseteq \{x \in C : kav \leq x \leq bv\}$, where*

$$k := 2\left(\sqrt{a/b} + \sqrt{b/a}\right)^{-1}.$$

(c) *M is a compact Hausdorff space and $(\mathcal{M}, \mu)$ is a measure space, $V = C(M)$, the space of continuous, real-valued functions on M, or $V = L^q(\mathcal{M}, \mu)$, $1 \leq q \leq \infty$, and C is the cone of nonnegative functions in V. For all $x, y \in G$ and $s \in [0, 1]$, $x^{1-s}y^s \in G$ for $0 \leq s \leq 1$ (where $(x^{1-s}y^s)(m) := (x(m))^{1-s}(y(m))^s$).*

Define T to be the set of lipschitz maps $\psi : [0, 1] \to (V, |\cdot|_u)$ such that $\psi(t) \in H$ for $0 \le t \le 1$, and define $S \subset T$ to be the set of $\psi \in T$ such that ψ is piecewise C^1. If p denotes the part metric on C, then for all $x_0, x_1 \in G$,

$$p(x_0, x_1; C) = \min\{\int_0^1 |\psi'(t)|_{\psi(t)} dt : \psi \in S, \psi(0) = x_0 \text{ and } \psi(1) = x_1\}.$$

If, in addition, V is finite dimensional, then

$$p(x_0, x_1; C) = \min\{\int_0^1 |\psi'(t)|_{\psi(t)} dt : \psi \in T, \psi(0) = x_0 \text{ and } \psi(1) = x_1\}.$$

Proof. Corollary 1.1 follows directly from Theorems 1.1 and 1.2 and Remark 1.3 if we can prove that whenever $x, y \in G$, there exists a C^1 minimal geodesic $\varphi \in S$ with $\varphi(0) = x$ and $\varphi(1) = y$

Case(a). Suppose that $G \subset P(u)$, $G \cup \{0\}$ is convex and $x, y \in G$. We can assume $x \neq y$, and we define

$$\beta = \max(M(y/x; C), (m(y/x; C))^{-1}).$$

We note that $\beta > 1$ ($\beta \neq 1$ because $x \neq y$). If we define $\alpha = \beta^{-1}$, Lemma 1.3 implies that (in the notation of equation (1.21)).

$$\varphi(t) := \varphi(t; x, y, \beta^{-1}, \beta) = (\frac{\beta^t - \beta^{-t}}{\beta - \beta^{-1}})y + (\frac{\beta^{1-t} - \beta^{-(1-t)}}{\beta - \beta^{-1}})x \tag{1.35}$$

gives a C^1 minimal geodesic (with respect to p) from x to y. The coefficients of x and y in equation (1.35) are nonnegative and not both zero for $0 \le t \le 1$, and we assume that $G \cup \{0\}$ is convex. Thus, to prove that $\varphi(t) \in G$ for $0 \le t \le 1$, it suffices to prove that, for $0 \le t \le 1$,

$$(\frac{\beta^t - \beta^{-t}}{\beta - \beta^{-1}}) + (\frac{\beta^{1-t} - \beta^{-(1-t)}}{\beta - \beta^{-1}}) := g(t) \le 1. \tag{1.36}$$

A calculation gives

$$(\beta - \beta^{-1})g'(t) = [\log(\beta)][\beta^t - \beta^{1-t}][1 - (\beta^t \beta^{1-t})^{-1}] := I_1 I_2(t) I_3(t).$$

Clearly, $I_1 := \log(\beta) > 0$ and $I_3(t) > 0$ for $0 \le t \le 1$. It is also clear that $I_2(t) < 0$ for $0 \le t < \frac{1}{2}$ and $I_2(t) > 0$ for $\frac{1}{2} < t \le 1$. It follows that g is strictly decreasing on $[0, \frac{1}{2}]$ and strictly increasing on $[\frac{1}{2}, 1]$. Since $g(0) = g(1) = 1$, we have proved equation (1.36) and have also shown that

$$\min_{0 \le t \le 1} g(t) = g(\tfrac{1}{2}) = (\frac{2}{\sqrt{\beta} + \sqrt{\beta^{-1}}}) < 1. \tag{1.37}$$

Case(b). $G = \{x : av \le x \le bv\}$. If $x, y \in G$ and $\varphi(t)$ is given by equation (1.35), we know that $\varphi(t)$ is a minimal geodesic. For $0 \le t \le 1$ and $g(t)$ as in equation (1.36) we have

$$ag(t)v \le \varphi(t) \le bg(t)v,$$

so equation (1.36) and (1.37) imply

$$2(\sqrt{\beta} + \sqrt{\beta^{-1}})^{-1} av \le \varphi(t) \le bv. \tag{1.38}$$

On the other hand we have

$$(\frac{a}{b})x \le (\frac{a}{b})(bv) \le y \le (\frac{b}{a})(av) \le (\frac{b}{a})x,$$

which implies that $\beta \le \sqrt{\frac{b}{a}}$. Using this estimate for β in equation (1.38) and defining k as in case (b), we see that

$$kav \le \varphi(t) \le bv,$$

so $\varphi(t) \in H$ for $0 \le t \le 1$.

Case(c). If V is as in Case(c) and $x, y \in P(u)$, the reader can directly verify that, for $0 \le s \le 1$,

$$\varphi(s) := x^{1-s}y^s$$

gives a C^1 minimal geodesic from x to y and, by assumption, $\varphi(s) \in G$ for $0 \le s \le 1$. Compare Proposition 1.8, p. 24, in [36]. Thus we are in the situation of Theorem 1.1. □

Definition 1.5. If G and V are as Case (c) of Corollary 1.1, we shall say that "G is logarithmically convex."

An important example of Definition 1.5 is when $M = \{1, 2, \dots, n\}$, $V = C(M) = \mathbb{R}^n$ and $C = K^n := \{x \in \mathbb{R}^n : x_i \ge 0 \text{ for } 1 \le i \le n\}$. Then $G \subset \overset{\circ}{C}$ is logarithmically convex if, whenever $x, y \in G$, $x^{1-s}y^s \in G$ for $0 \le s \le 1$, where $z = x^{1-s}y^s$ is the vector with $z_i = x_i^{1-s}y_i^s$.

Corollary 1.1 shows that, in cases (a) and (c), the set G is geodesically convex in $(P(u), p)$ and that a C^1 minimal geodesic can be chosen.

If C_1 and C_2 are cones, $G \subset \overset{\circ}{C}$, and $f : G \to \overset{\circ}{C}_2$ is a map, we wish to compute the Lipschitz constant of f with respect to the part metrics p_1 and p_2 on C_1 and C_2 respectively. Recall that f is "order-preserving on G" if $f(x) \le_{C_2} f(y)$ for all $x, y \in G$ with $x \le_{C_1} y$; f is "order-reversing on G" if $f(x) \ge_{C_2} f(y)$ whenever $x, y \in G$ and $x \le_{C_1} y$. Results related to the following Proposition can be found in [42] and Chapters 2 and 3 of [36].

Proposition 1.3. *Let C_i, $i = 1, 2$, be a normal Archimedean cone with nonempty interior in a normed linear space $(V_i, \|\cdot\|_i)$ and let p_i denote the part metric on C_i. Assume that $G \subset \overset{\circ}{C}$, and that $f : G \to \overset{\circ}{C}_2$ and $g : G \to \overset{\circ}{C}_2$ are maps.* (a) *Assume that there exists $c > 0$ such that for all $x, y \in G$*

$$p_2(f(x), f(y)) \le cp_1(x, y) \text{ and } p_2(g(x), g(y)) \le cp_1(x, y). \tag{1.39}$$

If $h(x) := f(x) + g(x)$, it follows that for all $x, y \in G$.

$$p_2(h(x), h(y)) \le cp_1(x, y). \tag{1.40}$$

If strict inequality, always holds in at least one of the inequalities in (1.39) *whenever $x, y \in G$ and $x \ne y$, it follows that for all $x, y \in G$ with $x \ne y$*

$$p_2(h(x), h(y)) < cp_1(x, y)$$

(b) *If $tG \subset G$ for $0 < t \le 1$, $\gamma > 0$, and f is order-preserving (respectively, order-reversing) and $f(tx) \ge t^\gamma f(x)$ (respectively, $f(tx) \le t^{-\gamma} f(x)$) whenever $0 < t \le 1$ and $x \in G$, then for all $x, y \in G$*

$$p_2(f(x), f(y)) \le \gamma p_1(x, y). \tag{1.41}$$

Proof. If $x, y \in G$ and $x \neq y$ and $p_1(x, y) = \log(R)$, $R > 1$, then

$$R^{-1}x \leq y \text{ and } R^{-1}y \leq x.$$

If f satisfies the conditions in case (b) and we use $t = R^{-1}$, this implies that

$$R^{-\gamma} f(x) \leq f(y) \leq R^{\gamma} f(x),$$

which gives equation (1.41).

If f and g are as in case (a), we have

$$R^{-c} f(x) \leq f(y) \leq R^{c} f(x) \ \text{ and } \ R^{-c} g(x) \leq g(y) \leq R^{c} g(x), \tag{1.42}$$

which immediately gives equation (1.40). If, for example,

$$p_2(g(x), g(y)) < cp_1(x, y),$$

we must have, for some $c_1 < c$,

$$R^{-c_1} g(x) \leq g(y) \leq R^{c_1} g(x).$$

Adding inequalities gives

$$R^{-c} f(x) + R^{-c_1} g(x) \leq h(y) \leq R^{c} f(x) + R^{c_1} g(x). \tag{1.43}$$

Because $f(x)$ and $g(x)$ are comparable, there exists c_2 with $c_1 < c_2 < c$ and

$$(R^{c} - R^{c_2}) f(x) \leq (R^{c_2} - R^{c_1}) g(x) \ \text{ and } \ (R^{-c_2} - R^{-c}) f(x) \leq (R^{-c_1} - R^{-c_2}) g(x).$$

For this choice of c_2 one obtains from equation (1.42) that

$$R^{-c_2} h(x) \leq h(y) \leq R^{c_2} h(x) \ \text{ and } \ p_2(h(x), h(y)) \leq c_2 p_1(x, y). \qquad \square$$

If C is a cone in a vector space V, V is a lattice (with respect to the partial ordering from C) if, for all $x, y \in V$, $\{z \in V : z \geq x \text{ and } z \geq y\} = U(x, y)$ is nonempty and there exists $\zeta \in U(x, y)$ with $\zeta \leq z$ for all $z \in U(x, y)$. Obviously, ζ is unique, and we write $\zeta = x \vee y$. The existence of $x \vee y$ implies that there exists $\zeta_* = x \wedge y$ with $\zeta_* \in L(x, y) = \{z : z \leq x \text{ and } z \leq y\}$ and $\zeta_* \geq z$ for all $z \in L(x, y)$ and $\zeta_* = -[(-x) \vee (-y)]$.

Proposition 1.4. *Let C_i, V_i and p_i be as in Proposition* 1.3 *and assume that V_2 is a lattice with respect to the partial ordering from C_2. Assume that $G \subset \overset{\circ}{C}_1$ and that $f : G \to \overset{\circ}{C}_2$ and $g : G \to \overset{\circ}{C}_2$ satisfy equation* 1.39). (a) *If $h : G \to \overset{\circ}{C}_2$ is defined by $h(x) = f(x) \vee g(x)$ or $h(x) = f(x) \wedge g(x)$, then h satisfies equation* (1.40). (b) *If $V = \mathbb{R}^n$ and $C_2 = \{y \in \mathbb{R}^n : y_i \geq 0 \text{ for } 1 \leq i \leq n\}$, let $f_i(x)$ and $g_i(x)$, $1 \leq i \leq n$, denote the components of $f(x)$ and $g(x)$ respectively. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be a given vector with $\varepsilon_i = \pm 1$ for each i, $1 \leq i \leq n$, and for $x \in G$ define $h_i(x) = f_i(x) \vee g_i(x)$ if $\varepsilon_i = 1$ and $h_i(x) = f_i(x) \wedge g_i(x)$ if $\varepsilon_i = -1$. Then, $h(x) = (h_1(x), h_2(x), \ldots, h_n(x))$ satisfies equation* (1.40).

Proof. If $x, y \in G$ and $\log R = p_1(x, y)$, equation (1.39) gives equation (1.42), from which we obtain

$$R^{-c}(f(x) \vee g(x)) \leq f(y) \vee g(y) \leq R^{c}(f(x) \vee g(x)),$$

and similarly for $f(x) \wedge g(x)$. This immediately gives equation (1.40).

To prove case (b), we work coordinatewise and use the same argument; details are left to the reader. $\square$

The process of taking the max or min of two functions does not preserve continuous differentiability, but does preserve a local Lipschitz property. Also, many examples of interest in applications arise by taking the max or min of a collection of functions. See, for example, work in [4], [41], where a function obtained by taking the maximum of certain affine linear maps in $\mathbb{R}^n$ is discussed at length.

One motivation for proving Theorems 1.1 and 1.2 and Corollary 1.1 is to determine the Lipschitz constant (with respect to p_1 and p_2) of a map $f : G \subset \overset{\circ}{C}_1 \to \overset{\circ}{C}_2$. To explain this we need another definition.

Definition 1.6. Let C_i, $i = 1, 2$, be a normal Archimedean cone with nonempty interior in a normed linear space V_i. Let H be a subset of $\overset{\circ}{C}_1$ and assume that H_1 is an open neighborhood of H in $\overset{\circ}{C}_1$ and that $f : H_1 \subset \overset{\circ}{C}_1 \to \overset{\circ}{C}_2$ is a map. If $x \in H$ and f is Frechet differentiable at x, we define $c(x)$ by

$$c(x) = \inf\{\lambda \geq 0 : |f'(x)(v)|_{f(x)} \leq \lambda |v|_x \text{ for all } v \in V_1\}, \tag{1.44}$$

where $|\cdot|_x$ and $|\cdot|_{f(x)}$ are norms on V_1 and V_2 respectively and are given by equation (1.9).

Obviously, $c(x)$ is just the norm of the linear map $f'(x) : (V_1, |\cdot|_x) \to (V_2, |\cdot|_{f(x)})$.

Corollary 1.2. *Let C_i, $i = 1, 2$, be a normal, Archimedean cone with nonempty interior in a normed linear space V_i. Let G and H be subsets of $\overset{\circ}{C}_1$ with $G \subset H$ and assume that for any points $x, y \in G$ there exists a piecewise C^1 minimal geodesic φ (with respect to p_i) with $\varphi(0) = x$, $\varphi(1) = y$ and $\varphi(t) \in H$ for $0 \leq t \leq 1$. (This condition will be satisfied if G and H are as in Corollary* 1.1*). Let H_1 be an open neighborhood of H, $H_1 \subset \overset{\circ}{C}_1$, and suppose that $f : H_1 \to \overset{\circ}{C}_2$ is continuously Fréchet differentiable on H_1 and that*

$$c_0 = \sup\{c(x) : x \in H\} < \infty,$$

where $c(x)$ is given by equation (1.44)*. If p_i denotes the part metric on C_i, we have, for all $x, y \in G$,*

$$p_2(f(x), f(y)) \leq c_0 p_1(x, y). \tag{1.45}$$

If $c(x) < c_0$ except for countably many $x \in H$, then strict inequality holds in equation (1.45) *for $x \neq y$.*

Proof. Given $x, y \in G$, there exists a piecewise C^1 minimal geodesic $\varphi : [0, 1] \to H$ (with respect to p_1) with $\varphi(0) = x$ and $\varphi(1) = y$. It follows that $f(\varphi(t))$ gives a piecewise C^1 map from $f(x)$ to $f(y)$ with $f(\varphi(t)) \in \overset{\circ}{C}_2$ for $0 \leq t \leq 1$. Using Theorem 1.1 and equation (1.44) we see that

$$p_2(f(x), f(y)) \leq \int_0^1 \left|\left(\frac{d}{dt}\right) f(\varphi(t))\right|_{f(\varphi(t))} dt \leq \int_0^1 c(\varphi(t)) |\varphi'(t)|_{\varphi(t)} dt \leq c_0 p_1(x, y).$$

If $c(z) < c_0$ except for countable many $z \in H$ and if $x \neq y$, this calculation also shows that

$$p_2(f(x), f(y)) < c_0 p_1(x, y). \qquad \square$$

Remark 1.6. Let notation be as in Corollary 1.2 and let $S = \left\{\left(\frac{u-v}{t}\right) : t \in \mathbb{R}-\{0\}, u, v \in H\right\}$. For each $x \in H$, define

$$\tilde{c}(x) = \inf\{\lambda \geq 0 : |f'(x)(v)|_{f(x)} \leq \lambda |v|_x \text{ for all } v \in S\} \text{ and } \tilde{c}_0 = \sup\{\tilde{c}(x) : x \in H\}.$$

Since $\varphi'(t)$ lies in the closure of S whenever $\varphi : [0, 1] \to H$ is a piecewise C^1 map, the proof of Corollary 1.2 actually shows that for all $x, y \in G$,

$$p_2(f(x), f(y)) \leq \tilde{c}_0 p_1(x, y).$$

It frequently happens that, in the notation of Corollary 1.2, c_0 is the best possible Lipschitz constant for the map f.

Specifically, we have

Corollary 1.3. *Let C_i and V_i be as in Corollary 1.2. Assume that G is an open subset of C_1, and that $f : G \to \overset{\circ}{C}_2$. If f is Fréchet differentiable at $x \in G$ and $c(x)$ is defined by equation (1.44), then*

$$c(x) = \lim_{\varepsilon \to 0^+} \left(\sup\left\{\frac{p(f(y), f(x); C_2)}{p(y, x; C_1)} : 0 < p(y, x; C_1) \leq \varepsilon\right\}\right). \tag{1.46}$$

IF $k_0 = \inf\{k : p(f(u), f(v); C_2) \leq kp(u, v; C_1)$ for all $u, v \in G\}$, then

$$k_0 \geq \sup\{c(\xi) : \xi \in G \text{ and } f \text{ is Fréchet differentiable at } \xi\}.$$

If, for all $u, v \in G$, there exists a piecewise C^1 minimal geodesic φ (with respect to the part metric) with $\varphi(0) = u$, $\varphi(1) = v$ and $\varphi(t) \in G$ for $0 \leq t \leq 1$, and if f is continuously Fréchet differentiable on G, then

$$k_0 = \sup\{c(x) : x \in G\}.$$

Proof. Suppose that f is Fréchet differentiable at a given point $x \in G$ and let $S = \{y \in V_1 : |y|_x = 1\}$. If $\|\cdot\|_i$ is the given norm on V_i, recall that $\|\cdot\|_1$ is equivalent to $|\cdot|_x$ and $\|\cdot\|_2$ is equivalent to $|\cdot|_{f(x)}$. Thus, by definition of Fréchet differentiability we have for all $v \in S$,

$$f(x+tv) = f(x)+tf'(x)(v)+R(tv) \text{ and } \lim_{t\to 0^+}(\sup\{t^{-1}|R(tv)|_{f(x)} : v \in S\}) = 0. \tag{1.47}$$

By definition of $c(x)$ we have

$$c(x) = \sup\{\max(M(f'(x)v/f(x)), \ -m(f'(x)v/f(x))) : v \in S\}. \tag{1.48}$$

If $v \in S$, we have that

$$\max(M(v/x), -m(v/x)) = 1.$$

Since we also know that

$$M((x+tv)/x) = 1 + tM(v/x) \text{ and } m((x+tv)/x) = 1 + tm(v/x)$$

we conclude that for $0 < t < 1$

$$p(x+tv, x) = \max(\log(1+tM(v/x)), -\log(1+tm(v/x))) = t + R_1(tv), \tag{1.49}$$

where $\lim_{t\to 0^+}(\sup\{t^{-1}|R_1(tv)| : v \in S\}) = 0$.

A similar argument can be used to estimate $p(f(x+tv), f(x))$ for $t > 0$ small and $v \in S$. By using equation (1.47) we see that for $v \in S$ and $0 < t < 1$

$$\begin{aligned} M(f(x+tv)/f(x)) &= M((f(x) + t(f'(x)v + t^{-1}R(tv)))/f(x)) \\ &= 1 + tM((f'(x)v + t^{-1}R(tv))/f(x)) \\ &= 1 + tM(f'(x)v/f(x)) + R_2(tv), \end{aligned}$$

where $\lim_{t\to 0^+}(\sup\{t^{-1}|R_2(tv)| : v \in S\}) = 0$. A similar argument shows that

$$m(f(x+tv)/f(x)) = 1 + tm(f'(x)v/f(x)) + R_3(tv),$$

where $\lim_{t\to 0^+}(\sup\{t^{-1}|R_3(tv)| : v \in S\}) = 0$. It follows that

$$\begin{aligned} p(f(x+tv), f(x)) &= \max(\log(1 + tM(f'(x)v/f(x)) + R_2(tv)), \\ &\quad - \log(1 + tm(f'(x)v/f(x)) + R_3(tv))) \\ &= t\max(M(f'(x)v/f(x)), -m(f'(x)v/f(x)) + R_4(tv), \end{aligned} \tag{1.50}$$

where $\lim_{t\to 0^+}(\sup\{t^{-1}|R_4(tv)| : v \in S\}) = 0$. If we use equations (1.48), (1.49) and (1.50), we find that

$$c(x) \geq \lim_{t\to 0^+}(\sup\{\frac{p(f(x), f(y))}{p(x,y)} : 0 < p(x,y) \leq t\}). \tag{1.51}$$

On the other hand, given any $\varepsilon > 0$, there exists $v = v_\varepsilon \in S$ with

$$c(x) - \varepsilon \leq \max(M(f'(x)v/f(x)), -m(f'(x)v/f(x))) \leq c(x).$$

For this choice of v, equations (1.49) and (1.50) imply that

$$\lim_{t\to 0^+} \frac{p(f(x+tv), f(x))}{p(x+tv, x)} = \max(M(f'(x)v/f(x)), -m(f'(x)v/f(x))) \geq c(x) - \varepsilon,$$

and this implies that equality holds in equation (1.51).

Equation (1.46) immediately implies that

$$k_0 \geq \sup\{c(\xi) : \xi \in G \text{ and } f \text{ is Fréchet differentiable at } \xi\}.$$

If G is geodesically convex as described and f is C^1 on G, Corollary 1.2 implies that $k_0 \leq \sup\{c(x) : x \in G\}$, so we have equality. $\square$

As we have already remarked, it is important to give a version of Corollary 1.2 for the case that f is locally Lipschitzian.

Corollary 1.4. *Let C_i, $i = 1, 2$, V_i, G and H be as in Corollary 1.2. Let $H_1 \subset \overset{\circ}{C_1}$ be an open neighborhood of H and suppose that $f : H_1 \to \overset{\circ}{C_2}$ is locally Lipschitzian. Assume that there exists $c_0 \geq 0$ for which the following condition is satisfied: For every piecewise C^1 map $\psi : [0,1] \to H_1$ and every $h \in C_2^* - \{0\}$ there exists a sequence $(a_j) \subset V_1$, dependent on h and ψ, such that $\lim_{j\to\infty} a_j = 0$ and*

$$\overline{\lim}_{j\to\infty}\big(\text{ess sup}\big\{\frac{\left|\frac{d}{dt}h(f(\psi(t)+a_j))\right|}{h(f(\psi(t)+a_j))}\big(\frac{1}{|\psi'(t)|_{\psi(t)}}\big) : \psi'(t) \neq 0\big\}\big) \leq c_0. \tag{1.52}$$

Then it follows that for all $x, y \in G$

$$p(f(x), f(y); C_2) \leq c_0 p(x, y; C_1).$$

Proof. Take points $x, y \in G$, $x \neq y$, and let $\varphi : [0,1] \to H$ be a piecewise C^1 minimal geodesic (with respect to the part metric) with $\varphi(0) = x$ and $\varphi(1) = y$. Define $\beta = \exp(p(f(x), f(y); C_2)) \geq 1$, so

$$\beta^{-1} f(x) \leq f(y) \leq \beta f(x).$$

We know that either (1) $\beta f(x) - f(y) \in \partial C_2$ or (2) $f(y) - \beta^{-1} f(x) \in \partial C_2$. The Hahn-Banach theorem implies that there exists $h \in C_2^* - \{0\}$ with $h(\beta f(x) - f(y)) = 0$ (if case (1) holds) or $h(f(y) - \beta^{-1} f(x)) = 0$ (case(2)). By using Proposition 1.1 and the definition of minimal geodesic one can see that $\varphi'(t) \neq 0$ wherever $\varphi'(t)$ is defined. By assumption, there exists a sequence $a_j \to 0$ with

$$\overline{\lim}_{j\to\infty}\Big(\text{ ess } \sup\Big\{\frac{|\frac{d}{dt}h(f(\varphi(t)+a_j)|}{h(f(\varphi(t)+a_j))}\Big(\frac{1}{|\varphi'(t)|_{\varphi(t)+a_j}}\Big) : 0 \leq t \leq 1\Big\}\Big) \leq c_0. \tag{1.53}$$

(Recall that there exists $\varepsilon_j \to 0^+$ with $|v|_{\varphi(t)+a_j} \leq (1+\varepsilon_j)|v|_{\varphi(t)}$ and $|v|_{\varphi(t)} \leq (1+\varepsilon_j)|v|_{\varphi(t)+a_j}$ for all t, $0 \leq t \leq 1$, and all $v \in V_1$. Thus we can replace the term $|\psi'(t)|_{\psi(t)}$ in equation (1.52) by $|\psi'(t)|_{\psi(t)+a_j}$).

By definition of φ we have

$$p(x, y; C_1) = \int_0^1 |\varphi'(t)|_{\varphi(t)} dt = \lim_{j\to\infty} \int_0^1 |\varphi'(t)|_{\varphi(t)+a_j} dt.$$

Given $\varepsilon > 0$, equation (1.53) implies that for all j sufficiently large and almost all t we have

$$\Big|\frac{d}{dt}\log(h(f(\varphi(t)+a_j)))\Big| \leq (c_0+\varepsilon)|\varphi'(t)|_{\varphi(t)+a_j}.$$

Integrating this inequality, we find

$$\begin{aligned}\Big|\log\Big(\frac{h(f(y+a_j))}{h(f(x+a_j))}\Big)\Big| &= \Big|\int_0^1 \frac{d}{dt}\log(h(f(\varphi(t)+a_j)))dt\Big| \\ &\leq \int_0^1 \Big|\frac{d}{dt}\log(h(f(\varphi(t)+a_j)))\Big| dt \\ &\leq (c_0+\varepsilon)\int_0^1 |\varphi'(t)|_{\varphi(t)+a_j} dt.\end{aligned} \tag{1.54}$$

Taking the limit as j approaches ∞ in equation (1.54) gives $p(f(x), f(y); C_2) = \log(\beta) = \big|\log\big(\frac{h(f(y))}{h(f(x))}\big)\big| \leq (c_0+\varepsilon)p(x, y; C_2)$, and since $\varepsilon > 0$ was arbitrary, we have the desired estimate.

Remark 1.7. An examination of the proof of Corollary 1.4 shows that it would actually suffice to know that equation (1.52) is satisfied for all $h \in \Gamma$, where Γ is some subset of $C_2^* - \{0\}$ which is dense in C_2^* in the weak * topology on V_2^*.

If V_1 and V_2 are finite dimensional, so that f is Fréchet differentiable almost everywhere on H_1, Corollary 1.4 takes a much simpler form.

Corollary 1.5. *Let C_i, $i = 1, 2$, V_i, G and H be as in Corollary 1.2 and assume that V_1 and V_2 are finite dimensional. Let $H_1 \subset \overset{\circ}{C_1}$ be an open neighborhood of H and $f : H_1 \to \overset{\circ}{C_2}$ a locally Lipschitzian map (so f is Fréchet differentiable almost everywhere and $c(x)$ as in equation (1.44) is defined for almost all x). Define c_0 by*

$$c_0 = \text{ess sup}\{c(x) : x \in H_1\} \tag{1.55}$$

and assume that $c_0 < \infty$. Then it follows that for all $x, y \in G$,

$$p_2(f(x), f(y); C_2) \leq c_0 p_1(x, y; C_1). \tag{1.56}$$

Proof. It suffices to verify equation (1.52). Let $N \subset H_1$ be a set of measure zero such that $f'(x)$ exists for all $x \in H_1 \backslash N$ and $c(x) \leq c_0$. Let $\psi : [0, 1] \to H_1$ be a piecewise C^1 map and select $\varepsilon > 0$ so that $\psi(t) + b \in H_1$ for $0 \leq t \leq 1$ and for all $b \in B_\varepsilon := \{y \in V : \|y\|_1 < \varepsilon\}$. The set $N \times [0, 1]$ has measure zero in $V_1 \times \mathbb{R}$. The map $(a, t) \in V_1 \times [0, 1] \to (a - \psi(t), t)$ is locally Lipschitz and thus takes sets of measure zero to sets of measure zero. Therefore, $E = \{(b - \psi(t), t) : b \in N, 0 \leq t \leq 1\}$ has measure zero, so

$$E_1 = \{(a, t) : a \in B_\varepsilon, t \in [0, 1], a + \psi(t) \in N\} \subset E$$

has measure zero. It follows from Fubini's theorem that for almost all $a \in B_\varepsilon$, $a + \psi(t) \notin N$ for almost all $t \in [0, 1]$. Thus there exists a sequence $a_j \in B_\varepsilon$ with $a_j \to 0$ and $a_j + \psi(t) \notin N$ for almost all t. By using the chain rule and the definition of c_0 we see that for almost all t

$$-c_0 f(\psi(t) + a_j)|\psi'(t)|_{\psi(t)+a_j} \leq f'(\psi(t) + a_j)(\psi'(t)) \leq c_0 f(\psi(t + a_j))|\psi'(t)|_{\psi(t)+a_j}.$$

If $h \in C_2^* - \{0\}$ we see that for almost all t

$$|h(f'(\psi(t) + a_j)(\psi'(t)))| = \left|(\frac{d}{dt})h(f(\psi(t) + a_j))\right| \leq c_0 h(f(\psi(t + a_j)))|\psi'(t)|_{\psi(t)+a_j}$$

from which one easily derives equation (1.52).

Remark 1.8. In Corollary 1.4 or 1.5 we could consider a decreasing sequence of open neighborhoods H_1^j, $j \geq 1$, of H and define c_0^j by equation (1.52) or (1.55) for $H_1^j = H_1$. If we define $c_0 = \lim_{j\to\infty} c_0^j$, equation (1.56) remains true.

Remark 1.9. Suppose that C_i, $i = 1, 2$, V_i, G and H are as in Corollary 1.2, that $H_1 \subset \overset{\circ}{C_1}$ is an open neighborhood of H and that $f : H_1 \to \overset{\circ}{C_2}$ is a locally Lipschitzian map. Suppose that $\gamma > 0$ and that f is order-preserving (respectively, order-reversing) on H, and $f(tx) \geq t^\gamma f(x)$ (respectively, $f(tx) \leq t^{-\gamma} f(x)$) whenever $0 < t \leq 1$, $x \in H_1$ and $tx \in H_1$. One can prove (details are left to the reader) that equation (1.52) is satisfied with $a_j = 0$ for all j and $\gamma = c_0$. Thus Corollary 1.4 implies that for all $x, y \in G$

$$p(f(x), f(y); C_2) \leq \gamma p(x, y; C_1),$$

and we obtain a refinement of Proposition 1.2, case (b).

In order to apply Corollary 1.2, one must estimate $c(x)$. For the case of the standard cone $K^n = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$, this evaluation is easy.

Corollary 1.6. *Let $C_1 = K^n \subset \mathbb{R}^n$ and $C_2 = K^m \subset \mathbb{R}^m$. Assume that $G \subset H \subset \overset{\circ}{C}_1$ and that for any two points $x, y \in G$ there exists a piecewise C^1 minimal geodesic φ (with respect to the part metric on C_1) with $\varphi(0) = x$, $\varphi(1) = y$ and $\varphi(t) \in H$ for $0 \le t \le 1$. (See Corollary 1.1). Let $H_1 \subset \overset{\circ}{C}_1$ be an open neighborhood of H and $f : H_1 \to \overset{\circ}{C}_2$ a locally Lipschitzian map with coordinate component maps f_i, $1 \le i \le m$. If $x \in H$, f is Fréchet differentiable at x, and $c(x)$ is defined by equation (1.44), then*

$$\max_{1 \le i \le m} (f_i(x))^{-1} \sum_{j=1}^{n} \Big|\Big(\frac{\partial f_i}{\partial x_j}\Big)(x)\Big| x_j = c(x). \tag{1.57}$$

If $\operatorname{ess\,sup}\{c(x) : x \in H_1\} = c_0$, *then, for all $x, y \in G$*

$$p(f(x), f(y); C_2) \le c_0 p(x, y; C_1). \tag{1.58}$$

If f is C^1 on H, equation (1.58) remains valid if c_0 in equation (1.58) is replaced by $\tilde{c} = \sup\{c(x) : x \in H\}$; *and if $c(x) < \tilde{c}_0$ except for countably many $x \in H$, then for all $x, y \in G$ with $x \neq y$*

$$p(f(x), f(y); C_2) < \tilde{c}_0 p(x, y; C_1).$$

Proof. By virtue of our previous results, it suffices to prove equation (1.57). Assume that $x \in H_1$, and f is Fréchet differential at x and define $\gamma(x)$ by

$$\gamma(x) = \max_{1 \le i \le m} (f_i(x))^{-1} \sum_{j=1}^{n} \Big|\frac{\partial f_i(x)}{\partial x_j}(x)\Big| x_j.$$

If $v \in \mathbb{R}^n$ and $|v|_x \le 1$, we have $|v_j| \le x_j$ for $1 \le j \le n$ and

$$\Big|\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x) v_j\Big| \le \sum_{j=1}^{n} \Big|\frac{\partial f_i}{\partial x_j}(x)\Big| x_j \le \gamma(x) f_i(x),\ 1 \le i \le m. \tag{1.59}$$

Equation (1.59) implies that

$$|f'(x)(v)|_{f(x)} \le \gamma(x),$$

so $c(x) \le \gamma(x)$. Conversely, select k, $1 \le k \le m$, so that

$$\sum_{j=1}^{m} \Big|\Big(\frac{\partial f_k}{\partial x_j}\Big)(x)\Big| x_j = \gamma(x) f_k(x).$$

Define $v \in \mathbb{R}^n$ by $v_j = \varepsilon_j x_j$, where $\varepsilon_j = \operatorname{sgn}\big(\frac{\partial f_k}{\partial x_j}(x)\big)$. Then we have that $|v|_x = 1$ and

$$\sum_{j=1}^{n} \Big(\frac{\partial f_k}{\partial x_j}(x)\Big) x_j = \gamma(x) f_k(x).$$

We conclude that

$$c(x) \ge |f'(x)(v)|_{f(x)} = \gamma(x),$$

which completes the proof. □

Corollary 1.6 generalizes Theorem 4.1 in [29].

In general the problem of estimating $c(x)$ is nontrivial. However, there is one class of examples which can be reduced to the situation of Corollary 1.6.

Lemma 1.4. *Let V be an n dimensional Banach space. Let h_i, $1 \leq i \leq n$, be continuous linear functionals on V and assume that the functionals are linearly independent. If $C = \{x \in V : h_i(x) \geq 0 \text{ for } 1 \leq i \leq n\}$, C is a closed cone with nonempty interior in V. If $L : V \to \mathbb{R}^n$ is defined by $L(x) = (h_1(x), h_2(x), \cdots, h_n(x))$, L is one-one and onto and $L(\overset{\circ}{C}) = \overset{\circ}{K^n}$. For all $x, y \in \overset{\circ}{C}$ we have*

$$p(Lx, Ly; k^n) = p(x, y; C) \text{ and } d(Lx, Ly; K^n) = d(x, y; C).$$

Proof. If L is not onto, then by taking a nonzero vector a in the orthogonal complement of the range of L, we get for all $z \in V$

$$\sum_{i=1}^{n} a_i h_i(z) = 0.$$

This implies that $\sum_{i=1}^{n} a_i h_i = 0$, which contradicts linear independence. It follows that L is onto, and since $\dim(V) = \dim(\mathbb{R}^n)$, L is necessarily one-one. By definition, $C = L^{-1}(K^n)$, where L^{-1} is one-one, continuous and linear, so C is easily seen to be a closed cone. Because L and L^{-1} are open maps, we have

$$L(\overset{\circ}{C}) \subset \overset{\circ}{K^n} \text{ and } L^{-1}(\overset{\circ}{K^n}) \subset \overset{\circ}{C},$$

which implies that $L(\overset{\circ}{C}) = \overset{\circ}{K^n}$.

Applying case (b) of Proposition 1.2 implies that for all $x, y \in C$

$$p(Lx, Ly; K^n) \leq p(x, y; C) \text{ and } p(L^{-1}(Lx), L^{-1}(Ly); C) \leq p(Lx, Ly; K^n),$$

which implies equality. The argument for the projective metric is the same. □

Corollary 1.7. *Let V_1 be a Banach space of dimension n and V_2 a Banach space of dimension m. Let $g_i : V_1 \to \mathbb{R}, 1 \leq i \leq n$, be n linearly independent, linear functionals and let $h_j : V_2 \to \mathbb{R}, 1 \leq j \leq m$, be m linearly independent, linear functionals. Let $C_1 := \{x \in V_1 : g_i(x) \geq 0 \text{ for } 1 \leq i \leq n\}$ and $C_2 := \{y \in V_2 : h_j(y) \geq 0, 1 \leq j \leq m\}$ and define $L_1 : V_1 \to \mathbb{R}^n$ and $L_2 : V_2 \to \mathbb{R}^m$ by $L_1(x) = (g_1(x), g_2(x), \ldots, g_n(x))$ and $L_2(y) = (h_1(y), h_2(y), \ldots, h_m(y))$. Then L_1 and L_2 are one-one and onto, C_1 and C_2 are cones, and $L_1(\overset{\circ}{C_1}) = \overset{\circ}{K^n}$ and $L_2(\overset{\circ}{C_2}) = \overset{\circ}{K^m}$. If G is a subset of $\overset{\circ}{C_1}$ and $f : G \to \overset{\circ}{C_2}$ a map, we have*

$$p(f(x), f(y); C_2) \leq \gamma p(x, y; C_1) \text{ for all } x, y \in G$$

if and only if

$$p(L_2 f L_1^{-1} u, L_2 f L_1^{-1} v; K_m) \leq \gamma p(u, v; K^n) \text{ for all } u, v \in L_1(G).$$

Proof. This follows easily from Lemma 1.4 and is left to the reader. □

Corollary 1.7 reduces a more general situation to the case of Corollary 1.6.

2. Finsler structure and Lipschitz maps for Hilbert's projective metric. In this section we shall present analogues for Hilbert's projective metric of our previous results for the part metric. We begin with some convenient definitions.

Definition 2.1. A subset S of a vector space V satisfies "condition R" ("R" for "radial") if, whenever $x \in S$, it follows that $\lambda x \notin S$ for $\lambda \geq 0$ and $\lambda \neq 1$. If S_1 and S_2 are subsets of V which satisfy condition R, we say that S_1 and S_2 are "radially isomorphic" if, for each $x \in S_1$, there exists $\lambda_x > 0$ with $\lambda_x x \in S_2$ and for each $y \in S_2$, there exists $\mu_y > 0$ with $\mu_y y \in S_1$. The one-one, onto map $x \to \lambda_x x$ of S_1 onto S_2 is called the "radial isomorphism."

Now suppose that C is an Archimedean cone in a real vector space V and that $u \in C - \{0\}$. If S_1 and S_2 are subsets of $P(u)$ and each satisfies condition R and if d_i denotes the restriction of Hilbert's projective metric d to $S_i \times S_i$, then Lemmas 1.1 and 1.2 imply that (S_i, d_i) is a metric space. If S_1 and S_2 are radially isomorphic by a radial isomorphism $\Phi : S_1 \to S_2$, then Φ is an isometry of (S_1, d_1) onto (S_2, d_2).

We wish to describe a Finsler structure for (S_i, d_i), at least when S_i is radially isomorphic to a convex set. Recall that by working in $(V_u, |\cdot|_u)$ we may as well assume initially that C is a normal, Archimedean cone in a normed linear space $(V, \|\cdot\|)$ and that $u \in \overset{\circ}{C}$. If $x \in \overset{\circ}{C}$ and $y \in V$, we abuse notation and write

$$\omega_x(y; C) := \omega(y/x; C), \tag{2.1}$$

where $\omega(y/x; C)$ is as in equation (1.3). We leave to the reader the verification that ω_x is a continuous seminorm on V and that $\omega_x(y; C) = 0$ if and only if $y = \lambda x$ for some $\lambda \in \mathbb{R}$. If C is obvious, we shall write $\omega_x(y)$ instead of $\omega_x(y; C)$.

Our next theorem gives the promised Finsler structure.

Theorem 2.1. *Let C be a normal, Archimedean cone with nonempty interior in a normed linear space $(V, \|\cdot\|)$. Let $G \subset \overset{\circ}{C}$ be a convex set and assume that $G \subset H \subset \overset{\circ}{C}$. Let Σ denote the set of piecewise C^1 maps $\varphi : [0, 1] \to \overset{\circ}{C}$ such that $\varphi(t) \in H$ for $0 \leq t \leq 1$. For any points $x, y \in G$ we have (see equation (2.1))*

$$d(x, y; C) = \min\{\textstyle\int_0^1 \omega_{\varphi(t)}(\varphi'(t))dt : \varphi \in \Sigma, \varphi(0) = x \ \textit{and} \ \varphi(1) = y\}. \tag{2.2}$$

Proof. To prove that the right hand side of equation (2.2) is less than or equal to the left hand side, consider $\varphi(t) = (1 - t)x + ty$. (Note that $\varphi \in \Sigma$ because G is convex). If we define $\alpha = m(y/x; C)$ and $\beta = M(y/x; C)$, the reader can verify (see p. 26 in [36]) that

$$m(\varphi'(t)/\varphi(t); C) = (\alpha-1)[1+t(\alpha-1]^{-1} \text{ and } M(\varphi'(t)/\varphi(t); C) = (\beta-1)[1+t(\beta.-1)]^{-1}.$$

It follows that

$$\begin{aligned}\int_0^1 \omega_{\varphi(t)}(\varphi'(t))dt &= \int_0^1 \{(\beta - 1)[1 + t(\beta - 1)]^{-1} - (\alpha - 1)[1 + t(\alpha - 1)]^{-1}\} \\ &= \log(\beta/\alpha) = d(x, y; C).\end{aligned} \tag{2.3}$$

Conversely, suppose that $\psi \in \Sigma$, $\psi(0) = x$ and $\psi(1) = y$. For convenience we assume that ψ is C^1; the argument for the piecewise C^1 case requires only minor changes. The reader can verify that $\gamma(t) := m(\psi'(t)/\psi(t); C)$ and $\delta(t) := M(\psi'(t)/\psi(t); C)$ give continuous functions of t, $0 \leq t \leq 1$. Because $\beta x - y \in \partial C$ and $y - \alpha x \in \partial C$, an application of the Hahn-Banach theorem as in the proof of Theorem 1.1 shows that there exist continuous linear functionals h_1 and h_2 on V with $h_i(y) > 0$ for all $y \in \overset{\circ}{C}$, $i = 1, 2$, $h_1(\beta x - y) = 0$ and

$h_2(y - \alpha x) = 0$. Let h denote any continuous linear functional which is positive on $\overset{\circ}{C}$. We know that

$$\gamma(t)\psi(t) \leq \psi'(t) \leq \delta(t)\psi(t),$$

which implies that

$$\gamma(t) \leq \frac{h(\psi'(t))}{h(\psi(t))} = \frac{d}{dt}\log(h(\psi(t)) \leq \delta(t), \quad 0 \leq t \leq 1.$$

Integrating this inequality gives

$$\int_0^1 \gamma(t)dt \leq \log(\frac{h(y)}{h(x)}) \leq \int_0^1 \delta(t)dt.$$

Taking $h = h_1$ or h_2 we conclude that

$$\log(\beta) \leq \int_0^1 \delta(t)dt \quad \text{and} \quad -\log(\alpha) \leq -\int_0^1 \gamma(t)dt.$$

It follows that

$$\log(\frac{\beta}{\alpha}) \equiv d(x, y; C) \leq \int_0^1 [\delta(t) - \gamma(t)]dt = \int_0^1 \omega_{\psi(t)}(\psi'(t))dt. \qquad \square$$

Remark 2.1 Wojtkowski [55] has observed that $(\overset{\circ}{K^n}, d)$ can be given a "Finsler structure" as in Theorem 2.1. (In this case, the "Finsler structure" amounts to a continuously varying seminorm $\omega_x, x \in \overset{\circ}{K^n}$). However, the argument in [55] depends on special features of K^n. In fact define $\Phi : \overset{\circ}{K^n} \to \mathbb{R}^n$ by

$$\Phi(x) := \log(x) - \log x_1, \log x_2, \ldots , \log x_n),$$

so Φ is C^∞, H and onto. Define a seminorm q on $\mathbb{R}^n$ by

$$q(y) = \max_i y_i - \min_i y_i.$$

It is observed in Proposition 1.7, p. 22, in [36] that for all $x, y \in \overset{\circ}{K^n}$,

$$d(x, y; K^n) = q(\Phi(x) - \Phi(y)).$$

Thus Φ is an "isometry" of $(\overset{\circ}{K^n}, d)$ onto $(\mathbb{R}^n, q)$. The obvious Finsler structure on $(\mathbb{R}^n, q)$ induces one on $(\overset{\circ}{K^n}, d)$ by using Φ; and for $C = K^n$, this is precisely the Finsler structure in Theorem 2. If $S = \{x \in \overset{\circ}{K^n} : x_n = 1\}$ and $W = \{y \in \mathbb{R}^n : y_n = 0\}$, (S, d) is a metric space the restriction of q to W is a norm, and $\Phi : (S, d) \to (W, q)$ gives an ordinary isometry; see Proposition 1.7 in [36].

As in the case of Theorem 1.2, it is useful to allow maps $\psi : [0, 1] \to \overset{\circ}{C}$ which are only Lipschitz, and this poses no difficulties if $\dim(V) < \infty$. The proof of the following theorem is very similar to that of Theorem 1.2 and is left to the reader.

Theorem 2.2. *Let C be a closed cone with nonempty interior in a finite dimensional Banach space $(V, \|\cdot\|)$. Let $G \subset \overset{\circ}{C}$ be a convex set and assume that $G \subset H \subset \overset{\circ}{C}$. Let Σ denote the set of Lipschitz maps $\varphi : [0,1] \to \overset{\circ}{C}$ with $\varphi(t) \in H$ for $0 \le t \le 1$. If $\varphi \in \Sigma$, φ is Fréchet differentiable almost everywhere, $t \to \varphi'(t)$ is essentially bounded and measurable and $t \to \omega_{\varphi(t)}(\varphi'(t))$ is essentially bounded and measurable. For any points $x, y \in G$ we have*

$$d(x, y; C) = \min\{\textstyle\int_0^1 \omega_{\varphi(t)}(\varphi'(t))dt : \varphi \in \Sigma, \varphi(0) = x \text{ and } \varphi(1) = y\}.$$

If C and V are as in Theorem 2.1 we know that there exist continuous linear functionals $h : V \to \mathbb{R}$ which are positive on $\overset{\circ}{C}$. As usual, we define C^* to be the set of continuous linear functionals which are nonnegative on C (so $h \in C^* - \{0\}$ implies $h(x) > 0$ for all $x \in \overset{\circ}{C}$). For $h \in C^* - \{0\}$ we define

$$S_h = \{x \in \overset{\circ}{C} : h(x) = 1\}, \tag{2.4}$$

and we note that S_h satisfies condition R and that if $h_1, h_2 \in C^* - \{0\}$, S_{h_1} and S_{h_2} are radially isomorphic.

As in the case of the part metric, we want to compute the Lipschitz constant of a map f with respect to Hilbert's projective metric. As a first step we have

Theorem 2.3. *Let C_i, $i = 1, 2$, be a normal Archimedean cone with nonempty interior in a normed linear space $(V_i, \|\cdot\|_i)$. Suppose that $h \in C_1{}^* - \{0\}$, S_h is given by equation (2.4), and $G \subset S_h$ is open in the relative topology on S_h. Assume that $f : G \to \overset{\circ}{C}_2$ is Fréchet differentiable at $x \in G$ and define*

$$\lambda(x) = \inf\{\lambda > 0 : \omega_{f(x)}(f'(x)(v)) \le \lambda\omega_x(v) \text{ for all } v \in V_1 \text{ with } h(v) = 0\} \tag{2.5}$$

and

$$\lambda_*(x) = \lim_{\varepsilon \to 0^+}\big(\sup\{\frac{d(f(y), f(x))}{d(y, x)} : 0 < d(y, x) \le \varepsilon, y \in S_h\}\big). \tag{2.6}$$

Then it is true that $\lambda_*(x) = \lambda(x)$.

Proof. Let $\Sigma = \{v \in V_1 : h(v) = 0, \omega_x(v) = 1\}$. If $v \in \Sigma$, define $\beta = M(v/x)$ and $\alpha = m(v/x)$, so $\beta - \alpha = 1$ and

$$\alpha \le \big(\frac{h(v)}{h(x)}\big) = 0 \le \beta.$$

It follows that for $0 < t < 1$ and $v \in \Sigma$, $M((x + tv)/x) = 1 + t\beta$, $m((x + tv)/x) = 1 + t\alpha$ and

$$\begin{aligned} &d(x + tv, x; C_1) = \log(1 + t\beta) - \log(1 + t\alpha) = t(\beta - \alpha) + R(tv) = t + R(tv), \\ &\lim_{t \to 0^+}(\sup\{t^{-1}R(tv) : v \in \Sigma\}) = 0. \end{aligned} \tag{2.7}$$

For convenience, write $\lambda_* = \lambda_*(x)$ and $\lambda = \lambda(x)$. If $v \in \Sigma$ and we write

$$\gamma = m((f'(x)v)/f(x)) \quad \text{and} \quad \delta = M((f'(x)v)/f(x)),$$

the definition of λ implies that $0 \le \delta - \gamma \le \lambda$. By definition of the Fréchet derivative (recalling that $|\cdot|_{f(x)}$ is equivalent to $\|\cdot\|_2$) we have for $v \in \Sigma$ and $0 < t < 1$

$$f(x + tv) = f(x) + f'(x)(v) + R_1(tv), \quad \lim_{t \to 0^+}(\sup\{t^{-1}|R_1(tv)|_{f(x)} : v \in \Sigma\}) = 0. \tag{2.8}$$

It follows from equation (2.8) that

$$M(f(x+tv)/f(x)) = 1 + t\delta + R_2(tv), \quad m(f(x+tv)/f(x)) = 1 + t\gamma + R_3(tv),$$
$$\lim_{t\to 0^+} (\sup\{t^{-1}R_j(tv) : v \in \Sigma\}) = 0, \quad j = 2, 3. \tag{2.9}$$

We conclude from equation (2.9)

$$d(f(x+tv), f(x); C_2) = \log\left(\frac{1+t\delta+R_2(tv)}{1+t\gamma+R_3(tv)}\right) = t(\delta-\gamma) + R_4(tv),$$
$$\lim_{t\to 0^+} (\sup\{t^{-1}R_4(tv) : v \in \Sigma\}) = 0. \tag{2.10}$$

It follows easily from equation (2.7) and equation (2.10) that given any $\varepsilon > 0$ there exists $\eta > 0$ such that for all $v \in \Sigma$ and $0 \le t \le \eta$,

$$\frac{d(f(x+tv), f(x))}{d(x+tv, x)} \le \lambda + \varepsilon,$$

and this implies that $\lambda_* \le \lambda$. On the other hand, given any $\varepsilon > 0$, there exists $v = v_\varepsilon \in \Sigma$ with $M(f'(x)v/f(x)) = \delta$, $m(f'(x)v/f(x)) = \gamma$ and $\delta - \gamma \ge \lambda - \varepsilon$. For this choice of v, equations (2.7) and (2.10) imply

$$\lim_{t\to 0^+} \frac{d(f(x+tv), f(x))}{d(x+tv, x)} = \delta - \gamma \ge \lambda - \varepsilon. \tag{2.11}$$

Since $\varepsilon > 0$ is arbitrary, we conclude from equation (2.11) that $\lambda_* \ge \lambda$.

With the aid of Theorem 2.3 we can describe precisely the Lipschitz constant (with respect to the projective metric) of a map $f : G \subset S_h \to \overset{\circ}{C}_2$.

Theorem 2.4. *Let C_i, $i = 1, 2$, be a normal Archimedean cone with nonempty interior in a normed linear space $(V_i, \|\cdot\|_i)$. For $h \in C_1{}^* - \{0\}$, let S_h be given by equation (2.4) and let $G \subset \overset{\circ}{C}$, be a convex subset of S_h and $\Gamma = \{u - v : u, v \in G\}$. Assume that $H \subset S_h$ is an open neighborhood of G (in the relative topology on S_h) and that $f : H \to \overset{\circ}{C}_2$ is a continuously Fréchet differentiable map. For $x \in H$ define $\lambda(x)$ by*

$$\lambda(x) = \inf\{\lambda > 0 : \omega_{f(x)}(f'(x)(v)) \le \lambda\omega_x(v) \text{ for all } v \in \Gamma\}. \tag{2.12}$$

If we define λ_0 and k_0 by

$$\begin{aligned} \lambda_0 &= \sup\{\lambda(x) : x \in G\} \\ k_0 &= \inf\{k > 0 : d(f(x), f(y)) \le kd(x, y) \text{ for all } x, y \in G\}, \end{aligned} \tag{2.13}$$

then $k_0 \le \lambda_0$. If the relative interior of G in S_h is nonempty, then $\lambda_0 = k_0$. If $\lambda(x) < \lambda_0$ except for countably many $x \in G$, then for all $x, y \in G$ with $x \ne y$,

$$d(f(x), f(y)) < \lambda_0 d(x, y). \tag{2.14}$$

Proof. Suppose that $x, y \in G$, $x \neq y$, define $\varphi(t) = (1-t)x + ty$, $0 \leq t \leq 1$, and note that $\psi(t) = f(\varphi(t))$ is a C^1 map from $f(x)$ to $f(y)$ in $\overset{\circ}{C}_2$. Theorem 2.1 implies that

$$d(f(x), f(y)) \leq \int_0^1 \omega_{f(\varphi(t))}(f'(\varphi(t))(y-x))dt \leq \int_0^1 \lambda(\varphi(t))\omega_{\varphi(t)}(y-x)dt. \quad (2.15)$$

If we recall that $\lambda(\varphi(t)) \leq \lambda_0$ and use equation (2.3) we find that

$$d(f(x), f(y)) \leq \lambda_0 \int_0^1 \omega_{\varphi(t)}(y-x)dt = \lambda_0 d(x, y), \quad (2.16)$$

so $k_0 \leq \lambda_0$. If $\lambda(z) < \lambda_0$ except for countably many z and $x \neq y$, we find that $\omega_{\varphi(t)}(y-x) > 0$ for $0 \leq t \leq 1$ and $\lambda(\varphi(t)) < \lambda_0$ except for countably many t, so equation (2.15) implies equation (2.14).

If G has nonempty relative interior $\overset{\circ}{G}$ in S_h, then $\cup_{t>0} t\Gamma = \{v \in V_1 : h(v) = 0\}$, so if $\lambda(x)$ is defined by equation (2.12) we have

$$\lambda(x) = \inf\{\lambda > 0 : \omega_{f(x)}(f'(x)v) \leq \lambda\omega_x(v) \text{ for all } v \in V_1 \text{ with } h(v) = 0\}.$$

It is well known that $(1-t)x + ty \in \overset{\circ}{G}$ whenever $x \in \overset{\circ}{G}$, $y \in G$ and $0 \leq t < 1$, so $\overset{\circ}{G}$ is certainly dense in G. Also, by using Proposition 1.1, one can see that $x \to \lambda(x)$ is continuous on G, so we conclude that

$$\lambda_0 := \sup\{\lambda(x) : x \in G\} = \sup\{\lambda(x) : x \in \overset{\circ}{G}\}.$$

Similarly, using Proposition 1.1, we see that

$$k_0 = \inf\{k > 0 : d(f(x), f(y)) \leq kd(x, y) \text{ for all } x, y \in \overset{\circ}{G}\}.$$

Thus, for purposes of proving that $\lambda_0 \leq k_0$, we may as well assume that $G = \overset{\circ}{G}$. However, using the notation of Theorem 2.3, we have

$$\lambda_0 = \sup\{\lambda(x) : x \in \overset{\circ}{G}\} = \sup\{\lambda_*(x) : x \in \overset{\circ}{G}\} \leq k_0,$$

which completes the proof. □

If V is a vector space and $x, y \in V - \{0\}$ we say that "x and y lie on the same linear ray" if there exists $s > 0$ with $y = sx$. If V_1 and V_2 are vector spaces and $U \subset V_1$, we say that $f : U \to V_2$ is "ray-preserving" if whenever $x, y \in U - \{0\}$ and x and y lie on the same linear ray, then $f(x)$ and $f(y)$ are nonzero and $f(x)$ and $f(y)$ lie on the same linear ray.

Corollary 2.1. *Suppose that C_i, $i = 1, 2$, is a normal, Archimedean cone with nonempty interior in a normed linear space $(V_i, \|\cdot\|_i)$ and that $U \subset \overset{\circ}{C}_1$ is a convex set with $\overset{\circ}{U} \neq \emptyset$ and $tU \subset U$ for all $t > 0$. Let $f : U \to \overset{\circ}{C}_2$ be a C^1 map which is ray preserving. For each $x \in U$ define $\tilde{\lambda}(x)$ and $\tilde{\lambda}_0$ by*

$$\tilde{\lambda}(x) = \inf\{c > 0 : \omega_{f(x)}(f'(x)v) \leq c\omega_x(v) \textit{ for all } v \in V_1\}, \quad \tilde{\lambda}_0 = \sup\{\tilde{\lambda}(x) : x \in U\}.$$

Define $\tilde{k}_0$ *by*

$$\tilde{k}_0 = \inf\{c > 0 : d(f(x), f(y); C_2) \leq cd(x, y; C_1) \textit{ for all } x, y \in U\}.$$

Then it follows that $\tilde{\lambda}_0 = \tilde{k}_0$.

Proof. Select $h \in C_1{}^* - \{0\}$ and define $G = \{x \in U : h(x) = 1\}$. If k_0 is defined by equation (2.13), we obviously have $k_0 \leq \tilde{k}_0$. On the other hand, if $x, y \in U$, there exist $s, t > 0$ with $sx, ty \in G$; and the ray-preserving property of f gives positive numbers σ and τ with $f(sx) = \sigma f(x)$ and $f(tx) = \tau f(x)$. Thus we find that

$$d(f(x), f(y)) = d(f(sx), f(ty)) \leq k_0 d(sx, ty) = k_0 d(x, y),$$

so $\tilde{k}_0 \leq k$ and $\tilde{k}_0 = k_0$.

By definition of ray-preserving, for $s > 0$ and $x \in U$, there exists a positive real $g(s, x)$ with

$$f(sx) = g(s, x) f(x).$$

We leave to the reader the exercise of proving that g is continuous on its domain. If $x \in U$ we have that

$$(f'(x))(x) = \lim_{t \to 0}\Big[\frac{f(x + tx) - f(x)}{t}\Big] = \lim_{t \to 0}\Big(\frac{g(1 + t, x) - 1}{t}\Big) f(x) = g'(1, x) f(x),$$

where $g'(t, x)$ denotes the derivative of $t \to g(t, x)$. (The fact that $f'(tx)$ exists implies that $g'(t, x)$ exists.) Let $\lambda(x)$ be defined by equation (2.5) for $x \in U$. If $u \in V_1$ and $x \in U$, we can write

$$u = v + sx, \quad s = \frac{h(u)}{h(x)},$$

so $h(v) = 0$. Using the properties of ω_x and $\omega_{f(x)}$ and writing $\tau = g'(1, x)$, we obtain

$$\begin{aligned}\omega_{f(x)}(f'(x)u) &= \omega_{f(x)}(f'(x)v + sf'(x)x) = \omega_{f(x)}(f'(x)v + \tau s f(x)) \\ &= \omega_{f(x)}(f'(x)v) \leq \lambda(x)\omega_x(v) = \lambda(x)\omega_x(u).\end{aligned}$$

This calculation shows that $\tilde{\lambda}(x) \leq \lambda(x)$. The opposite inequality is obvious, so $\tilde{\lambda}(x) = \lambda(x)$ for $x \in U$.

We claim that $\tilde{\lambda}(sx) = \tilde{\lambda}(x)$ for $s > 0$ and $x \in U$. Assuming this fact for the moment and using Theorem 2.4 we see that

$$\tilde{\lambda} = \sup\{\lambda(x) : x \in G\} = k_0 = \tilde{k}_0.$$

To prove that $\tilde{\lambda}(sx) = \tilde{\lambda}(x)$, we need an expression for $f'(sx)(s > 0, x \in U)$ in terms of $f'(x)$ and derivatives of g. For $v \in V_1$ we have

$$\begin{aligned}f'(sx)(v) &= s^{-1} \lim_{t \to 0}\Big[\frac{f(s(x + tv)) - f(sx)}{t}\Big] \\ &= s^{-1} \lim_{t \to 0}\Big\{g(s, x)\Big[\frac{f(x + tv) - f(x)}{t}\Big] + \Big[\frac{g(s, x + tv) - g(s, x)}{t}\Big] f(x + tv)\Big\}.\end{aligned}$$

It follows from this equation that

$$f'(sx)(v) = s^{-1}g(s,x)f'(x)(v) + s^{-1}\lim_{t\to 0}\Big\{\Big[\frac{g(s,x+tv)-g(s,x)}{t}\Big]f(x+tv)\Big\}.$$

Because g is continuous and f is differentiable at x we conclude that

$$\lim_{t\to 0}[g(s,x+tv)-g(s,x)]\Big[\frac{f(x+tv)-f(x)}{t}\Big] = 0 \quad \text{and}$$

$$\lim_{t\to 0}\Big[\frac{g(s,x+tv)-g(s,x)}{t}\Big]f(x+tv) = \lim_{t\to 0}\Big[\frac{g(s,x+tv)-g(s,x)}{t}\Big]f(x)$$
$$:= \varphi(s,x,v)f(x).$$

It follows that

$$f'(sx)(v) = s^{-1}g(s,x)f'(x)(v) + s^{-1}\varphi(s,x,v)f(x),$$

where $\varphi(s,x,v) = \lim_{t\to 0}\big[\frac{g(s,x+tv)-g(s,x)}{t}\big]$ and the limit exists. Using this formula for $f'(sx)$ we see that for $v \in V$, $x \in U$ and $s > 0$,

$$\omega_{f(sx)}(f'(sx)v) = s^{-1}\omega_{f(x)}(f'(x)v) \quad \text{and} \quad \omega_{sx}(v) = s^{-1}\omega_x(v),$$

which implies that $\tilde{\lambda}(sx) = \tilde{\lambda}(x)$. □

Corollary 2.2. *Let C_i $(i = 1,2)$, V_i and $U \subset \overset{\circ}{C}_1$ be as in Corollary 2.1. Assume that $L : V_1 \to V_2$ is a bounded linear map and that $L(U) \subset \overset{\circ}{C}_2$. Define numbers $N(L; U)$ and $k(L, U)$ by*

$$N(L;U) = \inf\{c \geq 0 : \omega(Ly/Lx; C_2) \leq c\omega(y/x; C_1) \text{ for all } x, y \in U\}$$
$$k(L;U) = \inf\{c \geq 0 : d(Ly, Ly; C_2) \leq cd(y,x; C_1) \text{ for all } x, y \in U\}.$$

Then $N(L;U) = k(L;U)$.

Proof. Because L is bounded and linear, we know that if $f(x) = L(x)$ for $x \in U$, then $f'(x)(v) = L(v)$ for $x \in U$, $v \in V_1$. The reader can verify with the aid of Proposition 1.1 that $N(L;U) = N(L;\overset{\circ}{U})$ and $k(L;U) = k(L;\overset{\circ}{U})$, so we may as well assume U is open. If we define $\hat{\lambda}(x)$ (for $x \in U$) by

$$\hat{\lambda}(x) = \inf\{c > 0 : \omega(Ly/Lx) \leq c\omega(y/x) \text{ for all } y \in U\},$$

then one can easily verify that $\hat{\lambda}(x) = \tilde{\lambda}(x)$, where $\tilde{\lambda}(x)$ is as in Corollary 2.1. If $\tilde{\lambda}_0$ and $\tilde{k}_0$ are as in Corollary 2.1, it follows that

$$\tilde{\lambda}_0 = \sup\{\hat{\lambda}(x) : x \in U\} = N(L;U) = \tilde{k}_0 = k(L;U),$$

which completes the proof. □

Remark 2.2 If $U \neq \overset{\circ}{C}_1$, the fact that $k(L;U) = N(L;U)$ is new. However, if $U = \overset{\circ}{C}$, one usually writes $k(L) = k(L;U)$ and $N(L) = N(L;U)$, and it has long been known that $N(L) = k(L)$. In fact, if one defines $\Delta(L)$ by

$$\Delta(L) = \sup\{d(Lx, Ly; C_2) : x, y \in \overset{\circ}{C}_1\},$$

beautiful classical results assert that

$$N(L) = k(L) = \tanh\big((\tfrac{1}{4})\Delta(L)\big), \tag{2.17}$$

where $\tanh(\infty) = 1$. Descriptions of these and related results and references to the literature are given in [36], pp. 42–45. An elementary, self-contained approach to equation (2.17) is given in [19] where it is shown that equation (2.17) is valid if V_i is a real vector space, $C_i \subset V_i$ is a cone which need not be almost Archimedean or have an interior, and $L : V_1 \to V_2$ is a linear map with $L(C_1) \subset C_2$.

Remark 2.3 In [19] the problem of evaluating $k(L) = N(L)$ for a general positive linear operator L is reduced to the case that $C_1 = C_2 = K^2 \subset \mathbb{R}^2$ and L is given by the matrix

$$\begin{pmatrix} \alpha & (1-\alpha) \\ (1-\alpha) & \alpha \end{pmatrix}, \quad \frac{1}{2} \le \alpha < 1.$$

More generally, for $\alpha \ge \frac{1}{2}$, define a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x_1, x_2) = (\alpha x_1 + (1-\alpha)x_2, (1-\alpha)x_1 + \alpha x_2)$. For $0 < \delta \le \frac{1}{2}$, define

$$G_\delta = \{(\tfrac{1}{2}+t, \tfrac{1}{2}-t) : |t| < \delta\} \subset \overset{\circ}{K^2} \quad \text{and} \quad U_\delta = \{sx : x \in G_\delta, s > 0\}$$

and note that

$$f(\tfrac{1}{2}+t, \tfrac{1}{2}-t) = (\tfrac{1}{2}+ct, \tfrac{1}{2}-ct), \quad c = 2\alpha - 1,$$

so $f(U_\delta) \subset \overset{\circ}{K^2}$ if $0 < \delta \le \frac{1}{2}$ and $0 < c\delta \le \frac{1}{2}$. If $x^t = (\frac{1}{2}+t, \frac{1}{2}-t)$ for $|t| \le \frac{1}{2}$ and $v = (1, -1)$, Theorem 2.4 and Corollary 2.2 imply that

$$\begin{aligned} k(f; U_\delta) = N(f; U_\delta) &= \sup\{\omega_{f(x^t)}(f'(x^t)v)(\omega_{x^t}(v))^{-1} : |t| < \delta\} \\ &= \sup\{c(1-4t^2)(1-4c^2t^2)^{-1} : |t| < \delta\}. \end{aligned}$$

It follows that for $0 \le \delta \le \frac{1}{2}$ and $0 \le c\delta \le \frac{1}{2}$, $c := 2\alpha - 1$,

$$N(f; U_\delta) = k(f; U_\delta) = \begin{cases} c := 2\alpha - 1, & \text{if } \frac{1}{2} \le \alpha \le 1 \\ c(1-4\delta^2)(1-4c^2\delta^2)^{-1}, & \text{if } \alpha > 1. \end{cases}$$

In the case that $\delta = \frac{1}{2}$ and $\frac{1}{2} \le \alpha \le 1$, one can check that $c = \tanh(\frac{1}{4}\Delta(f))$

As in the case of the part metric, it is useful to have versions of our theorems in which f is only assumed locally lipschitzian. For simplicity we restrict attention to the case that V_1 and V_2 are finite dimensional.

Theorem 2.5. *Let C_i, $i = 1, 2$, V_i, h, S_h and G be as in the statement of Theorem 2.4. Assume that $\dim(V_1) = n < \infty$, $\dim(V_2) = m < \infty$, that C_i is closed, and that $\overset{\circ}{G}$, the relative interior of G in S_h, is nonempty. Let $f : G \to \overset{\circ}{C_2}$ be locally lipschitzian, so f is Fréchet differentiable as a map from G to $\overset{\circ}{C_2}$ almost everywhere with respect to $(n-1)$ dimensional Lebesgue measure on S_h. For $x \in \overset{\circ}{G}$ such that f is Fréchet differentiable at x, define*

$$\lambda(x) = \inf\{c > 0 : \omega_{f(x)}(f'(x)(v)) \le c\omega_x(v) \text{ for all } v \in V_1 \text{ with } h(v) = 0\},$$

$$\lambda_0 = \text{ess sup}\{\lambda(x) : x \in \overset{\circ}{G}\}.$$

If we define k_0 by

$$k_0 = \inf\{k > 0 : d(f(y), f(x)) \leq kd(y, x) \textit{ for all } y, x \in G\},$$

then $\lambda_0 = k_0$.

Proof. By using Proposition 1.1 and recalling that $G \subset \overset{\circ}{C}_1$, one can see that

$$k_0 = \inf\{k > 0 : d(f(y), f(x)) \leq kd(y, x) \text{ for all } y, x \in \overset{\circ}{G}\}$$

Theorem 2.3 implies that (for $\lambda_*(x)$ as in equation (2.6))

$$k_0 \geq \sup\{\lambda_*(x) : x \in \overset{\circ}{G}\} \geq \lambda_0.$$

By definition, there exists a set $N \subset \overset{\circ}{G}$ of $(n-1)$-dimensional measure zero such that $f'(z)$ exists for $z \in G \backslash N$ and $\lambda(z) \leq \lambda_0$. If $x, y \in \overset{\circ}{G}$ and $x \neq y$, we know that $x - y \in C_1$ or $y - x \in C_1$ and C_1 is closed, so the Hahn-Banach theorem implies that there exists $h_1 \in C_1^*$ with $h_1(x) \neq h_1(y)$.

Select $r > 0$ and define

$$T_r = \{\xi \in S_h : \|\xi - x\| < r \text{ and } h_1(\xi) = h_1(x)\}$$

so T_r is a ball of radius r in an $(n-2)$-dimensional affine linear subspace $W_1 \subset V_1$ and $T_r \subset G$ for r small. For $(\xi, t) \in T_r \times [0, 1)$, we define

$$\Phi(\xi, t) = (1 - t)\xi + ty.$$

It is an easy exercise (left to the reader) to prove that Φ is one-one and C^∞ and that the Fréchet derivative of Φ (as a map from an open subset of $W_1 \times \mathbb{R}$ to S_h) is one-one for all $(\xi, t) \in T_r \times [0, 1)$. It follows by the change of variables formula that $\{(\xi, t) \in T_r \times [0, 1) : \Phi(\xi, t) \in N\}$ has measure zero. Fubini's theorem implies that for almost all $\xi \in T_r$, $\Phi(\xi, t) \notin N$ for almost all $t \in [0, 1]$. It follows that there exists $\xi_j \to x$, $\xi_j \in T_r$, and f is Fréchet differentiable at $\Phi(\xi_j, t)$ and $\lambda(\Phi(\xi_j, t)) \leq \lambda_0$ for almost all t. As in equation (2.15) and (2.16), this gives

$$d(f(\xi_j), f(y)) \leq \lambda_0 d(\xi_j, y).$$

Taking the limit as $j \to \infty$ implies that $d(f(x), f(y)) \leq \lambda_0 d(x, y)$. This shows that $k_0 \leq \lambda_0$ and completes the proof. □

The usefulness of Theorem 2.4 depends on estimating $\lambda(x)$ in equation (2.12). In general this seems a difficult problem, but in the special case $C_1 = K^n$, $V_1 = \mathbb{R}^n$, one can give an explicit formula. We begin with a simple lemma whose proof is left to the reader.

Lemma 2.1. *Let $(V, \|\cdot\|)$ be a normed linear space, and for $\psi \in V^* - \{0\}$ define $V_\psi = \{y : \psi(y) = 0\}$. If x is a fixed element of V with $\psi(x) \neq 0$, define $L_\psi : V \to V$ by*

$$L_\psi(y) = y - \left(\frac{\psi(y)}{\psi(x)}\right)x. \tag{2.18}$$

Then L_ψ is a continuous linear projection onto V_ψ, so $L_\psi(V) \subset V_\psi$ and $L_\psi(y) = y$ for all $y \in V_\psi$. If $\theta \in V^* - \{0\}$, $\theta(x) \neq 0$ and

$$L_\theta(y) = y - \Big(\frac{\theta(y)}{\theta(x)}\Big)x,$$

then $L_\psi L_\theta = L_\psi$ and $L_\theta L_\psi = L_\theta$, so $L_\theta \mid V_\psi$ is a one-one bounded linear map of V_ψ onto V_θ, with inverse $L_\psi \mid V_\theta$. If C is a normal Archimedean cone with nonempty interior in V and $x \in \overset{\circ}{C}$ then $\omega_x(y) := \omega(y/x; C)$ gives a seminorm on V. If θ, $\psi \in C^* - \{0\}$, the restriction of ω_x to V_θ (respectively, V_ψ) gives a norm on V_θ (respectively, V_ψ) which is equivalent to the restriction of $|\cdot|_x$ or $\|\cdot\|$ to V_θ (respectively, V_ψ). The map $L_\theta \mid V_\psi$ is an isometry of (V_ψ, ω_x) onto (V_θ, ω_x),

Remark 2.4. Proposition 1.1 implies that $|\cdot|_x$ and $\|\cdot\|$ are equivalent norms on V, so in proving Lemma 2.1 it suffices to show that the restrictions of ω_x and $|\cdot|_x$ to V_θ give equivalent norm. However, it is easy to check that for all $u \in V_\theta$,

$$|u|_x \leq \omega_x(u) \leq 2|u|_x.$$

Remark 2.5. If $E(x; \psi)$ denotes the set of extreme points of $\{y \in V_\psi : \omega_x(y) \leq 1\}$, the fact that $L_\theta \mid V_\psi$ is a linear isometry onto V_θ implies that $E(x; \theta) = L_\theta(E(x; \psi))$.

Before continuing, it is convenient to introduce some further notation. For positive integers n we define

$$I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}. \tag{2.19}$$

If $J \subset I_n$ and $J \neq \emptyset$, we define $P_J : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_J(x) = y, \ y_j = \begin{cases} x_j, & \text{if } j \in J \\ 0, & \text{if } j \notin J. \end{cases} \tag{2.20}$$

If $J \subset I_n$, J' will denote the complement of J in I_n.

Proposition 2.1. *Let $C := K^n \subset \mathbb{R}^n := V$ and assume that x is a given point in $\overset{\circ}{C}$. Suppose that $\psi \in C^* - \{0\}$ and define $V_\psi = \{y \in V : \psi(y) = 0\}$, $\omega_x(y) := \omega(y/x; C)$ and $B(x; \psi) = \{y \in V_\psi : \omega_x(y) \leq 1\}$. If $J \subset I_n$ and $J \neq \emptyset$, define*

$$\lambda^J = \frac{\psi(P_J x)}{\psi(x)} \quad \text{and} \tag{2.21}$$

$$x^J = P_J(x) - \lambda^J x = L_\psi(P_J(x)), \tag{2.22}$$

where P_J is given by equation (2.20). If $E(x; \psi)$ denotes the set of extreme points of $B(x; \psi)$, we have

$$E(x; \psi) = \{x^J : J \subset I_n, J \neq \emptyset \text{ and } J \neq I_n\}. \tag{2.23}$$

Proof. Define $\theta \in C^* - \{0\}$ by $\theta(y) = y_n$ for all $y \in \mathbb{R}^n$, so $V_\theta = \{y \in \mathbb{R}^n : y_n = 0\}$. We shall first determine $E(x; \theta)$. For any $y \in V_\theta$ we have

$$M(y/x) = \max_{1 \leq i \leq n} \Big(\frac{y_i}{x_i}\Big) \geq \Big(\frac{y_n}{x_n}\Big) = 0 \text{ and } m(y/x) = \min_{1 \leq i \leq n} \Big(\frac{y_i}{x_i}\Big) \leq \Big(\frac{y_n}{x_n}\Big) = 0.$$

We know that $\omega_x \mid V_\theta$ is a norm, so all extreme points y of $B(x;\theta)$ satisfy $\omega_x(y) = 1$. Thus, if y is an extreme point of $B(x;\theta)$ and $r = M(y/x)$ and $s = m(y/x)$, we know that $r \geq 0$, $s \leq 0$ and $r - s = 1$, so $0 \leq r \leq 1$ and $s = r - 1$. We define subsets J_1, J_2 and J_3 of I_n by

$$J_1 = \{j : (\frac{y_j}{x_j}) = r\},\ J_2 = \{j : (\frac{y_j}{x_j}) = r - 1\} \text{ and } J_3 = \{j : r - 1 < \frac{y_j}{x_j} < r\}.$$

Our remarks above show that J_1 and J_2 are nonempty subsets of I_n. If J_3 contains an element $k \neq n$, select $\varepsilon > 0$ and define points $\overline{y}$ and $\hat{y}$ in V_θ by $\overline{y}_i = \hat{y}_i = y_i$ for $i \neq k$ and $\overline{y}_k = y_k + \varepsilon$ and $\hat{y}_k = y_k - \varepsilon$, If ε is chosen sufficiently small, we have $\omega_x(\hat{y}) = 1$ and $\omega_x(\overline{y}) = 1$, so $\overline{y}, \hat{y} \in B(x;\theta)$. However, we also have that

$$y = (\tfrac{1}{2})(\hat{y} + \overline{y}),$$

which contradicts the assumption that y is an extreme point of $B(x;\theta)$. Thus J_3 is empty or $J_3 = \{n\}$.

We next claim that $r = 0$ or $r = 1$. If not, so $0 < r < 1$ and $n \in J_3$, select $\varepsilon > 0$ and define $\overline{y} \in V_\theta$ and $\hat{y} \in V_\theta$ by $\overline{y}_j = y_j = \hat{y}_j$ for $j \in J_3$, $\overline{y}_j = (r+\varepsilon)x_j$ for $j \in J_1$, $\overline{y}_j = (s+\varepsilon)x_j$ for $j \in J_2$, $\hat{y}_j = (r-\varepsilon)x_j$ for $j \in J_1$ and $\hat{y}_j = (s-\varepsilon)x_j$ for $j \in J_2$. For $\varepsilon > 0$ sufficiently small we have

$$0 < r \pm \varepsilon < 1$$

and

$$M(\overline{y}/x) = r + \varepsilon,\ \ m(\overline{y}/x) = s + \varepsilon,\ \ M(\hat{y}/x) = r - \varepsilon,\ \ m(\hat{y}/x) = s - \varepsilon.$$

It follows that, for $\varepsilon > 0$ small, $\overline{y}, \hat{y} \in B(x;\theta)$ and

$$y = (\tfrac{1}{2})(\overline{y} + \hat{y}).$$

This contradicts the assumption that y is an extreme point of $B(x;\theta)$. Thus we see that $r = 0$ or $r = 1$ and that $n \notin J_3$, so J_3 is empty.

It follows from the above remarks that if $y \in E(x;\theta)$, then $y = P_J x$ or $y = -P_J x$, where $J \subset I_n$, J is nonempty and $n \notin J$. Conversely, we claim that every such point $\pm P_J x$ is an extreme point of $B(x;\theta)$. For suppose that $P_J x = \frac{1}{2}(\overline{y} + \hat{y})$ for $\overline{y}, \hat{y} \in B(x;\theta)$. We know that if $y \in B(x;\theta)$ then $y_i \leq x_i$ for all i. In particular $\overline{y}_j \leq x_j$ and $\hat{y}_j \leq x_j$ for $j \in J$, and since we assume that

$$x_j = \tfrac{1}{2}(\overline{y}_j + \hat{y}_j) \quad \text{for } j \in J,$$

we must have that $\overline{y}_j = \hat{y}_j = x_j$ for $j \in J$. However, we also know that $\omega_x(\overline{y}) \leq 1$ and $\omega_x(\hat{y}) \leq 1$, so we must have that $\overline{y}_j \geq 0$ and $\hat{y}_j \geq 0$ for all j. We have

$$0 = \tfrac{1}{2}(\overline{y}_j + \hat{y}_j) \quad \text{for } j \notin J,$$

so $\overline{y}_j = \hat{y}_j = 0$ for $j \notin J$, and $P_J x = \overline{y} = \hat{y}$. This shows that $P_J x$ is an extreme point of $B(x;\theta)$, which implies that $-P_J x$ is also an extreme point.

It is convenient to describe the set of extreme point of $B(x;\theta)$ in a more symmetric way. We claim that

$$E(x;\theta) = \{P_J x - \big(\tfrac{\theta(P_J x)}{\theta(x)}\big)x : J \text{ is a proper, nonempty subset of } I_n\}. \tag{2.24}$$

If J is a nonempty subset of I_n and $n \notin J$, we find that

$$P_J x = P_J x - \Big(\frac{\theta(P_J x)}{\theta(x)}\Big)x.$$

If J is a proper subset of I_n and $n \in J$, then $\theta(P_J x) = \theta(x)$ and

$$P_J x - \Big(\frac{\theta(P_J x)}{\theta(x)}\Big)x = P_J x - x = -P_{J'}(x).$$

If we note that $n \notin J'$, we see that the right hand side of equation (2.24) is precisely $\{\pm P_J x : J \subset I_n,\ J \text{ nonempty },\ n \notin J\}$, which we have already seen equals $E(x;\theta)$.

In the notation of Lemma 2.1,

$$E(x;\theta) = \{L_\theta(P_J x) : J \text{ is a proper, nonempty subset of } I_n\}.$$

If $\psi \in C^* - \{0\}$ and if we use Lemma 2.1 and Remark 2.5 we see that

$$\begin{aligned} E(x;\psi) &= \{L_\psi(L_\theta(P_J x)) : J \subset I_n,\ J \neq \emptyset,\ J \neq I_n\} \\ &= \{L_\psi(P_J x) : J \subset I_n,\ J \neq \emptyset,\ J \neq I_n\}. \qquad \square \end{aligned}$$

With the aid of Proposition 2.1 we can give a useful formula for $\lambda(x)$ in the case that $C_1 = K^n$.

Corollary 2.1. *Suppose that $C_1 = K^n \subset \mathbb{R}^n$; and for $\psi \in C_1^* - \{0\}$, define $S_\psi = \{y \in \overset{\circ}{C}_1 : \psi(y) = 1\}$. Let C_2 be a normal, Archimedean cone with nonempty interior in a normed linear space V_2. Assume that $x \in S_\psi$, H is an open neighborhood of x and $f : H \to \overset{\circ}{C}_2$ is a map which is Fréchet differentiable at x. Define $\lambda(x)$ by*

$$\lambda(x) = \inf\{\lambda > 0 : \omega_{f(x)}(f'(x)v) \leq \lambda\omega_x(v) \text{ for all } v \in \mathbb{R}^n \text{ with } \psi(v) = 0\}.$$

If x^J is defined by equation (2.22), *then*

$$\lambda(x) = \max\{\omega_{f(x)}((f'(x)(x^J); C_2) : J \text{ a nonempty, proper subset of } I_n\}. \tag{2.25}$$

If $V_2 = \mathbb{R}^m$, $C_2 = K^m$ and $f_i(y)$, $1 \leq i \leq m$, denotes component i of $f(y)$, then

$$\begin{aligned} \omega_{f(x)}(f'(x)(x^J); C_2) = \max_{1\leq i,k\leq m} \Big\{&(1-\lambda^J)\Big(\frac{1}{f_i(x)}\Big)f_i'(x)(P_J x) - \lambda^J\Big(\frac{1}{f_i(x)}\Big)f_i'(x)(P_{J'}(x)) \\ &- (1-\lambda^J)\Big(\frac{1}{f_k(x)}\Big)f_k'(x)(P_J x) + \lambda^J\Big(\frac{1}{f_k(x)}\Big)f_k'(x)(P_{J'}x)\Big\}, \end{aligned} \tag{2.26}$$

where λ^J is given by equation (2.21). *If, in addition, we have either* (a) *$f'(x)(x) \leq \gamma f(x)$ and $c_{ij} := (f_i(x))^{-1}\frac{\partial f_i}{\partial x_j}(x) \geq 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$ or* (b) *$f'(x)(x) \geq -\gamma f(x)$ and $-c_{ij} := (f_i(x))^{-1}\frac{\partial f_i}{\partial x_j}(x) \leq 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$, then*

$$\lambda(x) \leq \gamma - \min_{i,k,J}\Big(\sum_{j\in J} c_{kj}x_j + \sum_{j\in J'} c_{ij}x_j\Big), \tag{2.27}$$

where the minimum in equation (2.27) *is taken over* $J \subset I_n$, $J \neq \emptyset$, $J \neq I_n$, *and integers* i, k *with* $1 \leq i, k \leq m$.

Proof. In the notation of Proposition 2.1, the extreme points of $B(x; \psi)$ are the points x^J, $J \subset I_n$, $J \neq \emptyset$ and $J \neq I_n$. Since the Krein-Milman theorem implies that every element of $B(x; \psi)$ is a convex combination of extreme points, equation (2.25) follows easily. Equation (2.26) follows from equation (2.25) and from the formula for $\omega_{f(x)}(y; C_2)$ when $C_2 = K^m$. If we substitute $x - P_{J'}x$ for $P_J x$ and $x - P_J x$ for $P_{J'}x$ in equation (2.26), we obtain

$$\omega_{f(x)}(f'(x)(x^J); C_2) = \max_{i,k}\Big\{(1-\lambda^J) f_i(x)^{-1} f_i'(x)(x) + \lambda^J f_k(x)^{-1} f_k'(x)(x) - f_i(x)^{-1} f_i'(x)(P_{J'}x) - f_k(x)^{-1} f_k'(x)(P_J x)\Big\}. \tag{2.28}$$

If condition (a) holds, we obtain from equation (2.28)

$$\omega_{f(x)}(f'(x)(x^J); C_2) \leq \gamma - \min_{i,k}\Big(\sum_{j\in J} c_{kj}x_j + \sum_{j \in J'} c_{ij}x_j\Big)$$

which gives equation (2.27). If condition (b) holds, a similar argument gives

$$\omega_{f(x)}(f'(x)(x^J); C_2) \leq \gamma - \min_{i,k}\Big(\sum_{j\in J'} c_{kj}x_j + \sum_{j \in J} c_{ij}x_j\Big),$$

which again gives equation (2.27). $\square$

Remark 2.6. If condition (a) or condition (b) in Corollary 2.1 is satisfied and C denotes the $m \times n$ matrix (c_{ij}), one can see from equation (2.27) that if CC^* has all positive entries, then $\lambda(x) < \gamma$.

3. Applications to ordinary differential equations. In this section we shall describe some applications of our previous results to ordinary differential equations in finite dimensional Banach spaces. The idea is first to show that a map T associated with translation along trajectories of an ordinary differential equation is nonexpansive (or even contractive) with respect to the part metric p and then to use some powerful general results concerning nonexpansive maps. We adopt the view that it is the nonexpansivity of T, rather than concavity or monotonicity of T, which is essential. Similar results hold for the projective metric, but we restrict attention to the part metric.

We begin by recalling some refinements of results from the literature. The following theorems are closely related to work of M. Ackoglu and U. Krengel [1], Shih-Kung Lo [32], D. Weller [53], P. Martus [34], R. Sine [49], R. Lyons and this author [33], and this author [38,39,40,41]. The reader is referred to [33] and [39] for a more detailed discussion.

Theorem 3.1 [See [32], [33], [34], [38], [49]]. *Let* $\|x\|_\infty$ *denote the sup norm on* $\mathbb{R}^n$, $\|x\|_\infty = \sup_{1\leq i\leq n} |x_i|$. *Let* D *be a compact subset of* $\mathbb{R}^n$ *and* $f : D \to D$ *a map which is nonexpansive with respect to the sup norm, so* $\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty$ *for all* $x, y \in D$. *For every* $x \in D$, *there exists a finite integer* $p = p(x)$ *and* $\xi = \xi(x) \in D$ *with* $f^p(\xi) = \xi$ *and* $\lim_{k\to\infty} f^{kp}(x) = \xi$. *The integer* $p(x)$ *satisfies* $p(x) \leq 2^n n!$; *and if* $1 \leq n \leq 3$, *then* $p(x) \leq 2^n$.

The following conjecture has been made by Nussbaum in [38, p. 525] and by Lyons and Nussbaum [33, p. 191].

Conjecture [The 2^n Conjecture]. *Let D be a compact subset of $\mathbb{R}^n$ and $f : D \to D$ a map which is nonexpansive with respect to the sup norm $\|\cdot\|_\infty$. If $\xi \in D$ and $f^p(\xi) = \xi$ for some minimal positive integer p, then $1 \le p \le 2^n$.*

The 2^n Conjecture has been proved by Lo [32] and Lyons and Nussbaum [33] for $n = 1, 2$ and by Lyons and Nussbaum for $n = 3$. An old result of Aronszajn and Panitchpakdi [3,54] asserts that if $f : D \to \mathbb{R}^n$ is nonexpansive with respect to the sup norm, then there exists an extension $F : \mathbb{R}^n \to \mathbb{R}^n$ which is nonexpansive with respect to the sup norm. Thus the map f in the conjecture can be considered as defined on $\mathbb{R}^n$. It is easy to show (see [33], [38]) that for every p, $1 \le p \le 2^n$, there is an f as in the Conjecture which has a periodic point of (minimal) period p. In some unpublished work this author has shown that such an f can even be taken to be piecewise linear.

A norm $\|\cdot\|$ on a finite dimensional Banach space X is called "polyhedral" if $\{x \in X : \|x\| \le 1\}$ is a polyhedron. Equivalently, a norm is polyhedral if there exist continuous linear functionals $\varphi_i : X \to \mathbb{R}$, $1 \le i \le m$, with

$$\|x\| = \max\{|\varphi_i(x)| : 1 \le i \le m\}. \tag{3.1}$$

A closed cone K in a finite dimensional Banach space Y is called "polyhedral" if there exist continuous linear functionals $\psi_i : Y \to \mathbb{R}$, $1 \le i \le m$, with

$$K = \{x \in Y : \psi_i(x) \ge 0 \text{ for } 1 \le i \le m\}. \tag{3.2}$$

If $\|\cdot\|$ is a polyhedral norm given by equation (3.1), then

$$L(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)) \tag{3.3}$$

gives a linear isometric imbedding of $(X, \|\cdot\|)$ into $(\mathbb{R}^m, \|\cdot\|_\infty)$. If K is a polyhedral cone given by equation (3.2) and $\overset{\circ}{K} \ne \emptyset$, the map

$$\Psi(x) = (\log(\psi_1(x)), \log(\psi_2(x)), \dots, \log(\psi_m(x))) \tag{3.4}$$

gives an isometry of $\left(\overset{\circ}{K}, p\right)$ into $(\mathbb{R}^m, \|\cdot\|_\infty)$. See [38], pages 524 and 530. In particular, if $K^n = \{x \in \mathbb{R}^n : x_i \ge 0 \text{ for } 1 \le i \le n\}$, the map

$$\Psi : \overset{\circ}{K^n} \to \mathbb{R}^n, \Psi(x) = (\log(x_1), \log(x_2), \dots, \log(x_n)), \tag{3.5}$$

is an isometry of $(\overset{\circ}{K^n}, p)$ onto $(\mathbb{R}^n, \|\cdot\|_\infty)$.

By using these isometries and Theorem 3.1 one immediately obtains

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a finite dimensional Banach space with a polyhedral norm given by equation* (3.1) *and let K be a closed, polyhedral cone given by equation* (3.2). *Assume either* (a) *D is a compact, nonempty subset of X and $f : D \to D$ is a nonexpansive map with respect to the polyhedral norm $\|\cdot\|$ or* (b) *$\overset{\circ}{K} \ne \emptyset$, $D \subset \overset{\circ}{K}$ is compact and nonempty and $f : D \to D$ is a nonexpansive map with respect to the part metric p on $\overset{\circ}{K}$. Then for every $x \in D$ there exists a finite integer $j(x) = j$ and $\xi = \xi(x) \in D$ with $\lim_{k\to\infty} f^{kj}(x) = \xi$ and $f^j(\xi) = \xi$. The integer j satisfies $1 \le j \le m!2^m$; and if $1 \le m \le 3$, then $j \le 2^m$.*

Theorem 3.2 applies in particular to the l_1-norm on $\mathbb{R}^n$; and indeed the first results of this type were obtained by Ackoglu and Krengel [1] for the l_1-norm. Further results for the l_1-norm case have been obtained by Scheutzow [45] and Nussbaum [39,40].

If (C, ρ) is a complete metric space and $T : C \to C$, recall that $\omega(x; T) = \omega(x)$, the omega limit set of x under T, is given by

$$\omega(x; T) = \cap_{k \geq 1} cl(\cup_{j \geq k} T^j(x)), \tag{3.6}$$

where $cl(A)$ denotes the closure of a set A. IF T is nonexpansive with respect to ρ and $\omega(x; T)$ is nonempty, it is known (see [38] for references) that $T|\omega(x; T)$ is an isometry of $\omega(x; T)$ onto $\omega(x; T)$ and that $\omega(y; T) = \omega(x; T)$ for all $y \in \omega(x; T)$. (Note that $\omega(x; T)$ is nonempty if $\gamma(x; T) := cl(\cup_{j \geq 1} T^j(x))$ is compact).

Now suppose that K is a closed, normal cone in a Banach space X, that $u \in K - \{0\}$ and that $P(u)$ is given by equation (3.4), so the part metric p is defined on $P(u)$ and $(P(u), p)$ is a complete metric space. Suppose that $B \subset P(u)$ is closed in the part metric topology. If $T : B \to P(u)$ is a map, we shall say that "T has the fixed point property on B" if, whenever $C \subset B$ is closed, bounded and convex (in the norm topology) and $T(C) \subset C$, then T has a fixed point in C. If T is norm-continuous and compact on every such set $C \subset B$, then T has the fixed point property on B.

If $C \subset P(u)$ and $R = \sup\{p(x, y) : x, y \in C\} < \infty$, we can associate a set $\tilde{C} \supset C$ by

$$\tilde{C} = \cap_{x \in C} V_R(x), \quad \text{where } V_R(x) := \{y \in P(u) : p(y, x) \leq R\}. \tag{3.7}$$

Our next theorem follows by the same argument used to prove Theorem 4.1 or Theorem 4.3 in Chapter 4 of [36].

Theorem 3.3 (Compare Theorem 4.1 and Theorem 4.3 in [36]). *Let K be a closed, normal cone in a Banach space X, and for $u \in K - \{0\}$ let $P(u)$ be given by equation* (3.4) *and let p denote the part metric on $P(u)$. Suppose that $B \subset P(u)$ is closed with respect to the part metric and that $T : B \to P(u)$ is nonexpansive with respect to p and has the fixed point property on B. Suppose that $C \subset B$ is nonempty and closed and bounded in the norm topology and that $T(C) = C$. If* (a) *$\tilde{C} \subset B$, where $\tilde{C}$ is given by equation* (3.7) *or* (b) *B is convex and $T(B) \subset B$, then T has a fixed point in $\tilde{C} \cap B$.*

Proof. Lemma 4.2 in [36] implies that $V_R(x)$ is closed and convex in the norm topology and the normality of K implies that $V_R(x)$ is norm bounded. It follows that in case (a) or case (b), $\tilde{C} \cap B \supset C$ is closed and bounded in the norm topology and convex. The same argument used in Theorem 4.1 of [36] shows that $T(\tilde{C} \cap B) \subset \tilde{C} \cap B$, so the fixed point property implies that T has a fixed point in $\tilde{C} \cap B$. $\square$

For the remainder of this section we shall deal with the situation that K is a closed normal cone with nonempty interior in a normed linear space $(X, \|\cdot\|)$ and that $A \subset \overset{\circ}{K}$. Proposition 1.1 implies that the norm topology and the part metric topology on $\overset{\circ}{K}$ are the same. We shall write $cl(A)$ to denote the closure of A in $(X, \|\cdot\|)$, and we shall write $p - cl(A)$ to denote the closure of A in $(\overset{\circ}{K}, p)$. Equivalently, $p - cl(A)$ is the closure of A in the relative norm topology on $\overset{\circ}{K}$. Thus if $A = \{x \in \overset{\circ}{K^n} : \|x\| < 1\}$, $cl(A) = \{x \in K^n : \|x\| \leq 1\}$ and $p - cl(A) = \{x \in \overset{\circ}{K^n} : \|\cdot\| \leq 1\}$.

In our next result recall that every closed, finite dimensional cone is normal.

Corollary 3.1. *Let K be a closed cone with nonempty interior in a finite dimensional Banach space $(X, \|\cdot\|)$. Assume that $B \subset \overset{\circ}{K}$ is closed in the relative topology on $\overset{\circ}{K}$ and B is convex.*

Let $T : B \to B$ be a map which is nonexpansive with respect to the part metric p. Suppose that there exists $x_0 \in B$ such that $cl(\cup_{j\geq 1} T^j(x_0))$ is a compact subset of B. Then T has a fixed point in B and for every $x \in B$, $cl(\cup_{j\geq 1} T^j(x))$ is a compact subset of B.

Proof. The assumption that $\gamma(x_0; T) = cl(\cup_{j\geq 1} T^j(x_0))$ is compact implies that $\omega(x_0; T) = C$ is compact and nonempty. Our previous remarks imply that $T(C) = C$ and $T|C$ is an isometry, so Theorem 3.3 implies that T has a fixed point $\xi \in B$. If $x \in B$ and $p(x, \xi) \leq R$, then $p(T^j x, \xi) \leq R$ for all $j \geq 1$, so $\cup_{j\geq 1} T^j(x) \subset V_R(\xi)$. Since $V_R(\xi)$ is a compact subset of $\overset{\circ}{K}$ and $T(B) \subset B$, we conclude that

$$cl(\cup_{j\geq 1} T^j(x)) \subset V_R(\xi) \cap B. \qquad \square$$

Remark 3.1. If, in the framework of Corollary 3.1, B is not convex but T has a fixed point ξ in B, then the same argument still shows that $\gamma(x; T)$ is a compact subset of B for all $x \in B$.

Our next theorem is a generalization of Theorem 4.4 in [36]; see also Theorem 4.2 in [36]. The proof is essentially the same and is omitted here.

Theorem 3.4 (Compare Theorem 4.4 in [36]). *Let K be a closed cone with nonempty interior in a finite dimensional Banach space X. Assume that $B \subset \overset{\circ}{K}$ is convex and closed in the relative topology on $\overset{\circ}{K}$. Suppose that $T : B \to B$ is nonexpansive with respect to the part metric p and that T has no fixed point in B. Given any compact sets $C \subset B$ and $D \subset B$, there exists an integer $N = N(C, D)$ such that $T^j(D) \cap C$ is empty for all $j \geq N$.*

Roughly speaking, Theorem 3.4 asserts that for any $x \in B$, $T^j(x)$ approaches $cl(B) - B$.

Theorem 3.4 treats the case that T has no fixed points. Our next theorem describes the structure of the fixed point set of T. The following result is essentially a very special case of Theorem 4.7 in [36], although B is taken to be $\overset{\circ}{K}$ in Theorem 4.7. The reader is referred to Theorem 4.7 on p. 128 in [36] for a proof.

Theorem 3.5 (Compare Theorem 4.7 in [36]). *Let K be a closed cone with nonempty interior in a finite dimensional Banach space X. Let $B \subset \overset{\circ}{K}$ be closed in the relative topology on $\overset{\circ}{K}$ and assume that B is convex. Let $T : B \to B$ be a map which is nonexpansive with respect to the part metric p and has a nonempty fixed point set S. Then there exists a retraction $r : B \to S$ of B onto S such that r is nonexpansive with respect to p.*

The following Theorem generalizes Theorem 3.2 in [29] and summarizes some of the preceding theorems in a form convenient for our applications. The proof follows immediately from the previous theorems.

Theorem 3.6 (Compare Theorem 3.2 in [29]). *Let K be a closed cone with nonempty interior in a finite dimensional Banach space X. Let $B \subset \overset{\circ}{K}$ be convex and closed in the relative topology on $\overset{\circ}{K}$ and assume that $T : B \to B$ is nonexpansive with respect to the part metric p. Then either* (a) *T has a fixed point in B or* (b) *T does not have a fixed point in B. If case* (a) *holds and $S^j = \{x \in B : T^j(x) = x\}$, then there exists a retraction r_j of B onto S_j such that r_j is nonexpansive with respect to p. If, in addition, K is a polyhedral cone given by m linear functionals (see equation* (3.2)*), then for every $x \in B$ there exists a minimal integer $\nu = \nu(x)$ and $\xi = \xi(x) \in B$ with $\lim_{k\to\infty} \|T^{k\nu}(x) - \xi\| = 0$. The integer ν satisfies $1 \leq \nu \leq 2^m m!$*

and $1 \le \nu \le 2^m$ *if* $m = 1, 2$ *or* 3. *If case* (b) *holds and* C *and* D *are any compact subsets of* B, *then there exists* $N = N(C, D)$ *such that* $f^j(D) \cap C$ *is empty for all* $j \ge N$.

In the framework of Theorem 3.6, one is interested in conditions which insure either that T^j has at most one fixed point in B for all $j \ge 1$ or (a stronger condition) that there exists $x_0 \in cl(B)$ with $\lim_{k\to\infty} \|T^k(x) - x_0\| = 0$ for all $x \in B$. The following Proposition provides answers which are satisfactory for many applications.

Proposition 3.1. *Let* K *be a closed, normal cone with nonempty interior in a normed linear space* X. *Le* $B \subset \overset{\circ}{K}$ *be a connected set which is closed in the relative topology on* $\overset{\circ}{K}$ *and suppose that* $T : B \to B$ *is nonexpansive with respect to the part metric* p. *Assume either* (a) $p(Tx, Ty) < p(x, y)$ *for all* $x, y \in B$ *with* $x \ne y$ *or* (b) *there exists* $x_0 \in cl(B)$ (*so possibly* $x_0 \in \partial K$) *and a norm open neighborhood* W *of* x_0 *such that* $\lim_{k\to\infty} \|T^k(x) - x_0\| = 0$ *for all* $x \in W \cap B$. *In case* (a), T^j *has at most one fixed point in* B *for all* $j \ge 1$, *and if* $T^j(x) = x$, *then* $T(x) = x$. *In case* (b), $\lim_{k\to\infty} \|T^k(x) - x_0\| = 0$ *for all* $x \in B$.

Proof. In case (a), we obtain that $p(T^j(x), T^j y) < p(x, y)$ for all $x \ne y$, $x, y \in B$. This clearly implies that T^j has at most one fixed point in B for every j. If $T^j(x) = x$ for some $x \in B$, then $T^j(Tx) = Tx$, so $Tx = x$. In case (b), we apply Lemma 2.3 on p. 66 of [36]. In the notation of Lemma 2.3 in [36], the metric σ comes from the norm on X and the metric ρ from the part metric p. Proposition 1.1 implies that σ and ρ give the same topology on B and that equation (2.46) on p. 67 in [36] is satisfied. Lemma 2.3 in [36] now implies that $\lim_{k\to\infty} \|T^k(x) - x_0\| = 0$.

Remark 3.2. The isometry $\Psi : (\overset{\circ}{K^n}, p) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ of equation (3.5) shows that $T : B \to B$ is nonexpansive with respect to p iff $\Psi T \Psi^{-1} : \Psi(B) \to \Psi(B)$ is nonexpansive with respect to $\|\cdot\|_\infty$. This observation, together with the results of Section 1 (e.g., Proposition 1.2) provides a convenient way of generating examples. Thus $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x_1, x_2) = (-x_2, -x_1)$ is nonexpansive with respect to $\|\cdot\|_\infty$ and has periodic points of period 1, 2 and 4. Thus $T : \overset{\circ}{K^2} \to \overset{\circ}{K^2}$, $T(y_1, y_2) = (y_2^{-1}, y_1^{-1})$ is nonexpansive with respect to p and has periodic points of period 1, 2, and 4.

The fixed point set of a nonexpansive map $f : (\mathbb{R}^n, \|\cdot\|_\infty) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ can be quite complicated and far from convex. For example, define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = (f_1(x), f_2(x))$ and

$$f_i(x) = \begin{cases} x_i, & \text{if } |x_i| \le |x_{i+1}| \quad (\text{ where } x_3 := x_1) \\ \operatorname{sgn}(x_i)|x_{i+1}| & \text{if } |x_i| \ge |x_{i+1}|. \end{cases}$$

The reader can verify that f is nonexpansive with respect to the sup norm and that f is a retraction onto $S = \{(x_1, x_2) : |x_1| = |x_2|\}$. It follows that $\Psi^{-1} f \Psi : \overset{\circ}{K^2} \to \overset{\circ}{K^2}$ is nonexpansive with respect to p and has fixed point set $T = \{y \in \overset{\circ}{K^2} : y_1 = y_2 \text{ or } y_1 = y_2^{-1}\}$.

If we strengthen the hypotheses of Theorem 3.6, we obtain a result which generalizes Theorem 3.3 in [29]. The proof is similar to that of Theorem 3.3 in [29] and is left to the reader.

Theorem 3.7 (Compare Theorem 3.3 in [29]). *Let notation and assumptions be as in Theorem* 3.6. *In addition assume that* T *extends continuously* (*in the norm topology*) *to* $cl(B)$ *and that* $T(cl(B) - \{0\}) \subset B$. *Then the following trichotomy holds*: (i) $\lim_{k\to\infty} \|T^k(x)\| = \infty$ *for all*

$x \in cl(B) - \{0\}$ or (ii) $\lim_{k\to\infty} \|T^k(x)\| = 0$ *for all $x \in cl(B)$ or* (iii) *T has a fixed point in B and all the conclusions of case* (a) *of Theorem 3.6 are satisfied.*

Before moving on to applications we need to recall one more result. If (M, d) is a metric space and $T(t) : M \to M$ is a collection of maps defined for $t \geq 0$, we shall say that $T(t)$ is a nonlinear, strongly continuous semigroups if $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, $T(0) = I$, the identity map, and $t \to T(t)x$ is continuous for all $x \in M$. We shall say that the semigroup is nonexpansive if

$$d(T(t)x, T(t)y) \leq d(x, y) \quad \text{for all } x, y \in M \text{ and } t \geq 0.$$

Theorem 3.8 (Theorem 4 in [38]). *Let (M, d) be a complete metric space and let $T(t) : M \to M$, $t \geq 0$ be a nonexpansive, nonlinear, strongly continuous semigroup. For $x_0 \in M$, assume that $C := cl(\{T(t)x_0 : t \geq 0\})$ is compact and that (C, d) is isometric to a subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$. Then there exists $z = z(x_0) \in C$ such that $\lim_{t\to\infty} T(t)(x_0) = z$.*

Remark 3.3. If M is a closed subset of a finite dimensional Banach space X and the metric d on M comes from a polyhedral norm on X, then we know that (M, d) is isometric to a subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$ for some n, so (C, d) is isometric to a subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$ for any $C \subset M$. Similarly, if K is a closed, polyhedral cone with nonempty interior in a finite dimensional Banach space X and M is a relatively closed subset of $\overset{\circ}{K}$ and p denotes the part metric on M, then (M, p) is isometric to a subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$ for some n. Thus Theorem 3.8 applies to these situations.

We wish to apply the previous theorems to study

$$x'(t) = f(t, x(t)), x(t_0) = x_0.$$

We shall always assume at least the following about f:

H3.1 K is a closed cone with nonempty interior in a finite dimensional Banach space X, B and B_1 are open subsets of $\overset{\circ}{K}$ with $B \subset B_1$ and $f : \mathbb{R} \times B_1 \to X$ is a continuous map which is locally lipschitzian in the x-variable. For each compact set $D \subset B$, there exists a compact set $D_1 \subset B_1$ with the following property: for every pair of points $x, y \in D$ there exists a piecewise C^1 minimal geodesic (with respect to the part metric p) $\varphi : [0, 1] \to \overset{\circ}{K}$ with $\varphi(0) = x$, $\varphi(1) = y$ and $\varphi(t) \in D_1$ for $0 \leq t \leq 1$. For each $x_0 \in B$ and $t_0 \geq 0$, there exists a solution $x(t) = x(t; t_0, x_0)$ of

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{3.8}$$

which is defined for all $t \geq t_0$ and satisfies $x(t; t_0, x_0) \in B$ for all $t \geq t_0$.

By "locally lipschitzian in the x-variable" we mean that for each $t_0 \in \mathbb{R}$ and $x_0 \in B_1$, there exists $\delta > 0$, $c > 0$ and an open neighborhood U of x_0 such that $f(t, \cdot) \mid U$ is a lipschitz map with lipschitz constant c for $|t - t_0| < \delta$.

Given $B \subset \overset{\circ}{K}$, the results of Section 1 (see Corollary 1.1) suggest how to choose B_1 so as to satisfy H3.1. The key assumption in H3.1 is that $x(t; t_0, x_0) \in B$ for all $t \geq t_0$ and all $x_0 \in B$ and $t_0 \geq 0$. We shall discuss later simple conditions which guarantee this.

Assuming H3.1, for each $t_0 \geq 0$ and $t \geq t_0$ we define $U(t, t_0) : B \to B$ by

$$U(t, t_0)(x_0) = x(t; t_0, x_0). \tag{3.9}$$

To apply our previous results we need to evaluate the Lipschitz constant of $U(t, t_0)$ with respect to the part metric p, but first we need some preliminary definitions and observations.

If $t \in \mathbb{R}$, $\xi \in B_1$, $\Delta > 0$, $\xi + \Delta f(t, \xi) \in \overset{\circ}{K}$ and $x \to f(t, x)$ is Fréchet differentiable at ξ with Fréchet derivative $f'(t, \xi)$, define

$$\tilde{c}(t, \Delta, \xi) = \inf\{\lambda > 0 : |v + \Delta f'(t, \xi)(v)|_{\xi+\Delta f(t,\xi)} \le \lambda |v|_\xi \text{ for all } v \in X\}, \tag{3.10}$$

where $|\cdot|_u$ is the norm on X given by equation (1.9). Thus $\tilde{c}(t, \Delta, \xi)$ is the norm of the linear map $I + \Delta f'(t, \xi) : (X, |\cdot|_\xi) \to (X, |\cdot|_{\xi+\Delta f(t,\xi)})$. Define $c(t, \Delta, \xi)$ by

$$c(t, \Delta, \xi) = \lim_{r \to 0^+} (\text{ess sup}\{\tilde{c}(t, \Delta, y) : \|y - \xi\| < r, y \in \overset{\circ}{K}\}). \tag{3.11}$$

If D_1 is a compact subset of B_1 (B_1 as in H3.1), $t \in \mathbb{R}$, $\Delta > 0$ and $\xi + \Delta f(t, \xi) \in \overset{\circ}{K}$ for all $\xi \in D_1$, we define $c(t, \Delta, D_1)$ by

$$c(t, \Delta, D_1) = \sup\{c(t, \Delta, \xi) : \xi \in D_1\}. \tag{3.12}$$

We must show that $c(t, \Delta, D_1) < \infty$. To prove this, let $D_2 \subset B_1$ be a compact set with $D_1 \subset \overset{\circ}{D_2}$ and $\{\xi + \Delta f(t, \xi) : \xi \in D_2\}$ a subset of $\overset{\circ}{K}$. It suffices to prove that

$$\text{ess sup}\{\tilde{c}(t, \Delta, \xi) : \xi \in D_2\} < \infty,$$

where the essential sup is taken over ξ such that $x \to f(t, x)$ is Fréchet differentiable at ξ. Recall that K is normal, so $\|\cdot\|$ and $|\cdot|_x$ are equivalent norms on X for any $x \in \overset{\circ}{K}$. As noted in Section 1, for any $\xi \in \overset{\circ}{K}$, there exists $r > 0$ and $M > 1$ such that

$$M^{-1}|v|_\xi \le |v|_x \le M|v|_\xi$$

for all $v \in X$ and all x with $\|x - \xi\| < r$. Using these facts, a simple compactness argument shows that there exists $M > 1$ with

$$M^{-1}\|v\| \le |v|_x \le M\|v\| \tag{3.13}$$

for all $v \in X$ and every $x \in D_2 \cup \{y + \Delta f(t, y) : y \in D_2\}$. Because f is locally lipschitzian a simple argument shows that there exists $M_1 > 0$ such that $\|f'(t, x)\| \le M_1$ for all $x \in D_2$ such that $y \to f(t, y)$ is Fréchet differentiable at x. If we combine these facts we obtain for $x \in D_2$ and $v \in X$

$$\begin{aligned} |v + \Delta f'(t, x)(v)|_{x+\Delta f(t,x)} &\le |v|_{x+\Delta f(t,x)} + M\Delta \|f'(t, x)(v)\| \\ &\le |v|_{x+\Delta f(t,x)} + M^2 M_1 \Delta |v|_x \le M^2 |v|_x + M^2 M_1 \Delta |v|_x, \end{aligned} \tag{3.14}$$

which proves that $c(t, \Delta, D_1)$ is finite.

However, we obtain more from equation (3.14). Select M_2 so

$$\sup\{\|f(tx)\| : x \in D_2\} = M_2.$$

Select $\rho > 0$ such that $\{y : \|y - x\| < \rho\} \subset \overset{\circ}{K}$ for all $x \in M_2$. If $0 < \Delta M_2 < \rho$, a simple argument as in Proposition 1.1 proves that for all $x \in D_2$

$$|v|_{x+\Delta f(t,x)} \leq (1 - \Delta M_2 \rho^{-1})^{-1} |v|_x. \tag{3.15}$$

It follows from (3.14) and (3.15) that for $0 < \Delta M_2 < \rho$

$$c(t, \Delta, D_1) \leq (1 - \Delta M_2 \rho^{-1})^{-1} + M^2 M_1 \Delta. \tag{3.16}$$

Using equation (3.16) we see that

$$\limsup_{\Delta \to 0^+} (c(t, \Delta, D_1) - 1)\Delta^{-1} := k(t, D_1) < +\infty. \tag{3.17}$$

A similar argument, which we leave to the reader, shows that $k(t, D_1) > -\infty$.

If D_1 is a compact subset of B_1, we shall need to know that $t \to c(t, \Delta, D_1)$ and $t \to k(t, D_1)$ are Lebesgue measurable (assuming that $\Delta > 0$ and $\{x + \Delta f(t, x) : x \in D_1\} \subset B_1$). For $\varepsilon > 0$ define $N_\varepsilon(D_1) = \{y \in \overset{\circ}{K} : \|y - x\| < \varepsilon$ for some $x \in D_1\}$. By using Corollaries 1.3 and 1.5 from Section 1, one can see that

$$\begin{aligned} c(t, \Delta, D_1) = \lim_{\varepsilon \to 0^+} (\sup\{&p(y + \Delta f(t, y), z + \Delta f(t, z))(p(y, z))^{-1} : y, z \in N_\varepsilon(D_1), \\ &0 < p(y, z) < \varepsilon\}). \end{aligned} \tag{3.18}$$

For fixed $\varepsilon > 0$ and $\Delta > 0$ let $\eta_j \to 0^+$ and define

$$\begin{aligned} \Theta_j(t) &= \sup\{p(y + \Delta f(t, y), z + \Delta f(t, z))(p(y, z))^{-1} : y, z \in N_\varepsilon(D_1), \eta_j \leq p(y, z) < \varepsilon\}, \\ \Theta(t) &= \sup\{p(y + \Delta f(t, y), z + \Delta f(t, z))(p(y, z))^{-1} : y, z \in N_\varepsilon(D_1), 0 < p(y, z) < \varepsilon\}. \end{aligned}$$

Assuming that $cl(N_\varepsilon(D_1)) \subset B_1$ and $\eta_j < \varepsilon$, one can see that Θ_j is continuous and $\Theta(t) = \lim_{j \to \infty} \Theta_j(t)$ for all t, so $\Theta(t)$ is Lebesgue measurable. It follows from (3.18) that $t \to c(t, \Delta, D_1)$ is Lebesgue measurable. If $\delta > 0$ and Δ_j, $1 \leq j < \infty$, is a dense set of positive reals in $(0, \delta)$, one can verify that

$$\sup_{0 < \Delta < \delta} [c(t, \Delta, D_1) - 1]\Delta^{-1} = \sup_{j \geq 1} [c(t, \Delta_j, D_1) - 1]\Delta_j^{-1}, \tag{3.19}$$

so the left hand side of (3.19) is measurable and $k(t, D_1)$ is the limit of a sequence of measurable functions and hence measurable.

Theorem 3.9. *Assume that hypothesis* **H3.1** *holds and let notation be as in* **H3.1.** *Let D_0 be compact subset of B, and define $U(t, t_0)$ by equation* (3.9) *and for $0 \leq t_0 \leq t \leq t_1$ define D by*

$$D = \{x(t; t_0, x_0) : x_0 \in D_0, t_0 \leq t \leq t_1\} = \{U(t, t_0)(x_0) : x_0 \in D_0, t_0 \leq t \leq t_1\}.$$

Since D is a compact subset of B, let D_1 be the corresponding compact subset of B_1 guaranteed by **H3.1.** *Select $\delta_0 > 0$ so small that $\xi + \Delta f(t, \xi) \in \overset{\circ}{K}$ for $0 < \Delta \leq \delta_0$, $\xi \in D_1$ and $t_0 \leq t \leq t_1 + \delta_0$. If (for $0 < \Delta < \delta_0$) $c(t, \Delta, D_1)$ is defined by equation* (3.12) *(or, equivalently, equation* (3.18)*) and $k(t, D_1)$ is defined by equation* (3.17), *then $t \to c(t, \Delta, D_1)$ and*

$t \to k(t, D_1)$ are bounded and Lebesgue measurable for $t_0 \le t \le t_1$. For any points $\xi_0, \eta_0 \in D_0$ and for $t_0 \le t \le t_1$ we have

$$p(U(t, t_0)\xi_0, U(t, t_0)\eta_0) \le \exp\Big(\int_{t_0}^{t} k(s, D_1)ds\Big)p(\xi_0, \eta_0), \tag{3.20}$$

where p denotes the part metric.

Proof. We have already shown that $t \to c(t, \Delta, D_1)$ and $t \to k(t, D_1)$ are Lebesgue measurable and an examination of the argument shows that the bounds on $c(t, \Delta, D_1)$ can be chosen independent of t with $t_0 \le t \le t_1$ and of Δ with $0 < \Delta < \delta_0$.

Select $\xi_0, \eta_0 \in D_0$ and for notational convenience write $x(t) = x(t; t_0, \xi_0)$ and $y(t) = x(t; t_0, \eta_0)$. We define $\Theta(t)$ by

$$\Theta(t) = p(x(t), y(t)), \quad t_0 \le t \le t_1.$$

Our first claim is that $t \to \Theta(t)$ is locally lipschitz, so that $\Theta(t)$ is differentiable almost everywhere. The triangle inequality for the part metric implies that

$$|\Theta(t) - \Theta(s)| \le p(x(t), x(s)) + p(y(t), y(s)).$$

If M_1 is chosen so that $\|f(t, \xi)\| \le M_1$ for $t_0 \le t \le t_1$, and $\xi \in D_1$, we obtain from the differential equation (3.8) that

$$\|x(t) - x(s)\| \le M_1|t - s| \text{ and } \|y(t) - y(s)\| \le M_1|t - s|. \tag{3.21}$$

If we use equation (3.21) and equation (1.13) in Proposition 1.13, we find that there exists $\eta > 0$ and $M > 0$ so that for all $t, s \in [t_0, t_1]$ with $|t - s| < \eta$ we have

$$p(x(t), x(s)) \le M\|x(t) - x(s)\| \text{ and } p(y(t), y(s)) \le M\|y(t) - y(s)\|. \tag{3.22}$$

A simple argument using equation (3.21) and (3.22) now shows that there exists a constant M_2 (depending on η, M_1 and M) with

$$p(x(t), x(s)) \le M_2|t - s| \text{ and } p(y(t), y(s)) \le M_2|t - s| \text{ for all } t, s \in [t_0, t_1], \tag{3.23}$$

and this proves that $t \to \Theta(t)$ is locally Lipschitz.

We next fix t, $t_0 \le t \le t_1$, such that $\Theta'(t)$ exists and seek to estimate $\Theta'(t)$. For notational convenience, write $\xi = x(t)$, $\eta = y(t)$, $g(\Delta) = \xi + \Delta f(t, \xi)$ and $h(\Delta) = \eta + \Delta f(t, \eta)$. In this terminology, we know that there exist functions $R_1(\Delta)$ and $R_2(\Delta)$ with

$$x(t + \Delta) = x(t) + \Delta f(t, x(t)) + R_1(\Delta) = g(\Delta) + R_1(\Delta), \tag{3.24}$$

$$y(t + \Delta) = y(t) + \Delta f(t, y(t)) + R_2(\Delta) = h(\Delta) + R_2(\Delta), \tag{3.25}$$

$$\lim_{\Delta \to 0} \Delta^{-1}\|R_j(\Delta)\| = 0. \tag{3.26}$$

By using Proposition 1.1 and equation (3.26) we find that

$$\lim_{\Delta \to 0} \frac{p(h(\Delta) + R_2(\Delta), h(\Delta))}{\Delta} = 0 \text{ and } \lim_{\Delta \to 0} \frac{p(g(\Delta) + R_1(\Delta), g(\Delta))}{\Delta} = 0. \tag{3.27}$$

By using the triangle inequality for p and equation (3.27) we obtain

$$\Theta'(t) = \lim_{\Delta\to 0^+} \frac{p(g(\Delta), h(\Delta)) - p(\xi, \eta)}{\Delta}.$$

By applying Corollary 1.5 and Remark 1.8 to the map $z \to z + \Delta f(t, z)$, $z \in D$, we find that

$$p(g(\Delta), h(\Delta)) = p(\xi + \Delta f(t, \xi), \eta + \Delta f(t, \eta)) \leq c(t, \Delta, D_1)p(\xi, \eta).$$

We conclude that

$$\Theta'(t) \leq \overline{\lim}_{\Delta\to 0^+} \frac{[c(t, \Delta, D_1) - 1]p(\xi, \eta)}{\Delta} := k(t, D_1)\Theta(t). \tag{3.28}$$

If we define $\psi(t)$ by

$$\psi(t) = \exp\Big(\int_{t_0}^{t} k(s, D)ds\Big)\Theta(t),$$

$\psi(t)$ is locally Lipschitzian, and equation (3.28) implies that $\psi'(t) \leq 0$ almost everywhere. It follows that $\psi(t) \leq \psi(t_0)$ for $t_0 \leq t \leq t_1$, and equation (3.20) is satisfied. □

In order to apply Theorem 3.9, one needs precise estimates for $k(t, D_1)$. In this case that $K = K^n \subset \mathbb{R}^n$, Corollary 1.6 gives a formula for $\tilde{c}(t, \Delta, \xi)$, which yields a formula for $k(t, D_1)$.

Theorem 3.10. *Assume that hypothesis* **H3.1** *is satisfied and that, in the notation of* **H3.1,** $X = \mathbb{R}^n$ *and* $K = K^n = \{x \in \mathbb{R}^n : x_i \geq 0,\ 1 \leq i \leq n\}$. *Let* D_0, D *and* D_1 *be as defined in Theorem* 3.1. *Let* E_t *denote the set of* $\xi \in B_1$ *such that* $x \to f(t, x)$ *is Fréchet differentiable at* ξ, *let* $f_i(t, \xi)$ *denote the* ith *coordinate of* $f(t, \xi)$, *and for* $\xi \in E_t$ *define* $g_i(t, \xi)$ *by*

$$g_i(t, \xi) = \xi_i^{-1}\Big[\frac{\partial f_i}{\partial \xi_i}(t, \xi)\xi_i + \sum_{j\neq i}\Big|\frac{\partial f_i}{\partial \xi_j}(t, \xi)\Big|\xi_j - f_i(t, \xi)\Big]. \tag{3.29}$$

Then, for $k(t, D_1)$ *given by equation* (3.17), *we have*

$$k(t, D_1) = \lim_{\varepsilon\to 0^+}\big(\sup_{1\leq i\leq n}(\text{ess sup}\{g_i(t, \xi) : \xi \in E_t \cap N_\varepsilon(D_1)\}\big), \tag{3.30}$$

where $N_\varepsilon(D_1)$ *is the* ε-*neighborhood of* D_1, *in the norm topology. Equation* (3.20) *is satisfied if* $k(t, D_1)$ *is defined by* (3.29) *and* (3.30).

Proof. We leave to the reader the verification that (in the general framework of H3.1 and for Δ small)

$$c(t, \Delta, D_1) = \lim_{\varepsilon\to 0^+}(\text{ess sup}\{\tilde{c}(t, \Delta, \xi) : \xi \in E_t \cap N_\varepsilon(D_1)\}). \tag{3.31}$$

Select $\varepsilon_0 > 0$ so that $cl(N_\varepsilon(D_1))$ is a compact subset of B_1 for $\varepsilon = \varepsilon_0$ and select $\Delta_0 > 0$ so that $\{x + \Delta f(t, x) : x \in cl(N_{\varepsilon_0}(D_1))\}$ is a compact subset of B_1 for $0 < \Delta \leq \Delta_0$ and

$$\Delta\Big|\frac{\partial f_i}{\partial x_i}(t, x)\Big| < 1, \text{ for } x \in N_{\varepsilon_0}(D_1) \text{ and } 0 < \Delta \leq \Delta_0. \tag{3.32}$$

If we apply Corollary 1.6 and equation (1.57) to the map $z \to z + \Delta f(t, z)$ for $z \in N_\varepsilon(D_1)$, we find (because of equation (3.32))

$$c(t, \Delta, D_1) = \lim_{\varepsilon \to 0^+} \Big(\sup_{1 \le i \le n} (\text{ess sup}\{h_i(t, \Delta, \xi) : \xi \in N_\varepsilon(D_1) \cap E_t\}) \Big),$$

$$\text{where } h_i(t, \Delta, \xi) := \xi_i + \Delta \frac{\partial f_i}{\partial \xi_i}(t, \xi)\xi_i + \Delta \sum_{j \ne i} \Big| \frac{\partial f_i}{\partial \xi_j}(t, \xi) \Big| \xi_j.$$

It follows that $0 < \Delta \le \Delta_0$ we have

$$\Delta^{-1}[c(t, \Delta, D_1) - 1] = \lim_{\varepsilon \to 0^+} \Big(\sup_{1 \le i \le n} (\text{ess sup}\{m_i(t, \Delta, \xi) : \xi \in N_\varepsilon(D_1) \cap E_t\}) \Big),$$

$$\text{where } m_i(t, \Delta, \xi) := (\xi_i + \Delta f_i(t, \xi))^{-1} \Big[\frac{\partial f_i}{\partial \xi_i}(t, \xi)\xi_i + \sum_{j \ne i} \Big| \frac{\partial f_i}{\partial \xi_j}(t, \xi) \Big| \xi_j - f_i(t, \xi) \Big].$$

We leave it to the reader to obtain equation (3.29) and equation (3.30) by taking the lim sup as $\Delta \to 0^+$. □

If H3.1 is satisfied and $k(t, D_1)$ is defined by equation (3.17), we define $k(t)$ by

$$k(t) = \sup\{k(t, D_1) : D_1 \text{ is a compact subset of } B_1\}. \tag{3.33}$$

A priori, it may happen that $k(t) = \infty$ for some t; but $k(t)$ is the supremum of a countable family of Lebesgue measurable functions $k(t, D_1^n)$, $n \ge 1$, so $k(t)$ is measurable. In the special case that $K = K^n$,

$$k(t) = \sup_{1 \le i \le n} (\text{ess sup}\{g_i(t, \xi) : \xi \in E_t \cap B_1\}, \tag{3.34}$$

where $g_i(t, \xi)$ is defined by equation (3.29).

With this terminology, we can state our next theorem.

Theorem 3.11. *Assume that hypothesis* **H3.1** *is satisfied. Assume that there is a number* $M \le \infty$ *with*

$$\int_{t_0}^{t} k(s) ds \le M \ \textit{for all } t \ge t_0, \tag{3.35}$$

where $k(t)$ *is defined by equation* (3.33). (1) *If there exists* $\xi_0 \in B$ *such that* $\lim_{t \to \infty} \|x(t; t_0, \xi_0)\| = 0$, *then* $\lim_{t \to \infty} \|x(t; t_0, \xi)\| = 0$ *for all* $\xi \in B$. (2) *If there exists* $\xi_0 \in B$ *with* $\lim_{t \to \infty} \|x(t; t_0, \xi_0)\| = \infty$, *then* $\lim_{t \to \infty} \|x(t; t_0, \xi)\| = \infty$ *for all* $\xi \in B$. (3) *If there exists* $\xi \in B$ *such that* $cl(\{x(t; t_0, \xi_0) : t \ge t_0\}) := \gamma(\xi_0)$ *is a compact subset of* $\overset{\circ}{K}$, *then* $cl(\{x(t; t_0, \xi) : t \ge t_0\}) := \gamma(\xi)$ *is a compact subset of* $\overset{\circ}{K}$ *for all* $\xi \in B$.

Proof. In case 1, 2, or 3, equation (3.20) and equation (3.35) imply that if $\xi \in B$ (so $p(\xi, \xi_0) < \infty$) there exists a positive constant M_1, independent of t, with

$$M_1^{-1} x(t; t_0, \xi_0) \le x(t; t_0, \xi) \le M_1 x(t; t_0, \xi_0). \tag{3.36}$$

If we recall that K is normal, we obtain from (3.36) that there is a constant A with

$$A^{-1} M_1^{-1} \|x(t; t_0, \xi_0)\| \le \|x(t; t_0, \xi)\| \le A M_1 \|x(t; t_0, \xi_0)\| \tag{3.37}$$

for all t, and this inequality completes the proof in case 1 or case 2. In case 3, equation (3.37) implies that $\gamma(\xi)$ is closed and bounded and hence (because X is finite dimensional) compact. It remains to show that $\gamma(\xi) \subset \overset{\circ}{K}$. If $\eta \in \gamma(\xi)$, there exists $t_j \geq t_0$ with

$$\lim_{j\to\infty} \|x(t_j; t_o, \xi) - \eta\| = 0.$$

By using the compactness of $\gamma(\xi_0)$, we can assume by taking a subsequence that there exists $\eta_0 \in \gamma(\xi_0) \subset \overset{\circ}{K}$ with

$$\lim_{j\to\infty} \|x(t_j; t_o, \xi_0) - \eta_0\| = 0.$$

It follows from equation (3.37) that

$$A^{-1}M_1^{-1}\eta_0 \leq \eta \leq AM_1\eta_0.$$

Recall that if C is any closed cone in a Hausdorff topological vector space and if $y \in \overset{\circ}{C}$ and $z \in C$, then $\left(\frac{1}{2}\right)(y+z) \in \overset{\circ}{C}$. If we apply this remark to $y = 2A^{-1}M_1^{-1}\eta_0 \in \overset{\circ}{K}$ and $z = 2\eta - y \in K$, we see that $\eta \in \overset{\circ}{K}$; and it follows that $\gamma(\xi) \subset \overset{\circ}{K}$. □

For simplicity we shall henceforth usually restrict ourselves to the case $X = \mathbb{R}^n$ and $K = K^n$, the standard cone in $\mathbb{R}^n$; but the reader will easily verify that our results hold in a much greater generality.

It remains to give explicit conditions which insure that H3.1 is satisfied. Here, some caution is necessary. In [37] it is observed that it may be desirable to build up the function $f(t, x)$ in equation (3.8) from functions like $\Theta_r(t, x) = \left(\sum_{i=1}^n \sigma_i(t)x_i^r\right)^{\frac{1}{r}}$ or $\Theta_0(t, x) = \Pi_{i=1}^n x_i^{\sigma_i(t)}$, where $\sigma_i(t) > 0$ and $\sum_{i=1}^n \sigma_i(t) = 1$. If r is any real, the map $x \to \Theta_r(t, x)$ is C^∞ on $\overset{\circ}{K^n}$ and extends continuously to K^n, but if $0 \leq r < 1$, the extended map is not locally Lipschitzian on K^n. It is usually assumed (see, for example, [50]) that (in the notation of H3.1) $f(t, x)$ extends to a continuous function on $cl(B)$ and that $x \to f(t, x)$ is locally Lipschitzian, but for certain examples neither of these conditions is satisfied.

We need to give conditions which insure that H3.1 is satisfied. We state below in hypotheses H3.2, H3.3 and H3.4 assumptions which are not meant to be definitive but to give examples where H3.1 or stronger assumptions are satisfied.

H3.2. Let c_i, $1 \leq i \leq n$, be positive reals and let $B \subset \overset{\circ}{K^n}$ be defined by

$$\begin{aligned} B &= \{x \in \overset{\circ}{K^n} : 0 < x_i < c_i \text{ for } 1 \leq i \leq n\}, \quad \text{so} \\ p - cl(B) &= \{x \in \overset{\circ}{K^n} : 1 < x_i \leq c_i \text{ for } 1 \leq i \leq n\}. \end{aligned} \tag{3.38}$$

Assume that $f : \mathbb{R}^x(p - cl(B)) \to \mathbb{R}^n$ is locally lipschitzian and bounded on norm-bounded subsets of $\mathbb{R} \times (p - cl(B))$. If $x \in p - cl(B)$ and $x_i = c_i$ for some i, then for all $t \in \mathbb{R}$,

$$f_i(t, x) < 0.$$

If $\xi \in cl(B) \cap (\partial K^n)$ and $t_0 \in \mathbb{R}$ and if $\xi_j = 0$ for $j \in J$ and $\xi_j > 0$ for $j \notin J$, then there exists $\delta > 0$, $C \geq 0$ and $i \in J$ such that if $|t - t_0| < \delta$ and $\|x - \xi\| < \delta$ and $x \in B$, then

$$x_i^{-1} f_i(t, x) > -C. \tag{3.39}$$

We shall also need a stronger version of H3.2.

H3.3. Let c_i, $1 \le i \le n$, and B be as in H3.2. Assume that $f : \mathbb{R} \times (cl(B)) \to \mathbb{R}^n$ is locally lipschitzian. If $x \in cl(B)$ and $x_i = c_i$ for some i, then for all $t \in \mathbb{R}$,

$$f_i(t, x) < 0.$$

If $\xi \in cl(B) \cap (\partial K^n)$, $\xi \neq 0$, and $t_0 \in \mathbb{R}$ and if $\xi_j = 0$ for $j \in J$ and $\xi_j > 0$ for $j \notin J$, then there exists $i \in J$ with $f_i(t_0, \xi) > 0$.

Remark 3.4. It is convenient in H3.2 and H3.3 sometimes to allow the possibility that $c_i = +\infty$ for some i. If $c_i = \infty$, we remove the assumption in H3.2 and H3.3 that $f_i(t, x) < 0$ if $x \in cl(B)$ and $x_i = c_i$.

It may happen in the framework of H3.2 that f is not bounded on $\mathbb{R} \times (p - cl(B))$ but that a condition even stronger than H3.1 is satisfied. We shall see that this follows from the next hypothesis.

H3.4. Let c_i, $1 \le i \le n$, and B be as in H3.2 and assume that $f : \mathbb{R} \times (p - cl(B)) \to \mathbb{R}^n$ is locally lipschitzian. Given $\varepsilon > 0$, define G_ε by

$$G_\varepsilon = \{x \in \overset{\circ}{K^n} : x_j \ge \varepsilon \text{ for } 1 \le j \le n\}. \tag{3.40}$$

If $x \in p - cl(B)$ and $x_i = c_i$ for some i, then for all $t \in \mathbb{R}$, $f_i(t, x) < 0$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \le \varepsilon_0$ and $x \in G_\varepsilon \cap B$ and $x_i = \varepsilon$ for some i, then $f_i(t, x) > 0$ for all t

If H3.2 or H3.4 holds, one can extend f to a map which is locally lipschitzian in the x-variable and defined on $\mathbb{R} \times \overset{\circ}{K^n}$; and if H3.3 holds, one can extend f to a map which is locally lipschitzian in the x-variable bounded and defined on $\mathbb{R} \times \mathbb{R}^n$. We shall always assume such an extension is to be made, so it makes sense to discuss the initial value problem equation (3.8) for $t_0 \in \mathbb{R}$ and $x_0 \in \overset{\circ}{K^n}$ if H3.2 or H3.4 holds and for $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ if H3.3 holds.

Lemma 3.1. *Assume that* **H3.2** *holds. If $t_0 \in \mathbb{R}$ and $x_0 \in p - cl(B)$ and if $x(t; t_0, x_0)$ denotes the solution of the corresponding initial value problem given by equation* (3.8) *then $x(t; t_0, x_0)$ is defined for all $t \ge t_0$ and $x(t; t_0, x_0) \in B$ for all $t > t_0$. If* **H3.3** *holds and $t_o \in \mathbb{R}$ and $x_0 \in cl(B) - \{0\}$, then $x(t; t_0, x_0)$ is defined for all $t \ge t_0$ and $x(t; t_0, x_0) \in B$ for all $t > t_0$. If $x_0 = 0$, $x(t; t_0, 0)$ is defined for all $t \ge t_0$ and there exists t_1, $t_0 \le t_1 \le \infty$, with $x(t; t_0, 0) = 0$ for $t_0 \le t \le t_1$ and $x(t; t_0, 0) \in B$ for $t > t_1$.*

Proof. Assume first that H3.2 holds. If $x_0 \in p - cl(B)$ and η_i, the ith coordinate of x_0, equals c_i, then $f_i(t_0, x_0) < 0$. It follows that there exists $\delta > 0$ with $x(t; t_0, x_0) \in B$ for $t_0 < t < t_0 + \delta$. Thus we may as well assume from the beginning that $x_0 \in B$. Define T by

$$T = \sup\{t \ge t_0 : x(s; t_0, x_0) \in B \text{ for } t_0 \le s \le t\}.$$

Because we assume that f is bounded on $\mathbb{R} \times B$, the standard existence and uniqueness theory for ordinary differential equations implies that if $T < \infty$, there exists $\xi \in cl(B)$, $\xi \notin B$, with

$$\lim_{t \to T^-} \|x(t; t_0, x_0) - \xi\| = 0.$$

If $\xi \in \partial K^n$, let $J = \{j : \xi_j = 0\}$. Hypothesis 3.2 implies that there exist $i \in J$, $\delta > 0$ and $C \ge 0$ so that equation (3.39) is satisfied for all (t, x) with $T - \delta < t < T$ and $\|x - \xi\| < \delta$ and $x \in B$. Select $0 < \delta_1 < \delta$ so that for $T - \delta_1 \le t < T$ we have

$$\|x(t; t_0, x_0) - \xi\| < \delta.$$

If we take $t - \delta_1 < t < T$ and use equation (3.39) we obtain

$$\log\left(\frac{x_i(t)}{x_i(T-\delta_1)}\right) = \int_{T-\delta_1}^{t} \left(\frac{x_i'(s)}{x_i(s)}\right) ds \geq \int_{T-\delta_1}^{t} -C ds \geq -C\delta_1.$$

Since the left hand side of this inequality approaches $-\infty$ as $t \to T^-$, we obtain a contradiction; and it follows that $\xi \notin \partial K^n$. On the other hand, once we know that $\xi \notin \partial K^n$, so $\xi \in p - cl(B)$, the same argument given at the start of the proof shows that $\xi_i \neq c_i$ for $1 \leq i \leq n$. Thus we conclude that $\xi \in B$, which is impossible.

Next assume that H3.3 holds. A simple continuity argument shows that $f_i(t,x) \geq 0$ for all t and for $x \in cl(B) \cap \{\xi : \xi_i = 0\}$. The reader can verify that the previously mentioned extension of f can be chosen so that $f_i(t,x) \geq 0$ for all t and for $x \in \{\xi : \xi_i = 0\}$. Standard results now imply that if $\eta = x_0 \in cl(B) \cap (\partial K^n)$ and $\eta \neq 0$, then $x(t; t_0, \eta) \in K^n$ for all $t \geq t_0$. If $J = \{j : \eta_j = 0\}$, H3.3 implies that there exists $i \in J$ with $f_i(t_0, \eta) > 0$, so (writing $x(t) = x(t; t_0, \eta)$) there exists $\delta > 0$ with $x_i(t) > 0$ for $t_0 < t < t_0 + \delta$. We claim that $x(t) \in \overset{\circ}{K^n}$ for $t_0 < t < t_0 + \delta$. If not, there exists τ, $t_0 < \tau < t_0 + \delta$, with $x(\tau) \in \partial K^n$, $x(\tau) \neq 0$. But then the same argument as before shows that there exists p, $1 \leq p \leq n$, with $f_p(\tau, x(\tau)) > 0$, However, this implies that $x_p(t) < 0$ for $t < \tau$ and $\tau - t$ small, a contradiction. A simpler argument, which we leave to the reader, shows that, by decreasing δ, we can also assume that $x_i(t) < c_i$ for $t_0 < t < t_0 + \delta$ and $1 \leq i \leq n$. It follows that $x(t) \in B$ for $t_0 < t < t_0 + \delta$, so our previous result for hypothesis H3.2 implies that $x(t) \in B$ for $t > t_0$. If $\eta = 0$ and $x(t_*; t_0, 0) \in cl(B) - \{0\}$ for some $t_* > t_0$, then the remarks above show that $x(t; t_0, 0) \in B$ for all $t > t_*$. The final statement in Lemma 3.1 follows easily from this fact.

Remark 3.5. Hypothesis 3.3 is a generalization of boundary conditions which have been used, for example, in [50]. To see this, recall that an $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \geq 0$ for $i \neq j$, is called "irreducible" if all entries of e^A are positive. Suppose that $B \subset \overset{\circ}{K^n}$ and that $f : \mathbb{R} \times cl(B)$ is continuous and that $\xi \in cl(B) \cap (\partial K^n)$, $\xi \neq 0$, and $t_0 \in \mathbb{R}$. Assume that there exists an irreducible matrix A with

$$f(t_0, \xi) \geq A\xi, \tag{3.41}$$

for ξ viewed as a column vector. (The existence of such an A follows easily from the assumptions made in [50]). If $\xi_j = 0$ for $j \in J$ and $\xi_j > 0$ for $j \notin J$, then there exists $i \in J$ with $f_i(t_0, \xi) > 0$. If not, we obtain from equation (3.41) that for all $i \in J$,

$$0 \geq f_i(t_0, \xi) \geq \sum_{j=1}^{n} a_{ij}\xi_j = \sum_{j \in J'} a_{ij}\xi_j. \tag{3.42}$$

Since $a_{ij} \geq 0$ for all $i \neq j$ and $\xi_j > 0$ for $j \notin J$, we obtain from equation (3.42) that $a_{ij} = 0$ for all $i \in J$, $j \in J'$. If $\mathbb{R}^n_J = \{x \in \mathbb{R}^n : x_j = 0 \text{ for } j \in J\}$ we conclude that $A(\mathbb{R}^n_J) \subset \mathbb{R}^n_J$, which contradicts the assumption that A is irreducible.

With these preliminaries we can give some further applications to differential equation. The following theorem is an easy consequence of our previous results, and the proof is left to the reader.

Theorem 3.12. *Assume that hypothesis* **H3.1** *is satisfied and that, in the notation of* **H3.1**, $X = \mathbb{R}^n$ *and* $K = K^n$ *and* $f(t+1, \xi) = f(t, \xi)$ *for all* $(t, \xi) \in \mathbb{R} \times B_1$. *Assume that* f *extends to a continuous map which is defined and locally lipschitzian in the* x*-variable on* $\mathbb{R} \times B_2$, *where* $B_2 \subset \overset{\circ}{K^n}$ *is an open neighborhood of* $p - cl(B_1)$. *(Note that these conditions are satisfied if* **H3.2** *is satisfied). Let* $k(t)$ *be defined by*

$$k(t) = \sup_{1 \leq i \leq n} (\text{ess sup}\{g_i(t, xi) : \xi \in E_t \cap B_1\}),$$

where $g_i(t, \xi)$ *is given by equation* (3.29) *and* E_t *denotes the set of* $\xi \in B_1$ *where* $x \to f(t, x)$ *is Fréchet differentiable. If* D_1 *is any compact subset of* B_1, *let* $k(t, D_1)$ *be defined by equation* (3.29) *and equation* (3.30). *Assume that* $k(t)$ *is bounded above and that*

$$\int_0^1 k(t)dt \leq 0. \tag{3.43}$$

For each $t_0 \in \mathbb{R}$ *and* $\xi \in p - cl(B)$, *let* $x(t; t_0, \xi) := U(t, t_0)(\xi)$ *denote the solution of the initial value problem equation* (3.8). *Then for each* $t \geq t_0$, *the map* $\xi \to U(t, t_0)(\xi) \in B$ *is defined and is Lipschitzian with respect to the part metric* p. *Thus* $\xi \to U(t, t_0)(\xi)$ *extends to a Lipschitzian map with respect to* p *on* $p - cl(B)$, *and for* $\xi \in p - cl(B)$ $x(t; t_0, \xi) = U(t, t_0)(\xi)$ *is defined for all* $t \geq t_0$ *and* $x(t; t_0, \xi) \in p - cl(B)$. *If* $T : p - cl(B) \to p - cl(B)$ *is defined by*

$$T(\xi) = U(1, 0)(\xi) = x(1; 0, \xi), \tag{3.44}$$

then T *is nonexpansive with respect to the part metric* p. *If* $T^j(\xi) = \xi$ *for some* $\xi \in p - cl(B)$, *then the corresponding solution* $x(t; 0, \xi)$ *of equation* (3.8) *is periodic and* $x(t + j; 0, \xi) = x(t; 0, \xi)$ *for all* t. *If there exists* $\xi_0 \in p - cl(B)$ *such that* $cl\{x(t; 0, \xi_0) : t \geq 0\} := \gamma(\xi_0)$ *is a compact subset of* $p - cl(B)$, *then* $cl\{x(t; 0, \xi) : t \geq 0\} := \gamma(\xi)$ *is a compact subset of* $p - cl(B)$ *for all* $\xi \in p - cl(B)$; *and if, in addition,* B *is convex, then* T *has a fixed point in* $p - cl(B)$. *If there exists* $\xi_0 \in p - cl(B)$ *such that* $cl\{x(t; 0, \xi_0) : t \geq 0\}$ *is a compact subset of* $p - cl(B)$, *then for every* $\xi \in p - cl(B)$, *there exists an integer* $\nu(\xi) = \nu$ *and* $\eta \in p - cl(B)$ *with*

$$\lim_{j \to \infty} \|T^{j\nu}(\xi) - \eta\| = 0 \text{ and } T^\nu(\eta) = \eta.$$

The integer ν *satisfies* $1 \leq \nu \leq 2^n n!$ *and* $1 \leq \nu \leq 2^n$ *if* $1 \leq n \leq 3$. *If* $\Sigma = \{\xi \in B : T(\xi) = \xi\}$ *and if* Σ *is nonempty and* B *is convex, then there exists a retraction* r *of* $p - cl(B)$ *onto* Σ *such that* r *is nonexpansive with respect to the part metric* p.

If, for every compact set $D_1 \subset B_1$, *it is also true that*

$$\int_0^1 k(t, D_1)dt < 0, \tag{3.45}$$

then T *has at most one fixed point in* B; *and if* T *has a fixed point* $\xi_0 \in B$, *then*

$$\lim_{j \to \infty} \|T^j(\xi) - \xi_0\| = 0 \text{ for all } \xi \in p - cl(B).$$

If there does not exist $\xi_0 \in p - cl(B)$ *such that* $\gamma(\xi_0)$ *is a compact subset of* $p - cl(B)$ *and if* C *and* D *are any compact subsets of* $p - cl(B)$, *then there exists an integer* $N = N(C, D)$ *with* $T^j(D) \cap C = \emptyset$ *for* $j \geq N$. *If there exists* $\xi_0 \in p - cl(B)$ *with* $\lim_{j \to \infty} \|T^j(\xi_0)\| = 0$

(respectively, $\lim_{j\to\infty} \|T^j(\xi_0)\| = \infty$*), then for all* $\xi \in p-cl(B)$, $\lim_{j\to\infty} \|T^j(\xi)\| = 0$ *(respectively,* $\lim_{j\to\infty} \|T^j(\xi)\| = \infty$*).*

If the stronger hypothesis **H3.3** *is satisfied (rather than* **H3.1**), *then* T *extends continuously to* $cl(B)$; *and if* T *has no fixed point in* B *and* $c_i < \infty$ *for* $1 \le i \le n$, *it follows that* $\lim_{j\to\infty} \|T^j(\xi)\| = 0$ *for all* $\xi \in cl(B)$.

Our previous results leave gaps as to how to verify H3.1, so it may be useful to mention another criterion.

Proposition 3.2. *Let* K *be a closed cone with nonempty interior in a finite dimensional Banach space* X *and let* B *and* B_1 *be open subsets of* $\overset{\circ}{K}$ *which satisfy the minimal geodesic condition of H3.1. Let* $B_2 \subset \overset{\circ}{K}$ *be an open neighborhood of* $(p-cl(B)) \cup B_1$. *Let* $f : \mathbb{R} \times B_2 \to X$ *be continuous and locally lipschitzian in the* x*-variable*; *and, if* $k(t)$ *is defined by equation* (3.33), *assume that* $k(t)$ *is bounded above on bounded subsets of* $\mathbb{R}$. *For all* $\tau \in \mathbb{R}$ *and* $\xi \in \overset{\circ}{K}$, *let* $x(t;\tau,\xi)$ *denote the solution of equation* (3.8); *and for every* $\tau \in \mathbb{R}$ *and* $\xi \in p-cl(B)$ *with* $\xi \notin B$, *assume that there exists* $\delta > 0$ *with* $x(t;\tau,\xi) \notin B$ *for* $\tau - \delta < t < \tau$. *Assume that there exists* $\xi_0 \in (p-cl(B))$ *and* $t_0 \in \mathbb{R}$ *such that* $x(t;t_0,\xi_0)$ *is defined for all* $t \ge t_0$ *and* $x(t;t_0,\xi_0) \in p-cl(B)$ *for all* $t \ge t_0$. *Then, for every* $\xi \in B$, $x(t;t_0,\xi)$ *is defined for all* $t \ge t_0$ *and* $x(t;t_0,\xi) \in B$ *for all* $t \ge t_0$.

Proof. Given $\xi \in B$, define $T(\xi) = T$ by $T = \sup\{t > t_0 : x(s;t_0,\xi)$ is defined and $x(s;t_0,\xi) \in B$ for $t_0 \le s \le t\}$. It suffices to assume that $T < \infty$ and obtain a contradiction. For $t_0 < t < T$, the same argument used in Theorem 3.9 still applies and shows that equation (3.20) is valid. A simple limiting argument extends equation (3.20) to points in $p-cl(B)$, so for $t_0 \le t < T$

$$p(x(t;t_0,\xi), x(t;t_0,\xi_0)) \le \exp\Big(\int_{t_0}^{t} k(s)ds\Big) p(\xi,\xi_0).$$

It follows that there is a constant M such that

$$p(x(t;t_0,\xi), x(t;t_0,\xi_0)) \le M, \quad t_0 \le t < T.$$

Since $\{x(t;t_0,\xi_0) : t_0 \le t \le T\}$ is a compact subset of $p-cl(B)$, we conclude (using Proposition 1.1) that $\{x(t;t_0,\xi) : t_0 \le t < T\}$ is contained in a compact subset of $\overset{\circ}{K}$. It follows from equation (3.8) that $x'(t;t_0,\xi)$ is uniformly bounded on $[t_0,T)$, so there exists $\eta \in p-cl(B)$ with

$$\lim_{t\to T^-} \|x(t;t_0,\xi) - \eta\| = 0.$$

If $\eta \in B$, we contradict the definition of T, so $\eta \notin B$. However, $x(t;t_0,\xi) = x(t;T,\eta)$ for $t_0 \le t < T$; and by assumption there exists $\delta > 0$ with $x(t;T,\eta) \notin B$ for $T-\delta < t < T$, which contradicts the fact that $x(t;t_0,\xi) \in B$ for $t_0 \le t < T$.

Remark 3.6. The key point in Theorem 3.12 is to verify that T is nonexpansive with respect to p, but T may be nonexpansive even if equation (3.43) is not satisfied. For example,

$$y_1'(t) = y_1(t)(\cos(2\pi t))\log(y_1(t)), \quad y_2'(t) = y_2(t)(\sin(2\pi t))\log(y_2(t))$$

gives a differential equation on $\overset{\circ}{K^2}$, and every point in $\overset{\circ}{K^2}$ gives an initial value with a corresponding periodic solution of period 1. However $k(t) = \max(\cos(2\pi t), \sin(2\pi t))$ and equation (3.43) is not satisfied. Nevertheless, it is sometimes possible, after a change of variables, to apply Theorem 3.12. For example, suppose $\alpha_i(t)$ is periodic of period 1, for $1 \le i \le n$, and $\int_0^1 \alpha_i(t)dt = 0$. Define $\beta_i(t) = \exp(\int_0^t \alpha_i(s)ds)$ and for given real numbers a_i and functions $h_i(t, w)$, $1 \le i \le n$, consider the differential equation on $\overset{\circ}{K^n}$

$$y_i' = y_i[a_i \log(y_i) + \alpha_i(t) \log y_i + \beta_i(t) h_i(t, \beta_{i+1}(t)^{-1} \log y_{i+1}), \tag{3.46}$$

where $y_{n+1} = y_1$. If one makes a change of variables $\beta_i \log(z_i) = \log(y_i)$, one obtains for $1 \le i \le n$

$$z_i' = z_i[a_i \log z_i + h_i(t, \log z_{i+1})]. \tag{3.47}$$

If one computes $k(t)$ for equation (3.47), one obtains

$$k(t) = \max_{1 \le i \le n} \Big(a_i + \sup_{w \in \mathbb{R}} \Big|\frac{\partial h_i(t, w)}{\partial w}\Big|\Big),$$

and it is easy to give examples where equation (3.43) is satisfied for this $k(t)$ but not for the $k(t)$ one obtains from equation (3.46).

In the autonomous case, when $f(t, x)$ is independent of t, one obtains directly from Theorem 3.8, Theorem 3.12 and Proposition 3.2 a much stronger form of Theorem 3.12. The following theorem, whose proof is left to the reader, generalizes results in Section 2 of [38].

Theorem 3.13 (Compare [38], Section 2). *Let K be a closed, polyhedral cone with nonempty interior in a finite dimensional Banach space X. Let B and B_1 be open subsets of $\overset{\circ}{K}$ and suppose that B and B_1 satisfy the minimal geodesic condition in hypothesis* **H3.1**. *Let $B_2 \subset \overset{\circ}{K}$ be an open neighborhood of $(p - cl(B)) \cup B_1$ and $f : B_2 \to X$ a locally lipschitzian map. Let $k(t) = c$ (independent of t) be defined by equation* (3.17) *and equation* (3.33). *If $X = \mathbb{R}^n$, $K = K^n$ and $E = \{\xi \in B_1 : x \to f(x)$ is Fréchet differentiable at $\xi\}$, then*

$$c = \max_{1 \le i \le n} (\text{ess sup}\{x_i^{-1}\big(x_i \frac{\partial f_i}{\partial x_i}(x) + \sum_{j \ne i} x_j \Big|\frac{\partial f_i}{\partial x_j}(x)\Big| - f_i(x)) : x \in E \cap B_1\}\big).$$

For every $\xi \in B_2$ let $x(t; \xi)$ denote the solution of

$$x'(t) = f(x(t)), \quad x(0) = \xi.$$

For every $\xi \in p - cl(B)$ with $\xi \notin B$, assume that there exists $\delta > 0$ with $x(t; \xi) \notin B$ for $-\delta < t < 0$. If $c < \infty$ and if there exists $\xi_0 \in p - cl(B)$ such that $x(t; \xi_0)$ is defined for $t \ge 0$ and $x(t; \xi_0) \in p - cl(B)$ for all $t \ge 0$, then for every $\xi \in B$, $x(t; \xi) \in B$ for all $t \ge 0$. If $c < \infty$ and if there exists $\xi_0 \in p - cl(B)$ such that $x(t; \xi_0) \in p - cl(B)$ for all $t \ge 0$ and $\lim_{t\to\infty} \|x(t; \xi_0)\| = 0$ (respectively, $\lim_{t\to\infty} \|x(t; \xi_0)\| = \infty$), then for all $\xi \in B$, $x(t; \xi) \in B$ for all $t \ge 0$ and $\lim_{t\to\infty} \|x(t; \xi)\| = 0$ (respectively, $\lim_{t\to\infty} \|x(t; \xi)\| = \infty$). If $c \le 0$ and if there exists $\xi_0 \in p - cl(B)$ with $f(\xi_0) = 0$, then for every $\xi \in p - cl(B)$, $x(t; \xi) \in p - cl(B)$ for $t \ge 0$ and there exists $\eta = \eta(\xi) \in p - cl(B)$ with $f(\eta) = 0$ and

$$\lim_{t\to\infty} \|x(t; \xi) - \eta\| = 0.$$

The map $\xi \to \eta(\xi)$ is nonexpansive with respect to the part metric and is a retraction of $p - cl(B)$ onto $\{\eta \in p - cl(B) : f(\eta) = 0\} := \Sigma$. If $k(t, D_1) := c(D_1)$ is defined *by equation* (3.17) *for compact subsets* D_1 *of* B_1 *and if* $c(D_1) < 0$ *for every compact set* $D_1 \subset B_1$, *then the equation* $f(\xi) = 0$ *has at most one solution in* B.

Remark 3.7. If $(f(x) = g(x) - \lambda x$ where $\lambda > 0$ and $g : B_2 \to \overset{\circ}{K}$ is nonexpansive with respect to the part metric, one can easily check that $c \leq 0$ for c as in Theorem 3.13. (See the argument on p. 536 in [38]). Using this observation and examples like that in Remark 3.2 or pp. 131–132 in [36] or p. 538 in [38], one can easily construct examples where the set Σ in Theorem 3.13 is complicated and far from convex.

As an example of the use of Theorem 3.12, we consider a variant of an equation studied by Aronsson and Mellander [2]:

$$\begin{aligned} y_i'(t) &= -\alpha_i(t)y_i + (c_i - y_i)\sum_{j=1}^{n} \beta_{ij}(t)y_j + \tilde{\alpha}_i(t)y_i^{-1} + (y_i^{-1} - c_i^{-1})\sum_{j=1}^{n} \tilde{\beta}_{ij}(t)y_j^{-1} \\ &:= f_i(t, y), \quad 1 \leq i \leq n. \end{aligned} \tag{3.48}$$

Theorem 3.14. *Assume that* c_i, $1 \leq i \leq n$, *are positive reals and that* $\alpha_i(t)$, $\tilde{\alpha}_i(t)$, $\beta_{ij}(t)$, *and* $\tilde{\beta}_{ij}(t)$, $1 \leq i, j \leq n$, *are nonnegative, continuous functions which are periodic of period 1. Assume that for all t and for* $1 \leq i \leq n$, $\alpha_i(t) > 0$, $\tilde{\alpha}_i(t) > 0$ *and*

$$-\alpha_i(t)c_i + \tilde{\alpha}_i(t)c_i^{-1} < 0. \tag{3.49}$$

Let $B = \{y \in \overset{\circ}{K}{}^n : 0 < y_i < c_i \text{ for } 1 \leq i \leq n\}$. *For* $\xi \in B$ *and* $t_0 \in \mathbb{R}$, *let* $y(t; t_0, \xi)$ *denote the solution of equation (3.48). Then, for every* $\eta \in B$ *and* $t_0 \in \mathbb{R}$, $y(t; t_0, \eta) \in B$ *for all* $t \geq t_0$ *and* $cl(\{y(t; t_0, \eta) : t \geq t_0\})$ *is a compact subset of* B. *If* $T : B \to B$ *is defined by*

$$T(\xi) = y(1; 0, \eta),$$

then T *has a unique fixed point* $\eta_0 \in B$; *and if* $\eta \in B$, *then* $\lim_{k\to\infty} \|T^k(\eta) - \eta_0\| = 0$. *In particular, equation (3.48) has a unique periodic solution* $y(t; 0, \eta_0) = y_0(t)$ *of period one whose orbit lies in* B.

Proof. For $\varepsilon > 0$, define $G_\varepsilon = \{y \in B : y_i > \varepsilon \text{ for } 1 \leq i \leq n\}$. Because $\tilde{\alpha}_i(t) > 0$, it is clear from equation (3.48) that there exists ε_0 such that if $\xi \in B$ and $\xi_i = \varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, then $f_i(t, y) > \varepsilon$ for all t. Also, equation (3.49) implies that if $\xi \in p - cl(B)$ and $\xi_i = c_i$ for some i, then $f_i(t, y) < 0$ for all t. It follows that if $\xi \in cl(G_\varepsilon)$ for some ε with $0 < \varepsilon \leq \varepsilon_0$, then $y(t; t_0, \xi) \in G_\varepsilon$ for all $t > t_0$. This implies that if $\xi \in p - cl(B)$, then $y(t; t_0, \xi) \in B$ for all $t > t_0$.

A calculation gives

$$\begin{aligned} y_i\left(\frac{\partial f_i}{\partial y_i}\right) = &- \alpha_i(t)y_i - y_i\sum_{j=1}^{n} \beta_{ij}(t)y_j - \tilde{\alpha}_i(t)y_i^{-1} - y_i^{-1}\sum_{j=1}^{n} \tilde{\beta}_{ij}(t)y_j^{-1} \\ &+ (c_i - y_i)\beta_{ii}(t)y_i - (y_i^{-1} - c_i^{-1})\tilde{\beta}_{ii}(t)y_i^{-1}. \end{aligned}$$

For $j \neq i$ and $y \in B$ we obtain

$$\begin{aligned} y_i\left|\frac{\partial f_i}{\partial y_j}\right| &= \left|(c_i - y_i)\beta_{ij}(t)y_j - (y_i^{-1} - c_i^{-1})\tilde{\beta}_{ij}(t)y_j^{-1}\right| \\ &\leq (c_i - y_i)\beta_{ij}(t)y_j + (y_i^{-1} - c_i^{-1})\tilde{\beta}_{ij}(t)y_j^{-1}. \end{aligned}$$

It follows that if $y \in p - cl(B)$, then

$$y_i^{-1}\big[y_i \frac{\partial f_i}{\partial y_i}(t, y) + \sum_{j \neq i} y_j \big|\frac{\partial f_i}{\partial y_j}(t, y)\big| - f_i(t, y)\big] \tag{3.50}$$

$$\leq -2(y_i^{-1} - c_i^{-1})\tilde{\beta}_{ii}(t)y_i^{-2} - 2\tilde{\alpha}_i(t)y_i^{-2} - \sum_{j=1}^{n} \beta_{ij}(t)y_j - y_i^{-2}\sum_{j=1}^{n} \tilde{\beta}_{ij}(t)y_j^{-1} < 0.$$

It follows from equation (3.50) that

$$k(t) \leq \max_{1 \leq i \leq n} [-2\tilde{\alpha}_i(t)c_i^{-2}] < 0,$$

so $T : p - cl(B) \to p - cl(B)$ is a strict contradiction with respect to p, and the contraction mapping principle implies the theorem. □

It is interesting to note that if one does not assume that $(\beta_{ij}(t))$ is irreducible, then our results provide new information even for the original equations studied in [2] and [31]. Thus consider for $1 \leq i \leq n$

$$y_i'(t) = -\alpha_i(t)y_i + (c_i - y_i)\sum_{j=1}^{n} \beta_{ij}(t)y_j(t). \tag{3.51}$$

Theorem 3.15. *Assume that c_i, $1 \leq i \leq n$, are positive constants, and that $\alpha_i(t)$, $1 \leq i \leq n$, are positive continuous functions which are periodic of period one. Assume that $\beta_{ij}(t)$, $1 \leq i$, $j \leq n$, are nonnegative, continuous periodic functions of period one and that*

$$\sum_{j=1}^{n} \beta_{ij}(t) > 0 \text{ for } 1 \leq i \leq n \text{ and all } t. \tag{3.52}$$

Let $B = \{y \in K^n : 0 < y_i < c_i, 1 \leq i \leq n\}$ and for $\eta \in \mathbb{R}^n$ let $y(t) = y(t; \eta)$ denote the solution of equation (3.51) with $y(0) = \eta$. For $\eta \in cl(B)$, $y(t; \eta)$ is defined for all $t \geq 0$ and $y(t; \eta) \in cl(B)$ for all $t \geq 0$; and if $\eta \in B$, $y(t; \eta) \in B$ for all $t \geq 0$. If $T : cl(B) \to cl(B)$ is defined by $T(\eta) = y(1; \eta)$, then T is norm-continuous, $T|B$ is nonexpansive with respect to the part metric p, and $T|D$ is a contraction mapping with respect to p for any compact set $D \subset B$. The map T has at most one fixed point $\eta_0 \in B$, and if $T(\eta_0) = \eta_0$ for some $\eta_0 \in B$, then $\lim_{k\to\infty} \|T^k(\eta) - \eta_0\| = 0$ for all $\eta \in B$. If there exists $\eta_0 \in B$ with $\lim_{k\to\infty} \|T^k(\eta_0)\| = 0$, then $\lim_{k\to\infty} \|T^k(\eta)\| = 0$ for all $\eta \in p - cl(B)$. If the spectral radius λ of $T'(0)$ satisfies $\lambda > 1$ and if 1 is not an eigenvalue of $T'(0)$ or if there exists $v \in \overset{\circ}{K^n}$ with $Tv = \lambda v$, then T has a nonzero fixed point in $cl(B)$.

For reasons of length, we omit the proof of Theorem 3.15. However, we note that the Perron-Frobenius theorem implies that there exists $v \in K^n - \{0\}$ with $T'(0)(v) = \lambda v$ and that $v \in \overset{\circ}{K^n}$ if $T'(0)$ is irreducible. We also remark that, in the generality, of Theorem 3.15, it is easy to construct examples where T may have several distinct nonzero fixed points in $cl(B)$ and hence several distinct periodic solutions.

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