FIR Filter Design via Semidefinite Programming and Spectral Factorization

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Abstract

We present a new semidefinite programming approach to FIR filter design with arbitrary upper and lower bounds on the frequency response magnitude. It is shown that the constraints can be expressed as linear matrix inequalities (LMIs), and hence they can be easily handled by recent interior-point methods. Using this LMI formulation, we can cast several interesting filter design problems as convex or quasi-convex optimization problems, *e.g.*, minimizing the length of the FIR filter and computing the Chebychev approximation of a desired power spectrum or a desired frequency response magnitude on a logarithmic scale.

1 Introduction

We consider the problem of designing a *finite impulse response* (FIR) filter with upper and lower bounds on its frequency response magnitude: given filter length N, find filter tap coefficients $x \in \mathbf{R}^N$, $x = (x(0), \ldots, x(N-1))$, such that the frequency response

$$X(\omega) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

satisfies the magnitude bounds

$$L(\omega) \le |X(\omega)| \le U(\omega), \quad \omega \in \Omega \subseteq [0, \pi] \tag{1}$$

over the frequency range Ω of interest.

One conventional approach to FIR filter design is Chebychev approximation of a desired filter response $D(\omega)$, *i.e.*, one minimizes the maximum approximation error over Ω. Different approaches have been proposed for FIR design via Chebychev approximation. To name a few, Chen and Park [6] use linear programming; Alkhairy *et al.* [1], Preuss [14] and Schulist [15] use Remez algorithm and its variations; Potchinkov and Reemtsen [13] use semi-infinite programming. Even though these methods work well in general, they cannot handle constraints of the form (1), unless certain linear phase constraints (*e.g.*, x_n symmetric around the middle index) are imposed; see [19],[7] and [5]. However, this approach leads to longer FIR filters than necessary if linear phase is not required.

In this paper, we present a new way of solving the proposed class of FIR filter design problems, based on magnitude design, *i.e.*, instead of designing the frequency response $X(\omega)$ of the filter directly, we design its power spectrum, $|X(\omega)|^2$ to satisfy the magnitude bounds (see [8] and [12, CH4]) Let $r(n)$ denote

$$r(n) = \sum_{k=-\infty}^{\infty} x(k)x(k+n), \tag{2}$$

where we take $x(k) = 0$ for $k < 0$ or $k > N-1$. The sequence $r(n)$ is symmetric around $n = 0$, zero for $n \le -N$ or $n \ge N$, and $r(0) \ge 0$. Note that the Fourier transform of $r(n)$,

$$R(\omega) = \sum_{n=-\infty}^{\infty} r(n) e^{-j\omega n} = |X(\omega)|^2,$$

is the power spectrum of $x(n)$. If we use r as our design variables, we can reformulate the FIR design problem in $\mathbf{R}^N$ as

$$\begin{array}{ll} \text{find} & r = (r(0), \ldots, r(N-1)) \\ \text{subject to} & L^2(\omega) \le R(\omega) \le U^2(\omega), \quad \omega \in \Omega \\ & R(\omega) \ge 0, \quad \omega \in [0, \pi]. \end{array} \tag{3}$$

The non-negativity constraint $R(\omega) \ge 0$ is a necessary and sufficient condition for the existence of x satisfying (2) by the Fejér-Riesz theorem (see §4). Once a solution of (3) is found, an FIR filter can be obtained via spectral factorization. An efficient method of minimum-phase spectral factorization is given in Section 4.

The reformulated FIR design problem (3) is a semi-infinite programming problem and many methods have been developed to solve (3) directly (see [9]). In Section 2 and 3, we present two relaxations of the problem that can be solved as a linear program (LP) or a semidefinite program (SDP). In an SDP, we minimize a linear objective subject to a *linear matrix inequality* (LMI) constraint:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = F_0 + \sum_{i=1}^m x_i F_i \geq 0, \end{array}$$

where $x \in \mathbf{R}^m$ is the optimization variable and the symmetric matrices $F_0, \dots, F_m$ are given. The problem of finding an x that satisfies the LMI constraint, or proving that no such x exists, is called an SDP *feasibility problem.* SDP feasibility problems can be cast as ordinary SDPs and solved [17].

The LP or SDP formulation of the filter design problems has several advantages. First, LPs and SDPs can be solved very efficiently and conveniently using recently-developed interior-point methods [10][17] and related tools [16][18]. Secondly, these methods produce a proof of infeasibility when the design specs are too tight. Thirdly, a wide variety of convex constraints can be expressed as LMIs, and hence easily included in the SDP problem.

2 LP formulation

A common practice of relaxing the semi-infinite program (3) is to solve a discretized version of it, *i.e.*, impose the constraints only on a finite subset of the $[0, \pi]$ interval and the problem becomes

$$\begin{array}{ll} \text{find} & r = (r(0), \dots, r(N-1)) \\ \text{subject to} & L^2(\omega_i) \leq R(\omega_i) \leq U^2(\omega_i), \quad \omega_i \in \Omega \\ & R(\omega_i) \geq 0, \quad i = 1, \dots, M, \end{array} \tag{4}$$

where $0 \leq \omega_1 < \omega_2 < \cdots < \omega_M \leq \pi$. Since $R(\omega_i)$ is a linear function in r for each i, (4) is in fact a linear program and can be efficiently solved. When M is sufficiently large, the LP formulation gives very good approximations of (3) in practice. A rule of thumb of choosing M, $M \approx 15N$, is recommended in [1].

However, no matter how large M is, if $R(\omega) \geq 0$ does not hold for *all* $\omega \in [0, \pi]$, no $x \in \mathbf{R}^N$ satisfies (2) and the spectral factorization fails. One way to resolve this problem is to solve (4) with the non-negativity constraint tightened to

$$R(\omega_i) \geq \epsilon, \quad i = 1, \dots, M$$

for an appropriate $\epsilon > 0$, so that even between frequency samples, $R(\omega) \geq 0$.

3 SDP formulation

In this section, we will show that the non-negativity of $R(\omega)$ for all $\omega \in [0, \pi]$ can be cast as an LMI constraint and imposed exactly at the cost of $N(N-1)/2$ auxiliary variables. We will use the following theorem.

Theorem 1 *Given a discrete-time linear system (A, B, C, D), A stable, (A, B, C) minimal and $D + D^T \geq 0$. The transfer function $H(z) = C(zI - A)^{-1}B + D$ satisfies*

$$H(e^{j\omega}) + H^*(e^{j\omega}) \geq 0 \quad \textit{for all} \quad \omega \in [0, 2\pi]$$

if and only if there exists real symmetric matrix P such that the matrix inequality

$$\begin{bmatrix} P - A^T P A & C^T - A^T P B \\ C - B^T P A & D + D^T - B^T P B \end{bmatrix} \geq 0 \tag{5}$$

is satisfied.

Proof: By the positive-real lemma [2][3, CH2.7.2], the existence of P that satisfies (5) implies that $H(z) + H^*(z) \geq 0$, for all $|z| \geq 1$. This provides the sufficient condition.

Let $\mathcal{C}$ denote the region $\{z \mid |z| > 1\}$ on the complex plane. Since $\exp(-H(z))$ is analytic in $\mathcal{C}$, $|\exp(-H(z))|$ assumes its maximum on the boundary of $\mathcal{C}$ by the maximum modulus principle. From the fact that

$$|\exp(-H(z))| = \exp(-\Re H(z)),$$

$\Re H(z) \geq 0$ for all $|z| = 1$ and $\lim_{|z| \to \infty} \Re H(z) = D + D^T \geq 0$, we conclude that $\Re H(z) \geq 0$ everywhere in $\mathcal{C}$. Thus, by the positive-real lemma, there exists P that satisfies (5) and the necessary condition is proved. ■

Observe that $R(\omega)$ has the form

$$R(\omega) = H(e^{j\omega}) + H^*(e^{j\omega}),$$

where

$$H(e^{j\omega}) = \frac{1}{2}r(0) + r(1)e^{-j\omega} + \cdots + r(N-1)e^{-j\omega(N-1)}.$$

In order to apply Theorem 1, we would like to define (A, B, C, D) in terms of r such that

$$\begin{aligned} & C(zI - A)^{-1}B + D \\ & \quad = \frac{1}{2}r(0) + r(1)z^{-1} + \cdots + r(N-1)z^{-(N-1)}. \end{aligned} \tag{6}$$

An obvious choice is the controllability canonical form:

$$A = \begin{bmatrix} 0 & 0 & & \cdots & 0 \\ 1 & 0 & & \cdots & 0 \\ & 1 & & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

$$C = \begin{bmatrix} r(1) & r(2) & \cdots & r(N-1) \end{bmatrix} \quad D = \frac{1}{2}r(0).$$

Of course the realization is not unique, *e.g.*, $(T^{-1}AT, T^{-1}B, CT, D)$ realizes the same transfer function.

It can be easily checked that (A, B, C, D) given by (7) satisfies (6) and all the hypotheses of Theorem 1. Therefore the existence of r and symmetric P that satisfy the matrix inequality (5) is the necessary and sufficient condition for $R(\omega) \geq 0$, for all $\omega \in [0, \pi]$, by Theorem 1.

Note that (5) depends affinely on r and P. Thus we can formulate the SDP feasibility problem:

$$\begin{array}{ll} \text{find} & r \in \mathbf{R}^N \quad \text{and} \quad P = P^T \in \mathbf{R}^{N-1 \times N-1} \\ \text{subject to} & L^2(\omega_i) \leq R(\omega_i) \leq U^2(\omega_i), \quad \omega_i \in \Omega \\ & \begin{bmatrix} P - A^T PA & C^T - A^T PB \\ C - B^T PA & D + D^T - B^T PB \end{bmatrix} \geq 0 \end{array} \quad (8)$$

with (A, B, C, D) given by (7). The SDP feasibility problem (8) can be cast as an ordinary SDP and solved efficiently.

4 Spectral factorization

Given $r \in \mathbf{R}^N$ be the solution of (3), (4) or (8), the desired N-tap FIR filter can be obtained via spectral factorization by the Fejér-Riesz theorem:

Theorem 2 (Fejér-Riesz) *If a complex function $W(z) : \mathbf{C} \to \mathbf{C}$ satisfies*

$$W(z) = \sum_{n=-m}^{m} w(n) z^{-n} \quad \textit{and} \quad W(z) \geq 0 \quad \forall\, |z| = 1,$$

then there exists $Y(z) : \mathbf{C} \to \mathbf{C}$ and $y(0), \ldots, y(m) \in \mathbf{C}$ such that

$$Y(z) = \sum_{n=0}^{m} y(n) z^{-n} \quad \textit{and} \quad W(z) = |Y(z)|^2 \quad \forall\, |z| = 1.$$

Y(z) is unique if we further impose the condition that all its roots be in the unit circle $|z| \leq 1$.

An efficient method for minimum-phase spectral factorization is as follows [11]. We denote the unique minimum-phase factor of $R(z)$ by $X_{\rm mp}(z)$. Denote $\log X_{\rm mp}(z)$ by

$$\log X_{\rm mp}(z) = \alpha(z) + j\varphi(z),$$

we have $\alpha(z) = (1/2) \log R(z)$ and is known. Since $X_{\rm mp}(z)$ is minimum-phase, $\log X_{\rm mp}(z)$ is analytic in the region $\{z \mid |z| \geq 1\}$ and has the power series expansion

$$\log X_{\rm mp}(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \quad |z| \geq 1.$$

Thus, for $z = e^{j\omega}$ we have

$$\alpha(\omega) = \sum_{n=0}^{\infty} a_n \cos \omega n \quad \text{and} \quad \varphi(\omega) = -\sum_{n=0}^{\infty} a_n \sin \omega n,$$

which implies $\alpha(\omega)$ and $\varphi(\omega)$ are Hilbert transform pairs. To determine $X_{\rm mp}(z)$, we first find $\varphi(\omega)$ from $\alpha(\omega)$ via the Hilbert transform. In practice, this step can be replaced by two Fourier transforms of order $\tilde{N}$, with $\tilde{N} \gg N$. Then we construct $X_{\rm mp}$ from $\alpha(\omega)$ and $\varphi(\omega)$. A third Fourier transform of order N yields the coefficients of $X_{\rm mp}(z)$, which gives the desired minimum-phase FIR filter coefficients.

5 Extensions

We have shown that FIR design with magnitude bounds can be cast as SDP feasibility problems. In fact, many extensions of the problem can be handled by simply adding a cost function and/or LMI constraints to our SDP formulation. We will give a few examples in this section.

5.1 Minimum-length FIR design

The length of an FIR-filter is a quasi-convex function of its coefficients [4]. Hence, the problem of finding the minimum-length FIR filter given magnitude upper and lower bounds

$$\begin{array}{ll} \text{minimize} & N \\ \text{subject to} & L(\omega_i) \leq |X_N(\omega_i)| \leq U(\omega_i), \quad i = 1, \ldots, M \end{array}$$

is quasi-convex and can be solved using bisection on N. Each iteration of the bisection involves solving an SDP feasibility problem (8).

5.2 Chebychev approximation on power spectrum

Another interesting extension is the Chebychev approximation of a desired power spectrum

$$\text{minimize} \quad \max_{i=1,\ldots,M} \left| |X(\omega_i)|^2 - |D(\omega_i)|^2 \right|, \quad (9)$$

which is *not* convex in the filter coefficients x. Using the technique developed in Section 3, problem (9) can be reformulated as a convex problem in r and P:

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,M} \left| R(\omega_i) - |D(\omega_i)|^2 \right| \\ \text{subject to} & \begin{bmatrix} P - A^T PA & C^T - A^T PB \\ C - B^T PA & D + D^T - B^T PB \end{bmatrix} \geq 0, \end{array}$$

with (A, B, C, D) given in (7). This problem can be further cast as an SDP:

$$\begin{array}{l} \text{minimize} \quad t \\ \text{subject to} \\ \quad |D(\omega_i)|^2 - t \leq R(\omega_i) \leq t + |D(\omega_i)|^2, \quad i = 1, \ldots, M \\ \quad \begin{bmatrix} P - A^T PA & C^T - A^T PB \\ C - B^T PA & D + D^T - B^T PB \end{bmatrix} \geq 0, \end{array}$$

where t is an auxiliary variable. Upper and lower bounds on the magnitude can also be added to the above SDP.

5.3 Log-Chebychev approximation on magnitude

Since the magnitude of the desired frequency response is usually represented in decibels, it is sometimes more natural to perform the Chebychev approximation on a logarithmic scale:

$$\text{minimize} \quad \max_{i=1,\ldots,M} \left| \log|X(\omega_i)| - \log|D(\omega_i)| \right|.$$

Again, the problem can be cast as a convex problem in r, P and t:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & 1/t \leq R(\omega_i)/|D(\omega_i)|^2 \leq t, \quad i = 1, \ldots, M \\ & \begin{bmatrix} P - A^T P A & C^T - A^T P B \\ C - B^T P A & D + D^T - B^T P B \end{bmatrix} \geq 0, \end{array}$$

which can be further reduced to an SDP using Schur complements:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & R(\omega_i)/|D(\omega_i)|^2 \leq t, \quad i = 1, \ldots, M \\ & \begin{bmatrix} R(\omega_i)/|D(\omega_i)|^2 & 1 \\ 1 & t \end{bmatrix} \geq 0, \quad i = 1, \ldots, M \\ & \begin{bmatrix} P - A^T P A & C^T - A^T P B \\ C - B^T P A & D + D^T - B^T P B \end{bmatrix} \geq 0. \end{array}$$

6 Examples

Example 1

We design a low-pass filter of minimum length, with passband $[0, 0.06]$ and stopband $[0.12, 0.5]$ in normalized frequency (Nyquist rate is equal to 1), that satisfies the magnitude bounds shown in Figure 1. The minimum filter length is 20. One of the solutions is shown in the figure. Note that the filter has roughly linear phase in the passband.

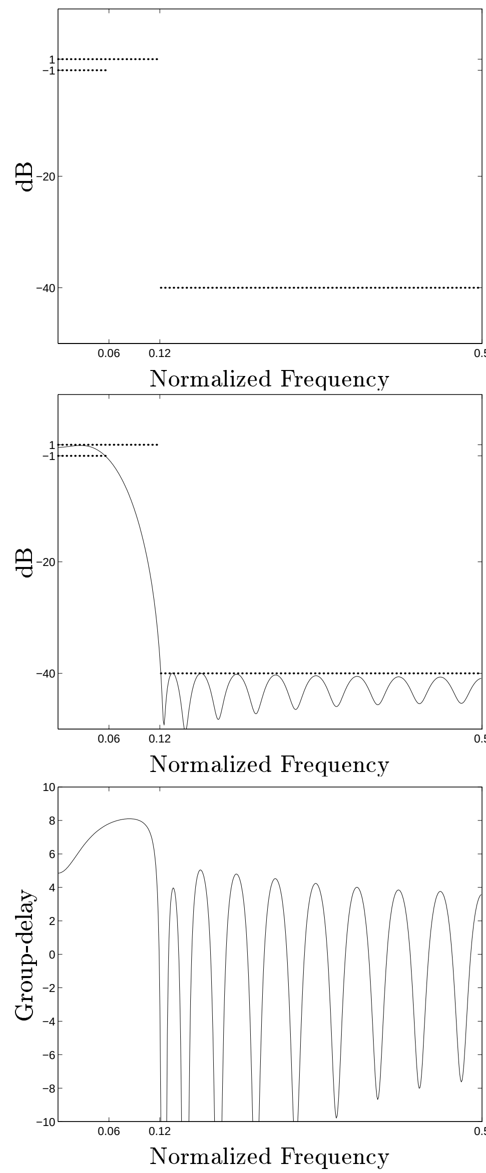

Figure 1: Magnitude bounds and filter response (magnitude and group-delay) of Example 1. The dotted line indicates the magnitude bounds and the solid line indicates the response.

Example 2

Consider the same passband and stopband specifications as in the previous example, we apply Chebychev approximation to the ideal lowpass power spectrum using a 25-tap filter. The magnitude response of the optimal filter is shown in Figure 2. Comparing to Example 1, this design has flatter passband response but higher stopband attenuation.

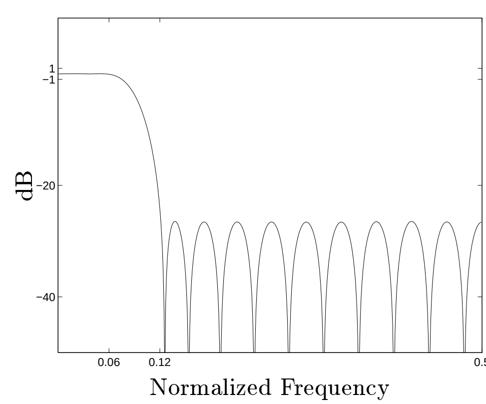

Figure 2: Filter response (magnitude) of Example 2

Example 3

We consider the same Chebychev approximation problem as in the previous example, but with the magnitude bounds from Example 1. The frequency response

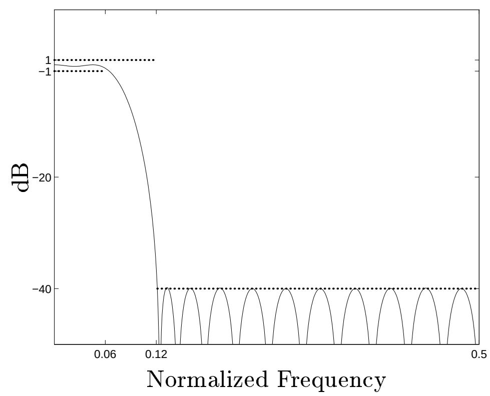

Figure 3: Filter response (magnitude) of Example 3

of the optimal 25-tap filter is shown in Figure 3. With the help of magnitude bounds, this design achieves the same stopband attenuation in Example 1.

Example 4
We design the minimum-length filter that satisfies the bandpass magnitude bounds shown in Figure 4. The result is a 24-tap filter with the frequency response magnitude shown in the same figure.

7 Concluding remarks

We have presented an SDP formulation of several FIR filter design problems:

- The feasibility problem: find an FIR filter that satisfies given upper and lower bounds on the frequency response magnitude, or show that no such filter exists.
- The problem of finding the minimum length filter that satisfies the upper and lower bounds.
- Chebychev-approximation of a desired power spectrum.
- Chebychev-approximation of a desired frequency response magnitude on a logarithmic scale.
- Chebychev-approximation with guaranteed magnitude upper/lower bounds.

Many other extensions that have not been discussed in the paper can be handled in the same framework, such as, maximum stopband attenuation or minimum transition-band width FIR design given magnitude bounds, or even linear array beam-forming. Recent interior-point methods for semidefinite programming can solve each of these problems very efficiently.

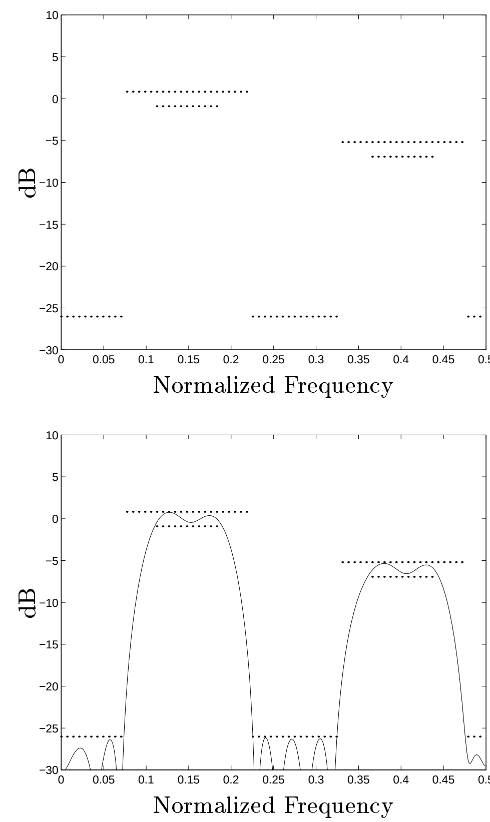

Figure 4: Filter specification and response (magnitude) of Example 3

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