Firm Size and Diversification: Asymmetric Multiproduct Firms under Cournot Competition*

Volker Grossmann†

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Abstract

A positive relationship between firm size and product diversification is a long-standing stylized fact. However, so far there is no appropriate theoretical model to explain the underlying forces of this observation. This paper analyzes an oligopoly model with asymmetric multiproduct firms, which is capable to address this issue. The model suggests that intangible assets of firms, which affect marginal costs or perceived quality of goods within a firm's product line, play a key role for the empirical regularity that larger firms are more diversified.

Key words: Asymmetric equilibrium; Diversification; Firm size; Intangible assets; Multiproduct firms.

JEL classification: L11; L13.

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† University of Zurich, Socioeconomic Institute, Zürichbergstr. 14, CH-8032 Zurich, Switzerland. Phone: +41-1-6342288; E-mail: volker.grossmann@wwi.unizh.ch.

1 Introduction

Product diversification is a dominant feature of firms in advanced countries. In particular, by using various indicators, empirical evidence for the manufacturing sector strongly suggests that there is a positive relationship between the size of firms/establishments and diversification, particularly regarding the number of products offered (e.g., Amey, 1964; Utton, 1977; Gollop and Monahan, 1991).[1] However, so far there is no appropriate theoretical analysis in the literature to explain the underlying forces of this important empirical regularity. In fact, due to the analytical complexity of oligopoly models with multiproduct firms, existing multiproduct models have focussed on the analysis of symmetric firms.[2] However, this modelling strategy does not allow to address the observed relationship between product diversification and firm size.

In contrast, this paper analyzes a simple *multiproduct oligopoly* model in which firms are *asymmetric* with respect to their production technology and consumers' valuation of varieties within a firm's product line. Firms produce differentiated goods, facing linear demand, and are engaged in a two-stage decision process. At stage 1, firms choose the number of products offered to the market. At stage 2, they enter Cournot competition.[3]

The main contribution of the paper is twofold. First, it derives basic properties of profit functions of multiproduct firms for the widely-used linear-demand model with

[1] For the U.S., Gollop and Monahan (1991, p. 327) conclude that "[q]uite clearly, large enterprises are more diversified than small ones". Moreover, their evidence suggests that a similarly strong result also holds at the establishment-level. In a recent study on Taiwanese firms, Aw and Batra (1998, p. 313) suggest that "[t]he positive relation between firm size and product diversification typically found in developed countries is limited to large exporting firms".

[2] For a theoretical analysis of various aspects of multiproduct firms, see, e.g., Raubitschek (1987), Shaked and Sutton (1990), Anderson and de Palma (1992), Anderson et al. (1992, ch. 7), Sutton (1998, ch. 2), and Ottaviano and Thisse (1999).

[3] For a similar modelling strategy, see, e.g., Raubitschek (1987), Sutton (1998), Ottaviano and Thisse (1999).

differentiated goods under Cournot competition, for given configurations of both the number of products and firm-specific characteristics (i.e., at stage 2 equilibrium). Second, from these properties, the analysis provides for the first time a rigorous theoretical explanation for the observation that size and product diversification of a firm are positively related. Both variables, firm size, measured by total sales of a firm, and its number of products are determined by firm-specific characteristics, i.e., by marginal costs and perceived quality of a firm's products. Thus, comparative-static analysis suggests that *intangible assets* of firms, like basic organizational or technological knowledge (affecting marginal costs) as well as consumer loyalty or trademarks (affecting perceived quality of a firm's products), play a key role for the empirical regularity that larger firms offer more diversified product lines.[4]

As well-known, applicability of standard tools for comparative-static analysis is very limited in the presence of strategic interactions of more than two players (e.g., Takayama, 1985; Dixit, 1986; Vives, 1999). However, focussing on the duopoly case is not very satisfying, as extension to the case of many firms adds complex interactions which need to be addressed. Fortunately, the structure of the model allows to apply a tool for comparative-statics in games with strategic substitutes which has recently been developed by Athey and Schmutzler (2001). By doing so, it is possible to derive a positive relationship between size and diversification for the $I-$firm model. Moreover, contrary to the common practice of ignoring the integer problem regarding the number of products, comparative-static results derived in this paper do not hinge on the treatment of product ranges as continuous variables.

The paper is organized as follows. Section 2 presents the basic model. Section 3 proves existence of a (pure-strategy) equilibrium and provides comparative-static

[4]Notably, in an informal way, Gorecki (1975) has already discussed the role of intangible assets for diversification, pointing out that "findings suggest that specific assets of a technological nature formed the basis of much diversification" (p. 134). For other motives of corporate diversification, see, e.g., the literature review by Montgomery (1994). Aydemir and Schmutzler (2002) provide a formal model in which mergers between big and small firms are driven by the motive of big firms to expand their product space, among other reasons.

results. Section 4 discusses the integer problem, allows for heterogeneity among firms also with respect to diversification costs, and briefly addresses the empirical regularity of an upward *trend* of corporate diversification in the last decades. Moreover, testable hypotheses emerging from the analysis are discussed in more detail. Section 5 provides concluding remarks. Some proofs are relegated to an appendix.

2 The Basic Model

Consider a market for differentiated goods with a finite set $\mathcal{I} = \{1, ..., I\}$ of firms, indexed by i. Let $\mathcal{N}_i$ be the set of goods produced by firm i, in (endogenous) number N_i, and let $\mathcal{K}$ be the set of all varieties in the market. For the sake of tractability (allowing for closed-form solutions) and its familiarity, the inverse demand function for variety $k \in \mathcal{K}$ has the simple linear form

$$p_k = A_k - \beta x_k - \gamma \sum_{l \neq k} x_l, \tag{1}$$

$A_k > 0$, $\beta > \gamma > 0$, where p_k and x_k denote the price and quantity of product k, respectively.[5] Moreover, marginal production costs of each variety k are constant and denoted by $c_k \geq 0$. Let $\alpha_k \equiv A_k - c_k > 0$, $k \in \mathcal{K}$. The parameter α_k summarizes the relationship between perceived quality of variety k, reflected by A_k, and unit production costs, c_k. It is assumed that for all varieties offered by the same multiproduct firm, this relationship is identical. Formally, this means the following.

A1. (Firm-specific characteristics). $\alpha_k = \alpha_i > 0$ for all $k \in \mathcal{N}_i$, $i \in \mathcal{I}$.

Assumption A1 implies that there is a positive relationship between perceived quality and unit production costs for all varieties supplied by a single firm. Moreover,

[5] That is, there is a representative consumer with quasi-linear preferences which are reflected by the utility function $U = \sum_{k \in \mathcal{K}} \left(A_k x_k - (\beta/2) x_k^2 \right) - \gamma \sum_k \sum_{l<k} x_k x_l + Y$, where Y is the quantity of the numeriare commodity. As a caveat, however, the linear demand model is not capable to address the notion that products offered by a single firm are closer substitutes for each other than for products sold by different firms (Anderson and de Palma, 1992).

firms are characterized by a single index α_i. For instance, the view of consumers regarding the quality of a firm's products is affected by the trademark associated with a product line. Firms may also differ in their organizational know-how or their internal human capital stock, affecting productivity and thus marginal costs. In other words, firms may differ in their *intangible assets*, which is reflected by differences in α_i in the model.[6] Consequently, α_i is called *quality* of intangible assets of firm i. The $I-$tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_I)$ is called a *configuration* of intangible asset qualities.

There are two stages, with decisions at each stage made non-cooperatively and simultaneously. At stage 1, firms choose their number of products N_i ("product range"). Let $C : [1, \bar{N}] \rightarrow \mathbb{R}_+$ be an increasing, twice continuously differentiable and convex function, $\bar{N} < \infty$. $C(N_i)$ denote the costs of firm i to introduce $N_i \in [1, \bar{N}]$ products in the market. (In section 4, choice sets at stage 1 are restricted to positive integers $\{1, 2, ..., \bar{N}\}$.) For instance, these include costs for marketing or designing products.[7] The $I-$tuple $\mathbf{N} = (N_1, N_2, ..., N_I)$ is called a *configuration of product ranges*. At stage 2, firms enter Cournot competition. This timing of events follows some existing literature on multiproduct firms (e.g., Raubitschek, 1987; Sutton, 1998; Ottaviano and Thisse, 1999). However, in contrast to this literature, the present set up allows for *asymmetry* of firms ex ante.

[6] The argument that a superior production technology or consumer loyality applies to *any* variety a firm offers also fits well into the common notion in the literature on multinational enterprises that intangible assets are of public good nature from the perspective of a single firm (see, e.g., Caves, 1971; Markusen, 2002).

[7] Assuming convexity of $C(\cdot)$ does not deny that there are economies of scope (or "subadditive costs") in marketing, designing or manufacturing multiple products within a firm (see, e.g., Baumol, 1977). However, one may think of increasing (Coasian) bureaucracy costs of product proliferation as a counteracting force. In fact, all that is needed is that $C(\cdot)$ is not "too concave" in the relevant range. For instance, Anderson and de Palma (1992) and Ottaviano and Thisse (1999) consider the special case of $C(N) = gN$, $g > 0$, which is included in the present analysis. Also restricting the choice set of firms at stage 1 to the *closed* interval $[1, \bar{N}]$, rather than to $[1, \infty)$ (relevant for proving existence of equilibrium) serves a purely technical purpose, as $\bar{N}$ can be arbitrarily large.

3 Equilibrium Analysis

In this section, the equilibrium of the model is derived by backwards induction. Moreover, it is examined whether there exists a systematic relationship between size and product diversification of a firm, and, if yes, what determines this relationship.

3.1 Cournot Competition (Stage 2)

First, consider the decision problem of firms at stage 2, for a given configuration $\mathbf{N}$. Taking output levels of rival firms as given, each firm $i \in \mathcal{I}$ solves

$$\max_{x_k \geq 0, k \in \mathcal{N}_i} \pi_i = \sum_{k \in \mathcal{N}_i} (p_k - c_k) x_k \text{ s.t. (1).} \tag{2}$$

Observing $\alpha_i = A_k - c_k$ for all $k \in \mathcal{N}_i$, according to A1, the following first result is obtained.

Proposition 1. (Equilibrium at stage 2). *Under A1. In an interior Cournot-Nash equilibrium at stage 2, each firm $i \in \mathcal{I}$ produces output level $\tilde{x}_k \equiv \tilde{x}_i$ for all $k \in \mathcal{N}_i$ with*

$$\tilde{x}_i = \frac{\Lambda_i}{\left(1 + \sum_i \Gamma_i\right) \left[2(\beta - \gamma) + \gamma N_i\right]} \equiv X_i(\mathbf{N}, \boldsymbol{\alpha}) \tag{3}$$

and earns profits

$$\tilde{\pi}_i = N_i(\beta - \gamma + \gamma N_i) X_i(\mathbf{N}, \boldsymbol{\alpha})^2 \equiv \Pi_i(\mathbf{N}, \boldsymbol{\alpha}), \tag{4}$$

where $\Gamma_i \equiv \gamma N_i / \left[2(\beta - \gamma) + \gamma N_i\right] \in (0, 1)$ *and* $\Lambda_i \equiv \alpha_i \left(1 + \sum_{j \neq i} \Gamma_j\right) - \sum_{j \neq i} \alpha_j \Gamma_j$.

Proof. See appendix. ■

The analysis focusses on configurations of intangible asset qualities, $\boldsymbol{\alpha}$, such that $\Lambda_i > 0$, and thus, $\tilde{x}_i > 0$ for all i. According to Proposition 1, a multi-product firm i produces equal output levels for all varieties $k \in \mathcal{N}_i$ which it offers. This is due to the symmetry of varieties in demand schedules (1), together with assumption A1. Moreover, output levels of single products (i.e., sales per variety) of firm i, $\tilde{x}_i$, are positively related to the intangible asset quality α_i, for any given configuration of product ranges $\mathbf{N}$.

Apart from own α_i, output levels and profits of a firm i depend on other firms' intangible asset qualities α_j, $j \neq i$, and on configuration $\mathbf{N}$. The following corollaries characterize profit functions $\tilde{\pi}_i = \Pi_i(\mathbf{N}, \boldsymbol{\alpha})$ in equilibrium at stage 2. In particular, these results will prove helpful for the comparative-static analysis of the next subsection, in which the firms' choice of product ranges (stage 1) is considered.

Corollary 1. (Properties of stage 2 profit functions). *Consider the set of profit functions* $\Pi_i : [1, \bar{N}]^I \times \mathbb{R}_+^I \rightarrow \mathbb{R}_+$, $i \in \mathcal{I}$. *For all* $i, j \in \mathcal{I}$, $j \neq i$, *we have*

(i) $\partial \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i > 0$ *and* $\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i^2 < 0$,

(ii) $\partial \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_j < 0$,

(iii) $\partial \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial \alpha_i > 0$ *and* $\partial \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial \alpha_j < 0$,

(iv) $\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_i > 0$ *and* $\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_j < 0$;

(v) if $\alpha_i \leq \alpha_j$ *or if* $(\alpha_i - \alpha_j)$ *sufficiently small, then* $\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial N_j < 0$.

Proof. See appendix. ■

To gain insight into Corollary 1, it is helpful to decompose $\Pi_i(\mathbf{N}, \boldsymbol{\alpha})$ into the product of total demand (or sales) of firm i in equilibrium at stage 2, $D_i(\mathbf{N}, \boldsymbol{\alpha}) \equiv N_i X_i(\mathbf{N}, \boldsymbol{\alpha})$, and its price-cost difference ("mark-up"), $M_i(\mathbf{N}, \boldsymbol{\alpha}) \equiv (\beta - \gamma + \gamma N_i) X_i(\mathbf{N}, \boldsymbol{\alpha})$. That is, $\Pi_i(\mathbf{N}, \boldsymbol{\alpha}) = D_i(\mathbf{N}, \boldsymbol{\alpha}) M_i(\mathbf{N}, \boldsymbol{\alpha})$, implying

$$\frac{\partial \Pi_i}{\partial N_j} = \frac{\partial D_i}{\partial N_j} M_i + D_i \frac{\partial M_i}{\partial N_j}, \tag{5}$$

$$\frac{\partial^2 \Pi_i}{\partial N_j \partial N_i} = \frac{\partial^2 D_i}{\partial N_j \partial N_i} M_i + \frac{\partial D_i}{\partial N_j} \frac{\partial M_i}{\partial N_i} + \frac{\partial D_i}{\partial N_i} \frac{\partial M_i}{\partial N_j} + D_i \frac{\partial^2 M_i}{\partial N_j \partial N_i}. \tag{6}$$

$i, j \in \mathcal{I}$. Total sales D_i of a firm are used as measure of firm size for the discussion of results below. The properties of the functions $D_i(\mathbf{N}, \boldsymbol{\alpha})$ and $M_i(\mathbf{N}, \boldsymbol{\alpha})$, which are used in the following discussion of Corollary 1, are formally derived in an appendix available from the author upon request.

First, according to part (i) of Corollary 1, the impact of an increase in product range N_i on both equilibrium demand, D_i, and on equilibrium mark-up, M_i, is positive.[8] Thus, $\partial \Pi_i / \partial N_i > 0$, according to (5) (for $i = j$), whenever varieties are imperfect substitutes (i.e., $\beta > \gamma$).[9] Moreover, strict concavity of profits at stage 2, Π_i, as function of product range N_i means that a firm's incentive to launch new varieties is weaker, the more diversified the firm is. Using (6) (for $i = j$) and the definitions of D_i and M_i, one can show that the underlying reason for this result is that the marginal gain of an increase in N_i regarding both D_i and M_i is decreasing, i.e., $\partial^2 D_i / \partial N_i^2 < 0$ and $\partial^2 M_i / \partial N_i^2 < 0$.

Part (ii) of Corollary 1 means that a firm's profits at stage 2, $\tilde{\pi}_i$, decline if any rival offers additional products, which reflects a conventional "business-stealing effect". In fact, an increase in N_j reduces both D_i and M_i of a firm $i \neq j$. Also unsurprisingly, part (iii) says that $\tilde{\pi}_i$ increases with its own intangible asset quality, α_i, but decreases with other firms' intangible asset qualities, α_j, $j \neq i$, holding the configuration of product ranges $\mathbf{N}$ constant. Again, the effects regarding both D_i and M_i go in the same direction.

According to part (iv), the profit gain of firm i from introducing additional varieties increases with α_i, but decreases with the strength of rivals α_j, $j \neq i$, all other things equal. It is easy to confirm that an increase in α_i raises the impact of an increase in product range N_i on both equilibrium demand D_i and on equilibrium

[8] The latter effect may be somewhat surprising at the first glance, but can easily understood as follows. Note that (1), together with A1, implies that, in stage 2 equilibrium, $p_k - c_k = \alpha_i - (\beta - \gamma)\tilde{x}_i - \gamma \tilde{Q} \equiv M_i$ for all $k \in \mathcal{N}_i$, $i \in \mathcal{I}$, where $\tilde{Q} \equiv \sum_{l \in \mathcal{K}} \tilde{x}_l$ is total equilibrium output in the market in stage 2 equilibrium. On the one hand, it is easy to check that an increase in N_i raises $\tilde{Q}$, all other things equal (see appendix). This has a negative effect on M_i. On the other hand, however, firm i reduces equilibrium output per variety, $\tilde{x}_i$, when increasing N_i, which has a positive effect on M_i. The second effect dominates the first one.

[9] Under the linear-demand structure (1), it is common to interpret the ratio β/γ as the degree of substitution between goods. Note that for $\gamma \to \beta$, i.e., if varieties are perfect substitutes, the limiting profit function of a firm i at stage 2 is given by $\lim_{\gamma \to \beta} \tilde{\pi}_i = \left[\left(I\alpha_i - \sum_{j \neq i} \alpha_j \right) / (1 + I) \right]^2 / \gamma$, according to (3) and (4). Obviously, it does not pay for firms to supply more than one variety in this limit case. In contrast, with imperfect substitutes, i.e., if $\beta > \gamma$, it may be optimal for firms to introduce more than one variety into the market, as will become apparent below.

mark-up M_i (i.e., $\partial^2 D_i/\partial N_i \partial \alpha_i > 0$ and $\partial^2 M_i/\partial N_i \partial \alpha_i > 0$), whereas an increase in α_j, $j \neq i$, has the opposite effect on $\partial D_i/\partial N_i$ and $\partial M_i/\partial N_i$, respectively.

As will become apparent below, the impact of an increase in a rival's product range N_j on the incentive of a firm $i \neq j$ to launch new varieties (i.e., how $\partial \Pi_i/\partial N_i$ changes with N_j, $j \neq i$) is of particular importance for the subsequent analysis. From the previous discussion of parts (i) and (ii), for $j \neq i$, one can conclude that the second and third summand of the right-hand side of (6) are both negative. However, one can also show that the first and last summand have ambiguous sign, i.e., an increase in N_j may increase or decrease both effects $\partial D_i/\partial N_i$ and $\partial M_i/\partial N_i$, $j \neq i$. Part (v) of Corollary 1 says that the profit gain of a firm i from increasing product diversification is reduced by an increase in a rival's product range N_j, $j \neq i$, if $\alpha_i \leq \alpha_j$ or if $(\alpha_i - \alpha_j)$ is sufficiently small, i.e., whenever firms are not "too heterogeneous" with respect to their quality of intangible assets. As will become apparent, in this case, the optimal response at stage 1 to an increase in a rival's product number is to decrease the own number of varieties, i.e., product ranges of firms are strategic substitutes. The subsequent analysis exclusively focusses on this case.

A2. (Strategic substitutes). *For all* $i, j \in \mathcal{I}$, $j \neq i$, *let* $\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial N_j < 0$.

It can be shown that, for some configurations $\mathbf{N}$, assumption A2 holds *for all* configurations $\boldsymbol{\alpha}$, i.e., even if firms are very heterogeneous. (See Remark 1 in appendix for a sufficient condition.)

Proposition 1 also implies the following useful result.

Corollary 2. (Exchangeability). *Under A1. Let* $(\hat{\mathbf{N}}, \hat{\boldsymbol{\alpha}})$ *be a permutation of* $(\mathbf{N}, \boldsymbol{\alpha})$ *so that, for any pair* $i, j \in \mathcal{I}$, $j \neq i$, $(\hat{N}_i, \hat{\alpha}_i) = (N_j, \alpha_j)$, $(\hat{N}_j, \hat{\alpha}_j) = (N_i, \alpha_i)$ *and* $(\hat{N}_h, \hat{\alpha}_h) = (N_h, \alpha_h)$ *for all* $h \in \mathcal{I} \backslash \{i, j\}$. *Then, we have* $\Pi_i(\hat{\mathbf{N}}, \hat{\boldsymbol{\alpha}}) = \Pi_j(\mathbf{N}, \boldsymbol{\alpha})$ *and* $\Pi_h(\hat{\mathbf{N}}, \hat{\boldsymbol{\alpha}}) = \Pi_h(\mathbf{N}, \boldsymbol{\alpha})$ *for all* $h \in \mathcal{I} \backslash \{i, j\}$.

Corollary 2 is directly implied by the fact that profits at stage 2 are exclusively determined by configurations $\mathbf{N}$ and $\boldsymbol{\alpha}$. Thus, exchanging both intangible asset qualities and product ranges of two firms also leads to an exchange of these firms' profits at stage 2, without affecting other firms' profits. Following Athey and Schmutzler (2001) (AS hereafter), we say that profit functions Π_i are "exchangeable" as functions of $(\mathbf{N}, \boldsymbol{\alpha})$. Exchangeability also plays a crucial role for the comparative-statics of the $I-$player case below.

3.2 Firms' Choice of Number of Products (Stage 1)

By analyzing the firms' choice of number of products (stage 1), this subsection characterizes the equilibrium configuration of product ranges, denoted $\mathbf{N}^*$, given the configuration of intangible asset qualities, $\boldsymbol{\alpha}$.

Given $\boldsymbol{\alpha}$, the profit maximization problem for each firm $i \in \mathcal{I}$ at stage 1 is to solve

$$\max_{N_i \in [1, \bar{N}]} \Psi_i(\mathbf{N}, \boldsymbol{\alpha}) \equiv \Pi_i(\mathbf{N}, \boldsymbol{\alpha}) - C(N_i). \tag{7}$$

Strategic interactions of firms at stage 1 are represented by the game $\left\{([1, \bar{N}], \Psi_i),\ i \in \mathcal{I}\right\}$. Concavity of stage 2 profits, Π_i, as function of N_i (part (i) of Corollary 1) ensures existence of equilibrium.

Proposition 2. (Existence of equilibrium). *Under A1. For any given $\boldsymbol{\alpha} \in \mathbb{R}_+^I$, a pure-strategy Nash equilibrium of the game $\left\{([1, \bar{N}], \Psi_i),\ i \in \mathcal{I}\right\})$ exists. Thus, a subgame-perfect equilibrium of the two-stage game exists.*

Proof. See appendix. ■

Let $N_i^*(\boldsymbol{\alpha})$ be an equilibrium product range offered by firm $i \in \mathcal{I}$. Without loss of generality (as $\bar{N}$ can be arbitrarily large), the analysis focusses on $N_i^* < \bar{N}$ for all i. Using (7), an equilibrium configuration of product ranges $\mathbf{N}^* = (N_1^*, ..., N_I^*)$ is then given by the following set of first-order conditions:

$$\frac{\partial \Pi_i(\mathbf{N}^*, \boldsymbol{\alpha})}{\partial N_i} \leq C'(N_i^*),\ i \in \mathcal{I}, \tag{8}$$

with strict equality if $N_i^* > 1$.

3.2.1 The Duopoly Case

We are now ready to address the question how differences in equilibrium product ranges, N_i^*, among firms (i.e., equilibrium diversification) depend on differences in intangible asset qualities, α_i. First, consider the duopoly case, $I = 2$.

Proposition 3. (Product range in duopoly). *Under A1 and A2. If $\mathcal{I} = \{1, 2\}$ and the equilibrium is unique, then $\alpha_i > \alpha_j$ implies $N_i^*(\boldsymbol{\alpha}) > N_j^*(\boldsymbol{\alpha})$.*

As will become apparent below, Proposition 3 is a special case of Proposition 4. Therefore, no formal proof of the two-player case is provided. Rather, the result is illustrated graphically in Fig. 1. Note from (8) and strict concavity of $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ as function of N_i that reaction functions are downward sloping under A2, i.e., N_1 and N_2 are strategic substitutes. Uniqueness of equilibrium requires that the reaction function of firm 1 is steeper than that of firm 2.[10] According to part (iv) of Corollary 1, an increase in, say, α_1 shifts the reaction function of firm 1 rightward and that of firm 2 downward, as shown in Fig. 1. That is, the marginal gain of firm 1 to extend its product range increases and the marginal gain of firm 2 decreases.

<Please insert **Figure 1** about here>

Fig. 1 depicts two kinds of a unique equilibrium (observing restriction $N_i \geq 1$, $i \in \mathcal{I}$). In panel (a), the equilibrium is interior, whereas $N_2^* = 1$ in panel (b). In panel (a), an increase in α_1 leads to an increase in N_1^* and a decrease in N_2^*.[11] In

[10] Note that any unique equilibrium is also "stable" if reaction functions are interpreted as describing dynamic behavior with alternate-period decisions. Using (8), it is easy to check that the reaction function of firm 1 is stepper than that of firm 2 if

$$\left(\left|\frac{\partial^2 \Pi_1}{\partial N_1^2}\right| + C''(N_1)\right)\left(\left|\frac{\partial^2 \Pi_2}{\partial N_2^2}\right| + C''(N_2)\right) > \left|\frac{\partial^2 \Pi_1(\mathbf{N}, \boldsymbol{\alpha})}{\partial N_1 \partial N_2}\right|\left|\frac{\partial^2 \Pi_1(\mathbf{N}, \boldsymbol{\alpha})}{\partial N_1 \partial N_2}\right|.$$

[11] In contrast, if the reaction function of firm 2 in $N_1 - N_2$ space is steeper than that of firm 1 (implying multiple equilibria), it is easy to check that for an interior equilibrium the opposite holds. See below for more discussion on the uniqueness requirement.

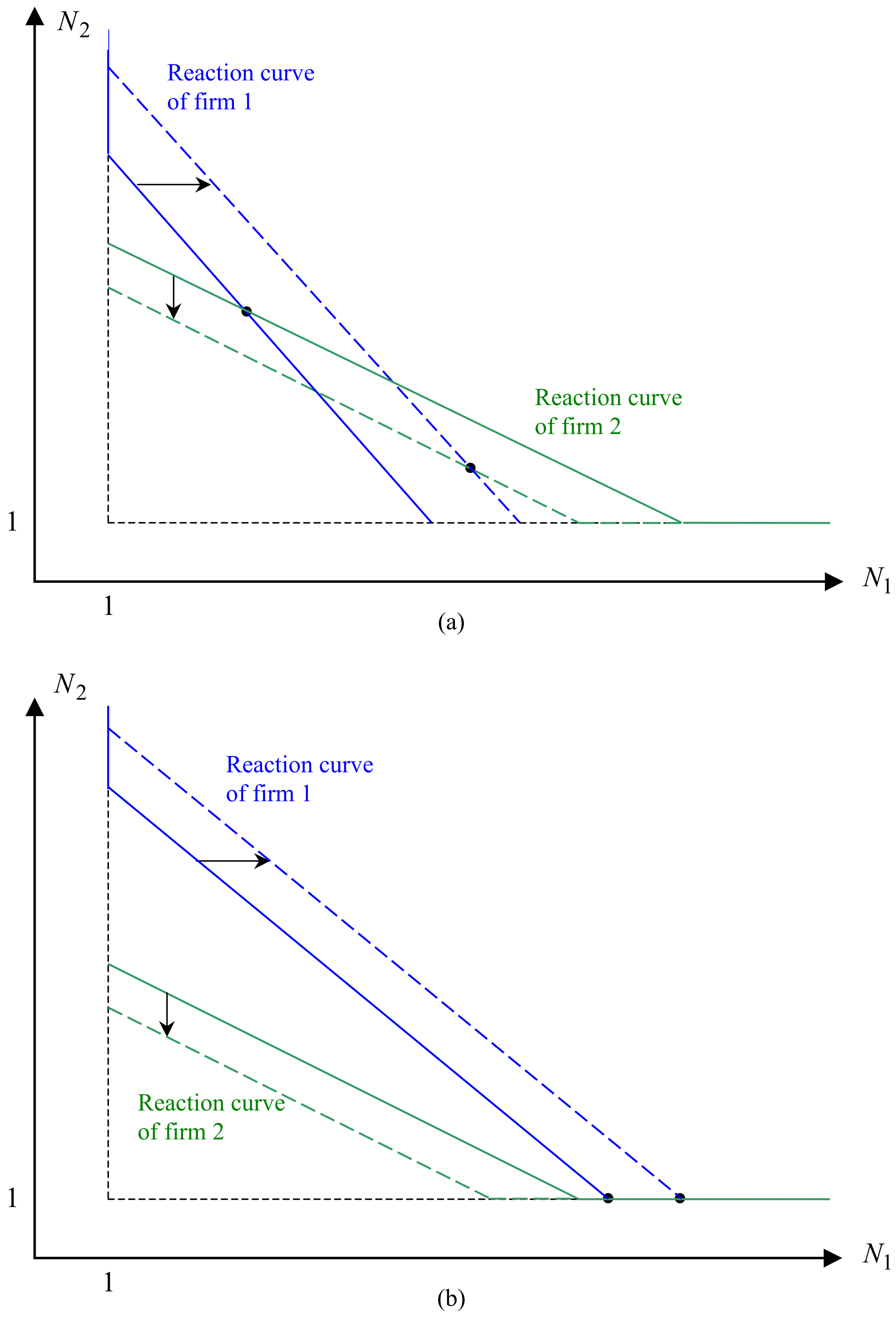

Figure 1: The impact of an increase in α_1 on N_1^* and N_2^* in the duopoly case.

panel (b), only N_1^* increases.[12]

3.2.2 The Case $I > 2$

In view of the duopoly case, it seems intuitive that firms with higher intangible asset qualities have more diversified product lines. However, as it is well-known from oligopoly theory, in general, comparative-static analysis of asymmetric equilibria with more than two players can be very messy and may require strong assumptions (e.g., Takayama, 1985; Dixit, 1986; Vives, 1999). To see the economic reason for this in the present context, consider, again, an increase in α_1. Recall that, in response to an increase in α_1, firm 1 has an incentive to increase N_1 and firm 2 has an incentive to decrease N_2. On the one hand, if product ranges are strategic substitutes, an actual rise in N_1 would have a negative impact on the marginal gain of a firm 3 (or higher) of increasing its product range. However, an actual decrease of N_2 would have the opposite effect on the behavior of firm 3. A priori, it is not clear which effect dominates. In fact, an analogous argument holds for firm 2 as well. If firm 3 decreases N_3 in response to an increase in α_1, firm 2 has an incentive to increase N_2. This even leaves the behavioral response of firm 1 ambiguous. Due to these complexities, generalization of the comparative-static results from the duopoly case to the case $I > 2$ is not straightforward.

Moreover, multiplicity of equilibria may create problems for comparative-static analysis. To ensure uniqueness of equilibrium, one could invoke standard "dominant-diagonal" conditions (see, e.g., Vives, 1999). In the present analysis,

$$\left|\frac{\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})}{\partial N_i^2}\right| > \sum_{j \neq i} \left|\frac{\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})}{\partial N_i \partial N_j}\right| \text{ for all } i \in \mathcal{I}, \tag{9}$$

would be sufficient. However, (9) may be a strong assumption if the number of players is high. Rather, following AS, it is assumed that the set of equilibrium strategies at stage 1 satisfies "conditional uniqueness". Formally, this means the following.

[12] Qualitatively, a decrease in α_2 has the same effects as an increase in α_1.

A3. (Conditional uniqueness). For each $i, j \in \mathcal{I}$ and each $\boldsymbol{\alpha} \in \mathbb{R}^I_+$, if there exist two equilibrium configurations of product ranges $\mathbf{N}^*(\boldsymbol{\alpha})$ and $\mathbf{N}^{**}(\boldsymbol{\alpha})$ which fulfill $N^*_h(\boldsymbol{\alpha}) = N^{**}_h(\boldsymbol{\alpha})$ for all $h \in \mathcal{I}\backslash\{i, j\}$, then $N^*_i(\boldsymbol{\alpha}) = N^{**}_i(\boldsymbol{\alpha})$ and $N^*_j(\boldsymbol{\alpha}) = N^{**}_j(\boldsymbol{\alpha})$.

For $I = 2$, conditional uniqueness reduces to uniqueness of equilibrium. However, if $I > 2$, the requirement of conditional uniqueness is considerably weaker. For instance, as argued in AS, a sufficient condition for conditional uniqueness of equilibria is

$$\left|\frac{\partial^2\Pi_i(\mathbf{N}, \boldsymbol{\alpha})}{\partial N_i^2}\right| > \left|\frac{\partial^2\Pi_i(\mathbf{N}, \boldsymbol{\alpha})}{\partial N_i \partial N_j}\right| \text{ for all } i, j \in \mathcal{I},\ j \neq i. \tag{10}$$

In the present context, condition (10) is a fairly weak requirement, as can be seen from the expressions for $\partial^2\Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i^2$ and $\partial^2\Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial N_j$, $j \neq i$, in appendix.

We are now ready to analyze comparative-statics for the $I-$player case. The next result is derived by applying a recent tool for comparative-static analysis of games with strategic substitutes, due to AS.

Proposition 4. (Product range in $I-$player case). *Under A1-A3. For all $i, j \in \mathcal{I}$, if $\alpha_i > \alpha_j$ then $N^*_i(\boldsymbol{\alpha}) > N^*_j(\boldsymbol{\alpha})$.*

Proof. Let $\mathbf{N}_{-i}$ denote $(N_1, ..., N_{i-1}, N_{i+1}, ..., N_I)$ and write $\mathbf{N} = (N_i, \mathbf{N}_{-i})$. Now consider the following definition.

Definition 1. (E.g., Vives, 1999). *$\Psi_i(N_i, \mathbf{N}_{-i}, \boldsymbol{\alpha})$ has (strictly) increasing differences in (N_i, α_j) if, for all $N_i > N'_i$, $\Psi_i(N_i, \mathbf{N}_{-i}, \boldsymbol{\alpha}) - \Psi_i(N'_i, \mathbf{N}_{-i}, \boldsymbol{\alpha})$ is (strictly) increasing in α_j, $i, j \in \mathcal{I}$; (strictly) decreasing differences are defined replacing "increasing" by "decreasing".*

If $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ is smooth and $\partial^2\Psi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_j > 0$, then $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ has strictly increasing differences in (N_i, α_j). (See, e.g., Vives, 1999). As a next step, note that the following holds.

Lemma 1. (Athey and Schmutzler, 2001). *Let $N_i \in S_i$ be a one-dimensional choice variable for player $i \in \mathcal{I}$. Suppose the set of equilibria of the game $\{(S_i, \Psi_i),\ i \in \mathcal{I}\}$*

is non-empty and fulfills conditional uniqueness. Suppose further that (i) the players' choices are strategic substitutes, (ii) $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ *is exchangeable in* $(\mathbf{N}, \boldsymbol{\alpha})$*, and (iii) for all* $i, j \in \mathcal{I}$*,* $j \neq i$*,* $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ *has increasing differences in* (N_i, α_i) *and decreasing differences in* (N_i, α_j)*. Let* $\mathbf{N}^*$ *be an equilibrium configuration. Then* $\alpha_i > \alpha_j$ *implies* $N_i^*(\boldsymbol{\alpha}) \geq N_j^*(\boldsymbol{\alpha})$*.*

In the present context, an equilibrium exists, according to Proposition 2, and the set of equilibria fulfills conditional uniqueness by assumption A3. It is now argued that conditions (i)-(iii) of Lemma 1 hold. Recalling $\Psi_i(\mathbf{N}, \boldsymbol{\alpha}) = \Pi_i(\mathbf{N}, \boldsymbol{\alpha}) - C(N_i)$, condition (i) holds by A2. Condition (ii) of Lemma 1 follows from Corollary 2, applying the same definition of "exchangeability" as in AS. Finally, note that $\partial^2\Psi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_i > 0$ and $\partial^2\Psi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_j < 0$, $j \neq i$, according to part (iv) of Corollary 1. Thus, also condition (iii) of Lemma 1 is fulfilled. Hence, $\alpha_i > \alpha_j$ implies $N_i^*(\alpha) \geq N_j^*(\alpha)$, according to Lemma 1.

However, in order to prove that $\alpha_i > \alpha_j$ indeed implies $N_i^*(\boldsymbol{\alpha}) > N_j^*(\boldsymbol{\alpha})$ (i.e., firm i has a *strictly* larger product range than firm j), Lemma 1 has to be modified slightly. In fact, the following can be deducted from the analysis of AS in a straightforward way.

Lemma 2. *Presume the same as in Lemma 1. Suppose further that* $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ *is smooth and for all* $i, j \in \mathcal{I}$*,* $j \neq i$*,* $\partial^2\Psi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_i > 0$ *and* $\partial^2\Psi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_j < 0$*. Then* $\alpha_i > \alpha_j$ *implies* $N_i^*(\boldsymbol{\alpha}) > N_j^*(\boldsymbol{\alpha})$*.*

As $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ are smooth and, as argued above, also all other presumptions of Lemma 2 hold, Proposition 4 is confirmed. ■

According to Proposition 4, the comparative-statics from the duopoly case carry over to the case $I > 2$. This is rather surprising in view of the complex additional interactions among firms (as outlined above) when there are more than two firms. One reason is that firms are symmetric in the sense that, ex ante, firms can be characterized by their intangible asset quality, α_i, only (recall the exchangeability property in Corollary 2). For instance, it does not matter for firm 3 (and higher) if firm 1 has α_1 and firm 2 has α_2 or if firm 1 has α_2 and firm 2 has α_1. As argued by AS (p. 10), such a situation allows "to hold fixed the behavior of players 3 and higher and focus on a two-player game". The preceding analysis has used this argument to examine the behavior of asymmetric multiproduct firms in the linear-demand Cournot model regarding the choice of the number of products.

3.3 Firm Size and Diversification

We are now ready to derive the relationship between firm size and equilibrium diversification. Equilibrium firm size is measured by total sales of a firm in equilibrium, i.e., by equilibrium demand $D_i^*(\boldsymbol{\alpha}) \equiv D_i(\mathbf{N}^*, \boldsymbol{\alpha})$, $i \in \mathcal{I}$. By making use of Proposition 4, one can show the following.

Proposition 5. (Firm size). *Under A1-A3. For all $i, j \in \mathcal{I}$, if $\alpha_i > \alpha_j$ then $D_i^*(\boldsymbol{\alpha}) > D_j^*(\boldsymbol{\alpha})$.*

Proof. See appendix. ■

As a corollary of Propositions 4 and 5, the following theorem emerges.

Theorem 1. (Firm size and diversification). *Under A1-A3. Firm size, measured by total equilibrium sales, and product diversification are positively related. The underlying determinant of both variables is the quality of intangible assets of a firm.*

Testable hypotheses implied by Theorem 1 as well as some modifications of the analysis are discussed in the next section.

4 Discussion

4.1 Testable Hypotheses

According to the preceding analysis, heterogeneity of firms in marginal production costs or perceived quality of goods within product lines account for differences of firms in both total sales (i.e., firm size) and product diversification. This suggests that intangible assets of firms are a key to understand the long-standing stylized fact of a positive size-diversification relationship (Theorem 1).

Interestingly, as far as differences in marginal production costs among firms are concerned, empirical evidence is also consistent with Proposition 5. For instance, Roberts and Supina (2000) report a negative correlation between firm size and marginal costs among U.S. manufacturing firms. Moreover, using micro-level data from the 'Longitudinal Research Database' (developed by the U.S. Bureau of the Census), Baily et al. (1992) find that the size of U.S. manufacturing firms is positively related to total factor productivity (see their Tab. 8 and 9). However, whereas in this literature productivity is the dependent variable in regression analysis, the theory developed in this paper suggests that productivity jointly determines both size and diversification.[13]

Moreover, the analysis suggests that consumer loyalty, which may be determined and thus proxied by past advertising effort, is not only related to sales of a firm but also to its product diversification. However, as pointed out by Gorecki (1975, footnote 16), one has to distinguish between advertising on brand names of single products and advertising which, in addition, refers to the name (or trademark) of an enterprise (as, e.g., in the automobile industry). Only the latter kind of advertising affects consumers' views on other products supplied by a firm, and thus, is related to consumers' valuation of a firms' trademark.

4.2 The Integer Problem

Existence of a pure-strategy equilibrium (Proposition 2) is ensured by the continuous choice sets at stage 1, $[1, \bar{N}]$. Obviously, however, the common practice in the theoretical literature on multiproduct firms of treating the N_i's as continuous variables is

[13] In fact, Baily et al. (1992, p. 223) carefully point out that their results should not be interpreted as causal effects but rather reflect correlations. See Bartelsman and Doms (2000) for an excellent survey of the empirical literature on productivity differences among firms.

problematic. Fortunately, in the present context, comparative-static analysis is also possible when choice sets at stage 1 are restricted to positive integers $\{1, 2, ..., \bar{N}\}$. (That is, differentiability of profits $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$ is not required.) However, there is an important caveat. Although every finite normal form game has at least one Nash equilibrium, existence of a *pure*-strategy equilibrium may not be guaranteed in the present context. If it exists, the following can be stated.

Proposition 6. (Integer choice sets). *Under A1-A3. Let strategy sets of firms at stage 1 be given by* $S_i \equiv \{1, 2, ..., \bar{N}\}$, $i \in \mathcal{I}$. *Suppose that a pure-strategy equilibrium for the game* $\{(S_i, \Psi_i),\ i \in \mathcal{I}\}$ *exists. Then,* $\alpha_i > \alpha_j$ *implies both* $N_i^*(\boldsymbol{\alpha}) \geq N_j^*(\boldsymbol{\alpha})$ *and* $D_i^*(\boldsymbol{\alpha}) > D_j^*(\boldsymbol{\alpha})$. *Thus, if anything, larger firms are more diversified.*

Proof. Note that Definition 1 and Lemma 1 (see the proof of Proposition 4) do not rely on differentiability of $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$, $i \in \mathcal{I}$. Thus, the result that $\alpha_i > \alpha_j$ implies $N_i^*(\boldsymbol{\alpha}) \geq N_j^*(\boldsymbol{\alpha})$ directly follows from Lemma 1, as it has already been established that all presumptions of Lemma 1 hold. Moreover, using $N_i^* \geq N_j^*$, the result that $\alpha_i > \alpha_j$ implies $D_i^*(\boldsymbol{\alpha}) > D_j^*(\boldsymbol{\alpha})$ can directly be deducted from the proof of Proposition 5. This concludes the proof. ■

With choice sets at stage 1 being restricted to positive integers, it is now possible that $N_i^* = N_j^*$ if $\alpha_i > \alpha_j$. Moreover, note that even in this case total sales of firm i are strictly larger than those of firm j, i.e., $D_i^* > D_j^*$ if $\alpha_i > \alpha_j$.

4.3 Heterogeneity in Diversification Costs

In the preceding analysis, the sunk cost schedule at stage 1, $C(\cdot)$, for introducing products in the market is identical among firms. This is a strong assumption. For instance, firms with higher perceived product quality may have lower marketing costs to introduce new varieties in the market, e.g., if their trademark is well recognized. Moreover, firms with low marginal production costs may be more efficient also in introducing additional products to the market, i.e., in marginal costs of designing or marketing. In order to capture these possibilities, replace $C(N_i)$ by $K(N_i, \alpha_i)$, assuming that $\partial^2 K(N_i, \alpha_i)/\partial N_i \partial \alpha_i \leq 0$. In this case, Proposition 4 remains valid, as implied by its proof. (Thus, both Proposition 5 and Theorem 1 still hold.) The same is true even if $\partial^2 K(N_i, \alpha_i)/\partial N_i \partial \alpha_i > 0$, as long as $\partial^2 \Pi_i(\mathbf{N}, \boldsymbol{\alpha})/\partial N_i \partial \alpha_i \geq \partial^2 K(N_i, \alpha_i)/\partial N_i \partial \alpha_i$.

4.4 Increasing Average Diversification

The main focus in this paper is the role of intangible assets for explaining *cross-section* evidence regarding the relationship between firm size and product diversification. This subsection briefly discusses another empirical regularity, namely a steady upward *trend* in product diversification of firms in the last decades. For instance, Goto (1981) presents evidence from large Japanese firms in various industries between 1963 and 1975 which suggests that firms became considerably more diversified with respect to their commodities over time. Gollop and Monahan (1991) analyze data from manufacturing firms for five points in time in the period between 1963 and 1982, concluding that "[d]iversification has replaced horizontal growth" (p. 318). In more recent times, the emergence of computer-aided design (CAD) has reduced costs of product design and development. Moreover, computer-aided manufacturing (CAM) has reduced production costs. As argued by Milgrom and Roberts (1990), these developments have increased the firms' incentives to extend product lines.

It is straightforward to capture these trends in the present context. As asymmetry of firms is not crucial for these arguments, for simplicity, suppose $\alpha_i = \hat{\alpha}$ for all $i \in \mathcal{I}$. Define the $I-$tuple $\hat{\boldsymbol{\alpha}} \equiv (\hat{\alpha}, ..., \hat{\alpha})$ and let $N_i^*(\hat{\boldsymbol{\alpha}}) \equiv \hat{N}^* \in (1, \bar{N})$, $i \in \mathcal{I}$. Also define the $I-$tuple $\hat{\mathbf{N}}^* \equiv (\hat{N}^*, ..., \hat{N}^*)$.

Let us start with a decrease in marginal production costs (e.g., due to the emergence of CAM), implying an increase in $\hat{\alpha}$. According to (8), if $\hat{N}^* > 1$ (which is presumed here), the equilibrium product range, $\hat{N}^*$, is given by $\partial \Pi_i(\hat{\mathbf{N}}^*, \hat{\boldsymbol{\alpha}})/\partial N_i =$

$C'(\hat{N}^*)$. Applying the implicit function theorem to this condition, one obtains

$$\frac{\partial \hat{N}^*}{\partial \hat{\alpha}} = -\frac{I \partial^2 \Pi_i(\hat{\mathbf{N}}^*, \hat{\boldsymbol{\alpha}})/\partial N_i \partial \hat{\alpha}}{\partial^2 \Pi_i(\hat{\mathbf{N}}^*, \hat{\boldsymbol{\alpha}})/\partial N_i^2 + \sum_{j \neq i} \partial^2 \Pi_i(\hat{\mathbf{N}}^*, \hat{\boldsymbol{\alpha}})/\partial N_i \partial N_j - C''(\hat{N}^*)}. \tag{11}$$

Using parts (i) and (iv) of Corollary 1, and observing both A2 and $C''(\cdot) \geq 0$, $\partial \hat{N}^* / \partial \hat{\alpha} > 0$ is implied. That is, if intangible asset qualities improve, then product ranges of firms increase. In an analogous way, a downward shift in the marginal sunk cost schedule C' (e.g., due to the emergence of CAD) leads to an increase in $\hat{N}^*$.

5 Concluding Remarks

This paper has analyzed an oligopoly model with asymmetric multiproduct firms, which is consistent with the long-standing empirical regularity that larger firms offer more diversified product lines. The analysis suggests that heterogeneity of enterprises with respect to intangible assets is a driving force behind a positive relationship between firm size, measured by total sales, and product diversification. This result has been derived for the familiar specification of linear demand schedules and differentiated goods under Cournot competition (under weak additional assumptions), by applying a recently developed tool for comparative-static analysis of games with strategic substitutes.

Admittedly, the focus of the present analysis on the number of products as measure of product diversification is quite narrow. For instance, Gollop and Monahan (1991) construct a diversification index which, in addition to the number of products supplied by an enterprise, also accounts for the distribution of sales from these products within a firm and differences in the heterogeneity of products. Applying this index to a large data set of U.S. manufacturing firms and establishments, they find that "[t]he number component is the dominant force" in explaining corporate diversification (p. 327). This gives some justification for focussing the theoretical analysis on the number of products, exogenously fixing the degree of product differentiation, and thus, leading to a uniform sales distribution within a firm.

Appendix

Proof of Proposition 1: First, note that $\pi_i = \sum_{k \in \mathcal{N}_i} (p_k - c_k) x_k$ implies

$$\frac{\partial \pi_i}{\partial x_k} = p_k - c_k + \sum_{l \in \mathcal{N}_i} \frac{\partial p_l}{\partial x_k} x_l, \tag{A.1}$$

where $\partial p_l / \partial x_l = -\beta$ and $\partial p_l / \partial x_k = -\gamma$ for $l \neq k$, according to demand structure (1). Thus, optimal behavior of firm $i \in \mathcal{I}$ at stage 2 is given by the following set of first-order conditions (presuming an interior solution): $\alpha_i - 2\beta x_k - \gamma \sum_{l \in \mathcal{K} \setminus \{k\}} x_l - \gamma \sum_{l \in \mathcal{N}_i \setminus \{k\}} x_l = 0$, $k \in \mathcal{N}_i$, where $\alpha_i = A_k - c_k$ for all $k \in \mathcal{N}_i$ has been used (assumption A1). Adding and subtracting $2\gamma x_k$ implies

$$\alpha_i - 2(\beta - \gamma) x_k - \gamma Q - \gamma \sum_{l \in \mathcal{N}_i} x_l = 0, \tag{A.2}$$

where $Q \equiv \sum_{l \in \mathcal{K}} x_l$ is total output in the market. Thus, $x_k = x_i$ for all $k \in \mathcal{N}_i$, which implies $\sum_{l \in \mathcal{N}_i} x_l = N_i x_i$. Hence, (A.2) can be rewritten as

$$x_i = \frac{\alpha_i - \gamma Q}{2(\beta - \gamma) + \gamma N_i}. \tag{A.3}$$

Note that $Q = \sum_i N_i x_i$. Multiplying both sides of (A.3) by N_i and summing over all $i \in \mathcal{I}$, one obtains the total output level $\tilde{Q} = \sum_i N_i \tilde{x}_i$ in Cournot-Nash equilibrium, given by

$$\gamma \tilde{Q} = \frac{\sum_i \alpha_i \Gamma_i}{1 + \sum_i \Gamma_i}, \tag{A.4}$$

where $\Gamma_i = \gamma N_i / \left[2(\beta - \gamma) + \gamma N_i\right]$. (A.4) implies

$$\alpha_i - \gamma \tilde{Q} = \frac{\Lambda_i}{1 + \sum_i \Gamma_i}, \tag{A.5}$$

where Λ_i is defined in Proposition 1. Using (A.3) and (A.5) yields (3).[14]

[14] Also note from (A.5), and the definitions of both Λ_i and Γ_i, that $\tilde{Q}$ is decreasing in N_i, $i \in \mathcal{I}$, as claimed in footnote 8.

To obtain (4), first, note that $p_k - c_k = \alpha_i - (\beta - \gamma)x_k - \gamma Q$ for all $k \in \mathcal{N}_i$, according to (1) and $\alpha_i = A_k - c_k$. Since $\tilde{x}_k = \tilde{x}_i$ for all $k \in \mathcal{N}_i$, (A.3) then implies equilibrium price-cost differences (or mark up's, respectively) $\tilde{p}_k - c_k = (\beta - \gamma + \gamma N_i)\tilde{x}_i \equiv M_i$ for all $k \in \mathcal{N}_i$. Finally, noting that $\tilde{\pi}_i = N_i \tilde{x}_i M_i$ confirms (4). This concludes the proof. □

Proof of Corollary 1: First, let us write $\sum_{h\in\mathcal{I}} \Gamma_h = 1 + \Phi_{-i} + \Gamma_i$, where $\Phi_{-i} \equiv \sum_{h\neq i} \Gamma_h$. Thus, using $\Gamma_i = \gamma N_i / [2(\beta - \gamma) + \gamma N_i]$, we have

$$\tilde{x}_i = \frac{\Lambda_i}{(1 + \Phi_{-i})(2(\beta - \gamma) + \gamma N_i) + \gamma N_i}, \tag{A.6}$$

according to (3). By substituting (A.6) into (4), we obtain

$$\tilde{\pi}_i = \frac{N_i(\beta - \gamma + \gamma N_i)\Lambda_i^2}{[(1 + \Phi_{-i})(2(\beta - \gamma) + \gamma N_i) + \gamma N_i]^2}. \tag{A.7}$$

Very tedious derivations reveal that

$$\frac{\partial \tilde{\pi}_i}{\partial N_i} = \frac{(\beta - \gamma)[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i}]\Lambda_i^2}{[(1 + \Phi_{-i})(2(\beta - \gamma) + \gamma N_i) + \gamma N_i]^3} > 0, \tag{A.8}$$

and

$$\begin{aligned} \frac{\partial^2 \tilde{\pi}_i}{\partial N_i^2} = & \frac{-2\gamma(\beta - \gamma)\Lambda_i^2}{[(1 + \Phi_{-i})(2(\beta - \gamma) + \gamma N_i) + \gamma N_i]^4} \times \\ & \left[\beta - \gamma + \gamma N_i + (\beta - \gamma + 5\gamma N_i)\Phi_{-i} + 3\gamma N_i \Phi_{-i}^2\right] < 0, \end{aligned} \tag{A.9}$$

respectively. Moreover, for $j \neq i$,

$$\frac{\partial \tilde{\pi}_i}{\partial N_j} = \frac{-4\gamma(\beta - \gamma)N_i(\beta - \gamma + \gamma N_i)(2(\beta - \gamma) + \gamma N_i)\Lambda_i \Lambda_j}{[2(\beta - \gamma) + \gamma N_j]^2 [(1 + \Phi_{-i})(2(\beta - \gamma) + \gamma N_i) + \gamma N_i]^3} < 0, \tag{A.10}$$

and

$$\begin{aligned} \frac{\partial^2 \tilde{\pi}_i}{\partial N_i \partial N_j} = & -\frac{4\gamma(\beta - \gamma)^2 \Lambda_i}{[(1 + \Phi_{-i})(2(\beta - \gamma) + \gamma N_i) + \gamma N_i]^4 [2(\beta - \gamma) + \gamma N_j]^2} \times \\ & \{(\alpha_i - \alpha_j)[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i}] \times \\ & [2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + \gamma N_i)\Phi_{-i}] - \\ & \Lambda_i[(2(\beta - \gamma) + \gamma N_i)(2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i} + 4(\beta - \gamma)(\beta - \gamma + \gamma N_i)]\}, \end{aligned} \tag{A.11}$$

respectively. (A.8) and (A.9) confirm part (i) of Corollary 1 and (A.10) confirms part (ii), respectively (recall that $\Lambda_i, \Lambda_j > 0$ in interior equilibrium). Moreover, note that for all $i, j \in \mathcal{I}$, $j \neq i$, $\partial\Lambda_i/\partial\alpha_i > 0$ and $\partial\Lambda_i/\partial\alpha_j < 0$ (recall $\Lambda_i = \alpha_i \left(1 + \sum_{j\neq i} \Gamma_j\right) - \sum_{j\neq i} \alpha_j \Gamma_j$). Thus, (A.7) and (A.8) also confirm parts (iii) and (iv), respectively. Part (v) follows from (A.11). This concludes the proof. □

Remark 1. Full derivations of (A.8)-(A.11) can be found in supplementary material to this paper, which is available from the author upon request. There, it is also shown that in order to obtain $\partial^2\tilde{\pi}_i/\partial N_i \partial N_j > 0$, $j \neq i$, it is necessary that both $\alpha_i > \alpha_j$ and $\Phi_{-i} > 2(\beta - \gamma)/\left[2(\beta - \gamma) + \gamma N_i\right]$ simultaneously hold. Thus, as claimed in the main text, even if firms are very heterogenous, assumption A2 may still be fulfilled. $\Phi_{-i} \leq 2(\beta - \gamma)/\left[2(\beta - \gamma) + \gamma N_i\right]$ is a sufficient condition for A2 to hold for all configurations $\boldsymbol{\alpha}$.

Proof of Proposition 2: Existence of equilibrium is proven by applying the following classical existence result.

Lemma A.1. (Debreu, 1952). *Let $S_i \subseteq \mathbb{R}^m$ denote the the set of feasible strategies of player $i \in \mathcal{I}$, with a typical element s_i. Moreover, let $U_i : \times_i S_i \rightarrow \mathbb{R}$ be the payoff function of player i. If, for all $i \in \mathcal{I}$, S_i is non-empty, compact and convex and U_i is continuous in $(s_1, s_2, ..., s_I)$ as well as quasiconcave in s_i, the game $\{(S_i, U_i),\ i \in \mathcal{I}\}$ possesses a pure-strategy Nash equilibrium.*

First, note that strategy sets $[1, \bar{N}]$ for the firms' choice at stage 1 (i.e., in maximization problem (7)), are nonempty, compact and convex subsets of $\mathbb{R}$. Second, note that objective functions of firms in (7), $\Psi_i(\mathbf{N}, \boldsymbol{\alpha})$, are continuous in $(N_1, N_2, ..., N_I)$. Third, according to part (i) of Corollary 1 and the convexity of $C(\cdot)$, $\Psi_i(\mathbf{N}, \boldsymbol{\alpha}) = \Pi_i(\mathbf{N}, \boldsymbol{\alpha}) - C(N_i)$ is strictly concave in N_i (and, thus, quasiconcave in N_i). Applying Lemma A.1, this proves existence of a Nash equilibrium for the firms' decision of product ranges. Existence of equilibrium for the entire two-stage game is directly implied. □

Proof of Proposition 5: Define $\Gamma_i^* \equiv \gamma N_i / [2(\beta - \gamma) + \gamma N_i]$ and

$$\Lambda_i^* \equiv \alpha_i \left(1 + \sum_{j \neq i} \Gamma_j^*\right) - \sum_{j \neq i} \alpha_j \Gamma_j^*, \tag{A.12}$$

$i \in \mathcal{I}$. Thus, one can write

$$D_i^*(\boldsymbol{\alpha}) = N_i^* X_i(\mathbf{N}, \boldsymbol{\alpha}) = \frac{N_i^* \Lambda_i^*}{(1 + \sum_i \Gamma_i^*) [2(\beta - \gamma) + \gamma N_i^*]}, \tag{A.13}$$

according to (3). Hence, we have $D_i^*(\boldsymbol{\alpha}) > D_j^*(\boldsymbol{\alpha})$ if and only if

$$\frac{N_i^* \Lambda_i^*}{2(\beta - \gamma) + \gamma N_i^*} > \frac{N_j^* \Lambda_j^*}{2(\beta - \gamma) + \gamma N_j^*}. \tag{A.14}$$

Recall from Proposition 4 that $N_i^* > N_j^*$ if $\alpha_i > \alpha_j$. Thus, using (A.14), Proposition 5 is confirmed if, for instance, $\alpha_i > \alpha_j$ implies $\Lambda_i^* > \Lambda_j^*$. To see that this is indeed the case, first, rewrite (A.12) as

$$\Lambda_i^* = \alpha_i \left(1 + \sum_{h \neq i,j} \Gamma_h^*\right) + (\alpha_i - \alpha_j)\Gamma_j^* - \sum_{h \neq i,j} \alpha_h \Gamma_h^*, \tag{A.15}$$

$i, j \in \mathcal{I}$. (A.15) then implies that

$$\begin{aligned} \Lambda_i^* - \Lambda_j^* &= (\alpha_i - \alpha_j) \left(1 + \sum_{h \neq i,j} \Gamma_h^*\right) + (\alpha_i - \alpha_j)\Gamma_j^* - (\alpha_i - \alpha_j)\Gamma_i^* \\ &= (\alpha_i - \alpha_j) \left(1 + \sum_{i \in \mathcal{I}} \Gamma_i^*\right), \end{aligned} \tag{A.16}$$

i.e., $\Lambda_i^* > \Lambda_j^*$ if $\alpha_i > \alpha_j$. This concludes the proof. □

References

Amey, L.R. (1964). Diversified Manufacturing Businesses, *Journal of the Royal Statistical Society* 27, series A, part 2.

Anderson, Simon P. and André de Palma (1992). Multi-Product Firms: A Nested Logit Approach, *Journal of Industrial Economics* 40 (3), 261-276.

Anderson, Simon P., André de Palma and Jacques-Francois Thisse (1992). *Discrete Choice Theory of Product Differentiation*, Cambridge, MIT Press.

Athey, Susan and Armin Schmutzler (2001). Investment and Market Dominance, *RAND Journal of Economics* 32 (1), 1-26.

Aw, Bee-Yan and Geeta Batra (1998). Firm Size and the Pattern of Diversification, *International Journal of Industrial Organization* 16, 313-331.

Aydemir, Zava and Armin Schmutzler (2002). Acquisitions versus Entry: The Evolution of Concentration, University of Zurich, Socioeconomic Institute, Working Paper No. 208.

Baily, Martin N., Charles Hulten, David Campbell, Timothy Bresnahan, and Richard E. Caves (1992). The Distribution of Productivity in Manufacturing Plants, *Brookings Papers of Economic Activity* 0, Microeconomics, 187-249.

Bartelsman, Eric J. and Mark Domes (2000). Understanding Productivity: Lessons from Longitudinal Microdata, *Journal of Economic Literature* 38, 569-594.

Baumol, William J. (1977). On the Proper Test for Natural Oligopoly in a Multiproduct Industry, *American Economic Review* 67 (5), 809-822.

Caves, Richard E. (1971). International Cooperations: The Industrial Economics of Foreign Investment, *Economica* 38, 1-27.

Debreu, Gérard (1952). A Social Equilibrium Existence Theorem, *Proceedings of the National Academy of Sciences* 38 (10), 886-893.

Dixit, Avinash (1986). Comparative Statics for Oligopoly, *International Economic Review* 27 (1), 107-22.

Gollop, Frank M. and James L. Monahan (1991). A Generalized Index of Diversification: Trends in U.S. Manufacturing, *Review of Economics and Statistics* 73 (2), 318-330.

Gorecki, Paul K. (1975). An Inter-Industry Analysis of Diversification in the U.K. Manufacturing Sector, *Journal of Industrial Economics* 24 (2), 131-146.

Goto, Akira (1981). Statistical Evidence on the Diversification of Japanese Large Firms, *Journal of Industrial Economics* 29 (3), 271-278.

Markusen, James R. (2002). *Multinational Firms and the Theory of International Trade*, Cambridge, MIT Press.

Milgrom, Paul and John Roberts (1990). The Economics of Modern Manufacturing: Technology, Strategy, and Organization, *American Economic Review* 80 (3), 511-528.

Montgomery, Cynthia A. (1994). Corporate Diversification, *Journal of Economic Perspectives* 8 (3), 163-178.

Ottaviano, Gianmarco I.P. and Jacques-Francois Thisse (1999). Monopolistic Competition, Multiproduct Firms and Optimum Product Diversity, CORE Discussion Paper No. 9919.

Raubitschek, Ruth S. (1987). A Model of Product Proliferation with Multiproduct Firms, *Journal of Industrial Economics* 35 (3), 269-279.

Roberts, Mark J. and Dylan Supina (2000). Output Price and Markup Dispersion in Producer Micro Data: The Roles of Producer Heterogeneity and Noise, in: Baye, M. (Ed.), *Advances in Applied Microeconomics*, Vol. 9, Industrial Organization (JAI Press).

Shaked, Avner and John Sutton (1990). Multiproduct Firms and Market Structure, *RAND Journal of Economics* 21 (1), 45-62.

Sutton, John (1998). *Technology and Market Structure*, Cambridge, MIT Press.

Takayama, Akira (1985). *Mathematical Economics*, 2nd edition, Cambridge, Cambridge University Press.

Utton, M.A. (1977). Large Firm Diversification in British Manufacturing Industry, *Economic Journal* 87, 96-113.

Vives, Xavier (1999). *Oligopoly Pricing: Old Ideas and New Tools*, Cambridge, MIT Press.

Supplementary Material to "Firm Size and Diversification: Asymmetric Multiproduct Firms under Cournot Competition"

Volker Grossmann*

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Abstract

This supplement contains three parts. First, details of derivations of equations (A.8)-(A.11) in appendix (proof of Corollary 1) are provided. Second, the claim in Remark 1 (implying that assumption A2 may hold even if firms are very heterogeneous) is proven. Third, basic properties of the functions D_i (total sales in stage 2 equilibrium) and M_i (mark-up in stage 2 equilibrium), which are used for the discussion of Corollary 1, are derived.

Available from: `http://www.wwi.unizh.ch/staff/grossmann/index.php`

* E-mail: volker.grossmann@wwi.unizh.ch.

Derivation of (A.8): From (A.7),

$$\tilde{\pi}_i = \frac{N_i(\beta - \gamma + \gamma N_i)\Lambda_i^2}{Z_i^2}, \tag{B.1}$$

where

$$Z_i \equiv 2(\beta - \gamma + \gamma N_i) + [2(\beta - \gamma) + \gamma N_i]\Phi_{-i}. \tag{B.2}$$

Note that

$$\Lambda_i = (1 + \Phi_{-i})\alpha_i - \sum_{j \neq i} \alpha_j \Gamma_j \tag{B.3}$$

and

$$\Phi_{-i} = \sum_{h \neq i} \Gamma_h = \sum_{h \neq i} \frac{\gamma N_h}{2(\beta - \gamma) + \gamma N_h} \tag{B.4}$$

are independent of N_i. Also note that $\partial Z_i / \partial N_i = \gamma(2 + \Phi_{-i})$. Thus,

$$\begin{aligned}
\frac{\partial \tilde{\pi}_i}{\partial N_i} &= \frac{\Lambda_i^2}{Z_i^4} \left\{ (\beta - \gamma + 2\gamma N_i) Z_i^2 - N_i(\beta - \gamma + \gamma N_i) 2 Z_i \frac{\partial Z_i}{\partial N_i} \right\} \\
&= \frac{\Lambda_i^2}{Z_i^3} \{ (\beta - \gamma + 2\gamma N_i) \left[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + \gamma N_i)\Phi_{-i} \right] - \\
&\quad 2\gamma N_i (\beta - \gamma + \gamma N_i)(2 + \Phi_{-i}) \} \\
&= \frac{\Lambda_i^2}{Z_i^3} \{ 2(\beta - \gamma + 2\gamma N_i)(\beta - \gamma + \gamma N_i) - 4\gamma N_i (\beta - \gamma + \gamma N_i) + \\
&\quad [(\beta - \gamma + 2\gamma N_i)(2(\beta - \gamma) + \gamma N_i) - 2\gamma N_i (\beta - \gamma + \gamma N_i)]\Phi_{-i} \} \\
&= \frac{\Lambda_i^2 (\beta - \gamma)}{Z_i^3} \left\{ 2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\, \Phi_{-i} \right\} > 0.
\end{aligned} \tag{B.5}$$

Substituting (B.2) into (B.5) confirms (A.8). □

Derivation of (A.9): Using (A.8), one obtains

$$\begin{aligned}
\frac{\partial^2 \tilde{\pi}_i}{\partial N_i^2} &= \frac{\Lambda_i^2(\beta-\gamma)}{Z_i^6}\left\{(2\gamma+3\gamma\Phi_{-i})Z_i^3 - [2(\beta-\gamma+\gamma N_i)+(2(\beta-\gamma)+3\gamma N_i)\,\Phi_{-i}]\,3Z_i^2\frac{\partial Z_i}{\partial N_i}\right\} \\
&= \frac{\Lambda_i^2(\beta-\gamma)\gamma}{Z_i^4}\{(2+3\Phi_{-i})(2(\beta-\gamma+\gamma N_i)+(2(\beta-\gamma)+\gamma N_i)\,\Phi_{-i}) - \\
&\quad 3\,[2(\beta-\gamma+\gamma N_i)+[2(\beta-\gamma)+3\gamma N_i]\Phi_{-i}(2+\Phi_{-i})]\} \\
&= \frac{\Lambda_i^2(\beta-\gamma)\gamma}{Z_i^4}\{-2(\beta-\gamma+\gamma N_i)+\Phi_{-i}[6(\beta-\gamma+\gamma N_i)+2(2(\beta-\gamma)+\gamma N_i) - \\
&\quad 6(2(\beta-\gamma)+3\gamma N_i)]+\Phi_{-i}^2[3(2(\beta-\gamma)+\gamma N_i)-3(2(\beta-\gamma)+3\gamma N_i)]\} \\
&= -\frac{2\Lambda_i^2(\beta-\gamma)\gamma}{Z_i^4}\{\beta-\gamma+\gamma N_i+(\beta-\gamma+5\gamma N_i)\Phi_{-i}+3\gamma N_i\Phi_{-i}^2\}<0. \qquad \text{(B.6)}
\end{aligned}$$

Substituting (B.2) into (B.6) confirms (A.9). □

Derivation of (A.10): First, note that, for all $i, j \in \mathcal{I}$, $j \neq i$,

$$\frac{\partial \Lambda_i}{\partial N_j} = \frac{2(\beta-\gamma)\gamma(\alpha_i-\alpha_j)}{[2\,(\beta-\gamma)+\gamma N_j]^2} \qquad \text{(B.7)}$$

and

$$\frac{\partial \Phi_{-i}}{\partial N_j} = \frac{\partial \Gamma_j}{\partial N_j} = \frac{2(\beta-\gamma)\gamma}{[2\,(\beta-\gamma)+\gamma N_j]^2}, \qquad \text{(B.8)}$$

according to (B.3) and (B.4), respectively; moreover, we have

$$\frac{\partial Z_i}{\partial N_j} = \frac{2(\beta-\gamma)\gamma[2(\beta-\gamma)+\gamma N_i]}{[2\,(\beta-\gamma)+\gamma N_j]^2}, \qquad \text{(B.9)}$$

according to (B.2) and (B.8). Hence, using

$$\tilde{\pi}_i = \frac{N_i(\beta-\gamma+\gamma N_i)\Lambda_i^2}{Z_i^2}, \qquad \text{(B.10)}$$

one obtains, for $j \neq i$,

$$\begin{aligned}
\frac{\partial \tilde{\pi}_i}{\partial N_j} &= \frac{N_i(\beta-\gamma+\gamma N_i)}{Z_i^4}\left\{2\Lambda_i\frac{\partial \Lambda_i}{\partial N_j}Z_i^2 - \Lambda_i^2 2Z_i\frac{\partial Z_i}{\partial N_j}\right\} \\
&= \frac{4N_i(\beta-\gamma+\gamma N_i)\Lambda_i(\beta-\gamma)\gamma Q_{i,j}}{[2(\beta-\gamma)+\gamma N_j]^2 Z_i^3}, \qquad \text{(B.11)}
\end{aligned}$$

where

$$\begin{aligned} Q_{i,j} &\equiv (\alpha_i - \alpha_j)\left[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + \gamma N_i)\Phi_{-i}\right] - (2(\beta - \gamma) + \gamma N_i)\,\Lambda_i \\ &= 2(\alpha_i - \alpha_j)(\beta - \gamma + \gamma N_i) + [2(\beta - \gamma) + \gamma N_i][(\alpha_i - \alpha_j)\Phi_{-i} - \Lambda_i]. \end{aligned} \tag{B.12}$$

Using (B.3), one finds

$$\begin{aligned} Q_{i,j} &= 2(\alpha_i - \alpha_j)(\beta - \gamma + \gamma N_i) - [2(\beta - \gamma) + \gamma N_i](\alpha_j \Phi_{-i} + \alpha_i - \sum_{j \neq i} \alpha_j \Gamma_j) \\ &= \gamma N_i(\alpha_i - \alpha_j) - [2(\beta - \gamma) + \gamma N_i]\left(\alpha_j(1 + \Phi_{-i}) - \sum_{j \neq i} \alpha_j \Gamma_j\right) \\ &= -[2(\beta - \gamma) + \gamma N_i]\left[(\alpha_j - \alpha_i)\Gamma_i + \alpha_j\left(1 + \Gamma_j + \sum_{h \neq i,j} \Gamma_h\right) - \alpha_j \Gamma_j - \sum_{h \neq i,j} \alpha_h \Gamma_h\right] \\ &= -[2(\beta - \gamma) + \gamma N_i]\left[\alpha_j\left(1 + \sum_{h \neq j} \Gamma_h\right) - \sum_{h \neq j} \alpha_h \Gamma_h\right] \\ &= -[2(\beta - \gamma) + \gamma N_i]\Lambda_j. \end{aligned} \tag{B.13}$$

Substituting (B.2) and (B.13) into (B.12) confirms (A.10). □

Derivation of (A.11): From (A.8), by making use of (B.7) and (B.9), one obtains, for $j \neq i$,

$$\begin{aligned} \frac{\partial^2 \tilde{\pi}_i}{\partial N_i \partial N_j} &= \frac{\beta - \gamma}{Z_i^6}\{\{(2(\beta - \gamma) + 3\gamma N_i)(\partial \Phi_{-i}/\partial N_j)\Lambda_i^2 + \\ &\quad [2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i}]2\Lambda_i \frac{\partial \Lambda_i}{\partial N_j}\}Z_i^3 - \\ &\quad \Lambda_i^2[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i}]3Z_i^2 \frac{\partial Z_i}{\partial N_j}\} \\ &= \frac{2(\beta - \gamma)^2 \gamma \Lambda_i}{Z_i^4[2(\beta - \gamma) + \gamma N_j]^2}\{2\,(\alpha_i - \alpha_j)\,[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i}]Z_i + \\ &\quad \Lambda_i Z_i(2(\beta - \gamma) + 3\gamma N_i) - \\ &\quad 3\Lambda_i[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i}](2(\beta - \gamma) + \gamma N_i)\} \\ &= \frac{2(\beta - \gamma)^2 \gamma \Lambda_i}{Z_i^4[2(\beta - \gamma) + \gamma N_j]^2} \times \\ &\quad \{2\,(\alpha_i - \alpha_j)\,[2(\beta - \gamma + \gamma N_i) + (2(\beta - \gamma) + 3\gamma N_i)\Phi_{-i}]Z_i + \Lambda_i T_{i,j}\}\,, \end{aligned} \tag{B.15}$$

where

$$
\begin{aligned}
T_{i,j} &\equiv [2(\beta-\gamma+\gamma N_i)+(2(\beta-\gamma)+\gamma N_i)\Phi_{-i}](2(\beta-\gamma)+3\gamma N_i)-3(2(\beta-\gamma)+\gamma N_i)\times \\
&\quad [2(\beta-\gamma+\gamma N_i)+(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}] \\
&= -2\left[(2(\beta-\gamma)+\gamma N_i)(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}+4(\beta-\gamma+\gamma N_i)(\beta-\gamma)\right], \qquad \text{(B.16)}
\end{aligned}
$$

i.e., $T_{i,j} < 0$. Substituting (B.16) into (B.15) yields (A.11). Hence, if $\alpha_i \leq \alpha_j$, then $\partial^2\tilde{\pi}_i/\partial N_i \partial N_j < 0$, $j \neq i$. In fact, if firms are not "too heterogeneous", then assumption A2 holds. □

Proof of claim in Remark 1: As will be shown in the following, even if $\alpha_i \gg \alpha_j$, A2 holds for some configurations $\mathbf{N}$. To derive a sufficient condition, first, note that, from the definition of Λ_i (and Λ_j) in (B.3), the following fact can easily be derived (see also (A.16) in appendix): For $j \neq i$,

$$(\alpha_i - \alpha_j)(1 + \sum_{h \in \mathcal{I}} \Gamma_h) - \Lambda_i = -\Lambda_j. \qquad \text{(B.17)}$$

(B.17) can be used in order to rewrite (A.11) in the following way:

$$\frac{\partial^2 \tilde{\pi}_i}{\partial N_i \partial N_j} = \frac{4(\beta-\gamma)^2\gamma\Lambda_i}{Z_i^4[2(\beta-\gamma)+\gamma N_j]^2} V_{i,j}, \qquad \text{(B.18)}$$

where

$$
\begin{aligned}
V_{i,j} &\equiv [2(\beta-\gamma+\gamma N_i)+(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}][2(\beta-\gamma+\gamma N_i)+(2(\beta-\gamma)+\gamma N_i)\Phi_{-i}]\times \\
&\quad (\alpha_i-\alpha_j)-\Lambda_i[(2(\beta-\gamma)+\gamma N_i)(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}+4(\beta-\gamma)(\beta-\gamma+\gamma N_i)] \\
&= (2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}(2(\beta-\gamma)+\gamma N_i)[(\alpha_i-\alpha_j)\Phi_{-i}-\Lambda_i]+2(\beta-\gamma+\gamma N_i)\times \\
&\quad \{(\alpha_i-\alpha_j)[2(\beta-\gamma+\gamma N_i)+(2(\beta-\gamma)+\gamma N_i)\Phi_{-i}+ \\
&\quad (2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}]-2(\beta-\gamma)\Lambda_i\} \\
&= (2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}(2(\beta-\gamma)+\gamma N_i)\times \\
&\quad [(\alpha_i-\alpha_j)(1+\sum_h\Gamma_h)-\Lambda_i-(\alpha_i-\alpha_j)(1+\Gamma_i)]+ \\
&\quad 4(\beta-\gamma+\gamma N_i)\left\{(\alpha_i-\alpha_j)(\beta-\gamma+\gamma N_i)(1+2\Phi_{-i})-(\beta-\gamma)\Lambda_i\right\} \\
&= (2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}(2(\beta-\gamma)+\gamma N_i)[-\Lambda_j-(\alpha_i-\alpha_j)(1+\Gamma_i)]+ \\
&\quad 4(\beta-\gamma+\gamma N_i)\left\{(\beta-\gamma)[(\alpha_i-\alpha_j)(1+\Phi_{-i})-\Lambda_i]+(\beta-\gamma+2\gamma N_i)\Phi_{-i}(\alpha_i-\alpha_j)\right\} \\
&= -\Lambda_j(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}(2(\beta-\gamma)+\gamma N_i)+ \\
&\quad 4(\beta-\gamma+\gamma N_i)(\beta-\gamma)[-\Lambda_j-(\alpha_i-\alpha_j)\Gamma_i]+ \\
&\quad (\alpha_i-\alpha_j)[4(\beta-\gamma+\gamma N_i)(\beta-\gamma+2\gamma N_i)\Phi_{-i}- \\
&\quad (1+\Gamma_i)(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}(2(\beta-\gamma)+\gamma N_i)] \\
&= -\Lambda_j[(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}(2(\beta-\gamma)+\gamma N_i)+4(\beta-\gamma+\gamma N_i)(\beta-\gamma)]- \\
&\quad (\alpha_i-\alpha_j)\{\Phi_{-i}[(2(\beta-\gamma)+3\gamma N_i)2(\beta-\gamma+\gamma N_i)-4(\beta-\gamma+\gamma N_i)(\beta-\gamma+2\gamma N_i)] \\
&\quad +4(\beta-\gamma+\gamma N_i)(\beta-\gamma)\Gamma_i\} \\
&= -\Lambda_j[(2(\beta-\gamma)+3\gamma N_i)\Phi_{-i}(2(\beta-\gamma)+\gamma N_i)+4(\beta-\gamma+\gamma N_i)(\beta-\gamma)]+ \\
&\quad (\alpha_i-\alpha_j)(\beta-\gamma+\gamma N_i)2[\gamma N_i\Phi_{-i}-2(\beta-\gamma)\Gamma_i] \qquad \text{(B.19)}
\end{aligned}
$$

Thus, if $\alpha_i > \alpha_j$ and $\gamma N_i \Phi_{-i} > 2(\beta-\gamma)\Gamma_i$, which, using $\Gamma_i = \dfrac{\gamma N_i}{2(\beta-\gamma)+\gamma N_i}$, is equivalent to $\Phi_{-i} > \dfrac{2(\beta-\gamma)}{2(\beta-\gamma)+\gamma N_i}$, then $\dfrac{\partial^2 \tilde{\pi}_i}{\partial N_i \partial N_j} > 0$, $j \neq i$, is possible. However, if

$$
\Phi_{-i} \leq \frac{2(\beta-\gamma)}{2(\beta-\gamma)+\gamma N_i},
$$

then A2 holds for *all* configurations $\boldsymbol{\alpha}$. (Recall that A2 always holds if $\alpha_i \leq \alpha_j$, according to part (v) of Corollary 1.) This confirms the claims made in Remark 1. □

Properties of $D_i(\mathbf{N}, \boldsymbol{\alpha})$ and $M_i(\mathbf{N}, \boldsymbol{\alpha})$: In section 3, we decomposed profits of a firm i in stage 2 equilibrium, $\Pi_i(\mathbf{N}, \boldsymbol{\alpha})$, into the product between equilibrium demand, $D_i(\mathbf{N}, \boldsymbol{\alpha}) = N_i X_i(\mathbf{N}, \boldsymbol{\alpha})$, and equilibrium mark-up, $M_i(\mathbf{N}, \boldsymbol{\alpha}) = (\beta - \gamma + \gamma N_i) X_i(\mathbf{N}, \boldsymbol{\alpha})$. The remainder of this supplement formally derives the properties of these two functions, $D_i(\mathbf{N}, \boldsymbol{\alpha})$ and $M_i(\mathbf{N}, \boldsymbol{\alpha})$, which have been used in the discussion of Corollary 1. We obtain the following results.

Corollary B.1. (Properties of $D_i(\mathbf{N}, \boldsymbol{\alpha})$). *For all $i, j \in \mathcal{I}$, $j \neq i$, we have*
(i) $\partial D_i / \partial N_i > 0$ and $\partial^2 D_i / \partial N_i^2 < 0$,
(ii) $\partial D_i / \partial N_j < 0$,
(iii) $\partial D_i / \partial \alpha_i > 0$, $\partial D_i / \partial \alpha_j < 0$,
(iv) $\partial^2 D_i / \partial N_i \partial \alpha_i > 0$ and $\partial^2 D_i / \partial N_i \partial \alpha_j < 0$, and
(v) if $\alpha_i \leq \alpha_j$ or if $(\alpha_i - \alpha_j)$ sufficiently small, then $\partial^2 D_i / \partial N_i \partial N_j < 0$.

Proof. First, note from (A.6) that $\partial \tilde{x}_i / \partial N_i = -\lambda_{i,j} \tilde{x}_i / N_i$, and thus, $\partial D_i / \partial N_i = \tilde{x}_i + N_i \partial \tilde{x}_i / \partial N_i = (1 - \lambda_{i,j}) \tilde{x}_i$, where

$$\lambda_{i,j} \equiv \frac{\gamma N_i (2 + \Phi_{-i})}{2(\beta - \gamma)(1 + \Phi_{-i}) + \gamma N_i (2 + \Phi_{-i})}. \tag{B.20}$$

Since $\lambda_{i,j} \in (0, 1)$, we have $\partial D_i / \partial N_i > 0$. Moreover, $\partial^2 D_i / \partial N_i^2 = (1 - \lambda_{i,j}) \partial \tilde{x}_i / \partial N_i - \tilde{x}_i \partial \lambda_{i,j} / \partial N_i < 0$, since $\partial \tilde{x}_i / \partial N_i < 0$, $\lambda_{i,j} \in (0, 1)$ and $\partial \lambda_{i,j} / \partial N_i > 0$, according to (B.20). This proves part (i) of Corollary B.1. To prove part (ii), note that $\partial D_i / \partial N_j = N_i \partial \tilde{x}_i / \partial N_j$, $j \neq i$. Using (A.6), (B.7) and (B.8), it is straightforward to show that this implies

$$\frac{\partial D_i}{\partial N_j} = -\frac{2\gamma(\beta - \gamma) N_i Q_{i,j}}{[2(\beta - \gamma) + \gamma N_j]^2 Z_i^2}, \tag{B.21}$$

$j \neq i$, where Z_i and $Q_{i,j}$ are given by (B.2) and (B.12), respectively. According to (B.13), we have $Q_{i,j} < 0$, thus, confirming $\partial D_i / \partial N_j < 0$, $j \neq i$. To prove part (iii) of Corollary B.1, first, note that $\tilde{x}_i = \Lambda_i / Z_i$, according to (A.6) and (B.2). Thus, part (iii) directly follows from $\partial \Lambda_i / \alpha_i > 0$ and $\partial \Lambda_i / \alpha_j < 0$, $j \neq i$, according to (B.3), and the fact that $\partial D_i / \partial \alpha_j = N_i \partial \tilde{x}_i / \partial \alpha_j$. Recalling $\partial D_i / \partial N_i = (1 - \lambda_{i,j}) \tilde{x}_i$, part (iv) follows by similar considerations, together with the fact that $\lambda_{i,j}$ is independent of α_i or α_j, respectively, according to (B.20). To prove part (v), first, note that $\partial D_i / \partial N_i = 2(\beta - \gamma) \Lambda_i (1 + \Phi_{-i}) / Z_i^2$, according to (B.2), (B.20) and $\tilde{x}_i = \Lambda_i/Z_i$. Using this together with (B.7)-(B.9), one obtains, after some manipulations, that, for $j \neq i$,

$$\frac{\partial^2 D_i}{\partial N_i \partial N_j} = \frac{4\gamma(\beta-\gamma)^2 N_i \{(\alpha_i - \alpha_j)(1+\Phi_{-i})Z_i - \Lambda_i [\Phi_{-i}(2(\beta-\gamma)+\gamma N_i) + 2(\beta-\gamma)]\}}{[2(\beta-\gamma)+\gamma N_j]^2 Z_i^3}. \tag{B.22}$$

Recalling that $\Lambda_i > 0$ in interior equilibrium confirms part (v). This concludes the proof of Corollary B.1. ■

Corollary B.2. (Properties of $M_i(\mathbf{N}, \boldsymbol{\alpha})$). *For all $i, j \in \mathcal{I}$, $j \neq i$, we have*

(i) $\partial M_i/\partial N_i > 0$ and $\partial^2 M_i/\partial N_i^2 < 0$,

(ii) $\partial M_i/\partial N_j < 0$,

(iii) $\partial M_i/\partial \alpha_i > 0$, $\partial M_i/\partial \alpha_j < 0$,

(iv) $\partial^2 M_i/\partial N_i \partial \alpha_i > 0$ and $\partial^2 M_i/\partial N_i \partial \alpha_j < 0$, and

(v) the sign of $\partial^2 M_i/\partial N_i \partial N_j$ is ambiguous.

Proof. First, note that we can write $M_i = (\beta - \gamma + \gamma N_i)\Lambda_i/Z_i$ since $\tilde{x}_i = \Lambda_i/Z_i$. Thus, using (B.2)-(B.4) leads to

$$\frac{\partial M_i}{\partial N_i} = \frac{\gamma \Lambda_i (\beta-\gamma)\Phi_{-i}}{Z_i^2} > 0. \tag{B.23}$$

Moreover, since $\partial Z_i/\partial N_i > 0$, according to (B.2), (B.23) implies $\partial^2 M_i/\partial N_i^2 < 0$. This confirms part (i) of Corollary B.2. In a similar fashion as in the proof of part (ii) of Corollary B.1, one can also show that

$$\frac{\partial M_i}{\partial N_j} = -\frac{2\gamma(\beta-\gamma)(\beta-\gamma+\gamma N_i)Q_{i,j}}{[2(\beta-\gamma)+\gamma N_j]^2 Z_i^2} < 0, \tag{B.24}$$

$j \neq i$. Part (iii) follows directly from recalling $\partial \Lambda_i/\alpha_i > 0$ and $\partial \Lambda_i/\alpha_j < 0$, $j \neq i$, together with $\partial M_i/\partial \alpha_j = (\beta - \gamma + \gamma N_i)\partial \tilde{x}_i/\partial \alpha_j$ and $\tilde{x}_i = \Lambda_i/Z_i$. Part (iv) follows from (B.23), and, again, $\partial \Lambda_i/\alpha_i > 0$ and $\partial \Lambda_i/\alpha_j < 0$, $j \neq i$. Finally, using (B.23), together with (B.7)-(B.9), one can show that, for $j \neq i$,

$$\frac{\partial^2 M_i}{\partial N_i \partial N_j} = \frac{2\gamma^2(\beta-\gamma)^2 \{(\alpha_i - \alpha_j)\Phi_{-i}Z_i + \Lambda_i [2(\beta-\gamma+\gamma N_i) - \Phi_{-i}(2(\beta-\gamma)+\gamma N_i)]\}}{[2(\beta-\gamma)+\gamma N_j]^2 Z_i^3}. \tag{B.25}$$

Unfortunately, for $j \neq i$, the sign of $\partial^2 M_i/\partial N_i \partial N_j$ is ambiguous even for $\alpha_i = \alpha_j$. This concludes the proof of Corollary B.2. ■