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Research Article

First Boundary Value Problem for Cordes-Type Semilinear Parabolic Equation with Discontinuous Coefficients

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For a class of semilinear parabolic equations with discontinuous coefficients, the strong solvability of the Dirichlet problem is studied in this paper. The problem $\sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_i x_j} - u_t + g(t,x,u) = f(t,x), u|_{\Gamma(Q_T)} = 0$, in $Q_T = \Omega \times (0,T)$ is the subject of our study, where Ω is bounded C^2 or a convex subdomain of $E_{n+1}, \Gamma(Q_T) = \partial Q_T \backslash \{t = T\}$. The function $g(x,u)$ is assumed to be a Caratheodory function satisfying the growth condition $|g(t,x,u)| \leq b_0 |u|^q$, for $b_0 > 0, q \in (0, (n+1)/(n-1)), n \geq 2$, and leading coefficients satisfy Cordes condition $b_0 > 0, q \in (0, (n+1)/(n-1)), n \geq 2$.

1. Introduction

Let E_n be an n-dimensional Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$ and Ω be a bounded domain in E_n with boundary $\partial\Omega$ of the class C^2 or simply a convex domain. Set $Q_T = \Omega \times (0,T)$ and $\Gamma(Q_T) = \partial Q_T \backslash \{t = T\}$. Consider in Q_T the Dirichlet problem:

$$\sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_i x_j} - u_t + g(t,x,u) = f(t,x), \ (t,x) \in Q_T, \tag{1}$$

$$u|_{\Gamma(Q_T)} = 0. \tag{2}$$

It is assumed that the coefficients $a_{ij}(t,x)$, $i,j = 1,2,\ldots,n$, of the operator

$$L = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}, \tag{3}$$

are bounded measurable functions satisfying the uniform parabolicity

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \leq \gamma^{-1}|\xi|^2, \tag{4}$$

for $\gamma \in (0,1), \forall (t,x) \in Q_T, \forall \xi \in E_n$, and the Cordes-type condition

$$\frac{\sum_{i,j=1}^{n} a_{ij}^2(t,x)}{\left(\sum_{i=1}^{n} a_{ii}(t,x)\right)^2} \leq \frac{1}{n-\mu^2} - \delta. \tag{5}$$

Here, $\mu = (\operatorname{ess\,inf} \sum_{i=1}^{n} a_{ii}(t,x))/(\operatorname{ess\,sup} \sum_{i=1}^{n} a_{ii}(t,x))$, and the number $\delta \in (0, (1/(n+1)))$. The nonlinear term, function $g(t,x,u)$: $Q_T \longrightarrow E_1$, satisfies the Caratheodory condition, that is, g is a measurable function with respect to variables $(t,x) \in \Omega$, and for almost all $(t,x) \in Q_T$ continuously depend on the variable $u \in E_1$. Also, the growth condition

$$|g(t,x,u)| \leq b_0|u|^q, \quad b_0 > 0, \tag{6}$$

is satisfied.

The space $\dot{W}_p^{2,1}(Q_T)$, $p > 1$, is a closure of function class $u \in C^\infty(\overline{Q}_T) \cap C(\overline{Q}_T), u|_{\Gamma(Q_T)} = 0$ with respect to norm

$$\|u\|_{\dot{W}_p^{2,1}(Q_T)} = \|u\|_{L_p(Q_T)} + \sum_{i=1}^{n} \left\|\partial_{x_i} u\right\|_{L_p(Q_T)} + \left\|\partial_t u\right\|_{L_p(Q_T)} + \sum_{i,j=1}^{n} \left\|\partial_{x_i}\partial_{x_j} u\right\|_{L_p(Q_T)}. \tag{7}$$

Here, u_i, u_t, and u_{ij} denote the weak derivatives u_{x_i}, u_t, and $u_{x_i x_j}$, respectively, $i, j = 1, \ldots, n$. The conjugate number is denoted by p', i.e., $1 < p < \infty$, $(1/p') + (1/p) == 1$. By the same letter C, we denote different positive constants, and the value of C is not essential for purposes of this study.

For $p \in [1, \infty]$, we denote by $\|v\|_{L_p(Q_T)}$ or simply $\|v\|_p$ the norm of a Banach space $L^p[0, T; L_p(\Omega)]$ defined as $\|g\|_p = (\int_0^T \|g(t, \cdot)\|_{L_p(\Omega)}^p \mathrm{d}t)^{1/p}$.

A function $u(t, x) \in \dot{W}_p^{2,1}(Q_T)$ is called the strong solution (almost everywhere) of problems (1) and (2) if it satisfies equation (1), a.e., in Q_T.

In this study, we will make essential use of the existence results given in Theorem 1.1 of [1] (see, also [2]) for Cordes-type parabolic equations satisfying (5). In [1], the estimate

$$\|u\|_{\dot{W}_2^{2,1}(Q_T)} \leq C\|Lu\|_{L_2(Q_T)}, \tag{8}$$

was proved for all $u \in \dot{W}_p^{2,1}(Q_T)$, and when $T \leq T_0$ with $T_0 = T_0(n, L, \Omega)$ to be sufficiently small and positive constant C depends on n, Ω, L.

In the stationary case, i.e., the solution does not depend on the time variable (the elliptic equation), from examples ([3], p. 48), it is followed that the equation $Lu = f$ is solvable in $\dot{W}_p^{2,1}(Q_T)$ for no $p > 1$ (see [3–8]) if the coefficients are discontinuous. In the absense of $g(t, x, u)$, the strong solvability of the Dirichlet problem for quasi-linear parabolic equations under more restrictive then (5) conditions see, e.g. [9, 10].

If the trace of matrix $\|a_{ij}(t, x)\|$ is constant, condition (5) is exactly Cordes condition (see, e.g., [7, 11–13]):

$$\frac{\sum_{i,j=1}^{n} a_{ij}^2(t, x)}{\left(\sum_{i=1}^{n} a_{ii}(t, x)\right)^2} \leq \frac{1}{n-1} - \delta. \tag{9}$$

For the strong solvability problem in $\dot{W}_p^2(\Omega)$ for any $p > 1$ for parabolic equations with discontinuous coefficients, we refer [8, 14, 15], where the leading coefficients are taken from the VMO class. We refer [16] on exact growth conditions for strong solvability of nonlinear elliptic equations $\Delta u = g(x, u, u_x)$ in $\dot{W}_p^2(\Omega)$ whenever $p > n$.

The aim pursued in this paper is to prove the strong solvability of Dirichlet problems (1) and (2) in the space $\dot{W}_2^{2,1}(Q_T)$ for T to be sufficiently small, the $\|f(t, x)\|_{L_2(Q_T)}$ norm to be sufficiently small, and the coefficients to satisfy (5).

2. Main Result

In order to carry out the proof of main Theorem 1, we need the following assertion from [1].

Lemma 1. *Let $u(t, x)$ be a $\dot{W}_2^{2,1}(Q_T)$ function in $Q_T = \Omega \times [0, T)$ and conditions (2), (4), and (5) be fulfilled for $u(t, x)$ and coefficients of the operator L; the domain Ω is of C^2 class or simply convex. Then, there exists sufficiently small T_0 depending on $\mathscr{L}, n, \Omega$ such that, for $T \leq T_0$, estimate (8) holds with the constant C depending on $\mathscr{L}, n, \Omega$.*

The following assertion is the main result of this paper.

Theorem 1. *Let $n > 4$, $0 < q < (n + 1/n - 1)$, and conditions (4)–(6) be fulfilled, and $\partial\Omega \in C^2$. Let T_0 be a number in Lemma 1 and $T \leq T_0$. Then, problems (1) and (2) have at least one strong solution in the space $\dot{W}_2^{2,1}(Q_T)$ for any $f(t, x) \in L_2(Q_T)$ satisfying*

$$\|f\|_{L_2(Q_T)} \leq C b_0^{-1/(q-1)} \mathrm{mes}_{n+1} Q_T^{(((q(n-1))/(n+1))-1)(1/2(q-1))}. \tag{10}$$

Proof. In order to get the solvability of problem (1) and (2), we apply the Schauder fixed point theorem on completely continuous mappings of a compact subset in the Banach space (see, e.g. [4], p. 257, or [17]).

Set $L^{2q}(Q_T)$ as a basic Banach space. In this space, we define the set $V_2 = \left\{u \in \dot{W}_2^{2,1}(Q_T) | \|u\|_{W_{2,1}^2(Q_T)} \leq K\right\}$, where the number K will be chosen later. Show that $V_2$is compact in $L^{2q}(Q_T)$. By using the condition $2q < 2(n+1)/(n-1)$ and Sobolev–Kondrachov's compact embedding theorem, the space $W_2^1(Q_T)$ is imbedded into $L^{2q}(Q_T)$ compactly. On the contrary, $W_2^{2,1}(Q_T) W_2^1(Q_T)$ is continuous. Therefore, $V_2 L^{2q}(Q_T)$ is compact.

Show V_2 is convex. For any $u_1, u_2 \in V_2$ and $t \in [0, 1]$, it holds $u = tu_1 + (1-t)u_2 \in V_2$:

$$\|u\|_{W_2^{2,1}(Q_T)} \leq t\left\|u_1\right\|_{W_2^{2,1}(Q_T)} + (1-t)\left\|u_2\right\|_{W_2^{2,1}(Q_T)} \leq K. \tag{11}$$

For $u(t, x) \in V_2$, denote $v(t, x) \in \dot{W}_2^{2,1}(Q_T)$ the solution of the Dirichlet problem:

$$Lv + g(t, x, u) = f(t, x), \quad (t, x) \in Q_T, \tag{12}$$

$$v|_{\Gamma(Q_T)} = 0. \tag{13}$$

For fixed $u(t, x) \in V_2$ and $f \in L_2(Q_T)$, problems (12) and (13) are uniquely solvable in the space $\dot{W}_2^{2,1}(Q_T)$; because of the assumptions on domain and q, we get the Dirichlet problem for equation (1) (for its solvability, we refer [1, 2, 9, 10]):

$$Lv = F(t, x), \quad (t, x) \in Q_T, \; u|_{\Gamma(Q_T)} = 0, \tag{14}$$

where $F = f(t, x) - g(t, x,) \in L_2(Q_T)$.

We have

$$\|F\|_{L_2(Q_T)} \le \|f\|_{L_2(Q_T)} + \|g\|_{L_2(Q_T)} \le \|f\|_{L_2(Q_T)} + b_0\left\||u|^q\right\|_{L_2(Q_T)}. \tag{15}$$

By using the chain of imbeddings, $W_2^{2,1}(Q_T)W_2^1(Q_T)L_{2q}(Q_T)$ and $u \in \dot{W}_2^{2,1}(Q_T)$, the norm $\||u|^q\|_{L_2(Q_T)}$ is finite.

Insert an operator A: $u \longrightarrow v$ acting on $L^{2q}(Q_T)$, where v is a solution of problems (12) and (13):

$$Au = v. \tag{16}$$

Show that operator A is completely continuous in $L^{2q}(Q_T)$. Let $\{u_m\}$ be a convergence sequence in $L_{2q}(Q_T)$ with $u_m \longrightarrow u_0$. Show that its image is convergent in $L_{2q}(Q_T)$ with $v_m \longrightarrow v_0$, where $v_0 = Au_0, v_m = Au_m$.

Then,

$$\begin{aligned} Lv_m &= -g(t,x,u_m) + f, \\ Lv_0 &= -g(t,x,u_0) + f. \end{aligned} \tag{17}$$

We have

$$L(v_m - v_0) = -(g(t,x,u_m) - g(t,x,u_0)). \tag{18}$$

Set $g_m = g(t,x,u_m)$, $g = g(t,x,u)$, and show that

$$\left\|g_m - g\right\|_{L_2(Q_T)} \longrightarrow 0 \text{ for } m \longrightarrow \infty. \tag{19}$$

For that, from $u_m \longrightarrow u_0$ in $L_{2q}(Q_T)$ follows the convergnce in measure in Q_T. This and the Caratheodory condition imply that the convergence in measure $(g_m - g_0)^2 \longrightarrow 0$. To prove (19), it remains to show the equicontinuity of g_m^2, which follows from equicontinuity of $|u_m|^{2q}$. The convergence $u_m \longrightarrow u_0$ in $L_{2q}(Q_T)$ implies equicontinuity of $|u_m|^{2q}$.

Applying Vitali's theorem, we get

$$\left\|g_m - g\right\|_{L_2(Q_T)} \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{20}$$

To show $v_m \longrightarrow v_0$ in $L_{2q}(Q_T)$, we use the estimate from Lemma 1 for sufficiently small T_0 with $T \le T_0$:

$$\left\|v_m - v_0\right\|_{W_2^{2,1}(Q_T)} \le C\left\|L(v_m - v_0)\right\|_{L_2(Q_T)} = C\left\|g_m - g\right\|_{L_2(Q_T)} \longrightarrow 0. \tag{21}$$

By virtue of $\dot{W}_2^{2,1}(Q_T) \hookrightarrow L_{2q}(Q_T)$, it follows that

$$\left\|v_n - v_0\right\|_{L_{r,2}(\Omega)} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{22}$$

The complete continuity of operator A in $L_{2q}(Q_T)$ has been shown.

Now, we have to show $u \in V_2$ implies $v = Au \in V_2$. For this, applying Lemma 1, it follows that

$$\|v\|_{W_2^{2,1}(Q_T)} \le C\|F\|_{L_2(Q_T)} \le C(\delta,\gamma,n)\left[\|g\|_{L_2(Q_T)} + \|f\|_{L_2(Q_T)}\right]. \tag{23}$$

Using Holder's inequality and the imbedding chain

$$W_2^{2,1}(Q_T) \hookrightarrow W_2^1(Q_T) \hookrightarrow L_{2q}(Q_T), \tag{24}$$

it follows that

$$\begin{aligned} \|g\|_{L_2(Q_T)} &\le \left(\int_{Q_T} b_0^2|u|^{2q}\mathrm{d}x\mathrm{d}t\right)^{1/2} = b_0\|u\|_{L_{2q}(Q_T)}^q \\ &\le Cb_0\|u\|_{2(n+1)/(n-1)}^q\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))} \\ &\le Cb_0\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))}\|u\|_{W_2^1(Q_T)}^q \\ &\le C_2b_0\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))}\|u\|_{W_2^{2,1}(Q_T)}^q. \end{aligned} \tag{25}$$

Using Lemma 1, this is exceeded:

$$C_1b_0\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))}\|\mathscr{L}u\|_{L_2(Q_T)}^q. \tag{26}$$

Using estimate (26) in (23), we get

$$\begin{aligned} &\|v\|_{W_2^{2,1}(Q_T)}\left[C_1b_0\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))}\|\mathscr{L}u\|_{L_2(Q_T)}^q \right.\\ &\left.+\|f\|_{L_2(\Omega)}\right] \le C_3\left[K^qb_0\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))}\|f\|_{L_2(\Omega)}\right]. \end{aligned} \tag{27}$$

Let K be such that

$$C_{2.5}\left[K^qb_0\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))} + \|f\|_{L_2(\Omega)}\right] \le K. \tag{28}$$

For such number K to exist, condition (10) is sufficient. To prove it, set the notation

$$\begin{aligned} a &= b_0\left(\mathrm{mes}_{n+1}Q_T\right)^{(1/2)-(q(n-1)/2(n+1))}, \\ b &= \|f\|_{L_2(\Omega)}. \end{aligned} \tag{29}$$

Inequality (28) takes the form

$$\begin{aligned} aK^q + b &\le K, \\ aK^q - K + b &\le 0, \\ K &> 0. \end{aligned} \tag{30}$$

The function $f(K) = aK^q - K$, $K \ge 0$, takes its minimal in $K_0 = (1/qa)^{1/(q-1)}$. Indeed, $df/dK = aqK^{q-1} - 1$; then, for $K_0^{q-1} = (1/qa)$, $(df/dK)(K_0) = 0$; $(d^2f/dK^2)(K_0) > 0$. Therefore, for $b \le f(K_0)$, inequality (30) is solvable with respect to K. To finish the proof, it remains to set sufficiently small T_0 so that condition (10) is satisfied. It is possible since $\mathrm{mes}_{n+1}Q_T = T\mathrm{mes}_n\Omega$, the power on $\mathrm{mes}_{n+1}Q_T$, is positive, i.e., $(1/2) - (q(n-1)/2(n+1)) > 0$.

This completes the proof of Theorem 1. □

3. Conclusion

In this paper, the strong solvability problem for a class of second-order semilinear parabolic equations is studied. For the strong solvability of the first boundary value problem for a class of parabolic equations having a nonlinear term, a sufficient condition is found for the power growth condition. In the proof, the Schauder fixed point theorem in the Banach space is used. Also, some a priori estimates are shown in order to realize the legitimate.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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