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FIRST EIGENVALUE AND MAXIMUM PRINCIPLE FOR FULLY NONLINEAR SINGULAR OPERATORS

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1. Introduction

In this paper we introduce the notion of *first eigenvalue* for fully nonlinear operators which are nonvariational but homogeneous. It is unnecessary to emphasize the importance of knowing the spectrum of a linear operator. When the operator is a uniformly elliptic operator of second order $Lu = tr(A(x)D^2u)$ associated with a Dirichlet problem in a bounded domain Ω the spectrum is a point spectrum bounded from below and the first eigenvalue $\bar\lambda$ is paramount. It is well known that $\bar\lambda$ is positive and it satisfies:

- There exists a positive function ϕ satisfying
$$\begin{cases} L\phi + \bar\lambda\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$
- For any $\lambda < \bar\lambda$ and for any $f \in L^N(\Omega)$ there exists a unique u such that
$$\begin{cases} Lu + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

See e.g. [13] for the proof of these results under suitable conditions on $A(x)$ and Ω. Berestycki, Nirenberg and Varadhan, in [3], have characterized the first eigenvalue of $-L$ in Ω by the fact that it is the supremum of the values λ such that $L+\lambda$ satisfies the maximum principle in Ω. Let us recall that $L+\lambda$ *satisfies the maximum principle in* Ω if any solution of $Lu+\lambda u \geq 0$ in Ω which is nonpositive on the boundary of Ω is nonpositive in Ω.

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Of course the notion of eigenvalue is really connected with the linearity of the operator; on the other hand it is possible to extend this notion to nonlinear operators F provided they still are homogeneous.

The idea is to find real values λ and some homogeneous operator $\tilde{I}$ such that $F + \lambda\tilde{I}$ has a nontrivial kernel.

In particular, when $Fu := \Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ this has been successfully done; see [2, 19]. More precisely the value

$$\bar{\lambda} = \inf \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

has been called the first eigenvalue of $-\Delta_p$ for the Dirichlet problem in Ω.

Indeed $\bar{\lambda}$ is such that

- There exists a positive function ϕ satisfying

$$\begin{cases} \Delta_p \phi + \bar{\lambda}\phi^{p-1} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

- For any $\lambda < \bar{\lambda}$ and for any $f \in L^{p'}(\Omega)$ there exists a unique u such that

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that if we call $\tilde{I}u = |u|^{p-2}u$, $\bar{\lambda}$ coincides with the extended notion of eigenvalue. It is important to remark that here the definition and the properties of $\bar{\lambda}$ are related to the variational nature of the p-Laplacian and its homogeneity.

Since we want to define the notion of *first eigenvalue* for singular fully nonlinear elliptic operators which are nonvariational but homogeneous we shall follow the idea of Berestycki, Nirenberg and Varadhan [3] and this so-called eigenvalue will be defined through the *maximum principle* as characterized above and it will coincide with the "eigenvalue" of $-\Delta_p$ when $F = \Delta_p$.

In the next section we shall give the precise conditions on the operator $Fu := F(\nabla u, D^2 u)$ for which we define the notion of eigenvalue, but for the sake of simplicity the results obtained in this paper will be stated in this introduction for

$$F(\nabla u, D^2u) = |\nabla u|^\alpha tr(D^2u) \quad \text{ or } \quad F(\nabla u, D^2u) = |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u),$$

where $\alpha > -1$ and $\mathcal{M}_{a,A}$ is one of the Pucci operators, i.e., either

$$\mathcal{M}_{a,A}(D^2u) = \mathcal{M}^+_{a,A}(D^2u) = \sup_{aId \le A(x) \le AId} tr(A(x)D^2u)$$

or

$$\mathcal{M}_{a,A}(D^2u) = \mathcal{M}^-_{a,A}(D^2u) = \inf_{aId \le A(x) \le AId} tr(A(x)D^2u).$$

In other words we are considering fully nonlinear elliptic operators which may be singular or degenerate as the p-Laplacian but are not variational. Of course the convenient notion of solution in this context will be that of viscosity solution. The precise definition of viscosity solution used will be given in the next section (see [8] for similar definitions).

Before going into details, let us mention that several interesting papers treat viscosity solutions for equations involving the p-Laplacian. In [16, 17] Juutinen, Lindqvist and Manfredi opened the way to this topic. We would like to emphasize that in those papers the variational structure of the p-Laplacian is used. This cannot be the case here since the operators that we consider are not in divergence form.

The main aim of this paper is to prove that $\bar{\lambda}$ defined by

$$\bar{\lambda} = \sup\{\lambda \in \mathbb{R} : \exists\, \phi > 0 \ \text{ in } \bar{\Omega},\\ F(\nabla\phi, D^2\phi) + \lambda\phi^{\alpha+1} \le 0 \text{ in the viscosity sense } \}.$$

is the first eigenvalue of $-F$ in Ω, with the meaning proposed above.

The first key ingredient is the following:

Theorem 1.1. *Suppose that Ω is a bounded open domain of $\mathbb{R}^N$. Suppose that for $\lambda \in \mathbb{R}$ there exists a function $v > 0$ such that*

$$F(\nabla v, D^2v) + \lambda v^{\alpha+1} \le 0 \quad \text{in } \Omega.$$

Suppose that $\tau < \lambda$, then every viscosity sub solution of

$$\begin{cases} F(\nabla\sigma, D^2\sigma) + \tau|\sigma|^\alpha\sigma \ge 0 & \text{in } \Omega \\ \sigma \le 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $\sigma \le 0$ in Ω.

In other words, this theorem states that if we denote by $I_\alpha(u) := |u|^\alpha u$, $\bar{\lambda}$ is the supremum of the values λ such that $F + \lambda I_\alpha$ satisfies the maximum principle in Ω. Clearly the set

$$E = \{\lambda \in \mathbb{R} : \exists\, \phi > 0 \ \text{ in } \bar{\Omega},\ F(\nabla\phi, D^2\phi) + \lambda\phi^{\alpha+1} \le 0 \text{ in the viscosity sense}\}$$

is an interval. Furthermore it is bounded from above since the following proposition gives a bound on E.

Proposition 1.2. *Suppose that R is the radius of the largest ball contained in the bounded set Ω. Then, there exists some constant C, which depends only on N and α, such that $\bar{\lambda} \le \frac{C}{R^{\alpha+2}}$.*

The next two theorems justify the name of "eigenvalue" given to $\bar\lambda$

Theorem 1.3. *There exists ϕ a continuous positive viscosity solution of*

$$\begin{cases} F(\nabla\phi, D^2\phi) + \bar\lambda\phi^{\alpha+1} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $F + \bar\lambda I_\alpha$ has a nontrivial kernel. Furthermore, we have:

Theorem 1.4. *Suppose that $\lambda < \bar\lambda$. If $f \le 0$ in Ω and bounded, then there exists u, a non-negative viscosity solution of*

$$\begin{cases} F(\nabla u, D^2u) + \lambda u^{\alpha+1} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

If moreover f is continuous and < 0 in Ω, the solution is unique.

Let us notice that in order to prove Theorems 1.3 and 1.4 we need to obtain some estimates which are interesting in their own right:

Theorem 1.5. *Suppose that f is a bounded function in $\bar\Omega$. Then if u is a bounded nonnegative viscosity solution of $F(\nabla u, D^2u) = f$ in Ω, it is Hölder continuous:*

$$|u(x) - u(y)| \le M|x-y|^\gamma.$$

The proof of this result uses some of the features used by Ishii and Lions in [15]. We also obtain local Lipschitz regularity under some additional assumptions on F that will be made explicit in section 4.

Finally, let us mention that it is possible to give a sort of "variational" characterization of $\bar\lambda$ that somehow recalls the definition of first eigenvalue for the p-Laplacian:

$$\bar\lambda = \inf_{\sigma>0} \sup_{x\in\Omega} \inf_{(p,X)\in J^{2,-}(\sigma(x)), p\neq 0} \Big(-\frac{F(p,X)}{\sigma(x)^{\alpha+1}} \Big),$$

where $J^{2,-}$ is the semi-jet as defined in the next section. This characterization will be justified by Proposition 3.2.

After the completion of this work we learned that the eigenvalue problem for Pucci's operators has been treated by Busca, Esteban and Quaas in [6]; this corresponds to the case $\alpha = 0$ here. In that paper the authors denote by μ_1^+ the eigenvalue here denoted $\bar\lambda$, but they also define

$$\mu_1^- = \sup\{\mu : \exists\psi < 0 \text{ in } \Omega : \mathcal{M}^+_{a,A}(\psi) + \mu\psi \ge 0\}.$$

This could be done in our case as well.

Indeed let

$$\underline{\lambda} = \sup\{\mu : \exists\ \phi < 0\ , F(\nabla\phi, D^2\phi) + \mu|\phi|^\alpha\phi \geq 0 \text{ in the viscosity sense}\}.$$

If $G(p,X) = -F(p,-X)$ then $\underline{\lambda} = \bar{\lambda}(G)$. Furthermore if F satisfies (H2) then so does G. Hence it is possible to prove for $\underline{\lambda}$ the same results as for $\bar{\lambda}$; i.e.:

(1) For any $\lambda < \underline{\lambda}$ any viscosity super solution of

$$F(\nabla u, D^2u) + \lambda|u|^\alpha u \leq 0 \text{ in } \Omega$$

with $u \geq 0$ on the boundary, is nonnegative in Ω.

(2) There exists ψ, a continuous negative viscosity solution of

$$\begin{cases} F(\nabla\psi, D^2\psi) + \underline{\lambda}|\psi|^\alpha\psi = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

(3) Suppose that $\lambda < \underline{\lambda}$. If $f \geq 0$ in Ω and bounded, then there exists u, a non-positive viscosity solution of

$$\begin{cases} F(\nabla u, D^2u) + \lambda|u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In general $F(p,-X) \neq -F(p,X)$ hence $\underline{\lambda} \neq \bar{\lambda}$. It is interesting to remark that there is a sort of "Fredholm alternative," even though very weak :

Remark 1.6. Suppose that $\underline{\lambda} \neq \bar{\lambda}$. Without loss of generality we can suppose $\underline{\lambda} < \bar{\lambda}$, then for $\mu \in (\underline{\lambda}, \bar{\lambda})$ if $f \geq 0$ with $f \not\equiv 0$ then there are no solutions

$$\begin{cases} F(\nabla u, D^2u) + \mu|u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed if such u exists, then $u \leq 0$ by the maximum principle; then by the strict maximum principle $u < 0$ and using the definition of $\underline{\lambda}$ we would get $\mu \leq \underline{\lambda}$, a contradiction.

Let us mention some open problems related to the results of this paper

• **Simplicity of the eigenfunction.** The eigenfunction ϕ is simple both for linear second-order elliptic operators and for the p-Laplacian. It is a natural question to ask if this is true also in the case treated here. More precisely, suppose that $\psi > 0$ is another eigenfunction. Does this imply that there exists $t \in \mathbb{R}^+$ such that $\psi = t\phi$?

• **Fredholm alternative.** Suppose that f is a continuous function which doesn't change sign in Ω and is not identically zero, then is it possible to prove that there exist no solutions for

$$\begin{cases} F(\nabla u, D^2u) + \bar{\lambda}|u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega? \end{cases} \tag{1.3}$$

Observe that when $f < 0$, just using the definition of $\bar{\lambda}$, there is no solution of (1.3).

On the other hand if $\bar{\lambda} < \underline{\lambda}$ then for $f \geq 0$ there exists a solution of (1.3), (see 3. above).

• $\bar{\lambda}$ **is isolated.** Suppose that $\lambda > \bar{\lambda}$ but sufficiently close. Does problem (1.1) have a solution?

In the next section we state the precise hypothesis on F and we give the notion of viscosity solution adapted to the operators considered here. In the third section we prove the maximum principle (Theorem 1.1) and a comparison principle. In the fourth section we give global Hölder and local Lipschitz estimates for the solutions. Finally the last section provides different existence results, including that of a first eigenfunction; i.e., we prove Theorem 1.3 and 1.4. Some properties of the distance function, required for the existence results, are proved in the appendix.

2. Preliminaries

Let F be a continuous function defined on $\mathbb{R}^N \setminus \{0\} \times S$, where S is the space of symmetric matrices in $\mathbb{R}^N$. In the whole paper, for some $\alpha > -1$, F satisfies:

(H1) $F(tp, \mu X) = |t|^\alpha \mu F(p, X)$, for all $t \in \mathbb{R}$, $\mu \geq 0$ and $F(p, X) \leq F(p, Y)$ for any $p \neq 0$, and $X \leq Y$.

We shall also suppose in most results that the operator satisfies also the following hypothesis

(H2) $a|p|^\alpha \mathrm{tr} N \leq F(p, M+N) - F(p, M) \leq A|p|^\alpha \mathrm{tr} N$ for $0 < a \leq A$, $\alpha > -1$ and $N \geq 0$.

When condition (H2) is required we shall state it explicitly.

Let us recall that (H2) implies that

$$|p|^\alpha \mathcal{M}^+_{a,A}(M) \geq F(p, M) \geq |p|^\alpha \mathcal{M}^-_{a,A}(M),$$

where, if e_i are the eigenvalues of M, $\mathcal{M}^+_{a,A}(M) = a\sum_{e_i<0} e_i + A\sum_{e_i>0} e_i$ and $\mathcal{M}^-_{a,A}(M) = A\sum_{e_i<0} e_i + a\sum_{e_i>0} e_i$ are the Pucci operators (see e.g. [7]). Let us also remark that the monotonicity of F is implied by (H2).

In [4] many examples of operators satisfying (H2) are given.

We shall now give the definition of viscosity sub or super solutions suited to operators that may be singular.

It is well known that in dealing with viscosity respectively sub and super solutions one works with

$$u^\star(x) = \limsup_{y,|y-x|\le r,\ r\to 0} u(y)$$

and

$$u_\star(x) = \liminf_{y,|y-x|\le r\ r\to 0} u(y).$$

It is easy to see that $u_\star \le u \le u^\star$ and $u^\star$ is uppersemicontinuous (USC) while $u_\star$ is lowersemicontinuous (LSC). See e.g. [9, 14].

Definition 2.1. *Let Ω be an open set in $\mathbb{R}^N$; then v bounded on $\overline{\Omega}$ is a viscosity super solution of $F(\nabla v, D^2 v) = g(x,v)$ in Ω if for all $x_0 \in \Omega$,*

-Either there exists an open ball $B(x_0,\delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $g(x,c) \ge 0$;

-Or for all $\varphi \in \mathcal{C}^2(\Omega)$, such that $v_\star - \varphi$ has a local minimum on x_0 and $\nabla\varphi(x_0) \neq 0$, one has

$$F(\nabla\varphi(x_0), D^2\varphi(x_0)) \le g(x_0, v_\star(x_0)). \tag{2.1}$$

Similarly u is a viscosity sub solution if for all $x_0 \in \Omega$,

-Either there exists a ball $B(x_0,\delta)$, $\delta > 0$ on which $u = cte = c$ and $g(x,c) \le 0$,

-Or for all $\varphi \in \mathcal{C}^2(\Omega)$, such that $u^\star - \varphi$ has a local maximum on x_0 and $\nabla\varphi(x_0) \neq 0$, one has

$$F(\nabla\varphi(x_0), D^2\varphi(x_0)) \ge g(x_0, u^\star(x_0)). \tag{2.2}$$

u is a viscosity solution if it is both a sub and a super viscosity solution.

See e.g. [8] for similar definition of viscosity solutions for equations with singular operators.

For convenience we recall the definition of semi-jets given e.g. in [9]

$$\begin{aligned} J^{2,+}u(\bar{x}) &= \{(p,X) \in \mathbb{R}^N \times S : u(x) \le u(\bar{x}) + \langle p, x-\bar{x}\rangle \\ &+ \frac{1}{2}\langle X(x-\bar{x}), (x-\bar{x})\rangle + o(|x-\bar{x}|^2)\} \end{aligned}$$

and

$$\begin{aligned} J^{2,-}u(\bar{x}) &= \{(p,X) \in \mathbb{R}^N \times S : u(x) \ge u(\bar{x}) + \langle p, x-\bar{x}\rangle \\ &+ \frac{1}{2}\langle X(x-\bar{x}), (x-\bar{x})\rangle + o(|x-\bar{x}|^2\}. \end{aligned}$$

In the definition of viscosity solutions the test functions can be replaced by the elements of the semi-jets in the sense that in the definition above one can restrict the function ϕ to $\phi(x) = u(\bar{x}) + \langle p, x - \bar{x}\rangle + +\frac{1}{2}\langle X(x-\bar{x}), (x-\bar{x})\rangle$ with $(p, X) \in J^{2,-}u(\bar{x})$ when u is a supersolution and $(p, X) \in J^{2,+}u(\bar{x})$ when u is a subsolution.

3. Maximum principle and comparison results

As we pointed out in the introduction, we want to generalize the concept of eigenvalue for the Dirichlet problem in a bounded domain Ω associated to the operator $F(u) = F(\nabla u, D^2 u)$ satisfying (H2) . It will be defined following the main ideas introduced in [3] for uniformly elliptic operators.

Throughout the paper we shall denote

$$E = \{\lambda \in \mathbb{R} : \exists\ \phi,\ \phi_\star > 0 \text{ in } \bar{\Omega},$$
$$F(\nabla\phi, D^2\phi) + \lambda\phi^{\alpha+1} \leq 0 \text{ in the viscosity sense }\}$$

and $\bar{\lambda} = \sup E$.

Remark 3.1. Of course E is nonempty, since any $\lambda \leq 0$ obviously belongs to E. In fact E is an interval since if $\lambda \in E$, every $\lambda' < \lambda$ is also in E.

Moreover, $\bar{\lambda} > 0$ if and only if there exists $X \geq 0$ and $p \neq 0$ such that $F(p, -X) < 0$, which is the case when F satisfies $(H2)$.

In fact one has a sharper estimate when F satisfies $(H2)$. Suppose that R is such that $\Omega \subset [-R, R] \times \mathbb{R}^{N-1}$, then there exists a constant $C > 0$ such that $\bar{\lambda} > \frac{C}{R^{\alpha+2}}$. Indeed, let us define $\varphi(x) = 3Rx_1 - x_1^2 + 5R^2$, $7R^2 \geq \varphi \geq 0$ in Ω. On the other hand

$$F(\nabla\varphi, D^2\varphi) \leq -2a|3R - 2x_1|^\alpha \leq -2aR^\alpha \sup(1, 4^\alpha).$$

Taking $\lambda = 2aR^\alpha \frac{\sup(1,4^\alpha)}{(7R^2)^{1+\alpha}} = \frac{C}{R^{2+\alpha}}$, one obtains that $\bar{\lambda} > \frac{C}{R^{2+\alpha}}$.

The next proposition proves that $\bar{\lambda} \neq +\infty$.

Proposition 3.2. *Suppose that R is the radius of the largest ball contained in Ω and that F satisfies* (H2). *Then, there exists some constant C which depends only on N and α, a and A such that $\bar{\lambda} \leq \frac{C}{R^{\alpha+2}}$.*

Proposition 3.2 is a consequence of the maximum principle stated in the following Theorem 3.3 and Lemma 3.5 below.

Theorem 3.3. *Suppose that Ω is a bounded open domain of $\mathbb{R}^N$. Suppose that $\tau < \bar{\lambda}$, then every viscosity solution of*

$$\begin{cases} F(\nabla\sigma, D^2\sigma) + \tau|\sigma|^\alpha\sigma \geq 0 & \text{in } \Omega \\ \sigma \leq 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $\sigma \leq 0$ in Ω.

Remark. In this theorem we do not require F to satisfy (H2), but only (H1).

An immediate consequence of Theorem 3.3 is

Corollary 3.4. *If $\lambda < \bar{\lambda}$ and $F(p, -X) = -F(p, X)$, then every solution of*

$$F(\nabla\phi, D^2\phi) + \lambda|\phi|^\alpha\phi = 0,$$

which is zero on the boundary, is identically zero.

Proof. Both ϕ and $-\phi$ are solutions of the equation and this implies that they are both nonpositive. This concludes the proof. □

Lemma 3.5. *Suppose that $\Omega = B(0, R)$, and let $q = \frac{\alpha+2}{\alpha+1}$ and*

$$\sigma(x) = \frac{1}{2q}(|x|^q - R^q)^2.$$

Let F satisfy (H2). *Then there exists some constant C which depends only on N, α and A, a, such that*

$$\sup_{x\in B(0,R)} \frac{-F(\nabla\sigma, D^2\sigma)}{\sigma^{\alpha+1}} \leq \frac{C}{R^{\alpha+2}}.$$

Proof of Proposition 3.2. Suppose that Theorem 3.3 and Lemma 3.5 hold. Without loss of generality we can suppose that $B(0, R) \subset \Omega$. We shall prove that

$$\bar{\lambda} \leq \sup_{x\in B(0,R)} \frac{-F(\nabla\sigma, D^2\sigma)}{\sigma^{\alpha+1}} = \tau;$$

by Lemma 3.5 this will end the proof.

Suppose by contradiction that $\tau < \bar{\lambda}$ and let $u = \sigma$ for $|x| \leq R$ and 0 elsewhere. Then one would have

$$F(\nabla u, D^2u) + \tau|u|^\alpha u \geq 0 \text{ in } \Omega.$$

Indeed, for $|x| \leq R$, u is a solution by the definition of τ; for $|x| > R$ the definition of viscosity solution gives the result immediately, and for $|x| = R$ all the test functions have zero gradient and so they don't need to be tested. Now since $u = 0$ on $\partial\Omega$, this would imply by Theorem 3.3 that $u \leq 0$ in Ω, a contradiction with the definition of σ which is nonnegative inside the ball. This ends the proof of Proposition 3.2.

□

Proof of Lemma 3.5. Let us first remark that for $x \neq 0$, σ is C^2 and hence $J^{2,+}\sigma(x) = J^{2,-}\sigma(x) = \{(\nabla\sigma(x), D^2\sigma(x))\}$, while $\nabla\sigma(0) = 0$ which implies that all test functions at the origin have their gradient equal to zero and hence need not be considered.

For simplicity, for $r = |x|$, let $g(r) = \sigma(x)$. Hence, $g'(r) = r^{2q-1} - r^{q-1}R^q$ and $g''(r) = (2q-1)r^{2q-2} - (q-1)r^{q-2}R^q$. Clearly, $g' \le 0$, while $g'' \le 0$ for $r \le (\frac{q-1}{2q-1})^{\frac{1}{q}} R := R_1$ and positive elsewhere.

By condition (H2) and using the fact that for radial functions the eigenvalues of the Hessian are $\frac{g'}{r}$ with multiplicity N-1 and g'' the following holds: For $r \le R_1$,

$$\begin{aligned} -A|g'|^\alpha \Big[g''(r) + (\frac{N-1}{r}) g'(r) \Big] &\ge -F(\nabla\sigma, D^2\sigma) \\ &\ge -a|g'|^\alpha \Big[g''(r) + (\frac{N-1}{r}) g'(r) \Big], \end{aligned}$$

while for $r \ge R_1$

$$\begin{aligned} -|g'|^\alpha \Big[Ag''(r) + a(\frac{N-1}{r}) g'(r) \Big] &\le -F(\nabla\sigma, D^2\sigma) \\ &\le -|g'|^\alpha \Big[ag''(r) + A(\frac{N-1}{r}) g'(r) \Big]. \end{aligned}$$

More precisely, for $r \le R_1$

$$|g'|^\alpha r^{q-2} a(-B_1 r^q + B_2 R^q) \le -F(\nabla\sigma, D^2\sigma) \le |g'|^\alpha r^{q-2} A(-B_1 r^q + B_2 R^q) \tag{3.1}$$

with $B_1 = (N + 2q - 2)$ and $B_2 = (N + q - 2)$; while for $r \ge R_1$

$$-F(\nabla\sigma, D^2\sigma) \le |g'|^\alpha r^{q-2}(-B_3 r^q + B_4 R^q) \tag{3.2}$$

with $B_3 = a(2q-1) + A(N-1)$ and $B_4 = a(q-1) + A(N-1)$.

Let us observe that $\frac{q-1}{2q-1} \le \frac{B_4}{B_3} < 1$ and then for $r \ge R_3 := (\frac{B_4}{B_3})^{\frac{1}{q}} R$, the quantity on the right-hand side of (3.2) is negative. But, by (3.1), $-F$ is positive for $r \le R_2 := (\frac{B_2}{B_1})^{\frac{1}{q}} R \le R_3$.

Hence the supremum is achieved for $R_1 \le r \le R_3$, from which one obtains that there exists a universal C such that

$$\frac{-F(\nabla\sigma, D^2\sigma)}{\sigma^{\alpha+1}} \le C R^q R^{q(\alpha+1)-\alpha-2} R^{-q(\alpha+2)} = C R^{-\alpha-2}.$$

This ends the proof of Lemma 3.5. □

Proof of Theorem 3.3. We assume that $\tau < \bar\lambda$. Then taking λ such that $\tau < \lambda < \bar\lambda$, there exists v, a viscosity sub solution of

$$F(\nabla v, D^2 v) + \lambda v^{1+\alpha} \le 0 \quad \text{in } \Omega,$$

with $v_\star > 0$ in Ω. Suppose that σ is a viscosity solution of

$$F(\nabla\sigma, D^2\sigma) + \tau|\sigma|^\alpha\sigma \geq 0 \text{ in } \Omega,$$

and $\sigma \leq 0$ on $\partial\Omega$. We need to prove that $\sigma \leq 0$ in Ω. It is sufficient to prove that $\sigma^\star \leq 0$. Using the definition of viscosity solution, one can assume without loss of generality that $\sigma \in USC$ and $v \in LSC$ and hence drop the stars.

Let us suppose by contradiction that $\frac{\sigma(x)}{v(x)}$ has a positive supremum inside Ω. For some $q > 2$, let us consider the function

$$\psi_j(x,y) = \frac{\sigma(x)}{v(y)} - \frac{j}{qv(y)}|x-y|^q$$

which is uppersemicontinuous. Then ψ_j also has a positive supremum, achieved on some pair of points $(x_j, y_j) \in \Omega^2$. One easily has that $(x_j, y_j) \to (\bar{x}, \bar{x})$, $\bar{x} \in \Omega$ which is a supremum for $\frac{\sigma}{v}$. One can also prove that $j|x_j - y_j|^q \to 0$, and that $\bar{x}$ is a continuity point for σ. To prove this last point let us note that

$$\frac{\sigma(x_j) - \frac{j}{q}|x_j - y_j|^q}{v(y_j)} \geq \frac{\sigma(\bar{x})}{v(\bar{x})}$$

and using the lowersemicontinuity of v on $\bar{x}$, together with $\lim \frac{j}{q}|x_j - y_j|^q = 0$ one gets that

$$\liminf \sigma(x_j) \geq \sigma(\bar{x}).$$

Assume for the moment that $x_j \neq y_j$ for j large enough. Take j large enough in order that

$$\sigma(x_j)^{1+\alpha} \geq \frac{3\sigma(\bar{x})^{1+\alpha}}{4}$$

and

$$\frac{j}{q}|x_j - y_j|^q \leq \frac{\sigma(\bar{x})^{1+\alpha}(\lambda - \tau)}{4\lambda}.$$

Using $\psi_j(x,y) \leq \psi_j(x_j, y_j)$, one gets that

$$\sigma(x)v(y_j) - v(y)\Big(\sigma(x_j) - \frac{j}{q}|x_j - y_j|^q\Big) \leq v(y_j)\frac{j}{q}|x-y|^q. \tag{3.3}$$

Then defining

$$\beta_j = \sigma(x_j) - \frac{j}{q}|x_j - y_j|^q$$

(3.3) becomes, after some simple calculation:

$$\big(\sigma(x + x_j) - \sigma(x_j) - j|x_j - y_j|^{q-2}(x_j - y_j.x)\big)\, v(y_j)$$

$$
\begin{aligned}
&-\Big(v(y+y_j)-v(y_j)-j|x_j-y_j|^{q-2}(x_j-y_j.y)\frac{v(y_j)}{\beta_j}\Big)\beta_j \qquad (3.4)\\
&\le v(y_j)\Big(\frac{j}{q}|x_j+x-y_j-y|^q-\frac{j}{q}|x_j-y_j|^q-j|x_j-y_j|^{q-2}(x_j-y_j,x-y)\Big).
\end{aligned}
$$

We define the functions

$$U(x)=\big(\sigma(x+x_j)-\sigma(x_j)-j|x_j-y_j|^{q-2}(x_j-y_j.x)\big)\,v(y_j)$$

and

$$V(y)=-\Big(v(y+y_j)-v(y_j)-j|x_j-y_j|^{q-2}(x_j-y_j.y)\frac{v(y_j)}{\beta_j}\Big)\beta_j.$$

With these notations (3.4) can be written as

$$U(x)+V(y)\le(x,y)A(x,y),$$

where

$$A=jv(y_j)\begin{pmatrix} D_j & -D_j\\ -D_j & D_j\end{pmatrix}$$

and

$$D_j=2^{q-3}q|x_j-y_j|^{q-2}\Big(I+\frac{(q-2)}{|x_j-y_j|^2}(x_j-y_j)\otimes(x_j-y_j)\Big).$$

Noting that

$$A\le 2jv(y_j)|D_j|\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}$$

and using Lemma 2.1 in [4], one gets that

$$\Big(j|x_j-y_j|^{q-2}(x_j-y_j),\frac{X_j}{v(y_j)}\Big)\in J^{2,+}\sigma(x_j)$$

and

$$\Big(j|x_j-y_j|^{q-2}(x_j-y_j)\frac{v(y_j)}{\beta_j},\frac{-Y_j}{\beta_j}\Big)\in J^{2,-}v(y_j)$$

with, for some $\varepsilon>0$,

$$\begin{pmatrix} X_j & 0\\ 0 & Y_j\end{pmatrix}\le A+\varepsilon A^2.$$

In particular, $X_j+Y_j\le 0$. We can conclude using the fact that v and σ are respectively a super and a sub solution and the properties of F.

$$-\tau\sigma(x_j)^{1+\alpha}\le F(j|x_j-y_j|^{q-2}(x_j-y_j),\frac{X_j}{v(y_j)})$$

$$\le F(j|x_j-y_j|^{q-2}(x_j-y_j),\frac{-Y_j}{v(y_j)})$$
$$\le \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}}F(j|x_j-y_j|^{q-2}(x_j-y_j)\frac{v(y_j)}{\beta_j},\frac{-Y_j}{\beta_j})$$
$$\le -\lambda\beta_j^{1+\alpha} = -\lambda[\sigma(x_j)-\frac{j}{q}|x_j-y_j|^q]^{1+\alpha}.$$

This gives a contradiction; indeed, by passing to the limit, the previous inequality yields

$$-\tau\sigma^{\alpha+1}(\bar{x}) \le -\lambda\sigma^{\alpha+1}(\bar{x}).$$

It remains to prove that $x_j \ne y_j$ for j large enough. If one assumes that $x_j = y_j$ one has

$$\sigma(x_j) \ge \sigma(x) - \frac{j}{q}|x_j - x|^q$$

and

$$v(x) \ge v(x_j) - \frac{jv(x_j)|x_j-x|^q}{q\sigma(x_j)}.$$

In that case one uses Lemma 2.2 in [4] to get a contradiction. This ends the proof of Theorem 3.3. □

Let us recall that in [4] we give a comparison principle for continuous viscosity solutions. It is not difficult to see that it can be extended to bounded viscosity solutions. We now prove a further extension adapted to our context.

Theorem 3.6. *Suppose that $\lambda < \bar{\lambda}$, $f \le 0$, f is upper semicontinuous and g is lower semicontinuous with $f \le g$ and*
- *either $f < 0$ in Ω,*
- *or $g(\bar{x}) > 0$ on every point $\bar{x}$ such that $f(\bar{x}) = 0$.*

Suppose that there exist v bounded and nonnegative, and σ bounded, respectively satisfying

$$F(\nabla v, D^2 v) + \lambda v^{1+\alpha} \le f, \quad F(\nabla\sigma, D^2\sigma) + \lambda|\sigma|^\alpha\sigma \ge g$$

in the viscosity sense, with $\sigma \le v$ on $\partial\Omega$. Then $\sigma \le v$ in Ω.

As a consequence one has

Corollary 3.7. *Suppose that $\lambda < \bar{\lambda}$; there exists at most one nonnegative viscosity solution of*

$$\begin{cases} F(\nabla v, D^2 v) + \lambda v^{1+\alpha} = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

for $f < 0$ and continuous.

Proof of Theorem 3.6. First, since $F(\nabla v, D^2 v) \le 0$ and $v \ge 0$ in Ω, using the strict maximum principle (see [5]) $v_\star > 0$ in Ω since it is not identically zero. Without loss of generality one can assume that σ and v are respectively USC and LSC.

Suppose by contradiction that $\sigma > v$ somewhere in Ω. Let $\bar{x}$ be a point such that

$$1 < \frac{\sigma(\bar{x})}{v(\bar{x})} = \sup_{x\in\overline{\Omega}} \frac{\sigma(x)}{v(x)}.$$

Clearly, $\bar{x} \in \Omega$ since $\frac{\sigma}{v} \le 1$ on $\partial\Omega$.

Doing exactly the same construction as in the proof of Theorem 3.3 we similarly get:

$$\begin{aligned} g(x_j) - \lambda\sigma(x_j)^{1+\alpha} &\le F(j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)}) \\ &\le \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} F(j|x_j - y_j|^{q-2}(x_j - y_j)\frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}) \\ &\le -\lambda\beta_j^{1+\alpha} + \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} f(y_j). \end{aligned}$$

Passing to the limit we obtain

$$g(\bar{x}) \le \Big(\frac{\sigma(\bar{x})}{v(\bar{x})}\Big)^{\alpha+1} f(\bar{x}).$$

Either $f(\bar{x}) = 0$, and then we have reached a contradiction because in that case by hypothesis $g(\bar{x}) > 0$, or, $f(\bar{x}) < 0$, and then we get

$$0 < f(\bar{x})\Big[1 - \Big(\frac{\sigma(\bar{x})}{v(\bar{x})}\Big)^{\alpha+1}\Big] \le f(\bar{x}) - g(\bar{x}) \le 0.$$

This concludes the proof. $\square$

Proof of Corollary 3.7. Let us consider u and v, two solutions of (3.5). Then $u_\epsilon = \frac{u}{1+\epsilon}$ satisfies

$$\begin{cases} F(u_\epsilon) + \lambda u_\epsilon^{1+\alpha} = \frac{f}{(1+\epsilon)^{1+\alpha}} & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Noting that $\frac{f}{(1+\epsilon)^{1+\alpha}} > f$ and applying the comparison theorem, one gets that $u_\epsilon \le v$. Passing to the limit when ϵ goes to zero, one obtains that $u \le v$, and exchanging u and v, that $u = v$.

4. Hölder and Lipschitz regularity

In all this section we assume that F satisfies (H2) and Ω is a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^N$.

Suppose that u is a viscosity solution of

$$\begin{cases} F(\nabla u, D^2 u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Theorem 4.1. *Suppose that Ω is a $\mathcal{C}^2$ domain. Let f be a bounded function in $\overline{\Omega}$. Let u be a nonnegative viscosity solution of* (4.1). *Then, for any $\gamma \in (0,1)$, there exists $C > 0$, such that*

$$|u(x) - u(y)| \le C|x - y|^\gamma.$$

Before proving Theorem 4.1 we shall state a Lipschitz regularity result which holds if F also satisfies a Hölder continuity hypothesis with respect to $p \ne 0$. Precisely:

(H3) There exists $\lambda \in]1/2, 1]$ and $\nu > 0$ such that for all $|p| = 1$, for all q, $|q| < \frac{1}{2}$, and for all $B \in \mathcal{S}$

$$|F(p + q, B) - F(p, B)| \le \nu |q|^\lambda |B|.$$

The following theorem holds:

Theorem 4.2. *Suppose that F satisfies* $(H1)$, $(H2)$, $(H3)$. *Suppose that f is bounded and let u be a nonnegative viscosity solution of equation* (4.1). *Then u is Lipschitz continuous inside Ω.*

Remark 4.3. Let us note that $(H3)$, together with the homogeneity with respect to p, implies that for all $|p| \ne 0$, q, $|q| < \frac{|p|}{2}$, for all $B \in \mathcal{S}$

$$|F(p + q, B) - F(p, B)| \le \nu |q|^\lambda |p|^{\alpha - \lambda} |B|.$$

It is under this form that we shall use assumption (H3) in order to prove the Lipschitz continuity result. In the rest of the paper we shall only use the Hölder continuity of the solution, hence condition (H3) is required only in Theorem 4.2.

Proof of Theorem 4.1. The proof relies on ideas used to prove Hölder and Lipschitz estimates in [15].

First we will prove that u is Hölder near the boundary using the regularity of the boundary and of the distance function near the boundary.

Let $d(x)$ be the distance from the boundary; i.e., $d(x) = \inf\{|x - y| : y \in \partial\Omega\}$.

Claim: *For any* $0 < \gamma < 1$ *there exist* $\delta > 0$, $M_o > 0$ *such that* $u(x) \leq M_o d(x)^\gamma$ *for* $d(x) \leq \delta$.

In order to prove the claim we need to show that $g(x) = d(x)^\gamma$ is a super solution of (4.1) in $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$. It is well known (see [11, 12, 18]) that d is $\mathcal{C}^2$ on Ω_δ for δ small enough since $\partial\Omega$ is $\mathcal{C}^2$. Furthermore the $\mathcal{C}^2$ norm of d is bounded. Then for δ small enough and $d(x) < \delta$,

$$F(\nabla g, D^2 g) \leq \gamma^{1+\alpha} d^{(\gamma(\alpha+1)-\alpha-2)}(\gamma - 1 + cd(x)|D^2 d(x)|_\infty) \leq -\epsilon < 0$$

for some constant c which depends on a and A and for some constant $\varepsilon > 0$ which depends on γ, N, α and $\partial\Omega$.

We now define M_o such that

$$M_o \delta^\gamma = \sup_{\partial\Omega_\delta \cap \Omega} u \quad \text{and} \quad M_0^{1+\alpha} > \frac{|f|_\infty}{\epsilon}.$$

By the comparison principle (Theorem 3.6) $u^\star \leq M_o d(x, \partial\Omega)^\gamma$ in Ω_δ and the claim is proved.

We now prove Hölder's regularity inside Ω.

We construct a function Φ as follows: Let M_o and γ be as in the Claim, $M = \sup(M_o, \frac{2 \sup u}{\delta^\gamma})$ and $\Phi(x) = M|x|^\gamma$.

We shall consider

$$\Delta_\delta = \{(x, y) \in \Omega^2 : |x - y| < \delta\}.$$

Claim 2: *For any* $(x, y) \in \Delta_\delta$,

$$u^\star(x) - u_\star(y) \leq \Phi(x - y). \tag{4.2}$$

If Claim 2 holds this completes the proof; indeed, taking $x = y$ we would get that $u^\star = u_\star$ and then u is continuous. Therefore, going back to (4.2)

$$u(x) - u(y) \leq \frac{2 \sup u}{\delta^\gamma} |x - y|^\gamma,$$

for $(x, y) \in \Delta_\delta$, which is equivalent to the local Hölder continuity.

Let us check first that (4.2) holds on $\partial\Delta_\delta$. On that set,

- either $|x - y| = \delta$ and then, since $M\delta^\gamma \geq 2u$, $u^\star(x) - u_\star(y) \leq M\delta^\gamma = \Phi(x - y)$

-or $(x, y) \in \partial(\Omega \times \Omega)$.

In that case, for $(x, y) \in (\Omega \times \partial\Omega)$ we have just proved that $u^\star(x) \leq M_o d^\gamma \leq M|y - x|^\gamma$, while for $(x, y) \in \partial\Omega \times \Omega$, $-u(y) \leq 0 \leq M|x - y|^\gamma$.

Now we consider interior points. Suppose by contradiction that $u^\star(x) - u_\star(y) > \Phi(x - y)$ for some $(x, y) \in \Delta_\delta$. Then there exists $(\bar{x}, \bar{y})$ such that

$$u^\star(\bar{x}) - u_\star(\bar{y}) - \Phi(\bar{x} - \bar{y}) = \sup(u^\star(x) - u_\star(y) - \Phi(x - y)) > 0.$$

Clearly, $\bar{x} \neq \bar{y}$. Then using Lemma 2.1 in [14] there exist X and Y in $\mathcal{S}$ such that

$$(\gamma M(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{\gamma-2}, X) \in J^{2,+}u^\star(\bar{x}),$$
$$(\gamma M(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{\gamma-2}, -Y) \in J^{2,-}u_\star(\bar{y})$$

with

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

and $B = D^2\Phi(\bar{x}-\bar{y})$.

In particular, this proves that $X+Y \leq 0$, while taking vectors of the form $(x,-x)$ one gets $X+Y \leq 4B$. We need a more precise estimate, as in [15]. For that aim let:

$$0 \leq P := \frac{(\bar{x}-\bar{y} \otimes \bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2} \leq I.$$

Remarking that $\mathrm{tr}AB \geq 0$ if A and B are symmetric semi-positive definite matrices then

$$tr(-(X+Y)(I-P)) \geq 0 \text{ and } \mathrm{tr}\left((4\mathrm{B} - (\mathrm{X}+\mathrm{Y}))\mathrm{P}\right) \geq 0.$$

Hence,

$$tr(X+Y) \leq tr(P(X+Y) \leq 4tr(PB) = 4\gamma M(\gamma-1)|\bar{x}-\bar{y}|^{\gamma-2} < 0. \quad (4.3)$$

Now we can use the fact that u is both a sub and a super solution of (4.1) and applying condition (H2)

$$\begin{aligned} f(\bar{x}) &\leq F(\nabla_x\Phi, X) \leq a|\nabla_x\Phi|^\alpha tr(X+Y) + F(\nabla_y\Phi, -Y) \\ &\leq f(\bar{y}) + a|\nabla_x\Phi|^\alpha tr(X+Y). \end{aligned}$$

This implies, using (4.3),

$$a|\nabla_x\Phi|^\alpha 4\gamma M(1-\gamma)|\bar{x}-\bar{y}|^{\gamma-2} \leq f(\bar{y}) - f(\bar{x}).$$

Recalling that $|\nabla_x\Phi| = \gamma M|\bar{x}-\bar{y}|^{\gamma-1}$ the previous inequality becomes:

$$aM^{\alpha+1}4\gamma^{1+\alpha}(1-\gamma)|\bar{x}-\bar{y}|^{\gamma(\alpha+1)-(\alpha+2)} \leq 2|f|_\infty. \quad (4.4)$$

Using $M \geq \frac{2(\sup u)}{\delta^\gamma}$ and $|\bar{x}-\bar{y}| \leq \delta$ one obtains

$$a(2\sup u)^{1+\alpha}4\gamma^{1+\alpha}(1-\gamma)\delta^{-(\alpha+2)} \leq 2|f|_\infty.$$

This is clearly false for δ small enough and it concludes the proof. $\square$

Proof of Theorem 4.2. The proof proceeds similarly to the proof given by Ishii and Lions in [15]. This proof requires use of the fact that we already know that u is Hölder continuous by Theorem 4.1, together with the additional assumption (H3):

For the sake of simplicity and without loss of generality we assume that in hypothesis (H2) $a = A = 1$. Let γ be in $(\frac{1}{2\lambda}, 1)$ and c such that by the Hölder continuity proved before

$$|u(x) - u(y)| \leq c|x - y|^\gamma.$$

Let μ be an increasing function such that $\mu(0) = 0$ and $\mu(r) \geq r$, let

$$l(r) = \int_0^r ds \int_0^s \frac{\mu(\sigma)}{\sigma} d\sigma,$$

and let us note that since $\mu \geq 0$ for $r > 0$, $l(r) \leq rl'(r)$. Let r_0 be such that $l'(r_0) = \frac{1}{2}$, M such that $Mr_0 = 4 \sup |u|$. Let also $\delta > 0$ be given, $K = \frac{r_0}{\delta}$, and z be such that $d(z, \partial\Omega) \geq 2\delta$.

We define $\varphi(x, y) = \Phi(x-y)+L|x-z|^k$, where $\Phi(x) = M(K|x|-l(K|x|))$, and

$$\Delta_z = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| < \delta, |x - z| < \delta\}.$$

We shall now choose all the constants above.

- k is such that $k > \frac{1}{1-\frac{1}{2\gamma\lambda}} > 2$;

- L is such that $M \geq \frac{2 \sup u}{r_0}$ and $L = c\delta^{\gamma-k}$; using the Hölder continuity of u, one has

$$u(x) - u(y) \leq \varphi(x, y)$$

on $\partial\Delta_z$.

Suppose by contradiction that for some point $(\bar{x}, \bar{y})$:

$$u(\bar{x}) - u(\bar{y}) > \varphi(\bar{x}, \bar{y}).$$

Clearly, $\bar{x} \neq \bar{y}$. Note that $L|\bar{x}-z|^k \leq c|\bar{x}-\bar{y}|^\gamma$. Proceeding as in the previous proof, there exist X, Y such that

$$(MK(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{-1}(1-l'(K|\bar{x}-\bar{y}|))+kL|\bar{x}-z|^{k-2}(\bar{x}-z), X) \in J^{2,+}u(\bar{x}),$$

and

$$(MK\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}(1-l'(K|\bar{x}-\bar{y}|)), -Y) \in J^{2,-}u(\bar{y}),$$

where the matrices X and Y satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} B+\tilde{L} & -B \\ -B & B \end{pmatrix} \tag{4.5}$$

with $B = D^2\Phi(\bar{x}-\bar{y})$ and

$$\tilde{L} = kL|\bar{x}-z|^{k-2}\Big(I + (k-2)\frac{(\bar{x}-z) \otimes (\bar{x}-z)}{|\bar{x}-z|^2}\Big).$$

Let us note that similarly to the Hölder case (4.5) implies that $X+Y-\tilde{L}\leq 4B$ and then

$$tr(X+Y-\tilde{L})\leq 4tr(PB)$$

with $P=\frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2}$. This gives

$$tr(X+Y-\tilde{L})\leq -\frac{MK\mu(K|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|}\leq -MK^2. \tag{4.6}$$

Let us note that

$$\nabla_x\varphi(\bar{x},\bar{y})=MK(1-l'(K|\bar{x}-\bar{y}|))\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}+kL|\bar{x}-z|^{k-2}(\bar{x}-z),$$

$$\nabla_y\varphi(\bar{x},\bar{y})=MK(1-l'(K|\bar{x}-\bar{y}|))\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}$$

and

$$L|\bar{x}-z|^{k-1}=O(\delta^{\gamma-k}\delta^{k-1})=O(K^{1-\gamma}).$$

From this we get in particular that

$$2MK\geq|\nabla_x\varphi(\bar{x},\bar{y})|,|\nabla_y\varphi(\bar{x},\bar{y})|\geq\frac{MK}{4}.$$

Finally, observe that $|\tilde{L}|\leq L|\bar{x}-z|^{k-2}\leq(C\delta^{\gamma-k})^{\frac{2}{k}}(\delta)^{\gamma(k-2)/k}=O(\delta^{\gamma-2})=O(K^{2-\gamma})$, from which we derive using (4.6) that for K large enough $tr(X+Y)\leq 0$ and

$$|tr(X+Y)|\geq C(K^2)$$

for some positive universal constant C, and $|\tilde{L}|\leq|tr(X+Y)|$ for K large enough.

In the following we shall need a bound from above for $|X|$.

For simplicity the constants C below will indicate constants that depend only on the data and they may vary from one line to the other. Note that Lemma 2.1 in [14] ensures the existence of some universal constant such that

$$|X-\tilde{L}|+|Y|\leq C(|B|^{\frac{1}{2}}|tr(X+Y-\tilde{L})|^{\frac{1}{2}}+|tr(X+Y-\tilde{L})|)$$

with $B=D^2\Phi$. The considerations on $\tilde{L}$ with respect to $|tr(X+Y)|$ also give

$$|X|+|Y|\leq C(|B|^{\frac{1}{2}}|tr(X+Y)|^{\frac{1}{2}}+|tr(X+Y)|).$$

Let us note that $|B|\leq\frac{CK}{|\bar{x}-\bar{y}|}$ and then with the assumptions on μ, and using $|tr(X+Y)|\geq CK|tr(X+Y)|^{\frac{1}{2}}$ one derives that

$$|X|\leq|tr(X+Y)|(1+\frac{1}{K^{\frac{1}{2}}|\bar{x}-\bar{y}|^{\frac{1}{2}}}).$$

We need to prove that

$$|\nabla_y\varphi(\bar{x},\bar{y})|^{\alpha-\lambda}(\tilde{L}|\bar{x}-z|^{k-1})^{\lambda}|X| = o(|tr(X+Y)||\nabla\varphi|^{\alpha}).$$

For that aim we write

$$K^{\alpha-\frac{\gamma\lambda}{k}}|\bar{x}-\bar{y}|^{\gamma(\lambda)(1-\frac{1}{k})}|X| = K^{\alpha-\frac{\gamma\lambda}{k}}|\bar{x}-\bar{y}|^{\frac{1}{2}}|tr(X+Y)|\Big(1+\frac{1}{K^{\frac{1}{2}}|\bar{x}-\bar{y}|^{\frac{1}{2}}}\Big)$$

$$\leq |tr(X+Y)|(K^{\alpha-\frac{\gamma\lambda}{k}-\frac{1}{2}}) = |tr(X+Y)|K^{\alpha}K^{-\gamma\lambda-\frac{1}{2}}$$
$$= o(|tr(X+Y)||\nabla\varphi|^{\alpha}).$$

We now write, using assumption (H3) and the calculations above

$$\begin{aligned}
f(\bar{x}) &\leq F(\nabla_x\varphi(\bar{x},\bar{y}),X)\\
&\leq F(\nabla_y\varphi(\bar{x},\bar{y}),X)+\nu[L|\bar{x}-z|^{k-1}]^{\lambda}|\nabla_x\varphi|^{\alpha-\lambda}|X|\\
&\leq F(\nabla_y\varphi(\bar{x},\bar{y}),-Y)+o(|tr(X+Y)||\nabla\varphi|^{\alpha})+|\nabla\varphi|^{\alpha}tr(X+Y)\\
&\leq f(\bar{y})-CK^{\alpha+2}+o(K^{\alpha+2}).
\end{aligned}$$

From this one gets a contradiction for K large.

We have proved that for all x such that $d(x,\partial\Omega)\geq 2\delta$ and for y such that $|x-y|\leq\delta$

$$u(x)-u(y)\leq \frac{2\sup u}{r_0}\frac{|x-y|}{\delta}.$$

Recovering the compact set Ω by a finite number of $\mathcal{C}^2$ sets Ω_i, $\Omega_i\subset\Omega_{i+1}$ such that $d(\partial\Omega_i,\partial\Omega_{i+1})\leq 2\delta$, the local Lipschitz continuity is proved.

5. Existence results

5.1. **The case $\lambda<\bar{\lambda}$.** The main result of this section is the following existence result in a bounded smooth domain:

Theorem 5.1. *Assume that F satisfies $(H1)$, $(H2)$. Suppose that f is bounded and $f\leq 0$ on $\overline{\Omega}$. Then, for $\lambda<\bar{\lambda}$ there exists u a nonnegative viscosity solution of*

$$\begin{cases} F(\nabla u,D^2u)+\lambda u^{1+\alpha}=f & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Moreover, if $f<0$ in Ω and continuous, the solution is unique.

To prove this theorem, we need the two following propositions.

Proposition 5.2. *Suppose that f is bounded and nonpositive, and $\lambda \in \mathbb{R}$. Suppose that there exists $v_1 \geq 0$ and $v_2 \geq 0$, respectively a sub solution and super solution of*

$$\begin{cases} F(\nabla v, D^2 v) + \lambda v^{1+\alpha} = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

with $v_1 \leq v_2$. Then there exists a viscosity solution v of (5.1)*, such that $v_1 \leq v \leq v_2$. Moreover, if $f < 0$ inside Ω the solution is unique.*

Proposition 5.3. *Suppose that F satisfies* $(H1)$, $(H2)$*. For any f bounded and nonpositive in $\overline{\Omega}$, there exists a viscosity solution w of*

$$\begin{cases} F(\nabla w, D^2 w) = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Of course w is nonnegative by the maximum principle and Hölder continuous. Moreover, if $f < 0$ and continuous in Ω the solution is unique.

By Proposition 5.2, Proposition 5.3 will be proved if we construct a sub and super solution for (5.2). Since the null function is clearly a sub solution, it is sufficient to construct a viscosity solution u of $F(\nabla u, D^2 u) \leq -1$ which is positive and zero on the boundary. Then multiplying by the right constant we get the required super solution of (5.2). This is what we do in the next proposition .

Proposition 5.4. *Let Ω be a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^N$. Assume that F satisfies* $(H1)$, $(H2)$*. Let $d(x) = d(x, \partial\Omega)$ be the distance to the boundary. Then for any $\beta < 0$, there exist $k \in \mathbf{N}$, $\gamma \in (0,1)$, such that*

$$u(x) = \Big(1 - \frac{1}{(1 + d(x)^\gamma)^k}\Big)$$

is a viscosity super solution of

$$F(\nabla u, D^2 u) \leq \beta.$$

This result is proved in the appendix, together with some properties of the distance function required here, while the proof of Proposition 5.2 is given at the end of this section.

Proof of Theorem 5.1. For $\lambda \leq 0$, one can apply directly Proposition 5.2, since 0 is a sub solution for (5.6) and the solution constructed in Proposition 5.3 is a super solution.

We now treat the case $\lambda > 0$.

We define the sequence $u_n = T_f^n(0)$ where $T_f(u)$ is defined as the unique viscosity solution of

$$\begin{cases} F(\nabla T_f(u), D^2 T_f(u)) = f - \lambda u^{1+\alpha} & \text{in } \Omega \\ T_f(u) = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 5.3 implies that $T_f u$ is well defined.

By the comparison principle and the maximum principle for F in [4], u_n is increasing and nonnegative. We want to prove that it is bounded. Suppose not; then, by the homogeneity of F $w_n := \frac{u_n}{|u_n|_\infty}$ satisfies

$$F(\nabla w_{n+1}, D^2 w_{n+1}) + \lambda\Big(\frac{u_n^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}}\Big) = \frac{f}{|u_{n+1}|_\infty^{1+\alpha}}.$$

Furthermore,

$$\begin{aligned} F(\nabla w_{n+1}, D^2 w_{n+1}) + \lambda w_{n+1}^{1+\alpha} &= \lambda\Big(\frac{u_{n+1}^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} - \frac{u_n^{1+\alpha}}{|u_{n+1}|^{1+\alpha}}\Big) + \frac{f}{|u_{n+1}|_\infty^{1+\alpha}} \\ &\geq \frac{f}{|u_{n+1}|_\infty^{1+\alpha}}. \end{aligned}$$

Clearly,

$$\Big|\lambda\Big(\frac{u_{n+1}^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} - \frac{u_n^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}}\Big) + \frac{f}{|u_{n+1}|_\infty^{1+\alpha}}\Big| \leq 2\lambda + \frac{|f|}{|u_1|_\infty^{1+\alpha}}$$

since $0 \leq \frac{u_n^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} \leq 1$.

By the Hölder estimates in the previous section, the sequence w_n is relatively compact in $\mathcal{C}(\bar\Omega)$; extracting a subsequence from (w_n) and passing to the limit one gets in particular

$$F(\nabla w, D^2 w) + \lambda w^{1+\alpha} \geq 0.$$

Moreover, $w = 0$ on the boundary .

We are under the hypothesis that $\lambda < \bar\lambda$, hence, we can apply the maximum principle and conclude that $w \leq 0$. We have reached a contradiction since $w \geq 0$ and $|w|_\infty = 1$.

So the sequence u_n is bounded. Since it is also increasing it converges and the convergence is uniform on $\overline{\Omega}$, by the Hölder estimates. Using the (obvious) properties of uniform limit of viscosity solutions, one gets that the limit u is a nonnegative solution of

$$F(\nabla u, D^2 u) + \lambda u^{1+\alpha} = f.$$

Proof of Proposition 5.2. The proof relies on Perron's method applied to viscosity solutions as is done in [14].

Let us define $v = \sup\{v_1 \le u \le v_2 : u \text{ is } a\ SUB\}$. We want to prove first that $v^\star$ is a sub solution. Let u_n be an increasing sequence of sub solutions, $v_1 \le u_n \le v_2$, u_n converging to $v^\star$.

Suppose that $\bar{x}$ is some point such that v is equal to a constant C on a ball $B(\bar{x}, r)$. Since $C \ge 0$, it satisfies the condition required in that case.

We now treat the points where v is not locally constant. Suppose by contradiction that v is not a sub solution, then there exist $\bar{x}$, a C^2 function φ, and $r > 0$, such that $\nabla\varphi(\bar{x}) \ne 0$ and

$$(v^\star - \varphi)(x) \le (v^\star - \varphi)(\bar{x}) = 0,$$

and

$$F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} \le f(\bar{x}) - r. \tag{5.3}$$

Let δ be small enough in order that the following inequalities hold, for $|\bar{x} - y| \le \delta$,

$$|F(\nabla\varphi, D^2\varphi)(y) - F(\nabla\varphi, D^2\varphi)(\bar{x})| \le \frac{r}{4}, \tag{5.4}$$

$$|\varphi(y)^{1+\alpha} - \varphi(\bar{x})^{1+\alpha}| \le \frac{r}{4\lambda}, \tag{5.5}$$

$$|f(y) - f(\bar{x})| \le \frac{r}{4}. \tag{5.6}$$

One can assume that the supremum of $v^\star - \varphi$ on $\bar{x}$ is strict, so that there exists $\alpha_\delta > 0$ with

$$\sup_{|y-\bar{x}|\ge\delta} (v^\star - \varphi) \le -\alpha_\delta.$$

Finally, take N large enough in order that by the simple convergence of $u_n(\bar{x})$ toward $v^\star(\bar{x})$ one has for $n \ge N$

$$u_n(\bar{x}) - v^\star(\bar{x}) \ge -\frac{\alpha_\delta}{4},$$

then

$$\sup_{|x-\bar{x}|\le\delta} (u_n - \varphi)(x) \ge \frac{-\alpha_\delta}{4} \ge -\alpha_\delta \ge \sup_{|x-\bar{x}|\ge\delta} (v^\star - \varphi)(x) \ge \sup_{|x-\bar{x}|\ge\delta} (u_n - \varphi)(x).$$

Furthermore, the supremum of $u_n - \varphi$ is achieved inside $B(\bar{x}, \delta)$, on some x_n. Then one has, using (5.3), (5.4), (5.5), (5.6),

$$\begin{aligned} f(\bar{x}) - r &\ge F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} \\ &\ge F(\nabla\varphi, D^2\varphi)(x_n) + \lambda\varphi(x_n)^{1+\alpha} - \frac{r}{2} \ge f(x_n) - \frac{r}{2} \ge f(\bar{x}) - \frac{3r}{4}, \end{aligned}$$

a contradiction.

We now prove that $v_\star$ is a super solution. If not, there would exist $\bar{x} \in \Omega$, $r > 0$ and $\varphi \in \mathcal{C}^2(B(\bar{x}, r)$, with $\nabla\varphi(\bar{x}) \neq 0$, satisfying

$$0 = (v_\star - \varphi)(\bar{x}) \leq (v_\star - \varphi)(x)$$

on $B(\bar{x}, r)$, and $\epsilon > 0$, such that

$$F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} > f(\bar{x}) + \epsilon.$$

We prove first that $\varphi(\bar{x}) < v_2(\bar{x})$. If not one would have $\varphi(\bar{x}) = v_\star(\bar{x}) = v_2(\bar{x})$ and then

$$(v_2 - \varphi)(x) \geq (v_\star - \varphi)(x) \geq (v_\star - \varphi)(\bar{x}) = (v_2 - \varphi)(\bar{x}) = 0,$$

hence since v_2 is a super solution and φ is a test function for v_2 on $\bar{x}$,

$$F(\nabla\varphi, D^2\varphi)(\bar{x}) + \lambda\varphi(\bar{x})^{1+\alpha} \leq f(\bar{x}),$$

a contradiction. Then $\varphi(\bar{x}) < v_2(\bar{x})$. We construct now a sub solution which is greater than v and less than v_2.

Let δ be such that for $|x - \bar{x}| \leq \delta$

$$|F(\nabla\varphi, D^2\varphi)(x) - F(\nabla\varphi, D^2\varphi)(\bar{x})| + |f(x) - f(\bar{x})| + \lambda|\varphi(x)^{1+\alpha} - \varphi(\bar{x})^{1+\alpha}| \leq \frac{\varepsilon}{2}.$$

Then

$$F(\nabla\varphi, D^2\varphi)(x) + \lambda\varphi^{1+\alpha}(x) \geq f(x) + \frac{\varepsilon}{2}.$$

One can assume that

$$(v_\star - \varphi)(x) \geq |x - \bar{x}|^4.$$

We take $r < \delta^4$ and such that $0 < r < \inf_{|x-\bar{x}|\leq\delta}(v_2(x) - \varphi(x))$ and define

$$w = \sup(\varphi(x) + r, v_\star);$$

w is LSC as the supremum of two LSC functions.

One has $w(\bar{x}) = \varphi(\bar{x}) + r$, and $w = v$ for $r < |x - \bar{x}| < \delta$. w is a sub solution, since when $w = \varphi + r$ one can use $\varphi + r$ as a test function, and since $\varphi(x) > 0$,

$$F(\nabla\varphi, D^2\varphi)(x) + \lambda(\varphi(x) + r)^{1+\alpha} \geq F(\nabla\varphi, D^2\varphi)(x) + \lambda\varphi^{1+\alpha}(x) \geq f(x) + \frac{\varepsilon}{2}.$$

Elsewhere, $w = v_\star$, hence it is a sub solution. Moreover, $w \geq v$, $w \neq v$ and $w \leq g$. This contradicts the fact that v is the supremum of the sub solutions. Using Hölder regularity we get that v is Hölder and hence $v^\star = v_\star$.

5.2. **The case $\lambda = \bar{\lambda}$.** In all this section we still assume that Ω is a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^N$.

Theorem 5.5. *Let F satisfy $(H1)$ and $(H2)$. Then, there exists $\phi > 0$ in Ω such that ϕ is a viscosity solution of*

$$\begin{cases} F(\nabla\phi, D^2\phi) + \bar{\lambda}\phi^{1+\alpha} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, ϕ is γ-Hölder continuous for all $\gamma \in (0,1)$ and locally Lipschitz if (H3) is satisfied by F.

Proof of Theorem 5.5. Let λ_n be an increasing sequence which converges to $\bar{\lambda}$. Let u_n be a nonnegative viscosity solution of

$$\begin{cases} F(\nabla u_n, D^2 u_n) + \lambda_n u_n^{1+\alpha} = -1 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 5.1 the sequence (u_n) is well defined. We shall prove that (u_n) is not bounded. Indeed suppose by contradiction that it is. Then by the Hölder estimate, there exists a subsequence, still denoted u_n, which tends uniformly to a nonnegative continuous function u which would be a viscosity solution of

$$F(\nabla u, D^2 u) + \bar{\lambda} u^{1+\alpha} = -1.$$

This contradicts the definition of $\bar{\lambda}$. Indeed $u > 0$ and one can choose $\varepsilon > 0$ small enough that

$$F(\nabla u, D^2 u) + (\bar{\lambda} + \varepsilon) u^{1+\alpha} \le -1 + \varepsilon u^{1+\alpha} \le 0.$$

We have obtained that the sequence $|u_n|_\infty \to +\infty$. Then defining $w_n = \frac{u_n}{|u_n|_\infty}$ one has

$$F(\nabla w_n, D^2 w_n) + \lambda_n w_n^{1+\alpha} = \frac{-1}{|u_n|^{1+\alpha}}$$

and then extracting as previously a subsequence which converges uniformly, one gets that there exists w, such that $|w|_\infty = 1$ and

$$F(\nabla w, D^2 w) + \bar{\lambda} w^{1+\alpha} = 0.$$

The boundary condition is given by the uniform convergence.

Clearly, w is Hölder and if (H3) is satisfied it is locally Lipschitz continuous.

6. Appendix: Properties of the distance function

In all this section Ω is a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^N$. We want to recall some known and new facts about the distance function in order to construct the sub solution requested in Proposition 5.4.

Proposition 6.1. *Suppose Ω is a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^N$.*

1. *d is differentiable at x if and only if there exists only one point $y = y(x) \in \partial\Omega$ such that $d(x) = |x-y|$. In that case $|\nabla d(x)| = 1$.*
2. *d is semi-concave; i.e., there exists C_1 such that $d(x) - C_1|x|^2$ is concave. This implies in particular that on a point where d is differentiable, then $(p, X) \in J^{2,-}d(x_0)$ implies that $p = \nabla d(x_0)$ and $X \leq C_1 Id$.*
3. *If the distance is achieved on at least two points, then $J^{2,-}d(x_0) = \emptyset$.*

Almost all these facts are contained in [1]. For completeness' sake we shall recall the proof of the last assertion.

Suppose that $x = 0$ and let y_1 and y_2 be two distinct points in $\partial\Omega$ such that $d(0, \partial\Omega) = d = |0-y_1| = |0-y_2|$. It is sufficient to prove that $J^{2,-}d^2(0)$ is empty. Suppose that e and E are in $\mathbb{R}^N \times S^N$ such that for all x in a neighborhood of 0

$$d^2 + e.x +^t xEx \leq d(x, \partial\Omega)^2.$$

In particular, this must be satisfied for all $x = ty_1$ and $|t| < r$ small enough. This implies in particular

$$d^2 + t(e.y_1) + t^2(Ey_1, y_1) \leq \inf_{|t|<r}(|ty_1 - y_1|^2, |ty_1 - y_2|^2).$$

In particular one gets first

$$(e.y_1)t \leq -2d^2 t + O(t^2)$$

which implies $a.y_1 = -2d^2$ and secondly one has

$$(e.y_1)t \leq -2(y_1.y_2)t + O(t^2)$$

which implies that $(e.y_1) = -2(y_1.y_2) = -2d^2$, a contradiction since $y_1 \neq y_2$ implies that $y_1.y_2 \neq d^2$.

Proposition 6.2. *Let Ω be a bounded open $\mathcal{C}^2$ set in $\mathbb{R}^N$. Suppose that F satisfies $(H1)$, $(H2)$. Then for any constant $\beta < 0$ there exists a function u which is a viscosity super solution of*

$$\begin{cases} F(\nabla u, D^2u) \leq \beta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof of Proposition 6.2. Let $K > diam\Omega$. Then $d \leq K$. Let $\gamma \in (0,1)$ and let k be large enough to be chosen later.

We construct the following function

$$u(x) = 1 - \frac{1}{(1 + d(x)^\gamma)^k}.$$

Clearly, $u = 0$ on the boundary, u is continuous and it is $\mathcal{C}^1$ on the points where d is achieved on a unique point and according to Proposition 6.1 in the other points $J^{2,-}u = \emptyset$. Hence we only have to test the points where the distance function is achieved only on one point. Let ϕ be a test function at $x_0 \in \Omega$; i.e.,

$$u(x) \geq \phi(x),\ u(x_0) = \phi(x_0).$$

Then clearly there exists a test function ψ defined by

$$\phi(x) = 1 - \frac{1}{(1 + \psi(x)^\gamma)^k}$$

which is a test function for d; i.e., $d(x) \geq \psi(x)$ and $d(x_0) = \psi(x_0)$. We shall now compute the gradient and the Hessian of ϕ in terms of ψ. Using the fact that $\nabla\psi(x_0) = \nabla d(x_0)$, one has

$$\nabla\phi(x_0) = \frac{k\gamma d^{\gamma-1}\nabla d}{(1 + d^\gamma)^{k+1}}$$

and

$$D^2\phi(x_0) = \frac{k\gamma d^{\gamma-2}}{(1 + d^\gamma)^{k+2}}\left[(\gamma - 1 - (k + 2 - \gamma)d^\gamma)\nabla d \otimes \nabla d + d(1 + d^\gamma)D^2\psi\right].$$

We need to study the eigenvalues of $D^2\phi$.

Clearly $\nabla d \otimes \nabla d \geq 0$ and $tr(\nabla d \otimes \nabla d) = 1$, while $D^2\psi \leq C_1 Id$. Using condition (H2) and these considerations, we obtain that

$$\begin{aligned}
&F(\nabla\phi(x_0), D^2\phi(x_0))\\
&\leq \left(\frac{kd^{\gamma-1}}{(1 + d^\gamma)^k}\right)^\alpha \frac{k\gamma d^{\gamma-2}}{(1 + d^\gamma)^{k+2}}\left[a(\gamma - 1 - d^\gamma(k + 2 - \gamma)) + AC_1Nd(1 + d^\gamma)\right]\\
&\leq k^{1+\alpha}\frac{d^{\gamma(\alpha+1)-(\alpha+2)}}{(1 + d^\gamma)^{k(\alpha+1} + 2}\left[a(\gamma - 1 - d^\gamma(k + 2 - \gamma)) + AC_1Nd(1 + d^\gamma)\right]\\
&\leq k^{1+\alpha}\frac{K^{\gamma(\alpha+1)-(\alpha+2)}}{(1 + K^\gamma)^{k(\alpha+1)+2}}\left[a(\gamma - 1 - d^\gamma(k + 2 - \gamma)) + AC_1Nd(1 + d^\gamma)\right]\\
&\leq \beta < 0
\end{aligned}$$

since the function $\frac{d^{\gamma(\alpha+1)-(\alpha+2)}}{(1+d^\gamma)^{k(\alpha+1}+2}$ is decreasing and choosing k large enough in order that $a(\gamma-1-d^\gamma(k+2-\gamma))+AC_1Nd(1+d^\gamma)<0$. Hence for $\beta<0$ fixed, and the right choice of k one gets

$$F(\nabla\phi, D^2\phi)\leq\beta.$$

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