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FIRST LAYER FORMULAS FOR CHARACTERS OF $SL(n, \mathbf{C})$

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ABSTRACT. Some problems concerning the decomposition of certain characters of $SL(n, \mathbf{C})$ are studied from a combinatorial point of view. The specific characters considered include those of the exterior and symmetric algebras of the adjoint representation and the Euler characteristic of Hanlon's so-called "Macdonald complex." A general recursion is given for computing the irreducible decomposition of these characters. The recursion is explicitly solved for the first layer representations, which are the irreducible representations corresponding to partitions of n. In the case of the exterior algebra, this settles a conjecture of Gupta and Hanlon. A further application of the recursion is used to give a family of formal Laurent series identities that generalize the (equal parameter) q-Dyson Theorem.

1. Introduction. The goal of this paper is to explore some of the combinatorial structure which is present in the characters of some particular representations of $SL(n, \mathbf{C})$ that have been the subject of much recent interest among both combinatorists and algebraists [**5, 7, 12, 15**]. This structure is found in the decomposition of representations of $SL(n, \mathbf{C})$ which are built out of the exterior and symmetric algebras of the adjoint representation. For convenience, we modify the usual sense of the adjoint representation in that we actually study the action of SL_n on the Lie algebra gl_n, rather than sl_n.

For the exterior algebra, we consider the irreducible decomposition of the characters of the exterior powers $\bigwedge^k(gl_n)$; for the symmetric algebra, we consider the irreducible decomposition of the characters of the symmetric powers $\mathrm{Sym}^k(gl_n)$. We also consider the decomposition of the Euler characteristic of P. Hanlon's so-called "Macdonald complex." This is equivalent to decomposing the character of the kth tensor power $T^k(\bigwedge(gl_n))$ when the submodules

$$\bigwedge^{\alpha_1}(gl_n) \otimes \cdots \otimes \bigwedge^{\alpha_k}(gl_n)$$

have been given special weightings. Each of these problems can be regarded as special cases of the problem of computing the decomposition of the formal character

$$\phi_n(z, q) = \prod_{1 \leqslant i,j \leqslant n} \prod_{k \geqslant 1} \frac{1 - q^k x_i x_j^{-1}}{1 - z q^{k-1} x_i x_j^{-1}}$$

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into irreducible characters, where z and q are indeterminates. Macdonald's affine analogue of the Weyl denominator formula [**11**] for the root system A_{n-1} may also be regarded as a special case (namely, $z = 0$) of this decomposition problem.

We give a recursion that can be used to compute the decomposition of $\phi_n(z, q)$ into irreducible characters. In the case $z = 0$, the recursion specializes to one which is essentially equivalent to one used by D. Stanton [**17**] to prove Macdonald's affine formula; we give an explicit solution in this case which is more combinatorial than those previously known. In the general case, we give an explicit solution of this recursion for the coefficients corresponding to "first layer representations." This explicit formula settles a conjecture of Gupta and Hanlon [**6**] about the decomposition of the exterior powers $\Lambda^k(gl_n)$.

It is known that the q-Dyson Theorem for equal parameters, which amounts to a formula for the constant term (with respect to $x_1, \ldots, x_n$) in the formal Laurent series

$$\prod_{1 \leqslant i < j \leqslant n} \prod_{l=1}^{k} \left(1 - q^{l-1} x_j x_i^{-1}\right)\left(1 - q^l x_i x_j^{-1}\right),$$

can be used to give a formula for the coefficient of the trivial character in the expansion of $\phi_n(z, q)$. We show that the first layer formula for $\phi_n(z, q)$ can be used to express coefficients of monomials in this series of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with all $\alpha_i \geqslant -1$ in terms of symmetric functions that are related to the Hall-Littlewood symmetric functions. In the case in which all of the exponents with $\alpha_i = -1$ occur consecutively, we give an explicit evaluation of this expression, thus giving a generalization of the q-Dyson Theorem in the case of equal parameters.

2. Definitions and background. Throughout this article, we will adhere to the conventions of [**13, 14**]; the reader may refer to them for any of the notation and terminology which we do not explicitly define.

Let Λ denote the ring of symmetric functions over $\mathbf{Z}$ in the variables $x_1, x_2, \ldots,$ and let Λ^k denote the symmetric functions that are homogeneous of degree k. Let Λ_n denote the ring of symmetric polynomials over $\mathbf{Z}$ in the variables $x_1, \ldots, x_n$. Consider the quotient Ω_n defined by

$$\Omega_n = \Lambda_n/(x_1 \cdots x_n - 1).$$

Recall that if V is a (complex, finite-dimensional) SL_n-module, then the character $\operatorname{char} V$ may be regarded as a member of Ω_n; the indeterminates x_i are identified as the eigenvalues of the linear transformations in SL_n.

It is well known that the Schur functions $s_\lambda(x_1, \ldots, x_n)$, where λ ranges over the partitions of length at most n, are the irreducible characters of the polynomial representations of $GL(n, \mathbf{C})$. If λ is restricted to partitions of length less than n, one obtains the irreducible characters of the continuous representations of $SL(n, \mathbf{C})$ (see [**14, 18**]). To be precise, one should regard the image of $s_\lambda(x_1, \ldots, x_n)$ in Ω_n as a character of SL_n, and not the polynomial itself. Hence, if V is an SL_n-module (on which the action of SL_n is continuous), there are unique nonnegative integers c_λ such that

$$\operatorname{char} V = \sum_\lambda c_\lambda s_\lambda(x_1, \ldots, x_n) \quad \text{modulo } x_1 \cdots x_n = 1,$$

where λ ranges over the partitions of length less than n. Throughout this paper, *all formal Laurent series in* $x_1, \dots, x_n$ *should be viewed modulo* $x_1 \cdots x_n = 1$, *unless stated otherwise*. The integer c_λ is the multiplicity of V_λ, the irreducible SL_n-module corresponding to λ, in the SL_n-module V.

One useful technique for computing the coefficients c_λ is equivalent to the extraction of coefficients from formal power series. If $\alpha \in \mathbf{Z}^n$, let $a_\alpha(x) = a_\alpha(x_1, \dots, x_n)$ denote the monomial alternating function; namely,

$$a_\alpha(x) = \sum_{w \in S_n} \varepsilon_w w(x_1^{\alpha_1} \cdots x_n^{\alpha_n}),$$

where ε_w denotes the sign of the permutation w. It is well known (see [**13**, I.3] or [**18**, §4]) that if $f \in \Omega_n$ and λ is a partition of length less than n, then the coefficient of the Schur function $s_\lambda(x_1, \dots, x_n)$ in the decomposition of f is given by $[x^{\lambda+\delta}]a_\delta(x)f(x)$, where δ is the partition $(n-1, n-2, \dots, 1, 0)$ and the notation $[x^\alpha]F(x)$ refers to the coefficient of x^α in the formal Laurent series F (which in this case is viewed modulo $x_1 \cdots x_n = 1$).

It is easy to see that if $X \in SL_n$ has eigenvalues $x_1, \dots, x_n$, then the adjoint action of X on the Lie algebra $gl_n = gl(n, \mathbf{C})$ has eigenvalues $x_i x_j^{-1}$ $(1 \leqslant i, j \leqslant n)$. Therefore, the character of the exterior power $\wedge^k(gl_n)$ is the coefficient of q^k in the generating function

$$\prod_{1 \leqslant i,j \leqslant n} \left(1 + q x_i x_j^{-1}\right). \tag{1}$$

If we define polynomials $E^n[\lambda](q)$ for each partition λ of length less than n via

$$\prod_{1 \leqslant i,j \leqslant n} \left(1 + q x_i x_j^{-1}\right) = \sum_\lambda E^n[\lambda](q) s_\lambda(x_1, \dots, x_n),$$

it follows that the coefficient of q^k in $E^n[\lambda](q)$ is the multiplicity of V_λ in $\wedge^k(gl_n)$.

Similarly, the character of the symmetric power $\mathrm{Sym}^k(gl_n)$ is the coefficient of q^k in the generating function

$$\prod_{1 \leqslant i,j \leqslant n} \frac{1}{1 - q x_i x_j^{-1}}, \tag{2}$$

and if we define formal power series $S^n[\lambda](q)$ for each partition λ of length less than n via

$$\prod_{1 \leqslant i,j \leqslant n} \frac{1}{1 - q x_i x_j^{-1}} = \sum_\lambda S^n[\lambda](q) s_\lambda(x_1, \dots, x_n),$$

it follows that the coefficient of q^k in $S^n[\lambda](q)$ is the multiplicity of V_λ in $\mathrm{Sym}^k(gl_n)$.

We will also consider a family of virtual characters of SL_n which are the Euler characteristics of certain complexes of SL_n-modules. In order to describe these complexes, we need to introduce a family of graded Lie algebras. An $\mathbf{N}$-graded Lie algebra L is a Lie algebra with a vector space decomposition $L = L_0 \oplus L_1 \oplus L_2 \oplus \cdots$ for which $[L_i L_j] \subseteq L_{i+j}$. If $A \in L_i$, then A is said to be homogeneous of degree i and we write $\deg(A) = i$.

Fix an integer $k \geqslant 1$. Consider the $\mathbf{N}$-graded Lie algebra $L = L_0 \oplus L_1 \oplus L_2 \oplus \cdots$, where

$$L_i \cong \begin{cases} gl_n & \text{if } 1 \leqslant i \leqslant k, \\ (0) & \text{otherwise,} \end{cases}$$

and the bracket [,] is defined in such a way that if $A \in L_i$, $B \in L_j$, and $i + j \leqslant k$, then $[A, B] = AB - BA \in L_{i+j}$, the usual commutator on gl_n.

The grading of L induces a grading on $\bigwedge(L)$: if $A_1, \dots, A_r \in L$ are homogeneous, then $A_1 \wedge \cdots \wedge A_r$ is homogeneous of degree $\sum \deg(A_i)$. Let $\bigwedge^r(L)_s$ denote the homogeneous submodule of $\bigwedge^r(L)$ of degree s. Using the obvious SL_n-module isomorphism between $\bigwedge(L)$ and $T^k(\bigwedge(gl_n))$ it follows that

$$\bigwedge^r(L)_s \cong \coprod_{\sum \alpha_i = r, \sum i\alpha_i = s} \bigwedge^{\alpha_1}(gl_n) \otimes \cdots \otimes \bigwedge^{\alpha_k}(gl_n) \tag{4}$$

as SL_n-modules.

The Macdonald complex of gl_n is the Koszul complex $M_k(gl_n)$ defined by

$$\cdots \to \bigwedge^{r+1}(L) \xrightarrow{\partial_{r+1}} \bigwedge^{r}(L) \xrightarrow{\partial_r} \bigwedge^{r-1}(L) \to \cdots,$$

where

$$\partial_r(A_1 \wedge \cdots \wedge A_r) = \sum_{i<j} (-1)^{i+j+1} \left[A_i, A_j\right] \wedge A_1 \wedge \cdots \wedge \hat{A}_i \wedge \cdots \wedge \hat{A}_j \wedge \cdots \wedge A_r.$$

We have used the notation $\hat{A}_i$ to denote deletion of the term A_i. Since ∂ is degree-preserving, this is actually a graded complex. This complex is a structure devised by P. Hanlon [**20**], in the more general context of an arbitrary semisimple Lie algebra, to aid in the study of Macdonald's root system conjectures [**12**].

Let $\chi_s(M_k(gl_n))$ denote the Euler characteristic of the degree s component of this complex of SL_n-modules, where the character is used as an Euler-Poincaré map (as in the language of [**9**, Chapter 4, §3]). We have

$$\chi_s(M_k(gl_n)) = \sum_r (-1)^r \operatorname{char} \bigwedge^r(L)_s.$$

Since the character of

$$\bigwedge^{\alpha_1}(gl_n) \otimes \cdots \otimes \bigwedge^{\alpha_k}(gl_n)$$

is given by

$$[y^\alpha] \prod_{1 \leqslant i, j \leqslant n} \left(1 + y_1 x_i x_j^{-1}\right) \cdots \left(1 + y_k x_i x_j^{-1}\right),$$

the decomposition (4) yields simple generating functions for the virtual characters $\chi_s(M_k(gl_n))$:

$$\begin{aligned} \sum_s q^s \chi_s\left(M^k(gl_n)\right) \\ &= \sum_{\alpha \in \mathbf{N}^k} (-1)^{\sum \alpha_i} q^{\sum i\alpha_i} [y^\alpha] \prod_{1 \leqslant i, j \leqslant n} \left(1 + y_1 x_i x_j^{-1}\right) \cdots \left(1 + y_k x_i x_j^{-1}\right) \\ &= \prod_{1 \leqslant i, j \leqslant n} \left(1 - q x_i x_j^{-1}\right) \cdots \left(1 - q^k x_i x_j^{-1}\right). \end{aligned} \tag{5}$$

As before, if we define formal power series $M_k^n[\lambda](q)$ so that

$$\prod_{1 \leqslant i, j \leqslant n} \left(1 - q x_i x_j^{-1}\right) \cdots \left(1 - q^k x_i x_j^{-1}\right) = \sum_{\lambda} M_k^n[\lambda](q) s_\lambda(x_1, \ldots, x_n),$$

it follows that the coefficient of q^s in $M_k^n[\lambda](q)$ can be interpreted as the coefficient of $s_\lambda(x_1, \ldots, x_n)$ in $\chi_s(M_k(gl_n))$

Notice that each of the generating functions (1), (2), and (5) is homogeneous of degree 0 with respect to $x_1, \ldots, x_n$. Hence, the only Schur functions that actually occur in their decompositions correspond to partitions of integers divisible by n; if λ is a partition of ln with length less than n, we say that λ belongs to the lth layer. If we use dominant weight vectors (rather than partitions) to index the irreducible characters that appear in their decompositions, the only such vectors which occur will be integral.

R. K. Gupta **[5]** has studied the decomposition of (2) into Schur functions in the limit as n tends to infinity. A certain amount of delicacy is required to do this, since it is not clear how to pass to a limit in the first place. If α and β are partitions of the same weight, let $[\alpha, \beta]_n$ denote the dominant weight vector for SL_n defined by

$$[\alpha, \beta]_n = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0, -\beta_s, \ldots, -\beta_1).$$

Gupta has shown that the limit

$$S_{\alpha\beta}(q) = \lim_{n \to \infty} S^n[\alpha, \beta]_n(q)$$

exists as a formal power series, and she conjectured that the $S_{\alpha\beta}$'s satisfy a number of remarkable properties.

In view of Gupta's results, Stanley **[15]** was led to consider the more general problem of computing the formal power series $c_\lambda^n(y) = c_\lambda^n(y_1, y_2, \ldots)$ defined by

$$\prod_{1 \leqslant i, j \leqslant n} \prod_{r \geqslant 1} \frac{1}{1 - y_r x_i x_j^{-1}} = \sum_{\lambda} c_\lambda^n(y) s_\lambda(x_1, \ldots, x_n) \tag{6}$$

in the limit as n tends to infinity. It should be mentioned that Stanley also introduced an additional set of indeterminates $z = (z_1, z_2, \ldots)$ and showed that the decomposition of

$$\prod_{1 \leqslant i, j \leqslant n} \prod_{r \geqslant 1} \frac{1 - z_r x_i x_j^{-1}}{1 - y_r x_i x_j^{-1}}$$

into Schur functions can be obtained from the decomposition of (6) without any extra work. As Gupta did for the symmetric algebra, Stanley showed that the limit

$$c_{\alpha\beta}(y) = \lim_{n \to \infty} c_{[\alpha, \beta]_n}^n(y)$$

exists as a formal power series and gave an explicit formula for the $c_{\alpha\beta}$'s in terms of the internal product of Schur functions.

The work of Gupta and Stanley led P. Hanlon **[7]** to study the limiting-case decompositions of the S_k-isotypic components of the tensor algebra for all of the classical Lie algebras (types A_n, B_n, C_n, and D_n). His methods, which are completely different from those of Stanley, rely on a mixture of combinatorial ideas and tools from the representation theory of semisimple Lie algebras. Hanlon was able to show that for each classical type (A, B, C, D), one may pass to a limit in a carefully chosen way and obtain a stable decomposition analogous to Stanley's for the exterior and symmetric algebras and the Macdonald complex of these Lie algebras.

For any nonnegative integer k and indeterminates z and q, let

$$(z; q)_k = \prod_{0 \leqslant i < k} (1 - q^i z).$$

By convention, $(z; q)_0 = 1$ and

$$(z; q)_\infty = \prod_{i \geqslant 0} (1 - q^i z).$$

In the following sections we will study the character decompositions of the exterior and symmetric algebras and the Euler characteristic of the Macdonald complex of gl_n, without passing to the limit as $n \to \infty$. Our point of attack will be through the decomposition of

$$\phi_n(z, q) = \prod_{1 \leqslant i, j \leqslant n} \frac{\left(q x_i x_j^{-1}; q\right)_\infty}{\left(z x_i x_j^{-1}; q\right)_\infty}$$

into Schur functions. Thus, let us define formal power series $C^n[\lambda](z, q)$ for each partition λ of length less than n so that

$$\phi_n(z, q) = \sum_\lambda C^n[\lambda](z, q) s_\lambda(x_1, \ldots, x_n) \tag{7}$$

is a formal power series identity in the ring $\mathbf{Z}[[z, q]] \otimes \Omega_n$.

Notice that as special cases,

• The decompositions of the exterior powers (1) are obtained via the specialization $z \to q^2$, $q \to -q$; we have

$$E^n[\lambda](q) = C^n[\lambda](q^2, -q).$$

• The decompositions of the symmetric powers (2) are obtained via the specialization $q \to 0$; we have

$$S^n[\lambda](z) = C^n[\lambda](z, 0).$$

• The decomposition of the Euler characteristic of the Macdonald complex (5) is obtained via the specialization $z \to q^{k+1}$; we have

$$M_k^n[\lambda](q) = C^n[\lambda](q^{k+1}, q).$$

At first glance, the problem of determining $C^n[\lambda]$ seems harder than determining $M_k^n[\lambda]$, but these problems should actually be considered equivalent. If one can find a formula for $M_k^n[\lambda](q)$ for all integers $k \geqslant 1$, then a formula for $C^n[\lambda](z, q)$ can be obtained by replacing every occurrence of q^k in this formula by z/q. This observation can be made rigorous by the following:

LEMMA 2.1. *Let $F(z, q)$ be a formal power series in the indeterminates z, q. If $F(q^k, q) = 0$ for infinitely many nonnegative integers k, then $F(z, q) = 0$.*

PROOF. Assume toward a contradiction that $F(z,q)\neq 0$ but $F(q^k,q)=0$ for arbitrarily large k. Certainly F has an expansion

$$F(z,q)=F_0(q)+zF_1(q)+z^2F_2(q)+\cdots \tag{8}$$

for suitable formal power series $F_r(q)$. We may assume that $F_0\neq 0$ by rescaling the counterexample F by a suitable power of z. Suppose that q^r is the smallest power of q for which $[q^r]F_0(q)\neq 0$. Choose $k>r$ so that $F(q^k,q)=0$. By (8),

$$[q^r]F(q^k,q)=[q^r]F_0(q)+[q^r]q^kF_1(q)+[q^r]q^{2k}F_2(q)+\cdots=[q^r]F_0(q),$$

which contradicts the choice of r. $\square$

3. The q-Dyson Theorem. For any nonnegative integer m, let

$$[m]!_q=(1-q)(1-q^2)\cdots(1-q^m).$$

The q-Dyson Theorem is the following constant term identity, originally conjectured by Andrews [**1**].

THEOREM 3.1 (ZEILBERGER AND BRESSOUD [**19**]). *Let $a_1,\dots,a_n$ be nonnegative integers. We have*

$$[x^0]\prod_{1\leqslant i<j\leqslant n}\left(x_jx_i^{-1};q\right)_{a_j}\left(qx_ix_j^{-1};q\right)_{a_i}=\frac{[a_1+\cdots+a_n]!_q}{[a_1]!_q\cdots[a_n]!_q}.$$

It is well known that this identity proves Macdonald's root system conjecture [**12**] for the root system A_{n-1}. Using the techniques in [**12**], it is not hard to show that the q-Dyson Theorem can also be used to evaluate the series $C^n[\varnothing](z,q)$, where $\varnothing$ denotes the void partition (which corresponds to the dominant weight vector 0). It is also possible to evaluate the series $C^n[\varnothing](z,q)$ directly from an identity due to Bressoud and Goulden [2], which is similar in flavor to the original q-Dyson Theorem.

THEOREM 3.2 (BRESSOUD AND GOULDEN [**2**, THEOREM 2.2]). *Let $a_1,\dots,a_n$ be nonnegative integers. We have*

$$[x^0]\prod_{1\leqslant i<j\leqslant n}\left(x_jx_i^{-1};q\right)_{a_j}\left(qx_ix_j^{-1};q\right)_{a_i-1}$$
$$=\frac{[a_1+\cdots+a_n]!_q}{[a_1]!_q\cdots[a_n]!_q}\cdot\prod_{1\leqslant i\leqslant n}\frac{1-q^{a_i}}{1-q^{a_n+\cdots+a_i}}.$$

The resulting formula for $C^n[\varnothing]$ is the following.

COROLLARY 3.3. *We have*

$$C^n[\varnothing](z,q)=\frac{(q;q)_\infty}{(qz^n;q)_\infty}\cdot\frac{1}{[n]!_z}.$$

PROOF. Consider the special case $z=q^k$. In this situation, definition (7) becomes

$$[k-1]!_q^n\cdot\prod_{i\neq j}\left(qx_ix_j^{-1};q\right)_{k-1}=\sum_\lambda C^n[\lambda](q^k,q)\cdot s_\lambda(x_1,\dots,x_n).$$

If the series $C^n[\varnothing]$ is expressed in terms of coefficient extraction, it follows that

$$\begin{aligned} C^n[\varnothing](q^k, q) &= [x^\delta]a_\delta(x) \cdot [k-1]!_q^n \cdot \prod_{i \neq j} \left(qx_ix_j^{-1}; q\right)_{k-1} \\ &= [k-1]!_q^n \cdot [x^0] \prod_{i<j} \left(x_jx_i^{-1}; q\right)_k \left(qx_ix_j^{-1}; q\right)_{k-1}. \end{aligned}$$

If we apply Theorem 3.2 in the case $a_1 = \cdots = a_n = k$, we find

$$C^n[\varnothing](q^k, q) = [k-1]!_q^n \cdot \frac{[nk]!_q}{[k]!_q^n} \cdot \frac{(1-q^k)^n}{[n]!_{q^k}}.$$

Therefore, the formal power series

$$C^n[\varnothing](z, q) \quad \text{and} \quad \frac{(q;q)_\infty}{(qz^n;q)_\infty} \cdot \frac{1}{[n]!_z}$$

agree for infinitely many of the special cases $z = q^k$. Apply Lemma 2.1. □

We remark that the resulting identities obtained by specializing Corollary 3.3 to the exterior and symmetric algebras of gl_n are well known and classical. Furthermore, as we remarked earlier, the resulting identity for the Macdonald complex of gl_n; namely,

$$M_k^n[\varnothing](q) = \frac{[n(k+1)]!_q}{[n]!_{q^{k+1}}},$$

is equivalent to Macdonald's root system conjecture for A_{n-1}.

In §7 we will use the theory of symmetric functions to show that formulas for the polynomials $C^n[\lambda]$ can be used to extract coefficients of the form

$$[x^\alpha] \prod_{1 \leqslant i<j \leqslant n} \left(x_jx_i^{-1}; q\right)_k \left(qx_ix_j^{-1}; q\right)_k,$$

for certain monomials x^α, thus generalizing the equal parameter version of the q-Dyson Theorem. Although Theorem 3.2 is needed to evaluate these coefficients, it is used only indirectly through Corollary 3.3; the identities we give are not special cases of the identities in [2].

4. Macdonald's identity for A_{n-1}. Ordinary root systems are finite subsets of a finite-dimensional real Euclidean space satisfying certain axioms. Macdonald [**11**] generalized this notion to certain (infinite) collections of affine-linear functionals on a real Euclidean space, and called them affine root systems. In addition to classifying these root systems, Macdonald proved a generalization of Weyl's denominator formula for affine root systems.

For the root systems of type A, the Weyl donominator formula is equivalent to the Vandermonde determinant identity. Macdonald's identites for the affine root systems of type A are essentially equivalent to a decomposition of the symmetric formal Laurent series

$$\prod_{1 \leqslant i, j \leqslant n} \left(qx_ix_j^{-1}; q\right)_\infty$$

into Schur functions $s_\lambda(x_1, \ldots, x_n)$. By our definition of the series $C^n[\lambda]$, we see that Macdonald's identities yield formulas for $C^n[\lambda](0, q)$.

In the following, we will give not only a precise statement of these formulas for $C^n[\lambda](0, q)$, but also a proof. We emphasize, however, that the proof we give is modeled on the technique developed by D. Stanton in [**17**], where he gives short, elementary proofs of Macdonald's identities. Our purpose in giving the proof is twofold. The statement of the formulas we give is not readily identifiable as the form of the identities given by Macdonald [**11**, (8.1)] or Stanton [**17**, (5.2)]; some sort of justification is warranted. Secondly, the proof technique will motivate the methods used in subsequence sections in our study of the full series $C^n[\lambda](z, q)$. Hanlon has shown that the form of the identities we give may also be derived from the theory in [**3**].

In order to state the formulas in a reasonable fashion, we introduce some notation. If $\alpha \in \mathbf{Z}^n$, let $|\alpha| = \sum \alpha_i$; if α is a partition, let $l(\alpha)$ denote the length of α; i.e., the number of nonzero terms. Let λ be a partition with $|\lambda|$ divisible by n and $l(\lambda) < n$. As we remarked earlier, these are the only partitions for which it is possible for $C^n[\lambda]$ to be nonzero. Let $v = (v_1, \ldots, v_n)$ denote the increasing sequence of integers defined by

$$v_i = \lambda_{n+1-i} + i - 1.$$

We shall refer to v as the vertical sequence of λ. This terminology is derived from the following geometric interpretation. View the diagram of λ as a lattice path from southwest to northeast, and number the horizontal and vertical steps consecutively from 0 to $n + \lambda_1 - 1$. The vertical steps are labeled $v_1, \ldots, v_n$. An example in which $n = 8$ and $\lambda = 764322$ is given in Figure 1. Notice that the terms of the vertical sequence are merely the terms of $\lambda + \delta$ taken in reverse order. Also notice that the initial term of the vertical sequence is always zero.

THEOREM 4.1. *Let λ be a partition with $l(\lambda) < n$ and $|\lambda|$ divisible by n.*

(a) *We have $C^n[\lambda](0, q) = 0$ unless the vertical sequence of λ, taken modulo n, is a permutation of $\mathbf{Z}/(n)$.*

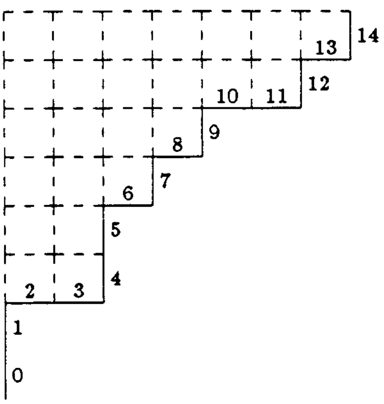

FIGURE 1. $v = (0, 1, 4, 5, 7, 9, 12, 14)$.

(b) *If the vertical sequence v does form a permutation of $\mathbf{Z}/(n)$, associate to v the sequence $\sigma \in \mathbf{Z}^n$, where $\sigma_i = \lfloor v_j/n \rfloor$ if $v_j \equiv i - 1 \bmod n$.*

We have

$$C^n[\lambda](0,q) = \varepsilon_v(-1)^{(n-1)|\sigma|} q^{\eta(\sigma)}(q;q)_\infty,$$

where

$$\eta(\sigma) = \sum_{1 \leqslant i < j \leqslant n} \binom{\sigma_i - \sigma_j}{2},$$

and ε_v denotes the sign of v as a permutation of $\mathbf{Z}/(n)$.

When v does form a permutation of $\mathbf{Z}/(n)$, we shall refer to the sequence σ defined above as the associated sequence for λ.

EXAMPLE 4.2. (a) In Figure 1, we have $n = 8$ and $\lambda = 764322$. Taken modulo 8, the vertical sequence is $(0,1,4,5,7,1,4,6)$, which is evidently not a permutation of $\mathbf{Z}/(8)$. We conclude that

$$C^8[764322](0,q) = 0.$$

(b) Consider $n = 8$ and let λ denote the partition corresponding to the dominant weight $[775, 8632]_8$. Taken modulo 8, the vertical sequence of λ is $(0,3,7,1,4,2,5,6)$, which is a permutation of $\mathbf{Z}/(8)$. The associated sequence for λ given by $\sigma = (0,1,2,0,1,2,2,0)$. We conclude that

$$C^8[775, 8632]_8(0,q) = q^{27}(q;q)_\infty.$$

PROOF OF THEOREM 4.1. Let λ be a partition in the lth layer with $l(\lambda) < n$, and $\alpha \in \mathbf{Z}^n$ the dominant weight vector corresponding to λ. Let the symmetric group S_n act on $\mathbf{Z}^n$ in the natural way:

$$w \circ (\gamma_1, \ldots, \gamma_n) = (\gamma_{w^{-1}(1)}, \ldots, \gamma_{w^{-1}(n)}).$$

Also, let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{Z}^n$, and let $1^n = \sum e_i$.

Our proof will be preceded by three technical lemmas:

LEMMA 4.3. *If $C^n[\alpha](0,q) \neq 0$, then the terms in the sequence $\alpha + \delta - 1^n + ne_n$ are distinct. When these terms are distinct, then there is a unique permutation $w \in S_n$ and dominant weight $\beta \in \mathbf{Z}^n$ such that*

$$\alpha + \delta - 1^n + ne_n = w \circ (\beta + \delta).$$

Moreover,

$$C^n[\alpha](0,q) = \varepsilon_w(-1)^{n-1} q^l \cdot C^n[\beta](0,q).$$

PROOF. Let

$$F_n(x_1, \ldots, x_n) = a_\delta(x) \prod_{1 \leqslant i, j \leqslant n} \left(q x_i x_j^{-1}; q\right)_\infty.$$

Recall that by the definition of $C^n[\lambda]$, we have

$$\prod_{1 \leqslant i, j \leqslant n} \left(q x_i x_j^{-1}; q\right)_\infty = \sum_\lambda C^n[\lambda](0,q) \cdot s_\lambda(x_1, \ldots, x_n).$$

It follows that

$$C^n[\alpha](0,q) = [x^{\alpha+\delta}]F_n(x_1,\ldots,x_n). \tag{9}$$

The key to this lemma is to find a functional equation satisfied by F_n; this is the idea that we have borrowed most explicitly from Stanton. Observe that

$$\begin{aligned}\frac{F_n(x_1,\ldots,x_{n-1},qx_n)}{F_{n-1}(x_1,\ldots,x_{n-1})} &= \prod_{1\leqslant i<n}(x_i - qx_n)\cdot(q^2x_nx_i^{-1};q)_\infty\cdot(x_ix_n^{-1};q)_\infty \\ &= (x_1\cdots x_{n-1})\prod_{1\leqslant i<n}(1-x_ix_n^{-1})\cdot(qx_nx_i^{-1};q)_\infty\cdot(qx_ix_n^{-1};q)_\infty \\ &= (-1)^{n-1}\frac{x_1\cdots x_n}{x_n^n}\cdot\frac{F_n(x_1,\ldots,x_n)}{F_{n-1}(x_1,\ldots,x_{n-1})}.\end{aligned}$$

Therefore,

$$F_n(x_1,\ldots,x_{n-1},qx_n) = (-1)^{n-1}(x_1\cdots x_n)x_n^{-n}\cdot F_n(x_1,\ldots,x_n).$$

Extracting the coefficient of $x^{\alpha+\delta}$ and comparing with (9), we find

$$q^{\alpha_n}C^n[\alpha](0,q) = (-1)^{n-1}\left[x^{\alpha+\delta-1^n+ne_n}\right]F_n(x_1,\ldots,x_n). \tag{10}$$

Note that $F_n(x_1,\ldots,x_n)$ is an alternating series. Therefore, any coefficient we care to extract, say x^γ, will vanish unless the terms in γ are distinct. When the terms are distinct, then there is a unique permutation $w\in S_n$ and dominant weight $\beta\in\mathbf{Z}^n$ so that $\gamma = w\circ(\beta+\delta)$, and in that case, we have

$$[x^\gamma]F_n = \varepsilon_w[x^{\beta+\delta}]F_n = \varepsilon_wC^n[\beta](0,q).$$

The lemma now follows by applying this observation to (10) and using the fact that the layer l of λ is $-\alpha_n$. □

LEMMA 4.4. *The recursion given in Lemma* 4.3 *uniquely determines the coefficients* $C^n[\lambda](0,q)$ *up to the initial condition* $C^n[\varnothing](0,q)$. *Moreover*, $C^n[\lambda](0,q)$ *is nonzero if and only if the vertical sequence of* λ, *taken modulo* n, *is a permutation of* $\mathbf{Z}/(n)$.

PROOF. Partially order dominant weights by insisting that $\alpha\geqslant\beta$ if and only if

$$\alpha_1+\cdots+\alpha_i\geqslant\beta_1+\cdots+\beta_i\quad\text{for } 1\leqslant i\leqslant n.$$

Thus, '$>$' is the usual partial order on the root lattice of A_{n-1}: $\alpha>\beta$ iff $\alpha-\beta$ is a sum of positive roots. Notice that $\varnothing$ is at the bottom of this partial order.

Let $\alpha>\varnothing$ be a dominant weight, and assume that the terms of $\alpha+\delta-1^n+ne_n$ are distinct. We must have

$$\alpha+\delta-1^n+ne_n = w\circ(\beta+\delta),$$

where β is the dominant weight given by

$$\beta = (\alpha_1-1,\ldots,\alpha_k-1,\alpha_n+k,\alpha_{k+1},\ldots,\alpha_{n-1}), \tag{11}$$

$w\in S_n$ is the permutation for which

$$w\circ(\gamma_1,\ldots,\gamma_n) = (\gamma_1,\ldots,\gamma_k,\gamma_{k+2},\ldots,\gamma_n,\gamma_{k+1}), \tag{12}$$

and k is the unique integer ($1\leqslant k<n$) for which

$$\alpha_k+n-k-1>\alpha_n+n-1>\alpha_{k+1}+n-k-2. \tag{13}$$

Such an integer k exists since $\alpha_n + n - 1 > \alpha_1 + n - 2$ would imply $\alpha = \varnothing$. Notice that the sign of w is $(-1)^{n-k-1}$. It is clear from (11) that $\alpha > \beta$. Therefore, the recursion in Lemma 4.3 expresses $C^n[\alpha](0,q)$ in terms of $C^n[\beta](0,q)$ for some $\beta < \alpha$. We may conclude that the recursion uniquely determines $C^n[\lambda](0,q)$ up to the initial condition $C^n[\varnothing](0,q)$.

To complete the proof of the lemma, it suffices to show that the vertical sequence of α is a permutation of $\mathbf{Z}/(n)$ if and only if the same is true for the vertical sequence of β. This is clear, since: (1) the terms of the vertical sequence of a partition ν are the same as the terms of $\nu + \delta$, and (2) each of the operations

$$\gamma \mapsto \gamma \pm 1^n, \quad \gamma \mapsto \gamma \pm ne_n, \quad \gamma \mapsto w \circ \gamma$$

preserve the property of being a permutation of $\mathbf{Z}/(n)$. □

Thus, we have proved part (a) of the theorem, and we may also conclude that if $C^n[\lambda](0,q) \neq 0$, then

$$C^n[\lambda](0,q) = (-1)^a q^b C^n[\varnothing](0,q)$$

for suitable integers a and b. Henceforth, let us assume that λ is a partition whose vertical sequence v, taken modulo n, is a permutation of $\mathbf{Z}/(n)$. Let α be the dominant weight corresponding to λ, and let σ denote its associated sequence. Notice that σ_i is characterized by the fact that $v_j = n\sigma_i + i - 1$, where v_j is the unique term of v for which $v_j \equiv i - 1 \bmod n$.

For example, if $n = 7$ and $\sigma = (0,1,1,0,2,0,1)$, this means that the terms of v whose residues mod 7 are 0, 3, 5 must be 0, 3, 5; the terms of v whose residues mod 7 are 1, 2, 6 must be 8, 9, 13; and the term of v whose residue mod 7 is 4 must be 18. Thus $v = (0,3,5,8,9,13,18)$.

Consider the effect of the recursion in Lemma 4.3 on σ. Let β be the dominant weight defined in (11), and w the permutation defined in (12). Notice that the integer k, defined in (13), is the number of terms of the form $\lambda_i + n - i$ which exceed n, since

$$\lambda_i + n - i = \alpha_i - \alpha_n + n - i \geqslant n$$

is equivalent to

$$\alpha_i + n - i - 1 \geqslant \alpha_n + n - 1.$$

But the terms $\lambda_i + n - i$ are those of v taken in reverse order, so we conclude

$$k = |\{i\colon 1 \leqslant i \leqslant n,\ \sigma_i > 0\}|.$$

As we remarked earlier, the sign of w is $(-1)^{n-k-1}$, so we have $\varepsilon_w \cdot (-1)^{n-1} = (-1)^k$. Furthermore, observe that

$$\binom{n}{2} + ln = |\lambda + \delta| = |v| = \binom{n}{2} + n\sum_i \sigma_i,$$

so we have $l = |\sigma| = \sum_i \sigma_i$.

Let u be the vertical sequence corresponding to the dominant weight β, and let τ be its associated sequence. Assume for the moment that $k < n - 1$. Since

$$(14) \qquad \beta + \delta = (\alpha_1 + n - 2, \ldots, \alpha_k + n - k - 1,$$
$$\alpha_n + n - 1, \alpha_{k+1} + n - k - 2, \ldots, \alpha_{n-1}),$$

it follows that u is obtained from (14) by subtracting $\alpha_{n-1}\cdot 1^n$ and reversing the elements. Hence, the unsorted terms of u are $\{n-v_2, 0, v_3-v_2,\dots,v_n-v_2\}$, since $v_2=\alpha_{n-1}+1-\alpha_n$.

Let $v_j\equiv i-1 \bmod n$, where $1\leqslant i\leqslant n$. Since $v_j=n\sigma_i+(i-1)$, we have

$$v_j-v_2=\begin{cases} n\sigma_i+(i-1-v_2) & \text{if } i\geqslant v_2+1,\\ n(\sigma_i-1)+(n-v_2+i-1) & \text{if } i<v_2+1.\end{cases}$$

Therefore, we may conclude that

$$\tau=(\sigma_r,\sigma_{r+1},\dots,\sigma_n,\sigma_1,\sigma_2-1,\dots,\sigma_{r-1}-1),$$

where $r=v_2+1$. Note that r is the least integer (>1) such that $\sigma_r=0$.

The analysis for the case $k=n-1$ is even easier. Since

$$\beta+\delta=(\alpha_1+n-2,\dots,\alpha_{n-1},\alpha_n+n-1),$$

the vertical sequence of β must be $u=(0,v_2-n,\dots,v_n-n)$ and the associated sequence for β must be $\tau=(\sigma_1,\sigma_2-1,\dots,\sigma_n-1)$.

We summarize the previous discussion by the following.

LEMMA 4.5. *We have*

$$C^n[\alpha](0,q)=(-1)^k q^l\cdot C^n[\beta](0,q),$$

where k is the number of positive terms in σ, $l=|\sigma|$, and the associated sequence for β is given by

$$\tau=(\sigma_r,\sigma_{r+1},\dots,\sigma_n,\sigma_1,\sigma_2-1,\dots,\sigma_{r-1}-1),$$

where r is the least integer (>1) for which $\sigma_r=0$. If no such integer exists, then

$$\tau=(\sigma_1,\sigma_2-1,\dots,\sigma_n-1).\quad\square$$

This lemma gives us an algorithm for computing $C^n[\alpha](0,q)$ in terms of the associated sequence σ. If we iterate the algorithm, we will obtain a series $\sigma=\sigma^0,\sigma^1,\sigma^2,\dots$ of associated sequences. The iterations cease as soon as $\sigma^m=\varnothing$ for some integer m. Let k_p and l_p denote the k- and l-statistics corresponding to the pth iterate σ^p. Lemma 4.5 tells us that

$$C^n[\alpha](0,q)=(-1)^{\Sigma k_p}q^{\Sigma l_p}C^n[\varnothing](0,q).$$

In view of Corollary 3.3, it suffices to show that

$$(n-1)|\sigma|+\operatorname{inv}(\sigma)\equiv\sum k_p \bmod 2\quad\text{and}\quad \eta(\sigma)=\sum l_p,$$

where $\operatorname{inv}(\sigma)$ denotes the number of inversions in σ; i.e.,

$$\operatorname{inv}(\sigma)=\left|\{(i,j)\colon 1\leqslant i<j\leqslant n,\ \sigma_i>\sigma_j\}\right|.$$

To prove this, it is conceptually easier to think of the operation $\sigma\mapsto\tau$ as being performed on an n-tuple of variables $(\sigma_1,\dots,\sigma_n)$. We obtain σ^p from σ^{p-1} by cyclically shifting the initial variable in σ^{p-1} to the end of the n-tuple repeatedly, until the initial variable is 0. All of the variables that were cyclically shifted to the end are decremented by 1, except for the previous initial 0.

In this scheme, suppose that we have the initial values $\sigma_i = a$ and $\sigma_j = b$, where $i < j$ and $a < b$. We want to examine the contributions of σ_j to the statistics k_p and l_p when the initial variable of σ^p is σ_i. Of course, σ_i will never occur as an initial variable until it has been decremented a times, and we have $\sigma_i = 0$ and $\sigma_j = b - a$. Thus, the first time that σ_i is an initial variable, σ_j will contribute $b - a$ to l_p and 1 to k_p. As soon as $\sigma_j = 0$, it will cease to contribute. Hence, σ_j contributes a total of $\binom{b-a+1}{2} = \binom{a-b}{2}$ to $\sum l_p$ and a total of $b - a$ to $\sum k_p$.

The analysis when $a > b$ is nearly the same. This time, we want to examine the contributions of σ_i to the statistics k_p and l_p when σ_j is the initial term of σ^p. Since the larger term a precedes the smaller term b, σ_i will be decremented before σ_j. The only rounds with $\sigma_j = 0$ will have $\sigma_i = a - b - 1, \ldots, 2, 1, 0$. Hence, σ_i contributes a total of $\binom{a-b}{2}$ to $\sum l_p$ and $a - b - 1$ to $\sum k_p$.

In summary, we have shown

$$\sum l_p = \sum_{i<j} \binom{\sigma_i - \sigma_j}{2} = \eta(\sigma)$$

and

$$\begin{aligned}\sum k_p &= \sum_{i<j} \begin{cases} \sigma_j - \sigma_i & \text{if } \sigma_i < \sigma_j \\ \sigma_i - \sigma_j - 1 & \text{if } \sigma_i > \sigma_j \end{cases} \equiv \operatorname{inv}(\sigma) + \sum_{i<j} (\sigma_i + \sigma_j) \mod 2 \\ &\equiv \operatorname{inv}(\sigma) + (n-1)|\sigma| \mod 2,\end{aligned}$$

which completes the proof. □

5. A recursion for computing $C^n[\lambda](z, q)$. In the previous section we found a functional equation satisfied by the formal Laurent series

$$F_n(x_1, \ldots, x_n) = a_\delta(x) \prod_{1 \leqslant i, j \leqslant n} \left(q x_i x_j^{-1}; q\right)_\infty,$$

and used it to find a recursion for the series $C^n[\lambda](0, q)$. In this section, we will extend this technique to obtain a functional equation for the formal Laurent series

$$G_n(x_1, \ldots, x_n) = a_\delta(x) \prod_{1 \leqslant i, j \leqslant n} \frac{\left(q x_i x_j^{-1}; q\right)_\infty}{\left(z x_i x_j^{-1}; q\right)_\infty}. \tag{15}$$

As before, the functional equation will yield a recursion for the series $C^n[\lambda](z, q)$. Of course, this recursion will simultaneously yield recursions for decomposing the characters of the exterior and symmetric algebras of gl_n, as well as the Euler characteristic of the Macdonald complex, by specialization. In the case of the symmetric algebra, this gives a recursion for computing the generalized exponents of SL_n.

Let us introduce additional notation and terminology. Let λ' denote the conjugate of the partition λ. Assume that λ is a partition with $l(\lambda) < n$ and $|\lambda|$ divisible by n. The horizontal sequence $h = (h_0, h_1, h_2, \ldots)$ of λ is the infinite sequence of integers defined by

$$h_i = \begin{cases} 0 & \text{if } i = 0, \\ n - \lambda'_i + i - 1 & \text{if } i > 0. \end{cases}$$

This terminology is derived from the same geometric considerations that motivated the introduction of the vertical sequence in §4. View the diagram of λ as a lattice path from southwest to northeast, and number the vertical and horizontal steps consecutively from 0, starting with a vertical step in the nth row. The horizontal steps are labeled $h_1, h_2, h_3, \ldots$. In the example given in Figure 1, we have $n = 8$, $\lambda = 764322$, and the corresponding horizontal sequence is $h = (0, 2, 3, 6, 8, 10, 11, 13, 15, 16, \ldots)$.

Recall that the rank of a partition λ is the size of the largest $r \times r$ square contained in λ. Let us define the rectangular rank of λ to be the largest integer r for which the $(r+1)$-row, r-column rectangle is contained in λ. For example, the rectangular rank of 764221 is 3.

Fix an integer $k \geqslant 0$. For any partition ν, let ν_L denote the partition whose diagram is the first k columns of ν, and let ν_R denote the partition whose diagram is the remainder; i.e.,

$$\nu'_L = (\nu'_1, \ldots, \nu'_k) \quad \text{and} \quad \nu'_R = (\nu'_{k+1}, \nu'_{k+2}, \ldots).$$

Let $\lambda \cup k$ denote the partition obtained by increasing the number of times the integer k occurs in λ by 1. Define

$$B_k(\lambda) = \{\mu \colon \mu \text{ satisfies rules } 1, 2, \text{ and } 3\}$$

1. $l(\mu) \leqslant n$, $|\mu| = |\lambda|$.
2. $\mu_R \subseteq (\lambda \cup k)_R$ and the skew diagram $(\lambda \cup k)_R/\mu_R$ is a vertical strip.
3. $(\lambda \cup k)_L \subseteq \mu_L$ and the skew diagram $\mu_L/(\lambda \cup k)_L$ is a vertical strip.

The reader is referred to [13] for the definitions of skew diagram and vertical strip.

Thus, rules 2 and 3 say that μ must be obtained from $\lambda \cup k$ by adding a vertical strip to the first k columns and deleting a vertical strip from the columns to the right of the first k. Notice that $B_0(\lambda) = \{\lambda\}$ and $B_k(\lambda)$ is nonempty if and only if $0 \leqslant k \leqslant r$, where r is the rectangular rank of λ. For example, let $n = 6$, $k = 2$, and $\lambda = 5553$. We have

$$B_2(5553) = \{54432, 55422, 54421, 444321\}.$$

The diagrams of these partitions are given in Figure 2. The cells of $\lambda \cup k$ have been marked with the symbol '$\cdot$'.

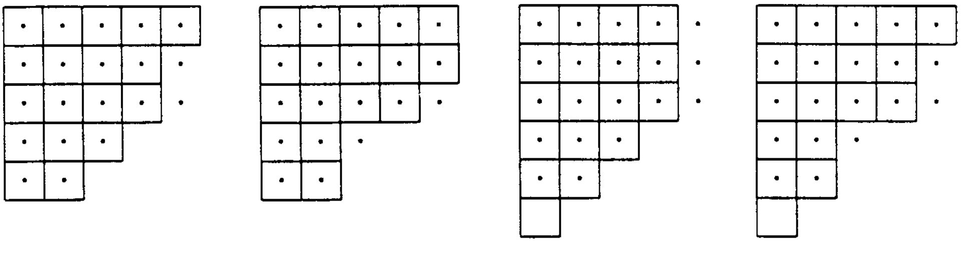

FIGURE 2. An illustration of the construction of $B_k(\lambda)$.

On occasion, we will violate our conventions and speak of the series $C^n[\mu]$ when μ is a partition of length n. In such circumstances, $C^n[\mu]$ should be considered synonymous with $C^n[\lambda]$, where $\lambda = (\mu_1 - \mu_n, \ldots, \mu_{n-1} - \mu_n, 0)$ is the partition obtained by removing all of the columns of length n from μ.

We are now ready to state a recursion for the computation of $C^n[\lambda](z,q)$. Although it can be stated in many forms, the following will suffice for the present.

THEOREM 5.1. (a) *Let λ be a partition of ln with $l(\lambda) < n$. Let h be the horizontal sequence and r the rectangular rank of λ. We have*

$$\sum_{0 \leqslant k \leqslant r} (-1)^k (z^{h_k} - q^l z^{n-h_{k+1}})(1 - z^{h_{k+1}-h_k}) \sum_{\mu \in B_k(\lambda)} C^n[\mu](z,q) = 0. \tag{16}$$

(b) *Moreover, the linear relations in* (16) *uniquely determine the series $C^n[\lambda]$ for all λ up to the initial value $C^n[\varnothing](z,q)$.*

EXAMPLE 5.2. (a) In the case $\lambda = \varnothing$, we have $r = l = h_0 = 0$, $h_1 = n$, and Theorem 5.1 becomes vacuous. A more typical example occurs if we take $n = 5$ and let λ be the second layer partition 4411. Its horizontal sequence is $(0, 1, 4, 5, 6, 9, \ldots)$ and λ has rectangular rank 2. One may verify that $B_1(\lambda) = \{43111\}$ and $B_2(\lambda) = \{33211\}$. By applying Theorem 5.1, we may deduce

$$C^5[4411](z,q) = \frac{(z - q^2z^4)(1 - z^3)}{(1 - q^2z^4)(1 - z)} C^5[32](z,q) - \frac{z^4 - q^2}{1 - q^2z^4} C^5[221](z,q).$$

(b) Consider the first layer partition $\lambda = 1^{n-2}2$. The rectangular rank of λ is 1, and we have $B_1(\lambda) = \{1^n\}$. Theorem 5.1 implies that

$$(1 - qz^{n-1})(1 - z) C^n[1^{n-2}2](z,q) = (z - q)(1 - z^{n-1}) C^n[\varnothing](z,q).$$

Therefore, by Corollary 3.3,

$$C^n[1^{n-2}2](z,q) = \frac{(z - q)(1 - z^{n-1})}{(1 - qz^{n-1})(1 - z)[n]!_z} \cdot \frac{(q;q)_\infty}{(qz^n;q)_\infty}.$$

In §6 we will use Theorem 5.1 to find a formula for $C^n[\lambda](z,q)$ for all first layer partitions.

PROOF OF THEOREM 5.1. To deduce part (b) from part (a) is straightforward. All of the partitions in $B_k(\lambda)$ can be obtained by removing at least k cells from the diagram of λ and inserting them into lower rows. Therefore, $\lambda \geqslant \mu$ for any $\mu \in B_k(\lambda)$, with equality only when $k = 0$. Since the coefficient

$$(z^{h_k} - q^l z^{n-h_{k+1}})(1 - z^{h_{k+1}-h_k})$$

of $C^n[\mu](z,q)$ in (16) can vanish only when $\lambda = \varnothing$, it follows that when $\lambda \neq \varnothing$, one can solve for $C^n[\lambda](z,q)$ in terms of $C^n[\mu](z,q)$ for some partitions $\mu < \lambda$. Part (b) thus follows by induction.

Notice that this also implies that each of the series $C^n[\lambda](z,q)$ is of the form $f(z,q) \cdot C^n[\varnothing](z,q)$ for some rational function f.

To prove part (a), we proceed by a series of lemmas, the first of which concerns a functional equation for the formal Laurent series G_n which we defined in (15).

LEMMA 5.3. *We have*

$$G_n(x_1,\dots,x_{n-1},qx_n)\cdot\prod_{1\leqslant i<n}\left(1-\frac{z}{q}x_ix_n^{-1}\right)=G_n(x_1,\dots,x_n)\cdot\prod_{1\leqslant i<n}\left(z-x_ix_n^{-1}\right).$$

PROOF. By the definition of G_n,

$$\begin{aligned}\frac{G_n(x_1,\dots,x_{n-1},qx_n)}{G_{n-1}(x_1,\dots,x_{n-1})}&=\prod_{1\leqslant i<n}(x_i-qx_n)\frac{(q^2x_nx_i^{-1};q)_\infty}{(qzx_nx_i^{-1};q)_\infty}\cdot\frac{(x_ix_n^{-1};q)_\infty}{(zq^{-1}x_ix_n^{-1};q)_\infty}\\&=(x_1\cdots x_{n-1})\cdot\prod_{1\leqslant i<n}(1-x_ix_n^{-1})\cdot\frac{(qx_nx_i^{-1};q)_\infty}{(zx_nx_i^{-1};q)_\infty}\\&\quad\cdot\frac{(qx_ix_n^{-1};q)_\infty}{(zx_ix_n^{-1};q)_\infty}\cdot\frac{1-zx_nx_i^{-1}}{1-zq^{-1}x_ix_n^{-1}}\\&=\frac{G_n(x_1,\dots,x_n)}{G_{n-1}(x_1,\dots,x_{n-1})}\cdot\prod_{1\leqslant i<n}\frac{z-x_ix_n^{-1}}{1-zq^{-1}x_ix_n^{-1}}.\end{aligned}$$

The lemma now follows immediately. □

Let $[n]$ denote the set $\{1,\dots,n\}$, and for any subset $S\subseteq[n]$ let $\kappa(S)\in\mathbf{Z}^n$ denote the characteristic vector of S; i.e., $\kappa(S)=(\kappa_1,\dots,\kappa_n)$, where

$$\kappa_i=\begin{cases}1 & \text{if } i\in S,\\ 0 & \text{if } i\notin S.\end{cases}$$

Let s denote the cardinality of S.

LEMMA 5.4. *Let λ be a partition of ln with $l(\lambda)<n$. We have*

$$\sum_{S\subseteq[n-1]}\varepsilon_{w(S)}(-z)^sC^n[\mu(S)](z,q)=q^lz^{n-1}\sum_{S\subseteq[n-1]}\varepsilon_{w(S)}(-z)^{-s}C^n[\mu(S)](z,q),$$

where S ranges over those subsets of $[n-1]$ for which the terms of $\lambda+\delta+se_n-\kappa(S)$ are distinct. (Even though $S\subseteq[n-1]$, we still regard $\kappa(S)\in\mathbf{Z}^n$.) For such S, $w(S)$ denotes the unique permutation and $\mu(S)$ the unique partition such that

$$\lambda+\delta+se_n-\kappa(S)=w(S)\circ(\mu(S)+\delta).$$

PROOF. By the definition of $C^n[\lambda]$, we know that

$$C^n[\lambda](z,q)=[x^{\alpha+\delta}]G_n(x_1,\dots,x_n),\tag{17}$$

where $\alpha\in\mathbf{Z}^n$ is the dominant weight corresponding to λ. By the functional equation in Lemma 5.3, we have

$$\begin{aligned}\sum_{S\subseteq[n-1]}\left(-\frac{z}{q}\right)^s\frac{x_S}{x_n^s}G_n(x_1,\dots,x_{n-1},qx_n)\\=z^{n-1}\sum_{S\subseteq[n-1]}(-z)^{-s}\frac{x_S}{x_n^s}G_n(x_1,\dots,x_n),\end{aligned}\tag{18}$$

where we have abbreviated $\prod_{i\in S} x_i$ by x_S. If we apply the fact that $[x^k]F(ax) = a^k[x^k]F(x)$ for any formal power series F, and extract the coefficient of $x^{\alpha+\delta}$ in (18), we obtain

$$(19)\quad q^{\alpha_n} \sum_{S\subseteq[n-1]} (-z)^s \left[x^{\alpha+\delta}\cdot\frac{x_n^s}{x_S}\right] G_n(x_1,\dots,x_n) \\ = z^{n-1} \sum_{S\subseteq[n-1]} (-z)^{-s} \left[x^{\alpha+\delta}\cdot\frac{x_n^s}{x_S}\right] G_n(x_1,\dots,x_n).$$

We have yet to exploit the fact, which is evident from (15), that G_n is an alternating series. This means that any coefficient we care to extract, say $[x^\gamma]G_n$, will vanish unless the terms of γ are distinct. If they are distinct, then there is a unique permutation $w \in S_n$ and dominant weight $\beta \in \mathbf{Z}^n$ for which $\gamma = w\circ(\beta+\delta)$. Furthermore, by (17),

$$[x^\gamma]G_n = \varepsilon_w[x^{\beta+\delta}]G_n = \varepsilon_w C^n[\beta](z,q).$$

The lemma now follows from this observation and (19). □

LEMMA 5.5. *Let h be the horizontal sequence of λ.*
(a) *If $h_k \leqslant s < h_{k+1}$, then $B_k(\lambda) = \{\mu(S) \colon |S| = s\}$.*
(b) *If $\mu(S) \in B_k(\lambda)$, then $\varepsilon_w(s) = (-1)^{s-k}$.*

PROOF. Let S be an s-subset of $[n-1]$ such that the terms of $\lambda+\delta+se_n-\kappa(S)$ are distinct.

First, we consider the possibility that $s = h_k$ for some $k \geqslant 0$. Since s is one of the horizontal steps in the lattice path of λ (or $s=0$), then the terms of $\lambda+\delta+se_n$ are distinct, and their relative order cannot be affected by $\kappa(S)$. The sorted elements of $\lambda+\delta+se_n$ are

$$(20)\qquad (\lambda_1+n-1,\dots,\lambda_{n-j-1}+j+1, s, \lambda_{n-j}+j,\dots,\lambda_{n-1}+1),$$

where j is the unique integer such that

$$(21)\qquad \lambda_{n-j-1}+j+1 > s > \lambda_{n-j}+j.$$

If $s=0$, define $j=0$. Notice that $w(S)$ is therefore a $(j+1)$-cycle, and so $\varepsilon_{w(S)} = (-1)^j$. By the definition of the horizontal sequence, $s = h_k$ is the kth smallest positive integer not among the terms of $\lambda+\delta$. Therefore, $h_k = j+k$ and, in particular, $\varepsilon_{w(S)} = (-1)^{s-k}$ in this case.

Let $E_j = \{n-1,\dots,n-j\}$. Notice that

$$\lambda+\delta+se_n-\kappa(E_j) = w(S)\circ((\lambda\cup k)+\delta).$$

This is clear from (20) if one decrements the last j terms by 1 and subtracts δ. If we choose to include all of E_j in S, the remaining k members of S must be chosen from the first $n-j-1$ integers. Clearly, in order to maintain distinct terms in (20), the rows of $\lambda\cup k$ which are chosen must be a collection of rows from which one can remove a vertical strip. Furthermore, the cells in such a vertical strip must occupy the columns $k+1, k+2, \dots$.

More generally, if we do not include all of E_j in S, we must add cells to the first k columns (or equivalently, the last j rows) of $\lambda \cup k$ in such a way that distinct terms are maintained; i.e., we must add a vertical strip. We conclude that $B_k(\lambda) = \{\mu(S)\colon |S| = h_k\}$.

The analysis for the case when $|S| = s = h_k + i$, where $h_k < s < h_{k+1}$, introduces only slight complications. Notice that each of the integers $h_k + 1, \ldots, h_k + i$ must appear among the terms of $\lambda + \delta$; say,

$$\lambda + \delta = \big(\lambda_1 + n - 1, \ldots, \lambda_{n-j-i-1} + j + i + 1, \\ h_k + i, \ldots, h_k + 1, \lambda_{n-j} + j, \ldots, \lambda_n\big), \tag{22}$$

where j is the integer defined in (21). In order to avoid having equal terms in $\lambda + \delta + se_n - \kappa(S)$, we are forced to include all of

$$T = \{n - j - i, \ldots, n - j - 1\}$$

in S. Notice that

$$\lambda + \delta + se_n - \kappa(T) = \big(\lambda_1 + n - 1, \ldots, \lambda_{n-j-i-1} + j + i + 1, \\ h_k + i - 1, \ldots, h_k, \lambda_{n-j} + j, \ldots, \lambda_{n-1} + 1, h_k + i\big). \tag{23}$$

Therefore, $w(S)$ must be a $(j + i + 1)$-cycle, and so

$$\varepsilon_{w(S)} = (-1)^{i+j} = (-1)^{h_k + i - k} = (-1)^{s-k},$$

as desired. Furthermore, we see from (22) that if $|S'| = h_k$ and the terms of $\lambda + \delta + h_k e_n - \kappa(S')$ are distinct, then $S' \cap T = \varnothing$. Since (23) implies that the terms of $\lambda + \delta + se_n - \kappa(T)$ are those of $\lambda + \delta + h_k e_n$, we deduce that the partitions $\mu(S)$ obtained for the s-subsets S coincide with the partitions $\mu(S')$ for h_k-subsets S'. This completes the proof of the lemma. □

We may now complete the proof of the theorem. Let S range over s-subsets of $[n-1]$ for which the terms of $\lambda + \delta + se_n - \kappa(S)$ are distinct. Let r be the rectangular rank of λ. Using the notation defined in Lemma 5.4 and applying Lemma 5.5 yields

$$\begin{aligned}(1 - z) &\sum_{S \subseteq [n-1]} \varepsilon_{w(S)} (-z)^s C^n[\mu(S)](z,q) \\ &= (1 - z) \sum_{0 \leqslant k \leqslant r} \sum_{\mu \in B_k(\lambda)} (-1)^k (z^{h_k} + \cdots + z^{h_{k+1}-1}) C^n[\mu](z,q) \\ &= \sum_{0 \leqslant k \leqslant r} \sum_{\mu \in B_k(\lambda)} (-1)^k z^{h_k} (1 - z^{h_{k+1} - h_k}) C^n[\mu](z,q),\end{aligned} \tag{24}$$

and

$$\begin{aligned}(1 - z) q^l z^{n-1} &\sum_{S \subseteq [n-1]} \varepsilon_{w(S)} (-z)^{-s} C^n[\mu(S)](z,q) \\ &= (1 - z) q^l \sum_{0 \leqslant k \leqslant r} \sum_{\mu \in B_k(\lambda)} (-1)^k (z^{n-1-h_k} + \cdots + z^{n-h_{k+1}}) C^n[\mu](z,q) \\ &= \sum_{0 \leqslant k \leqslant r} \sum_{\mu \in B_k(\lambda)} (-1)^k q^l z^{n-h_{k+1}} (1 - z^{h_{k+1} - h_k}) C^n[\mu](z,q).\end{aligned} \tag{25}$$

Compare (24) and (25) and apply Lemma 5.4. □

6. The first layer formula for $C^n[\lambda](z,q)$. We will now apply the recursion developed in the previous section to find an explicit formula for $C^n[\lambda](z,q)$ for first layer partitions λ. Recall that the hooklength $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ associated with the cell (i,j) of a partition λ is the number of cells directly below or to the right of (i,j), including (i,j).

THEOREM 6.1. *Let* λ *be a partition of* n. *We have*

$$C^n[\lambda](z,q) = \frac{(q;q)_\infty}{(qz^n;q)_\infty (q;z)_n} \cdot \prod_{(i,j)\in\lambda} \frac{z^{j-1} - qz^{i-1}}{1 - z^{h(i,j)}}.$$

The reader can easily verify that we recover the formula (Corollary 3.3) for $C^n[\varnothing](z,q)$ in the special case $\lambda = 1^n$. If we specialize to the exterior algebra $\Lambda(gl_n)$, we obtain an identity that was conjectured by Gupta and Hanlon [**6**]. This identity can also be proved by an intricate application of the Littlewood-Richardson rule [**18**].

COROLLARY 6.2. *Let* λ *be a partition of* n. *We have*

$$E^n[\lambda](q) = [n]!_{q^2} \prod_{(i,j)\in\lambda} \frac{q^{2i-1} + q^{2j-2}}{1 - q^{2h(i,j)}}.$$

If we specialize to the Macdonald complex, we find

COROLLARY 6.3. *Let* λ *be a partition of* n. *We have*

$$M_k^n[\lambda](q) = \frac{[n(k+1)]!_q}{(q;q^{k+1})_n} \cdot \prod_{(i,j)\in\lambda} \frac{q^{(k+1)(j-1)} - q^{(k+1)(i-1)+1}}{1 - q^{(k+1)h(i,j)}}.$$

If we specialize to the symmetric algebra, we obtain a formula that can be used to compute the first layer generalized exponents of SL_n. Gupta has shown that this identity can also be derived from the work of Macdonald [**13**, III, Examples 6.2,4], Hesselink [**8**], and D. Peterson.

COROLLARY 6.4. *Let* λ *be a partition of* n. *We have*

$$S^n[\lambda](z) = \prod_{(i,j)\in\lambda} \frac{z^{j-1}}{1 - z^{h(i,j)}}.$$

The first step in proving the first layer formula is to examine more carefully the recursion in Theorem 5.1. Notice that the partitions $\mu \in B_k(\lambda)$ are obtained by adding and removing vertical strips from the partition $\lambda \cup k$. This is formally similar to the process involved in certain instances of the Littlewood-Richardson rule. Specifically, we remark that a well-known consequence of the Littlewood-Richardson rule [**13**, I, (5.1,17)] is the following:

$$\begin{aligned} s_\lambda s_{1^k} &= \sum_\mu s_\mu\colon\ \mu/\lambda \text{ is a vertical } k\text{-strip}, \\ s_{\lambda/1^k} &= \sum_\mu s_\mu\colon\ \lambda/\mu \text{ is a vertical } k\text{-strip}. \end{aligned} \tag{26}$$

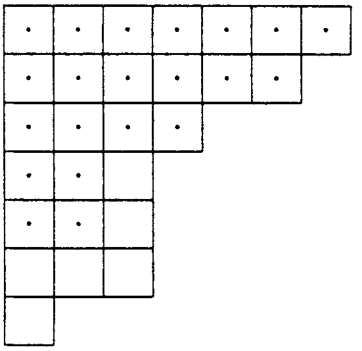

FIGURE 3. An illustration of the construction of $\lambda * k$.

In order to exploit this situation, let $\psi_n\colon \Lambda \to \mathbf{Z}[[z,q]]$ be an arbitrary $\mathbf{Z}$-linear transformation for which

1. $\psi_n(s_\lambda)$ vanishes unless $l(\lambda) \leqslant n$.
2. $\psi_n(s_\lambda) = \psi_n(s_\mu)$ if $\mu = \lambda + c \cdot 1^n$.

The practical reader will complain that we have given an overly elaborate presentation of what should be considered a group homomorphism $\Omega_n \to \mathbf{Z}[[z,q]]$. However, we will find it more convenient to allow an arbitrary symmetric function in the domain of ψ_n. Our goal is to impose constraints on ψ_n in such a way that to have $\psi_n(s_\lambda) = C^n[\lambda](z,q)$ is the only way to meet those constraints.

Let λ vary over partitions with $l(\lambda) < n$ and $|\lambda|$ divisible by n. For any nonnegative integer k, define a partition $\lambda * k$ via

$$(27)\qquad (\lambda * k)' = \begin{cases} (n, \lambda_1' + 1, \ldots, \lambda_{k-1}' + 1, \lambda_{k+1}', \lambda_{k+2}', \ldots) & \text{if } k \geqslant 0, \\ \lambda' & \text{if } k = 0. \end{cases}$$

For example, if $n = 7$, $k = 3$, and $\lambda = 76422$, then $\lambda * k = 7643331$. See Figure 3, in which the diagram of $\lambda * k$ is given, and the cells of λ have been marked. Notice that the partition $\lambda * k$ can also be defined as the partition obtained by adding a cell to each row of $\lambda \cup k$ which has fewer than k cells. Also note that $\lambda * k/\lambda$ is a border strip of size h_k; i.e., a connected skew diagram which contains no 2×2 square as a subdiagram.

The recursion in Theorem 5.1 can be reformulated as follows.

THEOREM 6.5. *Let $\psi_n\colon \Lambda \to \mathbf{Z}[[z,q]]$ be a $\mathbf{Z}$-linear transformation as described above. Let λ be a partition with $l(\lambda) < n$ and $|\lambda|$ divisible by n. Let l denote the layer, r the rectangular rank, and h the horizontal sequence of λ. If*

$$(28)\qquad \sum_{0 \leqslant k \leqslant r} (-1)^k \left(z^{h_k} - q^l z^{n-h_{r+1}}\right)\left(1 - z^{h_{r+1}-h_k}\right)\psi_n\left(s_{\lambda * k/1^{h_k}}\right) = 0$$

for all such λ, then $\psi_n(s_\lambda)$ must be, apart from multiplicative factors, $C^n[\lambda](z,q)$; i.e., there must exist a formal power series $c(z,q)$, independent of λ, such that

$$\psi_n(s_\lambda) = c(z,q)C^n[\lambda](z,q).$$

PROOF. For $k \geqslant 0$, define

$$g_k = \sum_{\mu \in B_k(\lambda)} s_\mu \quad \text{and} \quad f_k = s_{\lambda * k/1^{h_k}}.$$

Let $k \geqslant 1$ and assume that the lowest row of $\lambda \cup k$ of length k is the jth row. Recall that the partitions $\mu \in B_k(\lambda)$ are obtained by adding a vertical strip below the jth row, and deleting a vertical strip from the part of λ occupying columns $k+1,\ k+2, \ldots$. Since $\lambda * k$ is obtained by adding a cell to every row of $\lambda \cup k$ below the jth, we see that $\mu \in B_k(\lambda)$ if and only if μ can be obtained by deleting a vertical h_k-strip from $\lambda * k$, provided that this strip does not include the cell (j, k). If one removes a vertical h_k-strip from $\lambda * k$ which does include the cell (j, k), one obtains a partition in $B_{k-1}(\lambda)$, and conversely. By our previous remarks (see (26)), we conclude that $f_0 = g_0$ and $f_k = g_k + g_{k-1}$.

Hence,

$$\sum_{0 \leqslant k \leqslant r} (-1)^k \left(z^{h_k} - q^l z^{n-h_{r+1}}\right)\left(1 - z^{h_{r+1}-h_k}\right)\psi_n(f_k)$$
$$= \sum_{0 \leqslant k \leqslant r} (-1)^k \left(z^{h_k} - q^l z^{n-h_{k+1}}\right)\left(1 - z^{h_{k+1}-h_k}\right)\psi_n(g_k).$$

Therefore, we see that (28) is satisfied if and only if $\psi_n(s_\lambda)$ satisfies the recursion in Theorem 5.1. The uniqueness of the solution is guaranteed by part (b) of Theorem 5.1. $\square$

REMARK 6.6. Notice that a similar result holds even if we restrict our attention to the first layer. If λ is in the first layer, then only partitions of n appear in $B_k(\lambda)$. Therefore, if we have a $\mathbf{Z}$-linear map $\psi_n \colon \Lambda^n \to \mathbf{Z}[[z, q]]$ (defined only for symmetric functions of degree n) which satisfies (28) for all partitions λ of n, then the series $\psi_n(s_\lambda)$ satisfy (16) for all such λ. We may conclude that $\psi_n(s_\lambda)$ and $C^n[\lambda](z, q)$ differ by at most a multiplicative factor.

Recall that in the Frobenius notation for partitions, if λ has rank r, we write $\lambda = (\alpha \mid \beta)$, where $\alpha, \beta \in \mathbf{N}^r$ are the strictly decreasing r-tuples defined by

$$\alpha_i = \lambda_i - i, \quad \beta_i = \lambda'_i - i \qquad (1 \leqslant i \leqslant r).$$

LEMMA 6.7. *Let λ be a first layer partition of rectangular rank r, and let h be the horizontal sequence of λ. Define $\alpha \in \mathbf{N}^r$, $\beta \in \mathbf{N}^{r+1}$ via*

$$\begin{aligned} \alpha_i &= \lambda_i - i - 1, & 1 \leqslant i \leqslant r, \\ \beta_i &= n - h_i = \lambda'_i - i + 1, & 1 \leqslant i \leqslant r+1. \end{aligned}$$

If $1 \leqslant k \leqslant r+1$, then

$$s_{\lambda * k/1^{h_k}} = s_{1^{\beta_k}} \cdot s_{(\alpha \mid \beta_1, \ldots, \beta_{k-1}, \beta_{k+1}, \ldots, \beta_{r+1})}.$$

PROOF. We claim that the skew shape $\lambda * k/1^{h_k}$ is disconnected. Since the diagram of 1^{h_k} is a single column, this is equivalent to showing that $h_k \geqslant (\lambda * k)'_2$. By the definition of $\lambda * k$, we see that we must show

$$h_k \geqslant \begin{cases} \lambda'_2 & \text{if } k = 1, \\ \lambda'_1 + 1 & \text{if } k > 1. \end{cases}$$

However, these are immediate from the fact that $h_k = n - \lambda'_k + k - 1$ and λ is a partition of n. Therefore $\lambda * k/1^{h_k}$ is indeed disconnected.

Since the first column of $\lambda * k/1^{h_k}$ has length $n - h_k = \beta_k$, we have

$$s_{\lambda * k/1^{h_k}} = s_{1^{\beta_k}} \cdot s_\mu,$$

where μ is the partition obtained by deleting the first column of $\lambda * k$. Notice that μ is of rank r. Inspection of (27) reveals that $\mu = (\alpha \mid \beta_1, \ldots, \beta_{k-1}, \beta_{k+1}, \ldots, \beta_{r+1})$ is the Frobenius notation for μ. □

The following beautiful identity expresses the Schur function s_λ as a determinant of Schur functions corresponding to hooks; i.e., Schur functions of the form $s_{(a \mid b)}$ with a, $b \in \mathbf{N}$. An elementary proof is given by Macdonald [**13**, I, Example 3.9], who attributes the result to Giambelli [**4**].

PROPOSITION 6.8. *Let $(\alpha \mid \beta)$ be the Frobenius notation for a partition λ of rank r. We have*

$$s_\lambda = \det\left[s_{(\alpha_i \mid \beta_j)} \colon 1 \leqslant i, j \leqslant r\right].$$

LEMMA 6.9. *Let λ be a first layer partition with notation as in Lemma 6.7. Let M_λ denote the matrix of order $r + 1$ defined by*

$$M_\lambda = \begin{bmatrix} (1 - z^{n-\beta_1})(1 - qz^{\beta_1}) s_{1^{\beta_1}} & s_{(\alpha_1 \mid \beta_1)} & \cdots & s_{(\alpha_r \mid \beta_1)} \\ \vdots & \vdots & & \vdots \\ (1 - z^{n-\beta_{r+1}})(1 - qz^{\beta_{r+1}}) s_{1^{\beta_{r+1}}} & s_{(\alpha_1 \mid \beta_{r+1})} & \cdots & s_{(\alpha_r \mid \beta_{r+1})} \end{bmatrix}.$$

The identity $\psi_n(\det M_\lambda) = 0$ is equivalent to (28).

The pedantic reader may wish to enhance the coefficient ring of Λ to include all of $\mathbf{Z}[[z, q]]$ before applying ψ_n to $\det(M_\lambda)$.

PROOF. The identity (28) is equivalent to

$$\begin{aligned}(1 - qz^{n-h_{r+1}})&(1 - z^{h_{r+1}})\psi_n(s_\lambda) \\ &= \sum_{1 \leqslant k \leqslant r} (-1)^{k-1}(z^{h_k} - qz^{n-h_{r+1}})(1 - z^{h_{r+1}-h_k})\psi_n(s_{\lambda * k/1^{h_k}}).\end{aligned}$$

By Lemma 6.7, this is equivalent to

(29)

$$\begin{aligned}(1 - qz^{\beta_{r+1}})&(1 - z^{n-\beta_{r+1}})\psi_n(s_\lambda) \\ &= \sum_{k=1}^{r+1} (-1)^{k-1}(z^{n-\beta_k} - qz^{\beta_{r+1}})(1 - z^{\beta_k - \beta_{r+1}})\psi_n(s_{1^{\beta_k}} s_{(\alpha \mid \beta_1, \ldots, \beta_{k-1}, \beta_{k+1}, \ldots, \beta_{r+1})}).\end{aligned}$$

Let N_λ denote the matrix of order $r + 1$ whose kth row $(1 \leqslant k \leqslant r + 1)$ is

$$\left((z^{n-\beta_k} - qz^{\beta_{r+1}})(1 - z^{\beta_k - \beta_{r+1}}), s_{(\alpha_1 \mid \beta_k)}, \ldots, s_{(\alpha_r \mid \beta_k)}\right).$$

If we expand the determinant of N_λ by minors along the first column and apply Proposition 6.8, we see that (29) is equivalent to

(30) $$(1 - qz^{\beta_{r+1}})(1 - z^{n-\beta_{r+1}})\psi_n(s_\lambda) = \psi_n(\det N_\lambda).$$

On the other hand, suppose for the moment that $\beta_{r+1} > 0$; i.e., $\lambda'_{r+1} > r$. If so, then λ is of rank $r + 1$ and the Frobenius notation for λ must be

$$(\alpha_1 + 1, \ldots, \alpha_r + 1, 0 \mid \beta_1 - 1, \ldots, \beta_{r+1} - 1).$$

By Proposition 6.8, it follows that s_λ is the determinant of the matrix of order $r + 1$ whose kth row is

$$(31) \qquad \left(s_{(\alpha_1+1|\beta_k-1)}, \ldots, s_{(\alpha_r+1|\beta_k-1)}, s_{1^{\beta_k}}\right).$$

If it should happen that $\beta_{r+1} = 0$, then λ is only of rank r, but we may still assert that s_λ is the determinant of the matrix defined by (31), provided that we define $s_{(a|-1)} = 0$ for $a > 0$, since the last row of the matrix would become $(0, \ldots, 0, 1)$ in this case.

Notice that by (26), we have

$$s_{(a+1)} \cdot s_{1^b} = s_{(a|b)} + s_{(a+1|b-1)}.$$

Therefore, by subtracting suitable multiples of the $(r + 1)$th column of (31) from the other columns, we deduce that s_λ is the determinant of the matrix whose kth row is

$$\left(-s_{(\alpha_1|\beta_k)}, \ldots, -s_{(\alpha_r|\beta_k)}, s_{1^{\beta_k}}\right)$$

or equivalently,

$$(32) \qquad \left(s_{1^{\beta_k}}, s_{(\alpha_1|\beta_k)}, \ldots, s_{(\alpha_r|\beta_k)}\right).$$

This remains valid even when $\beta_{r+1} = 0$.

It is now clear that $\psi_n(\det M_\lambda) = 0$ is equivalent to (28); replace s_λ in (30) by the determinant of the matrix in (32) and use the linearity of the determinant in the first column. □

If f is a symmetric function, let the notation $f(p_r \to a_r)$ be an abbreviation for the expression obtained by writing f as a polynomial in the power-sum symmetric functions p_r, and replacing each occurrence of p_r by a_r, where the terms a_r are chosen from some suitable commutative ring.

We are now sufficiently prepared to give the

PROOF OF THEOREM 6.1. We claim that if we define

$$\psi_n(s_\mu) = s_\mu\left(p_k \to (-1)^{k-1}\frac{1 - q^k}{1 - z^k}\right)$$

for every partition μ of n, then all of the identities $\psi_n(\det M_\lambda) = 0$ are satisfied for first layer partitions λ. By an identity due to D. E. Littlewood [**10**, Chapter VII] (see also [**13**, I, Examples 2.5, 3.3]), we can give an explicit formula for $\psi_n(s_\mu)$; namely,

$$(33) \qquad \psi_n(s_\mu) = \prod_{(i,j)\in\mu} \frac{z^{j-1} - qz^{i-1}}{1 - z^{h(i,j)}}.$$

Notice that since the substitution

$$p_k \to (-1)^{k-1}\frac{1 - q^k}{1 - z^k}$$

defines a ring homomorphism $\Lambda \to \mathbf{Z}[[z, q]]$, we may verify that $\psi_n(\det M_\lambda)$ vanishes by showing that

$$\det\left[M_\lambda\left(p_k \to (-1)^{k-1}\frac{1-q^k}{1-z^k}\right)\right] = 0. \tag{34}$$

By (33), we have

$$s_{(a|b)}\left(p_k \to (-1)^{k-1}\frac{1-q^k}{1-z^k}\right) = \frac{(q;z)_{b+1} f_a(z,q)}{[a]!_z [b]!_z} \cdot \frac{1}{1-z^{a+b+1}},$$

where

$$f_a(z,q) = (z-q)(z^2-q)\cdots(z^a-q).$$

Therefore, the kth row of the matrix in (34) is

$$\frac{(q;z)_{\beta_k+1}}{[\beta_k]!_z}\left(1-z^{n-\beta_k}, \frac{f_{\alpha_1}(z,q)}{[\alpha_1]!_z}\cdot\frac{1}{1-z^{\alpha_1+\beta_k+1}}, \dots, \frac{f_{\alpha_r}(z,q)}{[\alpha_r]!_z}\cdot\frac{1}{1-z^{\alpha_r+\beta_k+1}}\right),$$

where α and β are as defined in Lemma 6.7. By rescaling the rows and columns of this matrix, we deduce that the vanishing of $\psi_n(\det M_\lambda)$ is equivalent to the vanishing of the determinant of the matrix whose kth row is

$$\left(1-z^{n-\beta_k}, \frac{1}{1-z^{\alpha_1+\beta_k+1}}, \dots, \frac{1}{1-z^{\alpha_r+\beta_k+1}}\right). \tag{35}$$

Consider the determinant of the matrix whose kth row is

$$\left(1, \frac{1}{1-z^{\alpha_1+\beta_k+1}}, \dots, \frac{1}{1-z^{\alpha_r+\beta_k+1}}\right). \tag{36}$$

Subtracting the first column from the remaining r columns yields

$$\left(1, \frac{z^{\alpha_1+\beta_k+1}}{1-z^{\alpha_1+\beta_k+1}}, \dots, \frac{z^{\alpha_r+\beta_k+1}}{1-z^{\alpha_r+\beta_k+1}}\right).$$

Extracting common factors from the rows and columns of this matrix shows that the determinant of (36) is the same as the determinant of the matrix whose kth row is

$$\left(z^{n-\beta_k}, \frac{1}{1-z^{\alpha_1+\beta_k+1}}, \dots, \frac{1}{1-z^{\alpha_r+\beta_k+1}}\right). \tag{37}$$

Comparison of (36) and (37) shows that the matrix defined in (35) must indeed be singular, so our claim is verified.

Now we are virtually finished. By Lemma 6.9, we deduce that the choices of $\psi_n(s_\mu)$ for partitions μ of n which we made in (33) actually satisfy the linear relations in (28). By Theorem 6.5 (see Remark 6.6), it follows that there must be a formal power series $c(z, q)$ such that

$$C^n[\lambda](z,q) = c(z,q)\cdot\prod_{(i,j)\in\lambda}\frac{z^{j-1}-qz^{i-1}}{1-z^{h(i,j)}}$$

for all partitions λ of n. The series $c(z, q)$ can be determined by taking $\lambda = 1^n$ and applying Corollary 3.3. □

7. An extension of the q-Dyson Theorem. The first layer formula (Theorem 6.1) has essentially provided us with an explicit formula for the coefficients of the monomials $x^{\alpha+\delta}$ in the formal Laurent series

$$G_n(x_1,\ldots,x_n) = a_\delta(x) \prod_{1\leqslant i,j\leqslant n} \frac{(qx_ix_j^{-1};q)_\infty}{(zx_ix_j^{-1};q)_\infty}$$

for those dominant weights $\alpha \in \mathbf{Z}^n$ corresponding to first layer partitions; i.e., dominant weights with $\alpha_n = -1$. If we take $z = q^k$ (which suffers no genuine loss of generality) and delete those factors corresponding to terms with $i = j$, we may equivalently view the first layer formula as a formula for the coefficients of certain monomials in

$$\prod_{1\leqslant i<j\leqslant n} (x_jx_i^{-1};q)_k(qx_ix_j^{-1};q)_{k-1}.$$

Specifically, we have

$$[x^\alpha]\prod_{i<j}(x_jx_i^{-1};q)_k(qx_ix_j^{-1};q)_{k-1} = [k-1]!_q^{-n}\cdot C^n[\alpha](q^k,q).$$

Thus, the first layer formula gives a generalization of Bressoud and Goulden's result (Theorem 3.2) in the case of equal parameters ($a_1 = \cdots = a_n = k$). It should be pointed out, however, that we used their identity (via Corollary 3.3) to prove this generalization.

In this section, we will show that the first layer formula can also be used to find some coefficients of monomials (not just constant terms) in the formal Laurent series

$$H_n(x_1,\ldots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{(x_jx_i^{-1};q)_\infty}{(zx_jx_i^{-1};q)_\infty}\cdot\frac{(qx_ix_j^{-1};q)_\infty}{(qzx_ix_j^{-1};q)_\infty},$$

and, in particular ($z = q^k$), some coefficients of monomials in

$$\prod_{1\leqslant i<j\leqslant n} (x_jx_i^{-1};q)_k(qx_ix_j^{-1};q)_k.$$

Thus, we will find a generalization of the equal-parameter version of the q-Dyson Theorem. In order to compute these new coefficients, we need to introduce some additional tools from the theory of symmetric functions.

For each $\alpha \in \mathbf{N}^n$, define a symmetric function $R^n_\alpha(x_1,\ldots,x_n;q) \in \Lambda_n[q]$ via

$$R^n_\alpha(x_1,\ldots,x_n;q) = \sum_{w\in S_n} w\left(x^\alpha \prod_{1\leqslant i<j\leqslant n} \frac{x_i - qx_j}{x_i - x_j}\right) \tag{38}$$

$$= \frac{1}{a_\delta(x)} \sum_{w\in S_n} \varepsilon_w w\left(x^\alpha \prod_{1\leqslant i<j\leqslant n} (x_i - qx_j)\right). \tag{39}$$

Notice that R^n_α is homogeneous of degree $|\alpha|$ with respect to $x_1,\ldots,x_n$. It is clear from (39) that there must exist polynomials $a_{\alpha\lambda}(q) \in \mathbf{Z}[q]$ such that

$$R^n_\alpha(x_1,\ldots,x_n;q) = \sum_{|\lambda|=k} a_{\alpha\lambda}(q)s_\lambda(x_1,\ldots,x_n), \tag{40}$$

where $|\alpha| = k$. We mention that these functions R^n_α are only slightly more general than the symmetric functions R_λ considered by Macdonald [13, III.1].

If $|\alpha| \leqslant n$, then the partitions λ which appear in (40) all have length at most n, and so we conclude that there is a unique symmetric function in $\Lambda[q]$, which we denote by R_α, whose image in $\Lambda_n[q]$ is R^n_α. In fact, from (40), we see that

$$R_\alpha(x_1, x_2, \ldots; q) = \sum_{|\lambda| = k} a_{\alpha\lambda}(q) s_\lambda(x_1, x_2, \ldots). \tag{41}$$

These symmetric functions differ only slightly from the Hall-Littlewood symmetric functions [13, III]. When $\lambda \in \mathbf{N}^n$ is a partition, $R_\lambda(x; q)$ and the Hall-Littlewood symmetric function $P_\lambda(x; q)$ differ only by a polynomial in q. However, it should be emphasized that even though we regard as identical two representations of a partition which differ only in the number of trailing zeros, the symmetric function $R_\alpha(x; q)$ is defined only for finite sequences of nonnegative integers. In particular, $R_\lambda(x; q)$ depends not only on the partition λ but on the chosen integer $n \geqslant l(\lambda)$ for which $\lambda \in \mathbf{N}^n$.

The coefficients we will extract from $H_n(x_1, \ldots, x_n)$ are the coefficients of the monomials x^α where $\alpha \in \mathbf{Z}^n$ and $\alpha_i \geqslant -1$. Of course, one could apply the transformation $x_i \to x^{-1}_{n+1-i}$ and thus obtain identical results about the coefficients of x^α with $\alpha_i \leqslant 1$. The following result connects these coefficient extraction problems to the symmetric functions R_β.

THEOREM 7.1. *Let $\alpha \in \mathbf{Z}^n$ and assume $|\alpha| = 0$, $\alpha_i \geqslant -1$ $(1 \leqslant i \leqslant n)$. We have*

$$[x^\alpha] H_n(x_1, \ldots, x_n) = \frac{(z; q)^n_\infty}{(q; q)^{n-1}_\infty (qz^n; q)_\infty (q; z)_n} \cdot R_\beta\left(p_r \to (-1)^{r-1} \frac{1 - q^r}{1 - z^r}; z\right),$$

where $\beta = \alpha + 1^n = (\alpha_1 + 1, \ldots, \alpha_n + 1)$.

In the special case $z = q^k$, we find

COROLLARY 7.2. *Let α, β be as described above. We have*

$$[x^\alpha] \prod_{i<j} \left(x_j x_i^{-1}; q\right)_k \left(q x_i x_j^{-1}; q\right)_k = \frac{[kn]!_q}{[k-1]!^n_q (q; q^k)_n} \cdot R_\beta\left(p_r \to (-1)^{r-1} \frac{1 - q^r}{1 - q^{kr}}; q^k\right).$$

PROOF OF THEOREM 7.1. By the definitions of H_n and G_n, we have

$$H_n(x_1, \ldots, x_n) = x^{-\delta} \frac{(z; q)^n_\infty}{(q; q)^n_\infty} G_n(x_1, \ldots, x_n) \prod_{i<j} \left(1 - z x_i x_j^{-1}\right). \tag{42}$$

Let $\binom{[n]}{2} = \{(i, j)\colon 1 \leqslant i < j \leqslant n\}$. For each subset $S \subseteq \binom{[n]}{2}$ define a sequence $\gamma = \gamma(S) \in \mathbf{Z}^n$ via

$$\gamma_i = |\{ j\colon 1 \leqslant j \leqslant n, (j, i) \in S\}| - |\{ j\colon 1 \leqslant j \leqslant n, (i, j) \in S\}|.$$

Let $\alpha \in \mathbf{Z}^n$ and assume that $|\alpha| = 0$ and $\alpha_i \geqslant -1$. It follows from (42) that

$$[x^\alpha] H_n(x_1, \ldots, x_n) = \frac{(z; q)^n_\infty}{(q; q)^n_\infty} \sum_{S \subseteq \binom{[n]}{2}} (-z)^{|S|} [x^{\alpha + \delta + \gamma(S)}] G_n(x_1, \ldots, x_n). \tag{43}$$

If $S \subseteq \binom{[n]}{2}$ has the property that the terms of $\alpha + \delta + \gamma(S)$ are all distinct, let $\alpha(S)$ denote the unique dominant weight and $w(S) \in S_n$ the unique permutation such that

$$\alpha + \delta + \gamma(S) = w(S) \circ (\alpha(S) + \delta).$$

Since G_n is an alternating function, it follows from (43) and (17) that

$$(44) \qquad [x^\alpha]H_n(x_1, \ldots, x_n) = \frac{(z;q)_\infty^n}{(q;q)_\infty^n} \sum_{S \subseteq \binom{[n]}{2}} \varepsilon_{w(S)}(-z)^{|S|} C^n[\alpha(S)](z,q),$$

summed over those $S \subseteq \binom{[n]}{2}$ for which the terms of $\alpha + \delta + \gamma(S)$ are distinct. Notice that the terms of $\delta + \gamma(S)$ are nonnegative; therefore, the terms of $\alpha + \delta + \gamma(S)$ are all at least -1, and so the dominant weights $\alpha(S)$ which appear in (44) correspond to partitions in the 0th or first layers. Recall that when we proved the first layer formula, we showed that

$$C^n[\lambda](z,q) = \frac{(q;q)_\infty}{(qz^n;q)_\infty (q;z)_n} s_\lambda\left(p_r \to (-1)^{r-1} \frac{1-q^r}{1-z^r} \right)$$

for partitions λ of n. Using this information in (44) yields

$$(45) \quad [x^\alpha]H_n(x_1, \ldots, x_n) = \frac{(z;q)_\infty^n}{(q;q)_\infty^{n-1} (qz^n;q)_\infty (q;z)_n} \cdot \sum_{S \subseteq \binom{[n]}{2}} \varepsilon_{w(S)}(-z)^{|S|} s_{\lambda(S)}\left(p_r \to (-1)^{r-1} \frac{1-q^r}{1-z^r} \right),$$

where $\lambda(S) = \alpha(S) + 1^n$.

On the other hand, let us consider the symmetric function $R_\beta(x;q)$, where $\beta = \alpha + 1^n$. It follows from (39) that

$$\begin{aligned} a_\delta(x) R_\beta(x_1, \ldots, x_n; q) &= \sum_{w \in S_n} \varepsilon_w \cdot w\left(x^{\beta+\delta} \prod_{1 \leqslant i < j \leqslant n} \left(1 - q x_j x_i^{-1}\right) \right) \\ &= \sum_{S \subseteq \binom{[n]}{2}} (-q)^{|S|} a_{\beta+\delta+\gamma(S)}(x_1, \ldots, x_n). \end{aligned}$$

Therefore, since $s_\lambda = a_{\alpha+\delta}/a_\delta$,

$$(46) \qquad R_\beta(x;q) = \sum_{S \subseteq \binom{[n]}{2}} \varepsilon_{w(S)}(-q)^{|S|} s_{\lambda(S)}(x).$$

The theorem follows upon comparison of (45) and (46). □

We remark that (46) is essentially equivalent to an identity given by Macdonald [**13**, I, Example 2.3] for Hall-Littlewood symmetric functions.

Theorem 7.1 compels us to study the effect of substitutions of the form

$$p_k \to \frac{a^k - b^k}{1 - q^k}$$

on the symmetric functions R_α where $\alpha \in \mathbf{N}^n$ and $|\alpha| = n$. We know of no general result that gives explicit formulas for such substitutions. However, we will be able to give an explicit formula in the case where the 0's of α occur consecutively. The resulting formula is the following:

THEOREM 7.3. *Let $\gamma \in \mathbf{Z}^n$, $|\gamma| = 0$, and assume that there exist $\alpha \in \mathbf{N}^r$, $\beta \in \mathbf{N}^s$ such that γ is of the form*

$$\gamma = (\alpha_1, \dots, \alpha_r, -1, \dots, -1, \beta_s, \dots, \beta_1).$$

Let $t = n - r - s =$ the number of -1's in γ. We have

$$[x^\gamma] H_n(x_1, \dots, x_n) = (-1)^t q^{|\alpha|} z^{m(\alpha,\beta)} \cdot \frac{(qz;q)_\infty^n [t]!_z (q;z)_{r+s}}{(q;q)_\infty^{n-1} (qz^n;q)_\infty (q;z)_n},$$

where

$$m(\alpha, \beta) = st + \sum (i-1)\alpha_i - \sum i\beta_i.$$

The special case $z = q^k$ yields

COROLLARY 7.4. *Let α, β, γ be as described above. We have*

$$[x^\gamma] \prod_{1 \leqslant i < j \leqslant n} \left(x_j x_i^{-1}; q\right)_k \left(q x_i x_j^{-1}; q\right)_k$$
$$= (-1)^t q^{|\alpha| + km(\alpha,\beta)} \cdot \frac{[kn]!_q [t]!_{q^k} (q; q^k)_{r+s}}{[k]!_q^n (q; q^k)_n}.$$

In particular, when $\gamma = \varnothing$, we recover the q-Dyson Theorem for equal parameters. Also it is interesting to consider the limit $q \to 1$ and thus obtain a generalization of the original Dyson Theorem. In this case we need not worry about where the -1's occur in γ since the limiting series

$$\prod_{i \neq j} \left(1 - x_i x_j^{-1}\right)^k$$

is a symmetric function of $x_1, \dots, x_n$.

COROLLARY 7.5. *Let $\gamma \in \mathbf{Z}^n$, $|\gamma| = 0$, and suppose that $\gamma_i \geqslant -1$ $(1 \leqslant i \leqslant n)$. If there are t -1's among the terms of γ, then*

$$[x^\gamma] \prod_{i \neq j} \left(1 - x_i x_j^{-1}\right)^k = (-1)^t \frac{(nk)!}{(k!)^n} \cdot \frac{k^t \cdot t!}{(1 + k(n-1)) \cdots (1 + k(n-t))}.$$

As a first step toward proving Theorem 7.3, we show that the computation of formulas for

$$R_\beta\left(p_r \to (-1)^{r-1} \frac{1 - q^r}{1 - z^r}; z\right)$$

is no more difficult than the computation of formulas for

$$R_\beta(1, q, \dots, q^{m-1}; q).$$

PROPOSITION 7.6. *If $f \in \Lambda^k$ is a homogeneous symmetric function, and $F(x, y)$ is a formal power series such that $f(1, q, \ldots, q^{m-1}) = F(q^m, q)$ for all sufficiently large integers m, then*

$$f\left(p_r \to \frac{a^r - b^r}{1 - q^r}\right) = a^k F(ba^{-1}, q).$$

PROOF. Notice that

$$p_r(a, aq, \ldots, aq^{m-1}) = \frac{a^r - (aq^m)^r}{1 - q^r}.$$

Therefore, since f is homogeneous,

$$f\left(p_r \to \frac{a^r - (aq^m)^r}{1 - q^r}\right) = f(a, aq, \ldots, aq^{m-1}) = a^k F(q^m, q).$$

Application of Lemma 2.1 yields

$$f\left(p_r \to \frac{a^r - (ab)^r}{1 - q^r}\right) = a^k F(b, q),$$

and so the result follows. □

The computation of $R^n_\alpha(1, q, \ldots, q^{n-1}; q)$ is comparatively easy:

LEMMA 7.7. *If $\alpha \in \mathbf{N}^n$, then*

$$R^n_\alpha(1, q, \ldots, q^{n-1}; q) = q^{n(\alpha)} \frac{[n]!_q}{(1 - q)^n},$$

where $n(\alpha) = \Sigma(i - 1)\alpha_i$.

PROOF. The argument we present here is the same as the one given by Macdonald [**13**, III, Example 2.1]. (He makes no use of the fact, but assumes that α is a partition.)

Let $w \in S_n$ and consider the product

$$\prod_{1 \leqslant i < j \leqslant n} (q^{w(i)-1} - q^{w(j)}).$$

This will vanish if (say) $i + 1$ occurs before i among $w(1), \ldots, w(n)$. In other words, this product vanishes unless w is the identity. Thus, the only term in the definition (38) of $R^n_\alpha(x_1, \ldots, x_n; q)$ which survives under the substitution $x_i \to q^{i-1}$ is the term corresponding to $w = 1$. The result is now immediate. □

Unfortunately, Lemma 7.7 gives us no direct information about

$$R_\alpha(1, q, \ldots, q^{m-1}; q)$$

when $\alpha \in \mathbf{N}^n$ and $m > n$. In order to compute formulas in such cases we need the following result, which relates the symmetric functions R_α with $\alpha \in \mathbf{N}^n$ to R_β with $\beta \in \mathbf{N}^{n+1}$.

LEMMA 7.8. *Let $\alpha \in \mathbf{N}^{n+1}$ and assume that $|\alpha| \leqslant n$. Let Z be the set of zeros of α; i.e., $Z = \{i \colon 1 \leqslant i \leqslant n + 1, \alpha_i = 0\}$. We have*

$$R_\alpha(x; q) = \sum_{i \in Z} q^{n+1-i} R_{\alpha|i}(x; q),$$

where

$$\alpha | i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n+1}) \in \mathbf{N}^n.$$

PROOF. By definition (38) of R_α^{n+1}, we see that

$$R_\alpha^{n+1}(x_1, \ldots, x_{n+1}; q) = \sum_{w \in S_{n+1}} x_{w(1)}^{\alpha_1} \cdots x_{w(n+1)}^{\alpha_{n+1}} \prod_{1 \leqslant i < j \leqslant n+1} \frac{x_{w(i)} - q x_{w(j)}}{x_{w(i)} - x_{w(j)}}.$$

In particular, observe that when we substitute $x_{n+1} = 0$, the only terms which survive correspond to permutations $w \in S_{n+1}$ with $w(i) = n + 1$ for some $i \in Z$. The remainder of w,

$$w(1), \ldots, w(i-1), w(i+1), \ldots, w(n+1),$$

constitutes a permutation of $1, \ldots, n$. Since

$$\prod_{j<i} \frac{x_{w(j)} - q x_{n+1}}{x_{w(j)} - x_{n+1}} \cdot \prod_{i<j \leqslant n+1} \frac{x_{n+1} - q x_{w(j)}}{x_{n+1} - x_{w(j)}} \to q^{n+1-i}$$

under the substitution $x_{n+1} = 0$, we conclude that

$$R_\alpha^{n+1}(x_1, \ldots, x_n, 0; q) = \sum_{i \in Z} q^{n+1-i} R_{\alpha|i}^n(x_1, \ldots, x_n; q).$$

The lemma now follows from definition (41) of the symmetric functions R_α and the fact that $|\alpha| \leqslant n$. □

For any $\alpha \in \mathbf{N}^r$, $\beta \in \mathbf{N}^s$, and $m \geqslant r + s$, let

$$\langle \alpha, \beta \rangle_m = \langle \alpha_1, \ldots, \alpha_r, 0, \ldots, 0, \beta_s, \ldots, \beta_1 \rangle \in \mathbf{N}^m.$$

We are now sufficiently prepared to give the

PROOF OF THEOREM 7.3. Let $\gamma \in \mathbf{N}^n$, $\alpha \in \mathbf{N}^r$, $\beta \in \mathbf{N}^s$ be as defined in the statement of the theorem, and let $t = n - r - s$. Define $\alpha^+ = \alpha + 1^r$ and $\beta^+ = \beta + 1^s$. Notice that

$$\langle \alpha^+, \beta^+ \rangle_n = \gamma + 1^n.$$

The zeros of $\langle \alpha^+, \beta^+ \rangle_m$ occur in the positions i for which $r + 1 \leqslant i \leqslant m - s$. For any such i, $\langle \alpha^+, \beta^+ \rangle_m | i = \langle \alpha^+, \beta^+ \rangle_{m-1}$. Hence, for any $m > n$, Lemma 7.8 implies

$$R_{\langle \alpha^+, \beta^+ \rangle_m}(x; q) = (q^s + \cdots + q^{m-r-1}) R_{\langle \alpha^+, \beta^+ \rangle_{m-1}}(x; q).$$

Successive applications of this recursion yield

$$R_{\langle \alpha^+, \beta^+ \rangle_m}(x; q) = q^{s(m-n)} \frac{(1 - q^{m-r-s}) \cdots (1 - q^{t+1})}{(1 - q)^{m-n}} R_{\langle \alpha^+, \beta^+ \rangle_n}(x; q).$$

If we substitute $x_i \to q^{i-1}$ $(1 \leqslant i \leqslant m)$ and apply Lemma 7.7, we obtain

$$\begin{aligned} R_{\langle \alpha^+, \beta^+ \rangle_n}(1, q, \ldots, q^{m-1}; q) &= q^{n(\langle \alpha^+, \beta^+ \rangle_m) - s(m-n)} \frac{[m]!_q (1 - q)^{-n}}{(1 - q^{m-r-s}) \cdots (1 - q^{t+1})} \\ &= q^{m(\alpha, \beta) + \binom{r+s}{2}} q^{m|\beta|} (1 - q^m) \cdots (1 - q^{m-r-s+1}) \frac{[t]!_q}{(1 - q)^n}. \end{aligned}$$

Hence, by Proposition 7.6

$$R_{\langle\alpha^+,\beta^+\rangle_n}\left(p_k \to \frac{a^k - b^k}{1-q^k}; q\right)$$

$$= q^{m(\alpha,\beta)+\binom{r+s}{2}} a^n (ba^{-1})^{|\beta|} \frac{[t]!_q}{(1-q)^n} \prod_{i=0}^{r+s-1} (1 - ba^{-1}q^{-i}).$$

If we substitute $q \to z$, $a \to -q$, $b \to -1$, we find

$$R_{\langle\alpha^+,\beta^+\rangle_n}\left(p_k \to (-1)^{k-1}\frac{1-q^k}{1-z^k}; z\right) = (-1)^t q^{|\alpha|} z^{m(\alpha,\beta)} \frac{(q;z)_{n-t}[t]!_z}{(1-z)^n}.$$

Apply Theorem 7.1. □

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