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First order description of black holes in moduli space

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Abstract: We show that the second order field equations characterizing extremal solutions for spherically symmetric, stationary black holes are in fact implied by a system of first order equations given in terms of a prepotential W. This confirms and generalizes the results in [14]. Moreover we prove that the squared prepotential function shares the same properties of a c-function and that it interpolates between $M^2_{\rm ADM}$ and $M^2_{\rm BR}$, the parameter of the near-horizon Bertotti-Robinson geometry. When the black holes are solutions of extended supergravities we are able to find an explicit expression for the prepotentials, valid at any radial distance from the horizon, which reproduces all the attractors of the four dimensional $N > 2$ theories. Far from the horizon, however, for N-even, our ansatz poses a constraint on one of the U-duality invariants for the non-BPS solutions with $Z \neq 0$. We discuss a possible extension of our considerations to the non extremal case.

Keywords: Supergravity Models, Black Holes.

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Contents

1. Introduction

Recent progress in the understanding of extremal non-BPS black-hole solutions in extended supergravities (for a review on black holes in supergravity see for example [1, 2]) have revived the interest in the physics of extremal black holes.

The peculiar feature of these solutions is the attractor mechanism [3 – 13], according to which the scalar "hair" of the black hole runs into a fixed value on the horizon, independently of the boundary conditions at spatial infinity. For static, spherically symmetric black holes, the fixed values of the scalars at the horizon are determined in terms of the quantized electric and magnetic charges characterizing the solution, as extrema of an effective potential $V_{\rm BH}$ [5]. This induces to expect the radial dependence of the scalar fields in each extremal solution to admit a description in terms of a system of first order equations:

$$\dot{\Phi}^r \propto \partial^r W(\Phi)\,, \tag{1.1}$$

which implies the second order field equations, provided $V_{\rm BH}$ has a definite expression in terms of W and its derivatives. In this description, the attractor point $\dot{\Phi} = 0$ is given by the singular point $\partial_r W(\Phi_0) = 0$ which is also an extremum of $V_{\rm BH}$. We will call W the *prepotential* of the extremal solution. This is indeed the case for BPS black holes, whose associated first-order differential equations are implied by the Killing-spinor equations. For the $N = 2$ case one finds $W = |Z|$ where Z is the central charge of the supersymmetry

JHEP11(2007)032

algebra. Therefore, this poses the problem of finding the analogous first order differential equations, with the associated prepotential, describing the non-BPS extremal solutions. Another motivation for developing a first order formulation is that it provides a natural framework for defining a c-function associated with the radial flow of the fields in these solutions [19]. In fact, as we will show, the squared prepotential is a viable candidate for such a function, sharing with it the monotonicity property in the radial variable and the value taken at the horizon.[1]

This problem was first addressed in [14] where, by exploiting the formal analogy between extremal black holes and domain wall solutions, explicit examples of W corresponding to certain $N = 2$ non-BPS extremal solutions were found.

It is the aim of the present paper to give a general form of the prepotential in extended supergravity which will allow to reproduce the attractor behavior of all the known extremal black-hole solutions for $N \geq 3$. W will be given as a function of the U-duality invariants of the theory built in terms of the dressed charges. Different attractors will correspond to different choices of the coefficients in W. Note, however, that the general ansatz we give for W can be considered as a minimal one reproducing correctly all the attractor points of static extremal black-hole solutions in extended four dimensional supergravity. Indeed, if we consider the full black-hole solution outside the horizon, it turns out that our ansatz requires a restriction, in the N-even cases, on the duality invariants characterizing the non-BPS $Z \neq 0$ attractors. More precisely, for these solutions the above restriction amounts to fixing an invariant overall phase of the complex dressed charges at radial infinity to the value it takes on the horizon. We argue that a refined ansatz could relax this restriction.

The paper is organized as follows. In section 2 we recall the main facts about static, spherically symmetric black holes and introduce the prepotential W, proving, in the extremal case, that it is monotonic. In section 3, which contains the main results of the paper, the general expression for W in the extremal case is given for $N \geq 3$ extended supergravity, and also some examples of $N = 2$ solutions. Section 4, which includes the concluding remarks, contains a speculative discussion where the issue of a possible extension of the definition of the prepotential to the non extremal case is addressed. For a class of non extremal black holes we show that a first order formulation in terms of a prepotential W may exist and we find the corresponding description in terms of first order differential equations. This generalizes the results in [18] to the case of scalar-matter coupled gravity.

2. Black holes as solutions to first order differential equations

We will consider the class of theories described by the bosonic action [3]:

$$\mathcal{S} = \int \sqrt{-g}\, d^4x \left(-\frac{1}{2}\, R + \mathrm{Im}\mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma|\mu\nu} + \frac{1}{2\sqrt{-g}}\, \mathrm{Re}\mathcal{N}_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma}\, F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma} + \right. \\ \left. + \frac{1}{2}\, g_{rs}(\Phi) \partial_\mu \Phi^r \partial^\mu \Phi^s \right)\,, \tag{2.1}$$

[1] We thank Sandip Trivedi for drawing this to our attention.

where R is the scalar curvature, Φ^r are a set of scalar fields and F^Λ gauge field strengths. $g_{rs}(\Phi)$, $\mathcal{N}_{\Lambda\Sigma}(\Phi)$ $(r, s, \ldots = 1, \ldots, m)$ are matrices depending on the scalar fields.

The most general Ansatz for a spherically symmetric and stationary metric is

$$ds^2 = e^{2U}\, dt^2 - e^{-2U}\, \left(\frac{c^4}{\sinh^4(c\tau)}\, d\tau^2 + \frac{c^2}{\sinh^2(c\tau)}\, d\Omega^2\right) . \tag{2.2}$$

The evolution coordinate τ is related to the radial coordinate r by the following relation:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{c^2}{\sinh^2(c\tau)} = (r - r_0)^2 - c^2 = (r - r^-)\,(r - r^+) . \tag{2.3}$$

$r_\pm$ being the radii of the two event horizons, with $r_+ > r_-$. Here $c \equiv 2ST$ is the extremality parameter of the solution, with S the entropy and T the temperature of the black hole. In the extremal case $c \to 0$, eq. (2.3) reduces to $\tau = -\frac{1}{r - r_H}$, where r_H denotes the radius of the horizon.

It is known [5] that by eliminating the vector fields via their equations of motion this system may be reduced to the following set of field equations for the metric function $\mathrm{U}(\tau)$ and the scalar fields $\Phi^r(\tau)$ in terms of the evolution parameter τ:

$$\frac{d^2U}{d\tau^2} \equiv \ddot{U} = V_{\mathrm{BH}}(\Phi, p, q) e^{2U} , \tag{2.4}$$

$$\frac{D^2\Phi^r}{D\tau^2} \equiv \ddot{\Phi}^r + \Gamma^r{}_{st}\dot{\Phi}^s\,\dot{\Phi}^t = g^{rs}(\Phi)\,\frac{\partial V_{\mathrm{BH}}(\Phi, p, q)}{\partial\Phi^s} e^{2U} , \tag{2.5}$$

together with the constraint

$$\left(\frac{dU}{d\tau}\right)^2 + \frac{1}{2}\, g_{rs}(\Phi)\,\frac{d\Phi^r}{d\tau}\frac{d\Phi^s}{d\tau} - V_{\mathrm{BH}}(\Phi, p, q) e^{2U} = c^2 , \tag{2.6}$$

where $V_{\mathrm{BH}}(\Phi, p, q)$ is a function of the scalars and of the electric and magnetic charges of the theory defined by:

$$V_{\mathrm{BH}} = -\frac{1}{2}\, Q^t \mathcal{M}(\mathcal{N}) Q , \tag{2.7}$$

and Q is the symplectic vector of quantized magnetic and electric charges $Q^t = (p^\Lambda, q_\Lambda)$. $\mathcal{M}(\mathcal{N})$ is the symplectic matrix defined in terms of the gauge field-strengths kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\Phi)$:

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}\mathrm{Im}\mathcal{N}^{-1} \\ -\mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} . \tag{2.8}$$

The field equations (2.5) can be extracted from the effective one-dimensional lagrangian:

$$\mathcal{L}_{\mathrm{eff}} = \left(\frac{dU}{d\tau}\right)^2 + \frac{1}{2}\, g_{rs}\frac{d\Phi^r}{d\tau}\frac{d\Phi^s}{d\tau} + V_{\mathrm{BH}}(\Phi, p, q) e^{2U} , \tag{2.9}$$

constrained with equation (2.6).

We are going to show that the second order field equations (2.4), (2.5) can in fact be derived by a first order system, for a large class of extremal and non-extremal black holes, by performing the following Ansatz [14]:

$$\frac{dU}{d\tau} \equiv \dot{U} = e^U \, W(\Phi, \tau) \,, \tag{2.10}$$

where W is a function of the scalar fields (depending on the quantized charges and τ) and explicitly of τ; the derivative is performed with respect to the evolution parameter τ. We argue that the extremal case corresponds to

$$\partial_\tau W = 0 \quad \Rightarrow \quad W = W(\Phi) \,, \tag{2.11}$$

while for non extremal black holes, in those cases which admit a first order description, an explicit dependence of W on τ should be included.

2.1 Extremal case

Let us consider in detail the extremal case $c = 0$. In this case eq. (2.10) becomes

$$\dot{U} = W(\Phi) \, e^U \,. \tag{2.12}$$

Differentiating (2.12) with respect to τ gives

$$\ddot{U} = (\dot{U})^2 + \dot{W} e^U \equiv W^2 \, e^{2U} + \dot{W} e^U \,, \tag{2.13}$$

where

$$\dot{W} = \dot{\Phi}^r \partial_r W \,. \tag{2.14}$$

Comparing eq. (2.13) with (2.4) we find the following expression for $V_{\rm BH}$:

$$V_{\rm BH} = W^2 + e^{-U} \dot{W}. \tag{2.15}$$

Moreover from eqs. (2.4) and (2.6) one finds:

$$\ddot{U} - (\dot{U})^2 = \frac{1}{2} g_{rs} \dot{\Phi}^r \dot{\Phi}^s = \dot{\Phi}^r \partial_r W \, e^U \,, \tag{2.16}$$

while eq. (2.13) can be recast in the form:

$$\ddot{U} - (\dot{U})^2 = \dot{W} \, e^U = \dot{\Phi}^r \partial_r W \, e^U \,. \tag{2.17}$$

It follows that, for $\dot{\Phi}^r \neq 0$, eq. (2.16) is solved for:[2]

$$\dot{\Phi}^r \, = 2 \, e^U \, g^{rs} \, \partial_s W \,. \tag{2.21}$$

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[2] To be precise, the most general solution to the constraint (2.16) would be:

$$\dot{\Phi}^r \, = 2 \, e^U \, g^{rs} \, \partial_s W + \alpha^r(\Phi, \tau) \,, \tag{2.18}$$

where $\alpha^r = P^{rs} h_s$ and $P^r{}_s \equiv \left(\delta^r_s - \frac{\dot{\Phi}^r \dot{\Phi}_s}{\dot{\Phi}^\ell \dot{\Phi}_\ell} \right)$ is a projector orthogonal to $\dot{\Phi}^r$. In this more general case the effective potential would include one additional term:

$$V_{\rm BH} \, = W^2 + 2 \partial_r W \partial^r W - \frac{1}{2} e^{-2U} \alpha_r \alpha^r \,. \tag{2.19}$$

Eq. (2.21) is reminiscent of the BPS condition; for a given W it relates the evolution of the scalar fields on the corresponding configuration to the partial derivative of the prepotential W with respect to the scalar fields. Note in particular that the fixed points for Φ^r are in direct relation with the extrema of W. Together with (2.10) and (2.14), (2.21) allows to express the field-equations in terms of a first order system. Indeed, using (2.21) the effective potential reads [14]

$$V_{\rm BH} = W^2 + 2\, g^{rs}\, \partial_r W\, \partial_s W\,. \tag{2.22}$$

By inserting eqs. (2.21) and (2.22) in the second order evolution equation for the scalars, eq. (2.5), we find that it is *identically satisfied*. This shows that, as far as the scalar sector is concerned, the system of *second order* differential equations (2.5) is in fact a *first order* system once expressed in terms of the prepotential W. Moreover, given any explicit expression for W, also the space-time metric may be found as solution of a first-order equation (2.12). Furthermore, the effective potential (2.22) is extremized for

$$\frac{\partial V_{\rm BH}}{\partial \Phi^r} \equiv \partial_r V_{\rm BH} = 2\,\partial_s W\left(W\delta_r^s + 2g^{s\ell}\nabla_r\partial_\ell W\right) = 0\,, \tag{2.23}$$

that is the fixed points for the scalars (corresponding to extrema of W) are also extrema of the potential $V_{\rm BH}$. Since the black-hole horizon is identified as the fixed point of the scalars, in this formulation it is directly related to the extrema of W (that are in particular also extrema for $V_{\rm BH}$). The BH entropy then reads, in terms of W, as

$$S_{\rm BH} = V_{\rm BH}|_{\rm extr} = W^2|_{\rm extr} = W^2(\Phi|_{\rm hor})\,. \tag{2.24}$$

Furthermore, let us observe that from the evolution equations above, together with the boundary condition on U at spatial infinity ($\mathrm{U}(\tau = 0) = 0$), it is easy to deduce that W and W^2 are monotonic functions, both decreasing along the evolution from spatial infinity towards the horizon. Indeed, if we define the function $b(\tau) = -\frac{1}{\tau}\, e^{-U}$, as in [7], the conditions of regularity of the solution at the horizon and of flatness of space-time at radial infinity, imply the following limiting behaviors:

$$\tau \to -\infty\, : b(\tau) \longrightarrow r_H + \mathcal{O}(\tau^{-1}) \tag{2.25}$$

$$\tau \to 0^-\, : b(\tau) \longrightarrow -\frac{1}{\tau} + M_{\rm ADM} + \mathcal{O}(\tau)\,. \tag{2.26}$$

Since $e^{-U} = -\tau b$, using the first order equations we may write

$$W = -\frac{d}{d\tau}e^{-U} = \frac{d}{d\tau}(\tau b) = b + \tau\dot{b}\,, \tag{2.27}$$

For $\alpha^r \neq 0$, however, the attractor condition at the horizon becomes

$$\lim_{\tau\to-\infty}\partial_r W = -\frac{1}{2}e^{-U}\alpha_r(\tau\to-\infty) \neq 0\,. \tag{2.20}$$

As we will see in the following (see eq. (2.24)), such deformation is immaterial since it gives for the entropy the same value $V|_{\rm extr} = W^2|_{\rm extr}$ as for the case with $\alpha^r = 0$. It could instead play a role in more general situations, for example when considering black-holes out of extremality, where the effect of α^r would be to effectively deform the constant non-extremality parameter c^2 into a function $\mathcal{C}^2(\Phi,\tau) = c^2 + \frac{1}{2}\alpha_r\alpha^r$.

which implies the following asymptotic limits for W:

$$\lim_{\tau\to-\infty} W = r_H = M_{\rm BR}\,,$$
$$\lim_{\tau\to 0^-} W = M_{\rm ADM} \geq M_{\rm BR}\,, \tag{2.28}$$

where $M_{\rm BR}$ denotes the Bertotti-Robinson mass parameter associated with the near-horizon geometry. Let us now show that W is monotonic:

$$\frac{dW}{d\tau} = \left(\ddot{U} - (\dot{U})^2\right)\, e^{-U} = \left(V - W^2\right)\, e^{U} = 2\, g^{rs}\,\partial_r W \partial_s W\, e^{U} \geq 0\,, \tag{2.29}$$

where we have used eq. (2.22) and the first order equations. We conclude that W is a positive monotonic function decreasing from the value $M_{\rm ADM}$ at radial infinity, towards the value $M_{\rm BR} \leq M_{\rm ADM}$ at the horizon. In [19] it was shown that for static, spherically symmetric black holes a monotone function $A(r)$ always exists such that $A(r_H) = S_{\rm BH}$, with $A(r)$ decreasing towards the horizon. For the extremal case, W^2 then appears as the appropriate quantity to play the role of the c-function $A(r)$.

3. The prepotential for extremal solutions of extended supergravity

It is our purpose to show that when the extremal black hole is a solution of an $N > 2$ extended supersymmetric theory, where the scalar manifold is a coset G/H, it is possible to find a general expression for W which reproduces all the known results concerning the BPS and non-BPS attractor points of the theory (see [2] for a review collecting all the solutions in four dimensions). Our results may be extended to the $N = 2$ case when the special geometry is described by homogeneous spaces. For more general models a case by case inspection is necessary. Some $N = 2$ examples have been given in [14].

We propose the following:

$$W = \sum_M \alpha^M e_M\,. \tag{3.1}$$

where $e_M \in \mathbb{R}$ are related to the invariants of the isotropy subgroup H built in terms of the complex central and matter charges. Such invariants indeed can be expressed in terms of the skew eigenvalues of the matrix of central charges Z_{AB} and of the norm of the matter charge vectors Z_I that we collectively call $\{e_M\}$.

The real coefficients α^M can be computed by requiring that the potential (2.22), with W given by (3.1), reproduces the general form taken by the effective scalar potential for any extended supersymmetric theory:

$$V_{\rm BH} = \frac{1}{2} Z_{AB}\overline{Z}^{AB} + Z_I\overline{Z}^I\,. \tag{3.2}$$

Here and in the following A, B are SU(N) R-symmetry group indices while the indices I, J label the fundamental representation of the matter group when present (namely U(3) for $N = 3$ and SO(6) in the $N = 4$ case). Since (2.22) involves the gradient of the prepotential W, the evaluation of $V_{\rm BH}$ requires the knowledge of the differential relations among central

JHEP11(2007)032

and matter charges, for which we refer to [15, 2]. In general, since the equations in the α^M are quadratic, their sign is not fixed in principle, but it can be fixed by requiring that the prepotential W is extremized on the black-hole horizon $\partial_r W|_{\rm extr} = 0$.

It was observed in [14] that for any $V_{\rm BH}$ it should exist a multiple choice of W. Here, using the ansatz (3.1) we give the explicit expression and the precise number of independent prepotentials for any given extended theory. Indeed, as we are going to show by a case by case analysis, there are in general up to three independent choices of $\{\alpha^M\}$ all reproducing the same $V_{\rm BH}$. The various independent solutions for W will reproduce the known different BPS and non-BPS solutions for any given theory. Any independent choice of $\{\alpha^M\}$ would then parametrize a different black-hole solution.

We may then adopt two equivalent points of view to find the different extremal black-hole attractors: either we study the extrema of $V_{\rm BH}$, or alternatively we consider the possible inequivalent choices of W compatible with the expression (3.2) for $V_{\rm BH}$.

We first analyze the N-odd cases, which are easier because the central and matter charges in normal form can all be made real. The cases with N even, which in general also include a solution with complex charges (corresponding to a negative fourth-order invariant of the duality group G) will be analyzed afterwords.

3.1 The N-odd cases

Since these are the simplest cases, we shall describe the calculations in detail. In the N-even cases the relevant results will be given.

3.1.1 The $N = 3$ case

In the $N = 3$ theory the scalar manifold is $\mathrm{U}(3, n)/[\mathrm{U}(3) \times \mathrm{U}(n)]$ and the central charge matrix $Z_{AB} = -Z_{BA}$, $A = 1, 2, 3$, and matter charges Z_I, $I = 1, \ldots, n$; the central and matter charges obey the differential relations

$$\begin{cases} \nabla Z_{AB} &=& P_{IAB}\overline{Z}^I \\ \nabla Z_I &=& \frac{1}{2}P_{IAB}\overline{Z}^{AB} \end{cases}, \tag{3.3}$$

where $P_{IAB} = P_{IAB,i}dz^i$ ($i = 1, \ldots 3n$) is the holomorphic vielbein of $\mathrm{U}(3, n)/[\mathrm{U}(3)\times\mathrm{U}(n)]$, ∇ denotes the U(1)-Kähler covariant and H-covariant derivative (we generally denote by H the isotropy group of the symmetric spaces G/H representing the scalar manifold of the various $N \geq 3$ theories [2, 15]). By a $\mathrm{U}(3)$ rotation it is always possible to put Z_{AB} in normal form

$$Z_{AB} = e\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad e \in \mathbb{R} \tag{3.4}$$

while by a $\mathrm{U}(n)$ rotation the vector Z_I may by chosen to be real and pointing in a given direction, say

$$Z_I = \rho\delta^1_I\,; \qquad \rho \in \mathbb{R}\,. \tag{3.5}$$

We then propose the following general expression for W:

$$\begin{aligned} W &= a\,e + b\,\rho \\ &= a\sqrt{\frac{1}{2}Z_{AB}\overline{Z}^{AB}} + b\sqrt{Z_I\overline{Z}^I}\,. \end{aligned} \tag{3.6}$$

From the relation (2.22) one obtains:

$$V = \left(a^2+b^2\right)\left(e^2+\rho^2\right) + 4ab\,e\rho. \tag{3.7}$$

In order to reproduce the general result (3.2), that written in normal form takes the form

$$V = e^2+\rho^2\,, \tag{3.8}$$

we must have:

$$\begin{cases} a^2+b^2 = 1\,, \\ ab = 0\,. \end{cases} \tag{3.9}$$

There are two different solutions to the system (3.9), namely

1. $a=1$, $b=0$, implying

$$W_{(1)} = e = \frac{1}{2}\sqrt{Z_{AB}\overline{Z}^{AB}} \tag{3.10}$$

2. $a=0$ and $b=1$, which imply

$$W_{(2)} = \rho = \sqrt{Z_I\overline{Z}^I}\,. \tag{3.11}$$

Note that these two choices reproduce precisely the two independent solutions for the extremization of the black-hole solutions of $N=3$ supergravity [2]. The former solution, which implies:

$$\nabla_i W_{(1)} = \frac{1}{4\sqrt{\frac{1}{2}Z_{AB}\overline{Z}^{AB}}}\overline{Z}^I\overline{Z}^{AB}P_{IAB,i} \tag{3.12}$$

is extremized for $Z_I=0$, $Z_{AB}\neq 0$ and corresponds to the BPS solution, with entropy $S_{(1)} = W^2_{(1)}|_{\text{extr}} = \frac{1}{2}|Z_{AB}|^2$. The second one gives

$$\nabla_i W_{(2)} = \frac{1}{4\sqrt{Z_I\overline{Z}^I}}\overline{Z}^I\overline{Z}^{AB}P_{IAB,i} \tag{3.13}$$

and is extremized for $Z_{AB}=0$, $Z_I\neq 0$ corresponding to the non-BPS solution, with entropy $S_{(2)} = W^2_{(2)}|_{\text{extr}} = |Z_I|^2$.

Since we know that the potential (3.2) has two minima for the $N=3$ theory, $W_{(1)}$ and $W_{(2)}$ exhaust the possible minima of the general potential for this theory. This in particular implies that eq. (2.23) cannot have further solutions coming from the vanishing of the second factor, as it can be easily shown by an explicit calculation.

JHEP11(2007)032

3.1.2 The prepotential for the $N = 5$ case

In this case there are no matter multiplets and the scalar manifold is the Kähler manifold $\mathrm{SU}(1,5)/\,\mathrm{U}(5)$, spanned by the holomorphic vielbein $P_{ABCD} = P_{ABCD,i}\, dz^i = \epsilon_{ABCDE} P^E$ (with A, $i = 1, \ldots, 5$) and its complex conjugate $P^{ABCD} = P^{ABCD}_{,\bar\imath}\, d\bar z^{\bar\imath}$. The central charges $Z_{AB} = -Z_{BA}$ obey the differential relations:

$$\nabla Z_{AB} = \frac{1}{2} P_{ABCD} \overline{Z}^{CD} \,. \tag{3.14}$$

Via a $\mathrm{U}(5)$ rotation they may be put in the normal form:

$$Z_{AB} = \begin{pmatrix} 0 & e_1 & 0 & 0 & 0 \\ -e_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_2 & 0 \\ 0 & 0 & -e_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.15}$$

in terms of the two real (non negative) proper-values e_1 and e_2, which are related to the two $\mathrm{U}(5)$ invariants

$$\begin{cases} I_1 \;\equiv\; \frac{1}{2} Z_{AB} \overline{Z}^{AB} = (e_1)^2 + (e_2)^2 \\ I_2 \;\equiv\; \frac{1}{2} Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} = (e_1)^4 + (e_2)^4 \end{cases} \tag{3.16}$$

by the inverse relation

$$\begin{cases} e_1 \;=\; \sqrt{\frac{1}{2}\left[I_1 + \sqrt{2I_2 - I_1^2}\right]} \\ e_2 \;=\; \sqrt{\frac{1}{2}\left[I_1 - \sqrt{2I_2 - I_1^2}\right]} \end{cases} . \tag{3.17}$$

According to equation (3.1), we then propose for the prepotential W the form

$$W = a_1 e_1 + a_2 e_2 \,. \tag{3.18}$$

Writing (3.14) in normal form, that is

$$\begin{aligned} \nabla_i e_1 &= P_{,i}\, e_2 \,, \\ \nabla_i e_2 &= P_{,i}\, e_1 \,, \end{aligned} \tag{3.19}$$

its holomorphic gradient in normal form is

$$\partial_i W = \frac{1}{2} P_{,i} (a_1 e_2 + a_2 e_1) \,, \tag{3.20}$$

where $P_{,i} = P_{1234,i}$ is the component of the holomorphic scalar vielbein which appears in (3.14) when the central charge is in its normal form. Evaluating the potential using (3.14) gives, for the black-hole potential,

$$V_{\mathrm{BH}} = (a_1^2 + a_2^2) \left[(e_1)^2 + (e_2)^2\right] + 4 a_1 a_2 e_1 e_2 \,. \tag{3.21}$$

This reproduces the result for the black-hole potential of supersymmetric theories, $V = I_1$, for

$$\begin{cases} a_1^2 + a_2^2 & = & 1 \\ a_1 a_2 & = & 0 \end{cases} \tag{3.22}$$

This system has, essentially, only one independent solution that, with our choice (3.17) of proper-values (which implies $e_1 \geq e_2$) is

$$a_1 = 1\,, \quad a_2 = 0 \tag{3.23}$$

giving $W = e_1$, which is extremized for $\nabla_i W = e_2 = 0$. It is a BPS solution.

Note that the extremization of the scalar potential (3.21) gives

$$\partial_i V_{\rm BH} = 2\partial_k W \left(\delta_i^k W + 2g^{k\bar{\jmath}}\nabla_i \partial_{\bar{\jmath}} W\right) = 0, \tag{3.24}$$

since, from (3.20), $\nabla_i \partial_k W = 0$. It may be easily shown by explicit calculation that there are no other solutions to $\partial_i V_{\rm BH} = 0$ besides $\partial_i W = 0$ from which we conclude that also in this case the extrema of W give all the extrema of $V_{\rm BH}$.

3.2 The N-even cases

In the cases with $N = 3$ and $N = 5$ supercharges the central and matter charges $Z_M \equiv \{Z_{AB},\, Z_I\}$ in normal form may all be chosen real and non negative. On the other hand, in the N-even cases the normal form of the Z_M contains in general an overall phase. Since our choice (3.1) of the prepotential is given only in terms of the moduli of the charges, we will see that the solution for the coefficents α^M in (3.1) which reconstruct the effective potential $V_{\rm BH}$ implies in particular that the phase must be fixed at certain values all over the moduli space. This value is actually the one corresponding to the attractor condition at the horizon.

Our general ansatz (3.1) can then be considered as the minimal one reproducing correctly all the attractor points of static extremal black-hole solutions in extended four dimensional supergravity. We argue that a refined ansatz could relax the fixing of the phase before extremization.

3.2.1 The prepotential for the $N = 4$ attractors

In this case the scalar manifold is the coset space

$$G/H = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,n)}{\mathrm{SO}(6)\times \mathrm{SO}(n)} \tag{3.25}$$

and the relations among central and matter charges are:

$$\begin{cases} DZ_{AB} & = & \overline{Z}^I P_{ABI} + \frac{1}{2}\overline{Z}^{CD}\epsilon_{ABCD}\, P\,, \\ DZ_I & = & \frac{1}{2}\overline{Z}^{AB} P_{ABI} + \overline{Z}_I\, \overline{P}\,. \end{cases} \tag{3.26}$$

We recall that for this theory the vielbein P_{ABI} satisfies the reality condition $\overline{P}^{ABI} \equiv (P_{ABI})^\star = \frac{1}{2}\epsilon^{ABCD} P^I_{CD}$.

Using the $\mathrm{U}(1) \times \mathrm{SO}(6) \sim \mathrm{U}(4)$ symmetry of the theory we can bring the central charges in the normal form [16]

$$Z_{AB} = \begin{pmatrix} 0 & Z_1 & 0 & 0 \\ -Z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_2 \\ 0 & 0 & -Z_2 & 0 \end{pmatrix} \tag{3.27}$$

where the graviphoton skew-eigenvalues Z_1, Z_2 can be chosen real and non-negative, thus coinciding with their modulus $Z_{1,2} = |Z_{1,2}| = e_{1,2}$. Further, using an $\mathrm{SO}(n)$ transformation it is also possible to reduce the vector of matter charges in such a way that only one real and one complex matter charge are different from zero. Let us call them $Z_I = \rho_I e^{\mathrm{i}\theta_I}$, $I = 1, 2$ (with the proviso that one of the phases may be always put to zero).

We consider a prepotential W of the form:

$$W = a_1\, e_1 + a_2\, e_2 + \sum_{I=1}^{2} b_I\, \rho_I\, . \tag{3.28}$$

It encodes the H invariants:

$$\begin{cases} I_1 & \equiv & \frac{1}{2} Z_{AB}\overline{Z}^{AB} = (e_1)^2 + (e_2)^2\, , \\ I_2 & \equiv & \frac{1}{2} Z_{AB}\overline{Z}^{BC} Z_{CD}\overline{Z}^{DA} = (e_1)^4 + (e_2)^4\, , \\ I_3 & \equiv & Z_I \overline{Z}_I = \rho_I \rho_I\, , \\ I_4 & \equiv & \mathrm{Re}\,(Z_I Z_I)\ . \end{cases} \tag{3.29}$$

The potential is related to W by eq. (2.22). In order to compute the derivatives of W we rewrite, as usual, the differential relations in normal form, where $P = P_{,i} dz^i$ is the Kählerian vielbein of $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ while $P_{12I} \equiv P_I$ ($P_{34I} = \overline{P}_{12I}$) are the components of the (non Kählerian) vielbein $\mathrm{SO}(6,n)/\mathrm{SO}(6) \times \mathrm{SO}(n)$:

$$\begin{cases} \nabla Z_1 & = & \overline{Z}^I P_I + \overline{Z}_2\, P\, , \\ \nabla Z_2 & = & \overline{Z}^I \overline{P}_I + \overline{Z}_1\, P\, , \\ \nabla Z_I & = & \overline{Z}_1 P_I + \overline{Z}_2 \overline{P}_I + \overline{Z}_I\, \overline{P}\, . \end{cases} \tag{3.30}$$

We then find:

$$\begin{aligned} \nabla_i W &= P_{,i}\, A + P_{I,i}\, B_I + \overline{P}_{I,i}\, \overline{B}_I\, , \\ A &= \frac{1}{2}\left[(a_1\, e_2 + a_2\, e_1) + \sum_I b_I \rho_I\, e^{2\,\mathrm{i}\,\theta_I}\right]\, , \\ B_I &= \frac{1}{2}\left[(a_1 \rho_I + b_I e_1)\, e^{-\,\mathrm{i}\,\theta_I} + (a_2 \rho_I + b_I e_2)\, e^{\,\mathrm{i}\,\theta_I}\right]\, . \end{aligned} \tag{3.31}$$

In terms of the above quantities the potential reads:

$$\begin{aligned} W^2+2\,g^{rs}\partial_r W\partial_s W \;=\;& \left(||a||^2+||b||^2\right)(|Z_1|^2+|Z_2|^2+\sum_I \rho_I^2)\\ &+\sum_{I\neq J}\left(b_I^2-b_J^2+2\,a_1\,a_2\,\cos(2\,\theta_I)\right)\,\rho_I^2\\ &+2\,|Z_1||Z_2|\left(2\,a_1\,a_2+\sum_I b_I^2\,\cos(2\,\theta_I)\right)\\ &+4\sum_I |Z_1|\rho_I\,b_I\,(a_1+a_2\,\cos(2\,\theta_I))\\ &+4\sum_I |Z_2|\rho_I\,b_I\,(a_2+a_1\,\cos(2\,\theta_I))\\ &+2\,b_1\,b_2\,\rho_1\rho_2\,\left(\cos 2\,(\theta_1-\theta_2)+1\right)\,, \end{aligned} \tag{3.32}$$

where we have defined: $||a||^2=a_1^2+a_2^2$, $||b||^2=\sum_I b_I^2$. Since the potential in terms of the central charges in the normal form reads

$$V=|Z_1|^2+|Z_2|^2+\sum_I \rho_I^2\,. \tag{3.33}$$

the coefficients a_k, b_I and the phases have to be chosen in such a way that the cross terms in the central charges in (3.32) vanish and the coefficients of the square norm of the central charges be equal to one. This implies that:

$$\left\{\begin{array}{ll} ||a||^2+||b||^2=1\,, & \\ b_I^2-b_J^2+2\,a_1\,a_2\,\cos(2\,\theta_I)=0\,, & \forall I\neq J\,,\\ 2\,a_1\,a_2+\sum_I b_I^2\,\cos(2\,\theta_I)=0\,, & \\ b_I\,(a_1+a_2\,\cos(2\,\theta_I))=0\quad\forall I\,, & \\ b_I\,(a_2+a_1\,\cos(2\,\theta_I))=0\quad\forall I\,, & \\ b_1\,b_2\,\left(\cos 2\,(\theta_1-\theta_2)+1\right)=0\,. & \end{array}\right. \tag{3.34}$$

In the following we choose normal form of the matter charges so that $\theta_2=0$. The above conditions can be explicitly solved giving three independent solutions for the coefficients. Consequently we have three different prepotentials, each characterizing a different attractor solution. The inequivalent solutions are:

1. $a_1=1$, $a_2=0$, $b_I=0$ or $a_1=0$, $a_2=1$, $b_I=0$, $\forall I$.

   The prepotential reads:

   $$W_{(1)}=W_{\rm BPS}=e_1\,. \tag{3.35}$$

   The attractor condition for this solution gives indeed, from (3.31):

   $$\partial_i W_{(1)}=0\Rightarrow e_2=\rho_I=0. \tag{3.36}$$

   This corresponds to the BPS attractor, with entropy $S_1=W^2_{1|extr}=e_1^2$.

2. $a_1 = a_2 = \frac{1}{\sqrt{2}} b_1 = \frac{1}{2}, \quad b_2 = 0, \quad \theta_1 = \frac{\pi}{2}$. The complete choice of the prepotential, fixed by the attractor condition at the horizon, gives:

$$W_{(2)} = \frac{1}{2}(e_1 + e_2 + \sqrt{2}\rho_1)\,, \tag{3.37}$$

which is indeed extremized for:

$$e_1 = e_2 = e\,; \quad Z_{I=1} = \sqrt{2}\mathrm{i}e\,; \quad Z_{I=2} = 0\,. \tag{3.38}$$

The entropy is given by $S_2 = W^2_{2|extr} = 4\,e^2$.

3. $a_1 = a_2 = 0, \quad b_1 = b_2 = \frac{1}{\sqrt{2}}\;, \; \theta_1 = \frac{\pi}{2}$. The prepotential reads

$$W_{(3)} = \frac{1}{\sqrt{2}}\,(\,\rho_1 + \rho_2)\,. \tag{3.39}$$

The extremum condition for this solution is

$$e_1 = e_2 = 0 \quad Z_{I=2} = \mathrm{i}Z_{I=1} = \rho\,. \tag{3.40}$$

The entropy is $S_3 = W^2_{3|extr} = \rho^2$.

3.2.2 The prepotential for the $N = 6$ theory

In this case the scalar manifold is $SO^*(12)/\,\mathrm{U}(6)$, spanned by the holomorphic vielbein $P_{ABCD} = P_{ABCD,i}\,dz^i = \frac{1}{2}\epsilon_{ABCDEF}P^{EF}$ (with $A = 1, \dots 6$, $i = 1, \dots 15$) and its complex conjugate $P^{ABCD} = P^{ABCD}_{,\bar{\imath}}\,d\bar{z}^{\bar{\imath}}$. The central charges of this theory are split into an antisymmetric matrix $Z_{AB} = -Z_{BA}$ and a singlet X. They obey the differential relations

$$\begin{cases} \nabla Z_{AB} &= \frac{1}{2}P_{ABCD}\overline{Z}^{CD} + \frac{1}{4!}\epsilon_{ABCDEF}\overline{P}^{CDEF}\overline{X} \\ \nabla X &= \frac{1}{2!4!}\epsilon_{ABCDEF}\overline{P}^{CDEF}\overline{Z}^{AB} \end{cases}\,. \tag{3.41}$$

The singlet complex charge X may be parametrized as $X = \rho\,\mathrm{e}^{\mathrm{i}\alpha}$ (with $\rho \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$). On the other hand the antisymmetric matrix Z_{AB} may be put in the normal form via a $\mathrm{U}(6)$ rotation:

$$Z_{AB} = \begin{pmatrix} 0 & Z_1 & 0 & 0 & 0 & 0 \\ -Z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_2 & 0 & 0 \\ 0 & 0 & -Z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_3 \\ 0 & 0 & 0 & 0 & -Z_3 & 0 \end{pmatrix} \tag{3.42}$$

in terms of the three proper-values Z_1, Z_2, Z_3. In the normal form they may indeed be chosen real and non negative $Z_\alpha = |Z_\alpha| \equiv e_\alpha$ $(\alpha = 1,2,3)$.

The four parameters e_α, ρ are related to the four $\mathrm{U}(6)$ invariants:

$$\begin{cases} I_1 &\equiv \frac{1}{2}Z_{AB}\overline{Z}^{AB} = (e_1)^2 + (e_2)^2 + (e_3)^2 \\ I_2 &\equiv X\overline{X} = \rho^2 \\ I_3 &\equiv \frac{1}{2}Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA} = (e_1)^4 + (e_2)^4 + (e_3)^4 \\ I_4 &\equiv -\frac{1}{2}Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DE}Z_{EF}\overline{Z}^{FA} = (e_1)^6 + (e_2)^6 + (e_3)^6\,. \end{cases} \tag{3.43}$$

Writing the differential relations among the dressed charges (3.41) in normal form, we find a simple expression for the holomorphic derivatives of the skew-eigenvalues e_α, namely:

$$\begin{cases} \nabla_i e_1 &=& \frac{1}{2}\left(P_{1,i}e_2 + P_{2,i}\rho e^{i\alpha} + P_{3,i}e_3\right) \\ \nabla_i e_2 &=& \frac{1}{2}\left(P_{1,i}e_1 + P_{2,i}e_3 + P_{3,i}\rho e^{i\alpha}\right) \\ \nabla_i e_3 &=& \frac{1}{2}\left(P_{1,i}\rho e^{i\alpha} + P_{2,i}e_2 + P_{3,i}e_1\right) \\ \nabla_i \rho &=& \frac{1}{2}e^{i\alpha}\left(P_{1,i}e_3 + P_{2,i}e_1 + P_{3,i}e_2\right) \\ \nabla_i \alpha &=& \frac{i}{2\rho}e^{i\alpha}\left(P_{1,i}e_3 + P_{2,i}e_1 + P_{3,i}e_2\right) \end{cases} \tag{3.44}$$

where $P_{1,i} = P_{1234,i}$, $P_{2,i} = P_{3456,i}$, $P_{3,i} = P_{1256,i}$ are the components of the scalar vielbein appearing in (3.44) when the central charge is written in normal form. We then propose, for the prepotential W, the form:

$$W = a_1e_1 + a_2e_2 + a_3e_3 + b\rho\,, \tag{3.45}$$

giving, for its holomorphic gradient in normal form:

$$\begin{aligned} \nabla_i W = \frac{1}{2}\Big\{ & P_{1,i}\left[a_1e_2 + a_2e_1 + e^{i\alpha}(a_3\rho + be_3)\right] + \\ & + P_{2,i}\left[a_2e_3 + a_3e_2 + e^{i\alpha}(a_1\rho + be_1)\right] + \\ & + P_{3,i}\left[a_1e_3 + a_3e_1 + e^{i\alpha}(a_2\rho + be_2)\right]\Big\}\,. \end{aligned} \tag{3.46}$$

Using eqs. (3.44) and eq. (3.45), the right-hand side of (2.22) takes the following form:

$$\begin{aligned} W^2 + 4\,g^{i\bar{\jmath}}\partial_i W\partial_{\bar{\jmath}}W = & \left(a_1^2 + a_2^2 + a_3^2 + b^2\right)\left(e_1^2 + e_2^2 + e_3^2 + b^2\right) \\ & + 2e_1e_2(a_1a_2 + a_3b\cos\alpha) + 2e_3\rho(a_1a_2\cos\alpha + a_3b) \\ & + 2e_2e_3(a_2a_3 + a_1b\cos\alpha) + 2e_1\rho(a_2a_3\cos\alpha + a_1b) \\ & + 2e_3e_1(a_3a_1 + a_2b\cos\alpha) + 2e_2\rho(a_3a_1\cos\alpha + a_2b) \end{aligned} \tag{3.47}$$

which reproduces the result for the black-hole potential of supersymmetric theories,

$$V = I_1 + I_2 = e_1^2 + e_2^2 + e_3^2 + \rho^2 \tag{3.48}$$

if:

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 + b^2 &=& 1 \\ a_1a_2 + a_3b\cos\alpha &=& 0 \\ a_1a_2\cos\alpha + a_3b &=& 0 \\ a_1a_3 + a_2b\cos\alpha &=& 0 \\ a_1a_3\cos\alpha + a_2b &=& 0 \\ a_2a_3 + a_1b\cos\alpha &=& 0 \\ a_2a_3\cos\alpha + a_1b &=& 0\,. \end{cases} \tag{3.49}$$

This system, together with the requirement of the existence of an attractor at the horizon, allows three inequivalent solutions, all of which requiring the phase of the singlet charge to be fixed, when the system is in normal form, by $\cos^2\alpha = 1$.

1. $a_1 = 1$, $a_2 = a_3 = b = 0$ (or, equivalently, for $a_1 \leftrightarrow a_2 \leftrightarrow a_3$).

   In this case the prepotential encoding the solution has the form:

$$W_{(1)} = e_1 \,, \tag{3.50}$$

   which is extremized (see (3.46)), for $e_1 = e$, $e_2 = e_3 = \rho = 0$. The Bekenstein-Hawking entropy is:

$$V_{(1)}|_{\rm extr} = e^2 \,. \tag{3.51}$$

   This is the BPS solution.

2. $a_1 = a_2 = a_3 = 0$, $b = 1$.

   In this case the prepotential is:

$$W_{(2)} = \rho = \sqrt{X\overline{X}} \,. \tag{3.52}$$

   It is extremized for $e_1 = e_2 = e_3 = 0$, and it is a non-BPS solution. The corresponding entropy is:

$$V_{(2)}|_{\rm extr} = \rho^2 \,. \tag{3.53}$$

3. $a_1 = a_2 = a_3 = b = \frac{1}{2}$.

   This solution requires that the phase of the singlet charge X be fixed to $\alpha = \pi$: $X = -\rho$. We then have:

$$\left\{ \begin{array}{rcl} W_{(3)} & = & \frac{1}{2}(e_1 + e_2 + e_3 + \rho) \\ \alpha & = & \pi \,. \end{array} \right. \tag{3.54}$$

   W is extremized for: $e_1 = e_2 = e_3 = e$, $X = -e$. In this case the entropy is

$$V_{(3)}|_{\rm extr} = 4e^2 \tag{3.55}$$

   This is also a non-BPS extremal solution.

Note that since the bosonic sector of the $N = 6$ theory also describes an $N = 2$ model, the three solutions above are also solutions of the equivalent $N = 2$ model based on the coset $\mathrm{SO}^*(12)/\mathrm{U}(6)$. In this case, however, the singlet charge X is the central charge corresponding to the $N = 2$ graviphoton, while the Z_{AB} are matter charges, so that the first two solutions are interchanged in the $N = 2$ version: the first one is non-BPS while the second one is the BPS solution.

3.2.3 The prepotential for the $N = 8$ theory

The scalar manifold of the $N = 8$ theory is $\mathrm{E}_{7(-7)}/\mathrm{SU}(8)$. It is not a Kähler manifold, and it is spanned by the vielbein $P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}\overline{P}^{EFGH}$ (with $A = 1, \dots 8$). The central charges of this theory belong to an antisymmetric matrix $Z_{AB} = -Z_{BA}$. They obey the differential relations:

$$\nabla Z_{AB} = \frac{1}{2} P_{ABCD} \overline{Z}^{CD} . \tag{3.56}$$

Since for this theory, differently from the other four dimensional cases, the holonomy group does not contain a U(1) factor, when the antisymmetric matrix Z_{AB} is put in normal form via an SU(8) rotation, it still depends on an overall phase. Therefore we can write:

$$Z_{AB} = \mathrm{e}^{\mathrm{i}\frac{\alpha}{4}} \begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & e_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{3.57}$$

in terms of the four real (non negative) skew-eigenvalues $e_1,\ e_2,\ e_3,\ e_4$ ($e_r = |Z_r|, r = 1, \ldots, 4$) and of the phase α. The moduli e_r of the skew-eigenvalues are related to the following SU(8) invariants:

$$\begin{cases} I_1 & \equiv & \frac{1}{2} Z_{AB} \overline{Z}^{AB} = (e_1)^2 + (e_2)^2 + (e_3)^2 + (e_4)^2 \\ I_2 & \equiv & \frac{1}{2} Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} = (e_1)^4 + (e_2)^4 + (e_3)^4 + (e_4)^4 \\ I_3 & \equiv & -\frac{1}{2} Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DE} Z_{EF} \overline{Z}^{FA} = (e_1)^6 + (e_2)^6 + (e_3)^6 + (e_4)^6 \\ I_4 & \equiv & \frac{1}{2} Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DE} Z_{EF} \overline{Z}^{FG} Z_{GH} \overline{Z}^{HA} = (e_1)^8 + (e_2)^8 + (e_3)^8 + (e_4)^8 . \end{cases} \tag{3.58}$$

The differential relations among the dressed charges still have a simple expression when written in terms of the skew-eigenvalues. We find indeed:

$$\begin{cases} \nabla_r e_1 & = & \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} \left(P_{1,r} e_2 + P_{2,r} e_3 + P_{3,r} e_4\right)\right] \\ \nabla_r e_2 & = & \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} \left(P_{1,r} e_1 + P_{2,r} e_4 + P_{3,r} e_3\right)\right] \\ \nabla_r e_3 & = & \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} \left(P_{1,r} e_4 + P_{2,r} e_1 + P_{3,r} e_2\right)\right] \\ \nabla_r e_4 & = & \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} \left(P_{1,r} e_3 + P_{2,r} e_2 + P_{3,r} e_1\right)\right] \end{cases} \tag{3.59}$$

where $P_{1,r} = P_{1234,r}$, $P_{2,r} = P_{1256,r}$, $P_{3,r} = P_{3456,r}$ are the components of the scalar vielbein appearing in (3.59) when the central charge is written in normal form.

We then propose, for the prepotential W, the form

$$W = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 , \tag{3.60}$$

giving, for its gradient in normal form:

$$\begin{aligned} \nabla W = \mathrm{Re}\Big\{ & P_1 \left[\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} (a_1 e_2 + a_2 e_1) + \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} (a_3 e_4 + a_4 e_3)\right] + \\ & + P_2 \left[[\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} (a_1 e_3 + a_3 e_1) + \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} (a_2 e_4 + a_4 e_2)\right] + \\ & + P_3 \left[[\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} (a_1 e_4 + a_4 e_1) + \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} (a_2 e_3 + a_3 e_2)\right] \Big\} , \end{aligned} \tag{3.61}$$

where P_1, P_2, P_3 are the scalar vielbein 1-forms in normal form. Using (3.56), eq. (3.60) gives, for the right-hand side of (2.22):

$$\begin{aligned} W^2 + 2\,g^{rs}\partial_r W \partial_s W &= (a_1^2 + a_2^2 + a_3^2 + a_4^2)\left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) \\ &\quad + 2e_1e_2(a_1a_2 + a_3a_4\cos\alpha) + 2e_3e_4(a_1a_2\cos\alpha + a_3a_4) \\ &\quad + e_1 \to e_2 \to e_3 \to e_4 \to e_1\,. \end{aligned} \tag{3.62}$$

This reproduces the result for the black-hole potential of supersymmetric theories,

$$V = I_1 = e_1^2 + e_2^2 + e_3^2 + e_4^2\,, \tag{3.63}$$

if

$$\begin{cases} a_1^2 + a_2^2 + +a_3^2 + a_4^2 &= 1 \\ a_1a_2 + a_3a_4\cos\alpha &= 0 \\ a_1a_2\cos\alpha + a_3a_4 &= 0 \\ a_1a_3 + a_2a_4\cos\alpha &= 0 \\ a_1a_3\cos\alpha + a_2a_4 &= 0 \\ a_2a_3 + a_1a_4\cos\alpha &= 0 \\ a_2a_3\cos\alpha + a_1a_4 &= 0\,. \end{cases} \tag{3.64}$$

This system is formally equivalent to (3.49) if we replace a_4 with the singlet coefficient b and if the overall phase in Z_{AB} is reinterpreted as the $N = 6$ singlet phase. The solutions can be found following the $N = 6$ approach, paying attention to the fact that since here we consider the four charges on the same footing, the BPS and the first non-BPS solutions of the $N = 6$ case become equivalent in the $N = 8$ version and correspond both to the BPS solution of the $N = 8$ case. Therefore now the system allows only for two inequivalent solutions.

1. All the $a_i's$ vanish except one, say a_1: $a_1 = 1$, $a_2 = a_3 = a_4 = 0$.

   In this case the prepotential encoding the solution has the form:

   $$W_{(1)} = e_1\,, \tag{3.65}$$

   which is extremized (see (3.61)), for $e_1 = e$, $e_2 = e_3 = e_4 = 0$. The Bekenstein-Hawking entropy is:

   $$V_{(1)}|_{\rm extr} = e^2\,. \tag{3.66}$$

   This is the BPS solution.

2. $a_1 = a_2 = a_3 = a_4 = \frac{1}{2}$.

   This solution requires that the overall phase of the central charge in the normal form be fixed to $\alpha = \pi$. We then have

   $$\begin{cases} W_{(2)} &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \\ \alpha &= \pi \end{cases} \tag{3.67}$$

   which is extremized for $e_1 = e_2 = e_3 = e_4 = e$, $\alpha = \pi$. In this case the entropy is

   $$V_{(2)}|_{\rm extr} = 4e^2 \tag{3.68}$$

   This is a non-BPS extremal solution.

3.2.4 The $N = 2$ case

The vector multiplet moduli space of $N = 2$ theory is given by special geometry, and allows σ-models which are not in general homogeneous spaces. Since our ansatz for W is given in terms of invariants of the representation of the isotropy group of the scalar manifold which the dressed charges belong to, a general result here is not easy to obtain. However, we know that the scalar manifold is embedded in the coset $\mathcal{M}_{\rm SK} \subset \mathrm{Sp}(2n+2)/\mathrm{U}(n+1)$ due to the symplectic embedding. The $n+1$ dressed charges $(Z, Z_i) = \mathcal{Z}_A$ compose a vector of $\mathrm{U}(n+1)$. For a general special manifold there are many $\mathrm{U}(n+1)$-invariants (the only request is to build coordinate invariants of Kähler weight zero) which can be constructed in terms of Z and Z_i out of $g_{i\bar{\jmath}}$, C_{ijk}, and/or their derivatives and products.

For special manifolds which are coset spaces G/H, the invariants are built in terms of the invariant tensors of H, and the result is found by a case by case inspection.

In the minimal case $\mathrm{SU}(1,n)/\mathrm{U}(n)$, where the $C_{ijk} \equiv 0$, the procedure to find the prepotential is straightforward. Indeed in this case we only have the 2 possible invariants $e^2 = Z\overline{Z}$ and $\rho^2 = Z_i \overline{Z}_{\bar{\jmath}} g^{i\bar{\jmath}}$, so that

$$W = ae + b\rho. \tag{3.69}$$

Using the differential relations of special geometry on (3.69) we find

$$V_{\rm BH} = (a^2+b^2)(e^2+\rho^2) + 2abe\rho \tag{3.70}$$

which coincides with the supersymmetric one

$$V_{\rm BH} = |Z|^2 + Z_i \overline{Z}_{\bar{\jmath}} g^{i\bar{\jmath}} \tag{3.71}$$

for

$$\begin{aligned} a^2 + b^2 &= 1 \\ ab &= 0 \end{aligned} \tag{3.72}$$

This system has two independent solutions corresponding to the two attractors:

1. $a = 1$, $b = 0$. In this case the prepotential encoding the solution has the form $W_{(1)} = e$ which is extremized for $Z_i = 0$. The Bekenstein-Hawking entropy is

$$V_{(1)}|_{\rm extr} = e^2 \tag{3.73}$$

   This is the BPS solution.

2. $a = 0$, $b = 1$. $W_{(2)} = \rho$ which is extremized for $Z = 0$. In this case the entropy is

$$V_{(2)}|_{\rm extr} = \rho^2 \tag{3.74}$$

   This is a non-BPS extremal solution.

JHEP11(2007)032

As a second example we may analyze the special manifold $SO^*(12)/\,\mathrm{U}(6)$ from the $N=2$ point of view. In this case the flat index I is identified with the antisymmetric couple AB, and the C-tensor, with flat indices is identified with the invariant tensor ϵ_{ABCDEF} [15]. We expect the following invariants to be involved: $Z\overline{Z}$, $Z_I\overline{Z}^I$, $\overline{Z}^J\overline{Z}^K Z_L Z_M C_{IJK}C^{ILM}$, $|Z_I Z_J Z_K C^{IJK}|^2$. They are in fact related to the invariants of $\mathrm{SO}^*(12)$ introduced in (3.43) by:

$$\begin{cases} Z_I\overline{Z}^I & = & I_1 \\ Z\overline{Z} & = & I_2 \\ \overline{Z}^J\overline{Z}^K Z_L Z_M C_{IJK}C^{ILM} & \propto & \left(I_1^2 - I_3\right) \\ |Z_I Z_J Z_K C^{IJK}|^2 & \propto & \left(I_1^3 - 3I_1 I_3 + 4I_4\right) \; . \end{cases} \tag{3.75}$$

We may then give for the prepotential exactly the same Ansatz (3.45) as for the $N=6$ case with e_r and ρ given in terms of the $N=2$ invariants (3.75) through (3.43), finding exactly the same solutions as in section 3.2.2. As already discussed there, the only difference is that, since in the $N=2$ interpretation the singlet X is in fact the central charge while the $Z_{AB} = Z_I$ are the matter charges, the meaning of the first two attractor solutions enumerated in section 3.2.2 is now interchanged: the BPS one is the second solution, while the first is now non-BPS.

Other examples are given in [14].

4. Concluding remarks and speculations on the non extremal case

In this paper we have dealt with the problem of finding, in four dimensional extended supergravity, the analogue, for non-BPS extremal black holes, of the first order differential equations which encode the attractor mechanism for BPS black holes and which imply the second order field equations.

We have given a general Ansatz for the prepotential W which reproduces all the known attractors in $N \geq 3$ extended supergravity.

In this concluding section, we discuss a possible extension of our analysis to the non extremal case $c \neq 0$. In this more general situation, we may argue that a possible generalization of the expression for the prepotential W might include an explicit dependence on the evolution parameter τ, that is:

$$\dot{U} = W(\Phi,\tau)\, e^U \; . \tag{4.1}$$

Indeed, differentiating (2.12) with respect to τ, we find:

$$\ddot{U} = (\dot{U})^2 + \dot{W} e^U \; , \tag{4.2}$$

where now:

$$\dot{W} = \dot{\Phi}^r \partial_r W + \partial_\tau W \; . \tag{4.3}$$

The (on-shell) expression for V_{BH} is still formally the same as for the extremal case:

$$V_{\mathrm{BH}} = W^2 + e^{-U}\dot{W}. \tag{4.4}$$

However, when inserted in (2.6) the above expression now gives (using (2.4)):

$$\ddot{U} - (\dot{U})^2 = \left(\dot{\Phi}^r \partial_r W + \partial_\tau W\right) e^U = \frac{1}{2} g_{rs} \dot{\Phi}^r \dot{\Phi}^s - c^2 . \tag{4.5}$$

For $\dot{\Phi}^r \neq 0$, eq. (4.5) admits the particular solution

$$\begin{cases} \dot{\Phi}^r &= 2\, e^U g^{rs} \partial_s W \\ \partial_\tau W &= -c^2 e^{-U} \end{cases} . \tag{4.6}$$

The integration of the second equation in (4.6) gives

$$W^2(\Phi, U) = c^2 e^{-2U} + W_0^2(\Phi) . \tag{4.7}$$

Eq. (4.6) reproduces the correct description of non-extremal black holes near the horizon. Indeed, given the general form (2.2) for the space-time metric, for $\tau \to -\infty$ the leading behavior of a generic charged black-hole solution is

$$e^{-2U} \sim \frac{A}{4\pi} \left(\frac{\sinh(c\tau)}{c}\right)^2 \tag{4.8}$$

so that

$$W = -\frac{d}{d\tau} e^{-U} = -\cosh(c\tau) \sqrt{\frac{A}{4\pi}} \tag{4.9}$$

giving

$$\partial_\tau W = -c \sinh(c\tau) \sqrt{\frac{A}{4\pi}} = -c^2 e^{-U} . \tag{4.10}$$

Note that (4.9) may also be written as

$$W^2 = \frac{A}{4\pi} \left(1 + \sinh^2(c\tau)\right) = \frac{A}{4\pi} + c^2 e^{-2U} \tag{4.11}$$

which coincides with (4.7) for $W_0^2 = \frac{A}{4\pi}$.

For the non-extremal cases where eqs. (4.6) hold, the field equations for the scalar sector are in fact still first order as for the extremal case. To show this, it is however necessary to make a slight modification to the effective potential. Indeed, using (4.6) the effective potential reads

$$V_{\rm BH} = W^2 + 2\, g^{rs} \partial_r W \partial_s W - c^2 e^{-2U} . \tag{4.12}$$

By inserting eqs. (4.6) in the second order evolution equation for the scalars, eq. (2.5), we actually find an inconsistency. However, since the expression (4.4) for the black-hole potential is an on-shell relation, any expression for $V_{\rm BH}$ given by

$$V_{\rm BH} = W^2 + 2\, g^{rs} \partial_r W \partial_s W - c^2 e^{-2U} + \alpha\, e^{-U} (\partial_\tau W + c^2 e^{-U}) \tag{4.13}$$

is equivalent to (4.12). If we redo the calculation of the field equations for the scalars (2.5) with the parametric expression (4.13), we find that for $\alpha = 2$ it is automatically solved when we use the Ansatz (4.6) for $\dot{\Phi}^r$. For all the cases where eqs. (4.6) hold, we then have the following expression for the effective potential in terms of W:

$$V_{\rm BH} = W^2 + 2\, g^{rs}\, \partial_r W\, \partial_s W + 2\partial_\tau W e^{-U} + c^2 e^{-2U} \tag{4.14}$$

By inserting the explicit expression (4.7) in (4.14) we find

$$V_{\rm BH} = W_0^2 + 2\, g^{rs}\, \partial_r W_0\, \partial_s W_0 - \frac{c^2 e^{-2U}}{W_0^2 + c^2 e^{-2U}} 2\, g^{rs}\, \partial_r W_0\, \partial_s W_0\,. \tag{4.15}$$

Note that the extrema of the prepotential W do not extremize $V_{\rm BH}$, corresponding to the fact that for non-extremal black holes, the attractor mechanism is not expected to be at work nor the horizon to be a fixed point for the scalar fields.

However, the expression (4.15) for the effective potential has the feature of containing an explicit dependence on the evolution parameter τ. Such behavior could be acceptable for purely bosonic theories such as fake supergravity [17, 18]. For black holes in supersymmetric theories this clashes with the request that the effective potential be identified with the general expression (2.7), where it depends on τ only through the scalars fields. We then have to assume that, in the supersymmetric case, (4.15) is rigorously valid only for the "double non-extremal" cases of constant scalars. For more general solutions, at a finite distance from the black-hole horizon we then expect eq. (4.6) to receive corrections. For completely general non-extremal cases, we do not expect to have a first-order description in terms of a prepotential.

As a final remark, let us recall that in the BPS case the effective lagrangian (2.9) may be written in terms of a sum of squares, as discussed in [4, 14]. We want to give a similar treatment for general extremal black holes and for all the cases where (4.6) hold and the effective potential takes the form (4.14). To this aim, we consider the following quantity:

$$K = (\dot{U} - We^U)^2 + \frac{1}{2} g_{rs} (\dot{\Phi}^r - 2g^{r\ell}\partial_\ell W)(\dot{\Phi}^s - 2g^{sm}\partial_m W) \geq 0 \tag{4.16}$$

Using (4.14) we find:

$$K = \mathcal{L}_{\rm eff} - R\,, \tag{4.17}$$

where

$$R = -\left[\frac{d}{d\tau}(2e^U W) + c^2\right] = -\frac{d}{d\tau}(2e^U W + c^2\tau)\,. \tag{4.18}$$

Eq. (4.17) implies that the effective lagrangian $\mathcal{L}_{\rm eff}$ is bounded from below:

$$\mathcal{L}_{\rm eff} \geq -\frac{d}{d\tau}(2e^U W + c^2\tau) \tag{4.19}$$

The extremum value $\mathcal{L}_{\rm eff} = R$ is realized on-shell for

$$\begin{cases} \dot{U} &= We^U \\ \dot{\Phi}^r &= 2g^{r\ell}\partial_\ell W \\ \partial_\tau W &= -c^2 e^{-U} \end{cases} \tag{4.20}$$

Under our hypothesis, (4.6), eqs. (4.20) are always verified and (4.19) then implies, on-shell, that

$$\mathcal{L}_{\text{eff}} = -\frac{d}{d\tau}(2e^{U}W + c^2\tau) \tag{4.21}$$

is a topological quantity characterizing the extremal solution

$$S_{on-shell} = \int_{-\infty}^{0} d\tau \mathcal{L}_{\text{eff}} = -[2\dot{U} + c^2\tau]_{\tau\to-\infty}^{\tau=0} = -2(M_{\text{ADM}} - c) + 2c^2\tau|_{-\infty}\,. \tag{4.22}$$

The non extremal infinite contribution from c^2 may be understood as a "vacuum energy" contribution to the action.

Our result generalizes the argument for "non-extremal but BPS solutions" discussed in [18] to cases where the scalar fields have a non trivial radial evolution. In [18], it is shown that the effective two dimensional model describing the non-extremal but BPS black hole has a supersymmetric completion where the first order equations play the role of Killing spinor equations, even if there is an obstruction to the four dimensional uplift of this effective supersymmetric model. It would be interesting to perform the same analysis for the class of non-extremal black holes defined by the first order equations (4.6). This is left to a future investigation.

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JHEP11(2007)032