First order quasilinear equations with boundary conditions in the L^∞ framework

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Abstract

We study a class of first order quasilinear equations on bounded domains in the L^∞ framework. Using the "semi Kružkov entropy-flux pairs", we define a weak-entropy solution, state an existence and uniqueness result, and a maximum principle.

Key words: Semi Kružkov entropy-flux pair; Maximum principle; BV estimates; Vanishing viscosity method; Doubling variable method.

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Introduction

In this paper, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded smooth domain. Let us denote by $\partial\Omega$ the boundary of Ω and by n the outer normal vector to $\partial\Omega$. We denote $Q_T \equiv (0,T) \times \Omega$ and $\Sigma_T \equiv (0,T) \times \partial\Omega$. Let us consider this set of equations:

$$\frac{\partial u}{\partial t} + \nabla \cdot \Big(f(t,x,u) \Big) + g(t,x,u) = 0 \quad \text{on } Q_T \tag{1}$$

$$u(0,\cdot) = u^0 \quad \text{on } \Omega \tag{2}$$

$$\text{“}u = u^D\text{”} \quad \text{on } \Sigma_T \tag{3}$$

where the sense of the boundary condition will be precised further. We consider the following assumption:

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Assumption 1

(i) f *and* g *are two functions defined on* $[0,T]\times\overline{\Omega}\times\mathbb{R}$ *such that*

$$f\in\left(C^2([0,T]\times\overline{\Omega}\times[a,b])\right)^d,\quad g\in C^2([0,T]\times\overline{\Omega}\times[a,b])$$

(ii) f, $\nabla\cdot f$ *and* g *are Lipschitz continuous w.r.t.* u*, uniformly in* (t,x)*, the constants of Lipschitz continuity being respectively denoted* $\mathcal{L}_{[f]}$, $\mathcal{L}_{[\nabla\cdot f]}$, $\mathcal{L}_{[g]}$.

(iii) $(u^0,u^D)\in L^\infty(\Omega;[a,b])\times L^\infty(\Sigma_T;[a,b])$,

(iv) $(\nabla\cdot f+g)(\cdot,\cdot,a)\leq 0$ *and* $(\nabla\cdot f+g)(\cdot,\cdot,b)\geq 0$ *uniformly in* (t,x).

From a mathematical point of view, numerous works have approached or investigated this field. On unbounded domains, existence and uniqueness of a solution for quasilinear first order equations domains has been solved in the pioneering works of Oleĭnik [1], Volpert [2] and Kružkov [3] who introduced the concept of weak entropy solutions and related "Kružkov entropy-flux pairs"

$$\left(\left|u-k\right|,\operatorname{sgn}(u-k)(f(t,x,u)-f(t,x,k))\right).$$

When dealing with bounded domains, under some regularity assumptions on the data, Bardos, Le Roux and Nédélec [4] also proved existence and uniqueness of a weak entropy solution satisfying a "Kružkov entropy-flux pair" formulation including boundary terms; for this, they introduced an appropriate mathematical boundary condition that must be understood in a particular way. Nevertheless, when considering L^∞ data, the lack of regularity prevents from using the result of Bardos, Le Roux and Nédélec. This difficulty was overcome, at least in the case of autonomous scalar conservation laws on bounded domains, by Otto [5,6] who introduced "boundary entropy-flux pairs"

$$\left(H(u,k),Q_{[f]}(u,k)\right)$$

satisfying particular properties (to be recalled further), which enable to state existence and uniqueness of a so-called weak entropy solution and a maximum principle for this solution. Finally, using a lemma proposed by Vovelle [7], it appears that a formulation using "semi Kružkov entropy-flux pairs"

$$\left((u-k)^\pm,\operatorname{sgn}_\pm(u-k)(f(t,x,u)-f(t,x,k))\right)$$

is equivalent to a formulation based on "boundary entropy-flux pairs". Here, $u\mapsto(u-\kappa)^\pm$ are the so-called "semi Kružkov entropies" [8,9,7] defined by

$$(u-\kappa)^+=\begin{cases}u-\kappa, \text{ if } u\geq\kappa,\\ \quad 0, \text{ otherwise.}\end{cases}\quad \text{and}\quad (u-\kappa)^-=(\kappa-u)^+.$$

The functions $\operatorname{sgn}_\pm(u-\kappa)(f(\cdot,\cdot,u)-f(\cdot,\cdot,\kappa))$ are the corresponding "semi Kružkov fluxes", where $u\mapsto\operatorname{sgn}_\pm(u)$ is the derivative of the function $u\mapsto u^\pm$

with value 0 at point 0. Notice that the "semi Kružkov entropy-flux pairs" formulation is very similar to the initial one of Kružkov. But:

- What is the appropriate definition of a weak entropy solution for first order quasilinear equations (i.e. including non-autonomous fluxes and source terms) on bounded domains with L^∞ data ? Answering this question would draw a complete parallel with the results of Bardos, Le Roux and Nédélec [4] and those of Otto [5] and Vovelle [7]: indeed, the analysis of scalar conservation laws with L^∞ data, initiated by Otto, would be extended to quasilinear first order equations, studied by Bardos, Le Roux and Nédélec.
- What sufficient conditions lead to a maximum principle ? Indeed, such a property is crucial when studying some physical problems.

Thus, it is the purpose of this paper to give a general framework which is valid for first order quasilinear equations on bounded domains with L^∞ data. Among the difficulties, we can observe that, when dealing with non autonomous fluxes and source terms, a formulation with "boundary entropy-flux pairs" is not possible anymore. Fortunately, the concept of "semi Kružkov entropy-flux pairs" allows to overcome difficulties. This work is organized as follows: in **Section 1**, we state the definitions and establish a maximum principle; in **Section 2**, we prove the existence result; in **Section 3**, we prove the uniqueness result. Existence and uniqueness theorems are based on techniques that have been widely used in [3–6]. But we point out the fact that these arguments have never been gathered with the appropriate definition of a weak entropy solution in this general framework in order to establish an existence and uniqueness theorem along with a maximum principle: in fact, we deeply use the results detailed in [6], up to the following modifications: proofs for existence and uniqueness are adapted to the "semi Kružkov entropy-flux pairs", dealing with additional terms induced by the source term and non-autonomous property of the flux.

1 Definition, initial / boundary conditions, maximum principle

Definition 1 *Suppose that Assumption 1 holds. A function $u \in L^\infty(Q_T, [a,b])$ is said to be a weak entropy solution of problem (1)-(2)-(3) if it satisfies*

$$(\mathcal{P}_{SK}) \begin{cases} \displaystyle\int_{Q_T} \Big\{ (u-k)^\pm \frac{\partial \varphi}{\partial t} + \Big(\mathrm{sgn}_\pm(u-k)(f(t,x,u)-f(t,x,k))\Big)\ \nabla\varphi \\ \quad -\mathrm{sgn}_\pm(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\ \varphi \Big\}\, dx\, dt \\ \quad + \displaystyle\int_{\Omega} (u^0-k)^\pm \varphi(0,x)\, dx + \mathcal{L}_{[f]} \int_{\Sigma_T} (u^D-k)^\pm \varphi(t,r)\, d\gamma(r)\, dt \geq 0 \\ \forall \phi \in \mathcal{D}((-\infty,T)\times\mathbb{R}^d),\ \phi\geq 0,\ \forall k\in\mathbb{R} \end{cases}$$

Let us explain the way the boundary / initial conditions are satisfied for this problem. Interestingly, the concept of "boundary entropy-flux pairs" defined by Otto is still the key point. Thus, let us recall their definition:

Definition 2 *Let* $(H, Q_{[f]})$ *be in* $C^1(\mathbb{R}^2) \times \left(C^1((0,T) \times \overline{\Omega} \times \mathbb{R}^2)\right)^d$. *The pair* $(H, Q_{[f]})$ *is said to be a "boundary entropy-flux pair" (for the flux* f*) if:*

1. *for all* $w \in \mathbb{R}$, $s \mapsto H(s,w)$ *is a convex function,*
2. $\forall w \in \mathbb{R}$, $\partial_s Q_{[f]}(\cdot,\cdot,s,w) = \partial_s H(s,w) \dfrac{\partial f}{\partial s}(\cdot,\cdot,s)$,
3. $\forall w \in \mathbb{R}$, $H(w,w) = 0$, $Q_{[f]}(\cdot,\cdot,w,w) = 0$, $\partial_s H(w,w) = 0$.

Let us recall the lemma provided by Vovelle [7], which gives the link between "semi Kružkov entropy-flux pairs" and "boundary entropy-flux pairs":

Lemma 3
(i) Let $\eta \in C^1(\mathbb{R};\mathbb{R})$ *be a convex function such that there exists* $w \in [a,b]$ *with* $\eta(w) = 0$ *and* $\eta'(w) = 0$. *Then* η *can be uniformly approximated on* $[a,b]$ *by applications of the kind*

$$s \longmapsto \sum_1^p \alpha_i (s-\kappa_i)^- + \sum_1^q \beta_j (s-\tilde{\kappa}_j)^+$$

where $\alpha_i \geq 0$, $\beta_j \geq 0$, $\kappa_i \in [a,b]$ *and* $\tilde{\kappa}_j \in [a,b]$.
(ii) Conversely, there exists a sequence of "boundary entropy-flux pairs which converges to the "semi Kružkov entropy-flux pairs".

Lemma 4 (Boundary condition) *Let* $u \in L^\infty(Q_T)$ *satisfying* $(\mathcal{P}_{SK})$. *Then,*

$$\operatorname*{ess\,lim}_{\varrho \to 0^+} \int_{\Sigma_T} Q_{[f]}(t,r,u(t,r-\varrho\, n(r)), u^D(t,r)) \cdot n(r)\, \beta(t,r)\, d\gamma(r)\, dt \geq 0, \quad (4)$$

PROOF. We directly use the proof of Lemma 7.12 in [6], adapted to the particular case of the "semi Kružkov entropy-flux pairs". Thus, we easily state that if $u \in L^\infty(Q_T)$ satisfies $(\mathcal{P}_{SK})$, then, defining the quantity

$$\operatorname*{ess\,lim}_{\varrho \to 0^+} \int_{\Sigma_T} \Big\{ \operatorname{sgn}_\pm(u(t,r-\varrho\, n(r)) - v^D(t,r)) \big(f(t,r,u(t,r-\varrho\, n(r))) - f(t,r,v^D(t,r)) \big) \Big\} \cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt \quad (5)$$

exists for all $\beta \in L^1((0,T)\times\mathbb{R}^{d-1})$, $\beta \geq 0$ a. e., and all $v^D \in L^\infty((0,T)\times\mathbb{R}^{d-1})$. Moreover, we have:

$$\operatorname{ess}\lim_{\varrho\to 0^+} \int\limits_{\Sigma_T} \Big\{ \operatorname{sgn}_\pm(u(t,r-\varrho\, n(r)) - v^D(t,r)) \Big(f(t,r,u(t,r-\varrho\, n(r))) - f(t,r,v^D(t,r))) \Big) \Big\} \cdot n(r)\ \beta(t,r)\, d\gamma(r)\, dt \geq -\mathcal{L}_{[f]} \int\limits_{\Sigma_T} (u^D(t,r) - v^D(t,r))^\pm\ \beta(t,r)\, d\gamma(r)\, dt,$$

for all $\beta \in L^1((0,T)\times\mathbb{R}^{d-1})$, $\beta \geq 0$ a. e., and all $v^D \in L^\infty((0,T)\times\mathbb{R}^{d-1})$. Then, taking $v^D = u^D$, every "boundary flux" $Q_{[f]}$ is uniformly approximated by a linear combination of "semi Kružkov fluxes" (see Lemma 3), every coefficient being non-negative, which preserves the inequality and concludes the proof.

To complete the scope of boundary / initial conditions, we recall the following result, which is proved with the same arguments as in Lemma 7.41 of [6]:

Lemma 5 (Initial condition) *Let $u \in L^\infty(Q_T)$ satisfying $(\mathcal{P}_{SK})$. Then,*

$$\operatorname{ess}\lim_{t\to 0^+} \int\limits_{\Omega} \left| u(t,x) - u^0(x) \right|\, dx = 0 \tag{6}$$

Now we give some details on the way the boundary condition is satisfied:

Remark 6 *The boundary condition 4 is nothing less than the one obtained in [5,6], up to a generalization to non-autonomous fluxes and taking account of a source-term which does not interfere in the boundary condition. We have proved that it is satisfied, although working only with the "semi Kružkov entropy-flux pairs" formulation (let us recall that a "boundary entropy-flux pairs" formulation is not possible anymore). However the way to understand the boundary condition is given in [5–7]: generally speaking, the problem should be overdetermined and the boundary equality cannot be required to be assumed at each point of the boundary, even if the solution is a regular function. But, with additional assumptions, the more comprehensive "BLN" condition is recovered: if u admits a trace, i.e. there exists $u_{|\Sigma_T} \in L^\infty(\Sigma_T)$ such that*

$$\operatorname{ess}\lim_{\varrho\to 0^+} \int\limits_{\Sigma_T} |u(\tau, r - \varrho\, n(r)) - u_{|\Sigma_T}(\tau,r)|\ d\gamma(r)\ d\tau = 0$$

then Eq. (4) is equivalent to the following equation (see [10,5])

$$Q_{[f]}(\cdot,\cdot,u_{|\Sigma_T},u^D)\cdot n \geq 0, \quad \text{a.e. on } \Sigma_T.$$

Considering the particular boundary entropy / flux pairs

$$H_\delta^+(z,\kappa) = \sqrt{\{(z-\kappa)^+\}^2+\delta^2} - \delta, \quad H_\delta^-(z,\kappa) = \sqrt{\{(\kappa-z)^-\}^2+\delta^2} - \delta,$$
$$Q^{\pm}_{[f],\delta}(\cdot,\cdot,z,\kappa) = \int_\kappa^z \partial_1 H_\delta^{\pm}(\lambda,k)\,\frac{\partial f}{\partial u}(\cdot,\cdot,\lambda)\,d\lambda,$$

and letting $\delta \to 0$, we obtain the following uniform convergences:

$$Q^{\pm}_{[f],\delta}(\cdot,\cdot,z,\kappa) \to \operatorname{sgn}_\pm(z-\kappa)(f(\cdot,\cdot,z)-f(\cdot,\cdot,\kappa)).$$

Finally taking the boundary flux

$$\begin{aligned} Q_{[f]}(\cdot,\cdot,s,w) = \; & \operatorname{sgn}_+(s-\max(w,k))(f(\cdot,\cdot,s)-f(\cdot,\cdot,\max(w,k))) \\ & + \operatorname{sgn}_-(s-\min(w,k))(f(\cdot,\cdot,s)-f(\cdot,\cdot,\min(w,k))) \end{aligned}$$

yields the classical condition given by Bardos, Le Roux and Nédélec [4]:

$$\begin{aligned} & \textit{for a.e. } (t,r)\in\Sigma_T,\ \forall k\in[\min(u_{|\Sigma_T},u^D),\max(u_{|\Sigma_T},u^D)], \\ & \operatorname{sgn}(u_{|\Sigma_T}(t,r)-u^D(t,r))(f(t,r,u_{|\Sigma_T}(t,r))-f(t,r,k))\cdot n(r) \geq 0. \end{aligned} \quad (7)$$

Before stating existence and uniqueness results in next sections, we prove:

Theorem 7 (Maximum principle) *Under Assumption 1, if u satisfies $(\mathcal{P}_{SK})$, then $a \leq u \leq b$ a.e. on Q_T.*

PROOF. Set $k = a$ in $(\mathcal{P}_{SK})$. Since we have by Assumption 1 (iii) and (iv),

$$(u^0-a)^- = 0, \quad (u^D-a)^- = 0,$$

the boundary / initial terms vanish. Then if we choose a particular test-function which only depends on time t, we obtain:

$$\int_{Q_T} \left\{(u-a)^-\phi'(t) - \operatorname{sgn}_-(u-a)\Big(\nabla\cdot f(t,x,a)+g(t,x,u)\Big)\,\phi(t)\right\} dx\,dt \geq 0$$

for all $\phi \in \mathcal{D}([0,T[)$, $\phi \geq 0$. Now, using

$$\Big(\nabla\cdot f(t,x,a)+g(t,x,u)\Big) = \Big(\nabla\cdot f(t,x,a)+g(t,x,a)\Big) + g(t,x,u) - g(t,x,a)$$

and Assumption 1 (iv), we get

$$\int_{Q_T} (u-a)^-\phi'(t) - \operatorname{sgn}_-(u-a)\,(g(t,x,u)-g(t,x,a))\;\phi(t)\,dx\,dt \geq 0, \quad (8)$$

for all $\phi \in \mathcal{D}([0,T[)$, $\phi \geq 0$. Furthermore, we can check that:

$$-\mathcal{L}_{[g]}(u-a)^- \leq \mathrm{sgn}_-(u-a)\,(g(t,x,u)-g(t,x,a)) \leq \mathcal{L}_{[g]}(u-a)^-$$

and Inequality (8) implies $\int\limits_{Q_T} (u-a)^- \left(\phi' + \mathcal{L}_{[g]}\phi\right) \geq 0$. Defining the function

$$q_a(t) = e^{-\mathcal{L}_{[g]}t} \int\limits_{\Omega} (u-a)^-(t,x)\,dx, \tag{9}$$

the above inequality gives

$$\int\limits_0^T q_a(t) e^{\mathcal{L}_{[g]}t} \left(\phi'(t) + \mathcal{L}_{[g]}\phi(t)\right)\,dt \geq 0.$$

Denoting $\psi(t) = e^{\mathcal{L}_{[g]}t}\phi(t)$, we infer that for all $\psi \in \mathcal{D}([0,T[)$, $\psi \geq 0$,

$$\int\limits_0^T q_a(t)\;\psi'(t)\,dx\,dt \geq 0, \tag{10}$$

Let $\tau < T$, $\delta_\tau = T - \tau$ and $r \in \mathcal{D}([0,T[)$ be such that: r is non-increasing, $r \equiv 1$ on $[0,\tau]$, $r \equiv 0$ on $[\tau + \delta_\tau/2, T[$. Choosing $\psi(t) = r(t)\,(T-t)/T$ in Inequality (10) gives

$$-\frac{1}{T}\int\limits_0^T q_a(t) r(t)\,dt + \int\limits_0^T q_a(t)\;\frac{T-t}{T} r'(t)\,dt \geq 0$$

Since $r' \leq 0$, the second term of the left-hand side is negative. Since $r(t) = 1$, $\forall t \in (0,\tau)$ and $r \geq 0$, the first term is upper bounded by

$$-\frac{1}{T}\int\limits_0^\tau q_a(t)\,dt$$

which is consequently non-negative. But, q_a is obviously a non-negative function, so that $q_a \equiv 0$, on $(0,\tau)$.Therefore, we deduce from the definition of q_a (see Eq. (9)) that $(u-a)^- = 0$ on $\Omega \times (0,\tau)$. Letting $\tau \to T$, we have $u \geq a$ a.e. Similarly, by choosing $k = b$ in $(\mathcal{P}_{SK})$ (with the "semi Kružkov entropy" $u \mapsto (u-b)^+$), we prove $u \leq b$ a.e.

Remark 8 *Under Assumption 1 (iv), we check that only the restriction to the set $[a,b]$ of functions $s \mapsto f(t,x,s)$ and $s \mapsto g(t,x,s)$ plays an active role. Therefore, it is sufficient to consider functions $f(t,x,\cdot)$ and $g(t,x,\cdot)$ defined on $[a,b]$ instead of $\mathbb{R}$, as proposed in Assumption 1 (i).*

2 Existence

Existence is obtained from the vanishing viscosity method. We consider the following set of equations:

$$\frac{\partial u_\varepsilon}{\partial t} + \nabla\cdot\Big(f(t,x,u_\varepsilon)\Big) + g(t,x,u_\varepsilon) = \varepsilon\Delta u_\varepsilon \quad \text{on } Q_T \tag{11}$$

$$u_\varepsilon(0,\cdot) = u_\varepsilon^0 \quad \text{on } \Omega \tag{12}$$

$$u = u_\varepsilon^D \quad \text{on } \Sigma_T \tag{13}$$

where the following assumption holds:

Assumption 2

(i) u_ε^D *and* u_ε^0 *satisfy compatibility conditions on* $\overline{\Sigma}_T \cap \overline{Q}_T$*: in particular,* u_ε^0 *and* u_ε^D *should be a restriction, on the sets* $\{0\}\times\Omega$ *and* Σ_T *respectively, of a smooth function* ψ_ε *defined on* $\overline{Q}_T$ *satisfying*

$$\frac{\partial \psi_\varepsilon}{\partial t} + \nabla\cdot\Big(f(t,x,\psi_\varepsilon)\Big) + g(t,x,\psi_\varepsilon) = \varepsilon\Delta\psi_\varepsilon, \quad \text{on } \{0\}\times\partial\Omega$$

(ii) u_ε^D *and* u_ε^0 *are smooth functions:* $u_\varepsilon^D \in C^2(\Sigma_T;[a,b])$*,* $u_\varepsilon^0 \in C^2(\overline{\Omega};[a,b])$*.*

Under Assumption 2, the parabolic problem (11)–(13) admits a unique solution $u_\varepsilon \in C^2(\overline{Q}_T)$ (see Chapter V, §6 in [11]). We study the convergence of $\{u_\varepsilon\}$ when ε tends to 0. As in [6], we introduce the following tools:

Definition 9 *Let us consider* $\mu > 0$ *small enough. We define the functions:*

$$s(x) = \begin{cases} \min\left(\mathrm{dist}(x,\partial\Omega),\mu\right), \text{ if } x\in\Omega \\ -\min\left(\mathrm{dist}(x,\partial\Omega),\mu\right), \text{ if } x\in\mathbb{R}^d\setminus\Omega \end{cases}$$

$$\xi_\varepsilon(x) = 1-\exp\left(-\frac{\mathcal{L}_{[f]}+\varepsilon\mathcal{R}}{\varepsilon}\; s(x)\right), \quad \text{with } \mathcal{R} = \sup_{0<s(x)<\mu}|\Delta s(x)|.$$

Notice that s is Lipschitz continuous in $\mathbb{R}^d$ and smooth on the closure of the set $\{x\in\mathbb{R}^d,\ |s(x)|<\mu\}$. Moreover, it can be proved (see [6]):

Proposition 10 ξ_ε *being defined in Definition 9, for all* $\varphi\in\mathcal{D}(\mathbb{R}^d)$*,* $\varphi\geq 0$*,*

$$\mathcal{L}_{[f]}\int_\Omega \left|\nabla\xi_\varepsilon\right|\varphi \leq \varepsilon\int_\Omega \nabla\xi_\varepsilon\nabla\varphi + (\mathcal{L}_{[f]}+\varepsilon\mathcal{R})\int_{\partial\Omega}\varphi \tag{14}$$

Lemma 11 *Let* (u,u^D,u^0) *satisfy equations (11)–(13), the data satisfying Assumption 2 (subscripts are dropped for convenience). Then,* (*i*) *for all* $\varphi \in \mathcal{D}(]-\infty,T[\times\mathbb{R}^d)$, *for all* $k \in \mathbb{R}$,

$$\int_{Q_T} \Big\{ (u-k)^{\pm}\frac{\partial \varphi}{\partial t} + \mathrm{sgn}_{\pm}(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\nabla\varphi \tag{15}$$

$$-\mathrm{sgn}_{\pm}(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\varphi + \varepsilon\ (u-k)^{\pm}\Delta\varphi\Big\}\,\xi_\varepsilon$$

$$+\int_{\Omega}(u^0-k)^{\pm}\varphi(0,\cdot)\,\xi_\varepsilon$$

$$\geq -2\varepsilon\int_{Q_T}(u-k)^{\pm}\nabla\varphi\nabla\xi_\varepsilon - (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int_{\Sigma_T}(u^D-k)^{\pm}\varphi,$$

(*ii*) *the following maximum principle holds:* $a \leq u \leq b$.

PROOF. ■ *Proof of* (*i*)*:* Let us define the functions:

$$\mathrm{sgn}^{\eta}_{\pm}(z) = \begin{cases} H_\eta(z), & \text{if } z \in \mathbb{R}^{\pm} \\ -H_\eta(-z), & \text{if } z \in \mathbb{R}^{\mp} \end{cases}, \qquad I^{\pm}_{\eta}(z) = \int_0^z \mathrm{sgn}^{\eta}_{\pm}(t)\ dt$$

where the function H_η is a classical approximation of the Heaviside graph: $H_\eta(z) = z/\eta\chi_{[0,\eta[}(z) + \chi_{[\eta,+\infty[}(z)$. Obviously, the pairs

$$\Big(I^{\pm}_{\eta}(z,k), \mathrm{sgn}^{\eta}_{\pm}(z-k)\Big(f(t,x,z)-f(t,x,k)\Big)\Big)$$

mimick the behaviour of the "semi Kružkov entropy-flux pairs". Notice that $I^{\pm}_{\eta}(\cdot,k) \in C^1(\mathbb{R})$ is piecewise convex. Multiplying Eq. (11) by $\mathrm{sgn}^{\eta}_{\pm}(u-k)\ \varphi\ \xi_\varepsilon$, with $\varphi \in \mathcal{D}(]-\infty,T[\times\mathbb{R}^d)$, we obtain (after integration by parts):

$$\int_{Q_T} \Big\{ I^{\pm}_{\eta}(u,k)\frac{\partial \varphi}{\partial t} + \mathrm{sgn}^{\eta}_{\pm}(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\nabla\varphi$$

$$-\ \mathrm{sgn}^{\eta}_{\pm}(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\varphi + \varepsilon\ I^{\pm}_{\eta}(u,k)\Delta\varphi\Big\}\ \xi_\varepsilon$$

$$+\int_{Q_T}\mathrm{sgn}^{\eta}_{\pm}(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\varphi\ \nabla\xi_\varepsilon$$

$$+\int_{\Omega}\Big(\int_k^{u^0}\mathrm{sgn}^{\eta}_{\pm}(v-k)\ dv\Big)\varphi(0,\cdot)\ \xi_\varepsilon$$

$$+\int_{Q_T}\Big(f(t,x,u)-f(t,x,k)\Big)\cdot\nabla u\ \mathrm{sgn}^{\eta\,\prime}_{\pm}(u-k)\ \varphi\ \xi_\varepsilon$$

$$\geq \varepsilon\int_{Q_T}\Big\{\nabla\Big(I^{\pm}_{\eta}(u,k)\ \varphi\Big)\nabla\xi_\varepsilon - 2I^{\pm}_{\eta}(u,k)\nabla\varphi\nabla\xi_\varepsilon\Big\}$$

After some computation, we state that:

$$\left|\mathrm{sgn}^\eta_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\right| \le \mathcal{L}_{[f]}\ I^\pm_\eta(u,k) + \mathcal{L}_{[f]}\ \eta$$

Moreover, using Proposition 10 with $I^\pm_\eta(u,k)\varphi$ instead of φ, we get:

$$\mathcal{L}_{[f]} \int\limits_{Q_T} I^\pm_\eta(u,k)\varphi\left|\nabla\xi_\varepsilon\right| \le \varepsilon \int\limits_{Q_T} \nabla\left(I^\pm_\eta(u,k)\varphi\right)\nabla\xi_\varepsilon + (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int\limits_{\Sigma_T} I^\pm_\eta(u^D,k)\varphi$$

Using these two results in the previous inequality gives:

$$\begin{aligned}
&\int\limits_{Q_T}\Big\{I^\pm_\eta(u,k)\frac{\partial\varphi}{\partial t} + \mathrm{sgn}^\eta_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\nabla\varphi \\
&\quad - \mathrm{sgn}^\eta_\pm(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\varphi + \varepsilon\ I^\pm_\eta(u,k)\Delta\varphi\Big\}\xi_\varepsilon \\
&\quad + \int\limits_\Omega \Big(\int\limits_k^{u^0} \mathrm{sgn}^\eta_\pm(v-k)\ dv\Big)\varphi(0,\cdot)\xi_\varepsilon \\
&\quad + \int\limits_{Q_T}\Big(f(t,x,u)-f(t,x,k)\Big)\cdot\nabla u\,\mathrm{sgn}^{\eta\,\prime}_\pm(u-k)\varphi\xi_\varepsilon \\
&\quad \ge -2\varepsilon\int\limits_{Q_T} I^\pm_\eta(u,k)\nabla\varphi\nabla\xi_\varepsilon - (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int\limits_{\Sigma_T} I^\pm_\eta(u^D,k)\varphi - \mathcal{L}_{[f]}\eta\int\limits_{Q_T}\varphi\left|\nabla\xi_\varepsilon\right|
\end{aligned}$$

Now, let η tend to 0. The first and second terms of the left-hand side give:

$$\begin{aligned}
&\int\limits_{Q_T}\Big\{(u-k)^\pm\frac{\partial\varphi}{\partial t} + \mathrm{sgn}_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\nabla\varphi \\
&\quad - \mathrm{sgn}_\pm(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\varphi + \varepsilon\ (u-k)^\pm\Delta\varphi\Big\}\xi_\varepsilon \\
&\quad + \int\limits_\Omega (u^0-k)^\pm\varphi(0,\cdot)\ \xi_\varepsilon
\end{aligned}$$

The last term of the left-hand side tends to 0 by Lemma 2 in [4] [1]. Finally, the right-hand side tends to:

$$-2\varepsilon\int\limits_{Q_T}(u-k)^\pm\nabla\varphi\nabla\xi_\varepsilon - (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int\limits_{\Sigma_T}(u^D-k)^\pm\varphi$$

and the proof is concluded.

■ *Proof of (ii):* The result is obtained as in the proof of Theorem (7), by working with Eq. (15) instead of $(\mathcal{P}_{SK})$.

[1] This lemma, see Saks [12], says that if $v\in C^1(\overline{\Omega})$, then $\lim\limits_{\eta\to 0}\int\limits_\Omega |\nabla v|\,\mathbf{1}_{[|v(x)|\le\eta]} = 0$.

Now we propose the following L^1-stability result:

Lemma 12 *Let (u_1, u_1^D, u_1^0), (u_2, u_2^D, u_2^0), satisfy equations (11)–(13), the corresponding data satisfying Assumption 2. Then, for all $t \in (0,T)$,*

$$\int_\Omega \Big|u_1(t,\cdot) - u_2(t,\cdot)\Big|\,\xi_\varepsilon \le \left\{ \int_\Omega \Big|u_1^0 - u_2^0\Big|\,\xi_\varepsilon + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} \Big|u_1^D - u_2^D\Big| \right\} e^{\mathcal{L}_{[g]}T} \tag{16}$$

PROOF. The proof follows the idea used in [6] and only needs to be adapted to our problem: let us denote $w = u_1 - u_2$, $w^D = u_1^D - u_2^D$, $w^0 = u_1^0 - u_2^0$ and let us introduce $\varphi_\delta(z) = (z^2 + \delta^2)^{1/2}$. Multiplying

$$\frac{\partial w}{\partial t} + \nabla \cdot \Big(f(t,x,u_1) - f(t,x,u_2)\Big) + \Big(g(t,x,u_1) - g(t,x,u_2)\Big) - \varepsilon \Delta w = 0$$

by $\varphi_\delta{}'(w)\xi_\varepsilon$ and integrating over $(0,t) \times \Omega$, we get

$$\begin{aligned} &\int_\Omega \varphi_\delta(w(t,\cdot))\,\xi_\varepsilon - \int_\Omega \varphi_\delta(w^0)\,\xi_\varepsilon \\ &\quad - \int_0^t \int_\Omega \Big\{ \Big(f(\tau,x,u_1) - f(\tau,x,u_2)\Big)\Big(\varphi_\delta{}''(w)\,\nabla w\,\xi_\varepsilon + \varphi_\delta{}'(w)\,\nabla \xi_\varepsilon\Big) \\ &\qquad + \Big(g(\tau,x,u_1) - g(\tau,x,u_2)\Big)\varphi_\delta{}'(w)\xi_\varepsilon \\ &\qquad + \varepsilon \left[\Big|\nabla w\Big|^2 \varphi_\delta{}''(w)\xi_\varepsilon + \nabla\varphi_\delta(w)\nabla\xi_\varepsilon \right] \Big\} = 0 \end{aligned} \tag{17}$$

Now we study the behaviour of each term w.r.t δ: using the uniform Lipschitz continuity of f, Young's inequality and the fact that $z^2\varphi_\delta{}''(z) = z^2\delta^2(z^2 + \delta^2)^{-3/2} < \delta$, we get

$$\begin{aligned} &-\Big(f(\tau,x,u_1) - f(\tau,x,u_2)\Big)\ \nabla w\ \varphi_\delta{}''(w)\xi_\varepsilon + \varepsilon\Big|\nabla w\Big|^2 \varphi_\delta{}''(w)\xi_\varepsilon \\ &\quad \ge \left\{ -\mathcal{L}_{[f]}|w|\ |\nabla w| + \varepsilon\Big|\nabla w\Big|^2 \right\} \varphi_\delta{}''(w)\xi_\varepsilon \ge -\frac{\mathcal{L}_{[f]}^2}{4\varepsilon} w^2 \varphi_\delta{}''(w)\xi_\varepsilon \ge -\frac{\mathcal{L}_{[f]}^2\ \delta}{4\varepsilon}\xi_\varepsilon \end{aligned}$$

Moreover, observing that $|z|\varphi_\delta{}'(z) \le \varphi_\delta(z)$, we obtain

$$\begin{aligned} -\Big(f(\tau,x,u_1) - f(\tau,x,u_2)\Big)\varphi_\delta{}'(w)\ \nabla\xi_\varepsilon &\ge -\mathcal{L}_{[f]}\ |w|\ |\varphi_\delta{}'(w)|\ |\nabla\xi_\varepsilon| \\ &\ge -\mathcal{L}_{[f]}\ \varphi_\delta(w)|\nabla\xi_\varepsilon| \end{aligned}$$

Following the same idea, we get

$$\Big(g(\tau,x,u_1)-g(\tau,x,u_2)\Big)\varphi_\delta{}'(w)\xi_\varepsilon \geq -\mathcal{L}_{[g]}\ |w|\ |\varphi_\delta{}'(w)|\ \xi_\varepsilon \geq -\mathcal{L}_{[g]}\ \varphi_\delta(w)\xi_\varepsilon$$

Finally, using the previous inequalities, we state that

$$\int\limits_\Omega \varphi_\delta(w(t,\cdot))\xi_\varepsilon - \int\limits_\Omega \varphi_\delta(w^0)\xi_\varepsilon - \frac{\mathcal{L}_{[f]}^2\delta\ T}{4\varepsilon}\int\limits_\Omega \xi_\varepsilon - \mathcal{L}_{[f]}\int\limits_0^t\int\limits_\Omega \varphi_\delta(w)|\nabla\xi_\varepsilon|$$
$$-\mathcal{L}_{[g]}\int\limits_0^t\int\limits_\Omega \varphi_\delta(w)\xi_\varepsilon + \varepsilon\int\limits_0^t\int\limits_\Omega \nabla\left(\varphi_\delta(w)\right)\ \nabla(\xi_\varepsilon)\leq 0.$$

Then, putting these inequalities in Eq. (17) and using Inequality (14), we get

$$\int\limits_\Omega \varphi_\delta(w(t,\cdot))\xi_\varepsilon \int\limits_\Omega \varphi_\delta(w^0)\xi_\varepsilon$$
$$\leq \mathcal{L}_{[g]}\int\limits_0^t\int\limits_\Omega \varphi_\delta(w)\xi_\varepsilon + (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int\limits_0^t\int\limits_\Omega \varphi_\delta(w^D) + \frac{\mathcal{L}_{[f]}^2\delta T}{4\varepsilon}\int\limits_\Omega \xi_\varepsilon$$

Now let δ tend to 0. We obtain

$$\int\limits_\Omega |w(t,\cdot)|\ \xi_\varepsilon \leq \int\limits_\Omega |w^0|\ \xi_\varepsilon + (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int\limits_{\Sigma_T}|w^D| + \mathcal{L}_{[g]}\int\limits_0^t\int\limits_\Omega |w|\ \xi_\varepsilon.$$

Applying Gronwall's lemma concludes the proof.

Lemma 13 *Let (u,u^D,u^0) satisfy equations (11)–(13), the data satisfying Assumption 2. We suppose furthermore that u^D has a smooth extension to $\overline{Q}_T$, denoted $\overline{u}^D$. Then, there exists a constant λ which only depends on $\|u^0\|_\Omega$, $\|\overline{u}^D\|_{\Sigma_T}$, T, Ω, f and g such that*

$$\sup_{t\in(0,T)}\int\limits_\Omega \left\{\Big|\frac{\partial u}{\partial t}(t,\cdot)\Big| + \Big|\nabla u(t,\cdot)\Big|\right\}\leq\lambda \tag{18}$$

Here, we used the notation

$$\begin{aligned}\left\|u^0\right\|_\Omega &= \int\limits_\Omega \left|\Delta u^0\right| + \left|\nabla u^0\right| + \left|u^0\right|\\ \left\|\overline{u}^D\right\|_{\Sigma_T} &= \sup_{Q_T}\left\{\left|\Delta\overline{u}^D\right| + \Big|\frac{\partial\overline{u}^D}{\partial t}\Big| + \left|\nabla\overline{u}^D\right| + \left|\overline{u}^D\right|\right\}\\ &\quad + \int\limits_{Q_T}\Big|\nabla^2\frac{\partial\overline{u}^D}{\partial t}\Big| + \left|\nabla^3\overline{u}^D\right| + \Big|\frac{\partial^2\overline{u}^D}{\partial t^2}\Big| + \Big|\nabla\frac{\partial\overline{u}^D}{\partial t}\Big| + \left|\nabla^2\overline{u}^D\right|\end{aligned}$$

PROOF. In this proof, we will say that a constant "does not depend on ε" if it only depends on $\|u^0\|_\Omega$, $\|\overline{u}^D\|_{\Sigma_T}$, T, Ω, f and g. Moreover, for the sake of simplicity, $\overline{u}^D$ will be identified to u^D. The proof follows the idea developped in [6] (up to the source term g and non-autonomous property of f): in particular, we may observe the influence of the derivatives $\partial f/\partial t$, $\partial f/\partial x$, $\partial g/\partial t$, $\partial g/\partial x$ and $\partial g/\partial u$ in this statement. Indeed, this leads to the main differences with the proof stated in [6], since additional terms have to be treated in order to get a BV estimate. The proof is organized in two steps:

■ **Step 1: Boundness of** $\displaystyle\int_\Omega \left|\frac{\partial u}{\partial t}(t,\cdot)\right|$

Let us still denote u^D the smooth extension of u^D onto $\overline{Q}_T$. We introduce

$$v = u - u^D, \qquad e = \frac{\partial^2 u^D}{\partial t^2} + \nabla\cdot\left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\right) - \varepsilon\Delta\frac{\partial u^D}{\partial t} + \nabla\cdot\left(\frac{\partial f}{\partial t}(\cdot,\cdot,u)\right) + \frac{\partial g}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t} + \frac{\partial g}{\partial t}(\cdot,\cdot,u),$$

so that we easily get

$$\frac{\partial^2 v}{\partial t^2} + \nabla\cdot\left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial v}{\partial t}\right) + \frac{\partial g}{\partial u}(\cdot,\cdot,u)\,\frac{\partial v}{\partial t} - \varepsilon\Delta\left(\frac{\partial v}{\partial t}\right) = -e. \tag{19}$$

Multiplying Eq. (19) by $\varphi_\delta{}'\,(\partial v/\partial t)$, with $\varphi_\delta(z) = \sqrt{z^2+\delta^2}$, and integrating over $(0,t)\times\Omega$, we obtain

$$\begin{aligned}
&\int_\Omega \varphi_\delta\left(\frac{\partial v}{\partial t}(t,\cdot)\right) - \int_\Omega \varphi_\delta\left(\frac{\partial v}{\partial t}(0,\cdot)\right)\\
&\qquad - \int_0^t\int_\Omega \frac{\partial f}{\partial u}(\tau,x,u)\cdot\nabla\left(\frac{\partial v}{\partial t}\right)\,\frac{\partial v}{\partial t}\,\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right)\\
&\qquad + \int_0^t\int_\Omega \frac{\partial g}{\partial u}(\tau,x,u)\,\frac{\partial v}{\partial t}\,\varphi_\delta{}'\left(\frac{\partial v}{\partial t}\right)\\
&\qquad + \int_0^t\int_\Omega \varepsilon\left|\nabla\left(\frac{\partial v}{\partial t}\right)\right|^2\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right) + e\,\varphi_\delta{}'\left(\frac{\partial v}{\partial t}\right) = 0,
\end{aligned} \tag{20}$$

by using the property $\varphi_\delta{}'\,(\partial v/\partial t) = 0$ on Σ_T. Further, we have

$$-\frac{\partial f}{\partial u}(\tau,x,u)\cdot\nabla\left(\frac{\partial v}{\partial t}\right)\ \frac{\partial v}{\partial t}\ \varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right)+\varepsilon\Big|\nabla\left(\frac{\partial v}{\partial t}\right)\Big|^2\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right)$$
$$\geq -\frac{1}{4\varepsilon}\Big|\frac{\partial f}{\partial u}(\tau,x,u)\Big|^2\ \left(\frac{\partial v}{\partial t}\right)^2\ \varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right)\geq -\frac{\mathcal{L}_{[f]}^2\ \delta}{4\varepsilon}.$$

Thus, letting $\delta \to 0$ in Eq. (20) implies the following inequality

$$\int_\Omega \Big|\frac{\partial u}{\partial t}(t,\cdot)\Big| \leq \int_\Omega \Big|\frac{\partial u^D}{\partial t}(t,\cdot)\Big| + \int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big| + \int_0^t\int_\Omega |e| + \mathcal{L}_{[g]}\int_0^t\int_\Omega \Big|\frac{\partial v}{\partial t}\Big|. \quad (21)$$

Now, let us analyse each term of the right-hand side in the previous inequality:

► **(step 1)** *Analysis of* $\displaystyle\int_\Omega \Big|\frac{\partial u^D}{\partial t}(t,\cdot)\Big|$. It is obviously bounded by $c_1 = \|u^D\|_{\Sigma_T}$.

► **(step 1)** *Analysis of* $\displaystyle\int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big|$. We obtain from Eq. (11)

$$\int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big| = \int_\Omega \Big| -\nabla\cdot\Big(f(0,\cdot,u^0)\Big) - g(0,\cdot,u^0) + \varepsilon\Delta u^0 - \frac{\partial u^D}{\partial t}(0,\cdot)\Big|.$$

So far, we have:

$$\begin{aligned}\int_\Omega \Big| -\nabla\cdot\Big(f(0,\cdot,u^0)\Big)\Big| &= \int_\Omega \left\{\Big|\left(\nabla\cdot f\right)(0,\cdot,u^0)\Big)\Big| + \Big|\frac{\partial f}{\partial u}(0,\cdot,u^0)\ \nabla u^0\Big|\right\}\\ &\leq \quad \mathcal{L}_{[\nabla\cdot f]}\int_\Omega |u^0| \quad + \quad \mathcal{L}_{[f]}\int_\Omega |\nabla u^0|\\ &\leq \quad c_2^{(1)}\end{aligned}$$

where $c_2^{(1)}$ only depends on f and $\|u^0\|_\Omega$. Moreover,

$$\int_\Omega \Big|g(0,\cdot,u^0)\Big| \leq |\Omega|\ \sup\Big(\Big|g(t,x,s)\Big|,\ (t,x,s)\in\overline{Q}_T\times[a,b]\Big) \leq c_2^{(2)},$$

where $c_2^{(2)}$ only depends on g and Ω. Further, for ε bounded (which can be assumed, for instance $\varepsilon \leq 1$), we get

$$\int_\Omega \Big|\varepsilon\Delta u^0 - \frac{\partial u^D}{\partial t}(0,\cdot)\Big| \leq \int_\Omega \Big|\Delta u^0\Big| + |\Omega|\ \sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big| \leq c_2^{(3)},$$

where $c_2^{(3)}$ only depends on $\|u^0\|_\Omega$, $\|u^D\|_{\Sigma_T}$ and Ω. Thus, the sum satisfies:

$$\int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big| \le c_2 := c_2^{(1)} + c_2^{(2)} + c_2^{(3)}.$$

► **(step 1)** *Analysis of* $\int_0^t\int_\Omega |e|$. Let us recall that, from the definition of e:

$$\int_0^t\int_\Omega |e| \le \int_0^t\int_\Omega \Big\{\Big|\frac{\partial^2 u^D}{\partial t^2}\Big| + \Big|\nabla\cdot\Big(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big)\Big| + \Big|\nabla\cdot\Big(\frac{\partial f}{\partial t}(\cdot,\cdot,u)\Big)\Big| + \Big|\frac{\partial g}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big| + \Big|\frac{\partial g}{\partial t}(\cdot,\cdot,u)\Big| + \Big|\varepsilon\Delta\frac{\partial u^D}{\partial t}\Big|\Big\}.$$

Now, we have $\int_0^t\int_\Omega \Big|\frac{\partial^2 u^D}{\partial t^2}\Big| \le \int_{Q_T}\Big|\frac{\partial^2 u^D}{\partial t^2}\Big| \le c_3^{(1)}$ with $c_3^{(1)} = \|u^D\|_{\Sigma_T}$. Moreover,

$$\nabla\cdot\Big(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big) = \frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot\nabla\frac{\partial u^D}{\partial t} + \frac{\partial^2 f}{\partial u^2}(\cdot,\cdot,u)\cdot\nabla u\,\frac{\partial u^D}{\partial t} + \Big(\nabla\cdot\frac{\partial f}{\partial u}\Big)(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}.$$

Thus, each term can be controlled in the following way (the last term comes from the non-autonomous property of f):

$$\int_0^t\int_\Omega \Big|\frac{\partial^2 f}{\partial u^2}(\cdot,\cdot,u)\cdot\nabla u\,\frac{\partial u^D}{\partial t}\Big| \le c_3^{(2)}\int_0^t\int_\Omega \big|\nabla u\big|,\ \text{with}\ c_3^{(2)} := \sup_{Q_T\times[a,b]}\Big|\frac{\partial^2 f}{\partial u^2}\Big|\sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big|,$$

$$\int_0^t\int_\Omega \Big|\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot\nabla\frac{\partial u^D}{\partial t}\Big| \le c_3^{(3)} := \mathcal{L}_{[f]}\int_{Q_T}\Big|\nabla\frac{\partial u^D}{\partial t}\Big|$$

$$\int_0^t\int_\Omega \Big|\Big(\nabla\cdot\frac{\partial f}{\partial u}\Big)(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big| \le c_3^{(4)} := \sup_{Q_T\times[a,b]}\Big|\nabla\cdot\frac{\partial f}{\partial u}\Big|\ \sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big|\ T\ |\Omega|,$$

Further again, as f may explicitely depend on (t,x) and g may be a nonzero function, we state the additional estimates

$$\int_0^t\int_\Omega \Big|\nabla\cdot\Big(\frac{\partial f}{\partial t}(\cdot,\cdot,u)\Big)\Big| \le \int_0^t\int_\Omega \Big|\nabla\cdot\frac{\partial f}{\partial t}(\cdot,\cdot,u)\Big| + \Big|\frac{\partial^2 f}{\partial t\,\partial u}(\cdot,\cdot,u)\cdot\nabla u\Big|$$
$$\le c_3^{(5)} + c_3^{(6)}\int_\Omega\int_0^t \big|\nabla u\big|$$

with $c_3^{(5)} = \sup_{Q_T\times[a,b]} \left|\nabla \cdot \frac{\partial f}{\partial t}\right| T\,|\Omega|$ and $c_3^{(6)} = \sup_{Q_T\times[a,b]} \left|\frac{\partial^2 f}{\partial u^2}\right| \sup_{Q_T} \left|\frac{\partial u^D}{\partial t}\right|$ and also

$$\int_0^t \int_\Omega \left|\frac{\partial g}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\right| + \left|\frac{\partial g}{\partial t}(\cdot,\cdot,u)\right| + \left|\varepsilon\Delta\frac{\partial u^D}{\partial t}\right| \leq c_3^{(7)},$$

with $c_3^{(7)} = \left(\mathcal{L}_{[g]} \sup_{Q_T}\left|\frac{\partial u^D}{\partial t}\right| + \sup_{Q_T\times[a,b]}\left|\frac{\partial g}{\partial t}\right| + \sup_{Q_T}\left|\Delta\frac{\partial u^D}{\partial t}\right|\right) T\,|\Omega|$. Taking

$$c_3 = \max\left(c_3^{(1)} + c_3^{(2)} + c_3^{(4)} + c_3^{(5)} + c_3^{(7)}, c_3^{(3)} + c_3^{(6)}\right)$$

and using the previous inequalities gives:

$$\int_0^t \int_\Omega |e| \leq c_3 \left(1 + \int_\Omega \int_0^t \left|\nabla u\right|\right).$$

► **(step 1)** *Analysis of* $\mathcal{L}_{[g]} \int_0^t \int_\Omega \left|\frac{\partial v}{\partial t}\right|$. We have, obviously, the property:

$$\mathcal{L}_{[g]} \int_0^t \int_\Omega \left|\frac{\partial v}{\partial t}\right| \leq \mathcal{L}_{[g]} \left(\int_0^t \int_\Omega \left|\frac{\partial u}{\partial t}\right| + \int_0^t \int_\Omega \left|\frac{\partial u^D}{\partial t}\right|\right) \leq c_4 \left(1 + \int_0^t \int_\Omega \left|\frac{\partial u}{\partial t}\right|\right),$$

with

$$c_4 = \mathcal{L}_{[g]} \max\left(1, \sup_{Q_T}\left|\frac{\partial u^D}{\partial t}\right|\ |\Omega|\ T\right).$$

Thus, recalling Inequality (21) along with the previous results, we obtain

$$\int_\Omega \left|\frac{\partial u}{\partial t}(t,\cdot)\right| \leq c_5 \left(1 + \int_0^t \int_\Omega |\nabla u| + \int_0^t \int_\Omega \left|\frac{\partial u}{\partial t}\right|\right) \tag{22}$$

by taking, for instance, $c_5 = \sum_{i=1}^{4} c_i$ which does not depend on ε.

■ **Step 2: Boundness of** $\int_\Omega \left|\nabla u(t,\cdot)\right|$

For this, we proceed in two steps, namely Steps $2^{(a)}$ and $2^{(b)}$, which will be gathered in order to conclude Step 2. Let us first proceed to **Step** $2^{(a)}$. Recalling that $v = u - u^D$ and denoting

$$h_1 = \frac{\partial u^D}{\partial t} + \nabla\cdot(f(t,x,u^D)) + g(t,x,u^D) - \varepsilon\Delta u^D, \tag{23}$$

we have

$$\frac{\partial v}{\partial t} + \nabla \cdot (f(t,x,u) - f(t,x,u^D)) + g(t,x,u) - g(t,x,u^D) - \varepsilon \Delta v = -h_1. \quad (24)$$

We multiply Eq. (24) by ${\varphi_\delta}'(v)\ \beta$, where $\beta \in \mathcal{D}(\mathbb{R})$, $\beta \geq 0$, depends only on the space variable and $\varphi_\delta(z) = (z^2 + \delta^2)^{1/2} - \delta$.. After integration over $(0,t) \times \Omega$, and since ${\varphi_\delta}'(v) = 0$, $\varphi_\delta(v) = 0$, $\nabla \varphi_\delta(v) \cdot n = 0$ on Σ_T, we obtain

$$\begin{aligned}
&\int_\Omega \varphi_\delta(v(t,\cdot))\ \beta - \int_\Omega \varphi_\delta(v(0,\cdot))\ \beta - \varepsilon \int_0^t \int_\Omega \varphi_\delta(v)\, \Delta\beta \\
&- \int_0^t \int_\Omega {\varphi_\delta}'(v)(f(\tau,x,u) - f(\tau,x,u^D))\nabla\beta - (f(\tau,x,u) - f(\tau,x,u^D))\nabla v\, {\varphi_\delta}''(v)\, \beta \\
&+ \int_0^t \int_\Omega {\varphi_\delta}'(v)\,(g(\tau,x,u) - g(\tau,x,u^D))\beta + \varepsilon |\nabla v|^2\, {\varphi_\delta}''(v)\,\beta = - \int_0^t \int_\Omega {\varphi_\delta}'(v)\, h_1\, \beta.
\end{aligned}$$

We let $\delta \to 0$ and thus

$$\begin{aligned}
&\int_\Omega |v(t,\cdot)|\, \beta - \int_\Omega |v(0,\cdot)|\, \beta - \varepsilon \int_0^t \int_\Omega |v|\, \Delta\beta \\
&- \int_0^t \int_\Omega \operatorname{sgn}(u - u^D)(f(\tau,x,u) - f(\tau,x,u^D))\nabla\beta \qquad (25) \\
&+ \int_0^t \int_\Omega \operatorname{sgn}(u - u^D)(g(\tau,x,u) - g(\tau,x,u^D))\beta \leq - \int_0^t \int_\Omega \operatorname{sgn}(u - u^D)\, h_1\, \beta.
\end{aligned}$$

Now we choose

$$\beta(x) = \gamma\left(\frac{s(x)}{\rho}\right),$$

where $s(x)$ is defined as before, ρ is a strictly positive number and $\gamma \in \mathcal{D}(\mathbb{R})$ is a fixed non-negative function such that $\gamma(0) = 0$ and $\gamma(\sigma) = 1$, for $\sigma \geq 1$. Let us study the behaviour with respect to ρ of each term.

► **(step $2^{(a)}$)** *Behaviour w.r.t. ρ of* $\int_0^t \int_\Omega \operatorname{sgn}(u - u^D)\ (f(t,x,u) - f(t,x,u^D))\nabla\beta$*:*

Obviously, one has

$$\nabla\beta = \gamma'\left(\frac{s(x)}{\rho}\right)\ \frac{1}{\rho}\nabla s(x) \qquad \text{and} \qquad \nabla s(x) = 0, \text{ on } \Omega \setminus K_\mu$$

with $K_\mu = \{x \in \Omega,\ \mathrm{dist}(x, \partial\Omega) < \mu\}$. Thus, each point $x \in K_\mu$ (for μ small enough) can be described as $x = r(x) - s(x)\ n(r)$, where $r(x)$ is the nearest boundary point to x, and $n(r)$ is the outer vector to $\partial\Omega$ at point $r(x)$. Let us notice that $\nabla s(x) = -n(r)$, if $x \in K_\mu$. From the previous observations, we deduce the following equality (for the sake of simplicity, $F(u, u^D)(\tau, x)$ denotes the value of the function

$$\mathrm{sgn}(u - u^D)\ (f(\cdot,\cdot,u) - f(\cdot,\cdot,u^D))$$

at point $(\tau, x) \in Q_T$):

$$\begin{aligned}
&\int_0^t \int_\Omega F(u, u^D)(\tau, x)\ \nabla\beta(x)\ dx\ d\tau \\
&= \int_0^t \int_{K_\mu} F(u, u^D)(\tau, x)\ \gamma'\left(\frac{s(x)}{\rho}\right)\ \frac{1}{\rho}\nabla s(x)\ dx\ d\tau \\
&= \int_0^t \int_0^\mu \int_{\partial\Omega} F(u, u^D)(\tau, r - sn(r))\ \gamma'\left(\frac{s}{\rho}\right)\ \frac{1}{\rho}\ (-n(r))\ d\gamma(r)\ ds\ d\tau \\
&= -\int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} F(u, u^D)(\tau, r - \sigma\rho\ n(r))\ \gamma'(\sigma)\ \ n(r)\ d\gamma(r)\ d\sigma\ d\tau \\
&= -\int_0^{\mu/\rho} \gamma'(\sigma) \left(\int_0^t \int_{\partial\Omega} F(u, u^D)(\tau, r - \sigma\rho\ n(r))\ n(r)\ d\gamma(r)\ d\tau \right)\ d\sigma.
\end{aligned}$$

Thus, letting $\rho \to 0$, we obtain (recalling that $F(u, u^D) = 0$ on Σ_T):

$$\begin{aligned}
&\lim_{\rho\to 0} \int_0^t \int_\Omega F(u, u^D)(\tau, x)\nabla\beta(x)\, dx\, d\tau \\
&= -\int_0^{+\infty} \gamma'(\sigma) \left(\int_0^t \int_{\partial\Omega} F(u, u^D)(\tau, r)\, n(r)\, d\gamma(r)\, d\tau \right) d\sigma \\
&= -\int_0^{+\infty} \gamma'(\sigma)\ d\sigma\ \left(\int_0^t \int_{\partial\Omega} F(u, u^D)(\tau, r)\, n(r)\, d\gamma(r)\, d\tau \right) = 0.
\end{aligned}$$

► **(step $2^{(a)}$)** *Behaviour with respect to ρ of* $\int_0^t \int_\Omega |v|\ \Delta\beta$*:*

The particular choice of β gives $\Delta\beta(x) = \nabla \cdot \left(\frac{1}{\rho}\gamma' \left(\frac{s(x)}{\rho}\right) \nabla s(x) \right)$ and thus:

$$\Delta\beta(x) = \frac{1}{\rho} \sum_{i=1}^d \left\{ \frac{1}{\rho}\gamma'' \left(\frac{s(x)}{\rho}\right) \left(\frac{\partial s(x)}{\partial x_i}\right)^2 + \gamma' \left(\frac{s(x)}{\rho}\right) \frac{\partial^2 s(x)}{\partial x_i^2} \right\}.$$

Now, if $x \in K_\mu$, then $\dfrac{\partial s(x)}{\partial x_i} = -n_i(r)$, n_i being the ith component of n, so that

$$\frac{1}{\rho^2}\gamma''\left(\frac{s(x)}{\rho}\right)\sum_{i=1}^{d}\left(\frac{\partial s(x)}{\partial x_i}\right)^2 = \frac{1}{\rho^2}\gamma''\left(\frac{s(x)}{\rho}\right)\|n(r)\|^2 = \frac{1}{\rho^2}\gamma''\left(\frac{s(x)}{\rho}\right)$$

and, as a consequence,

$$\Delta\beta(x) = \begin{cases} \dfrac{1}{\rho^2}\gamma''\left(\dfrac{s(x)}{\rho}\right) + \dfrac{1}{\rho}\gamma'\left(\dfrac{s(x)}{\rho}\right)\ \Delta s(x), & \text{on } K_\mu, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, since $v(\tau, r(x)) = 0$ ($r(x)$ being a boundary point) and using the previous expression of $\Delta\beta$, we have

$$\begin{aligned} \int_0^t\int_\Omega \left|v\right|\ \Delta\beta &= \int_0^t\int_\Omega \left|v(\tau,x) - v(\tau,r(x))\right|\ \Delta\beta \\ &= \int_0^t\int_{K_\mu} \left|v(\tau,x) - v(\tau,r(x))\right| \left(\gamma''\left(\frac{s(x)}{\rho}\right)\frac{1}{\rho^2} + \frac{1}{\rho}\gamma'\left(\frac{s(x)}{\rho}\right)\Delta s(x)\right). \end{aligned}$$

Let us focus on the first right-hand side term of the previous equality:

$$\begin{aligned} &\int_0^t\int_0^\mu\int_{\partial\Omega} \left|v(\tau, r - s\ n(r)) - v(\tau,r)\right| \gamma''\left(\frac{s}{\rho}\right)\frac{1}{\rho^2}\ dr\ ds\ d\tau \\ &= \int_0^t\int_0^{\mu/\rho}\int_{\partial\Omega} \frac{\left|v(\tau, r - \sigma\rho\ n(r)) - v(\tau,r)\right|}{\rho}\ \gamma''(\sigma)\ dr\ d\sigma\ d\tau \\ &= \int_0^{\mu/\rho} \sigma\ \gamma''(\sigma)\ \left(\int_0^t\int_{\partial\Omega} \frac{\left|v(\tau, r - \sigma\rho\ n(r)) - v(\tau,r)\right|}{\sigma\rho}\ dr\ d\tau\right) d\sigma. \end{aligned}$$

Now let us focus on the second right-hand side: since $|\Delta s| \le \mathcal{R}$ on K_μ,

$$\begin{aligned} &\left|\int_0^t\int_{K_\mu} \left|v(\tau,x) - v(\tau,r(x))\right| \frac{1}{\rho}\ \gamma'\left(\frac{s(x)}{\rho}\right)\ \Delta s(x)\ dx\ d\tau\ \right| \\ &\le \mathcal{R}\int_0^t\int_0^\mu\int_{\partial\Omega} \left|v(\tau, r - s\ n(r)) - v(\tau,r)\right| \frac{1}{\rho}\left|\gamma'\left(\frac{s}{\rho}\right)\right| dr\ ds\ d\tau \\ &\le \mathcal{R}\int_0^t\int_0^{\mu/\rho}\int_{\partial\Omega} \left|v(\tau, r - \sigma\rho\ n(r)) - v(\tau,r)\right| \gamma'(\sigma)\ dr\ d\sigma\ d\tau \\ &= \mathcal{R}\ \rho\int_0^{\mu/\rho} \sigma\ \gamma'(\sigma)\ \left(\int_0^t\int_{\partial\Omega} \frac{\left|v(\tau, r - \sigma\rho\ n(r)) - v(\tau,r)\right|}{\sigma\rho}\ dr\ d\tau\right) d\sigma. \end{aligned}$$

Letting $\rho \to 0$ (notice that the second right-hand side term tends to 0) gives

$$\lim_{\rho\to 0}\int_0^t\int_\Omega \left|v\right| \,\Delta\beta = \int_0^{+\infty} \sigma\ \gamma''(\sigma)\ \left(\int_0^t\int_{\partial\Omega} \left|\nabla v(\tau,r)\cdot n(r)\right|\, dr\ d\tau\right) d\sigma$$
$$= -\int_0^t\int_{\partial\Omega} \left|\nabla v(\tau,r)\cdot n(r)\right|\, dr\ d\tau$$

As a consequence, Inequality (25) becomes

$$\int_\Omega |v(t,\cdot)| + \varepsilon\int_{\partial\Omega} \left|\nabla v\cdot n\right| \le \int_\Omega |v(0,\cdot)| + \mathcal{L}_{[g]}\int_0^t\int_\Omega |v| + \int_0^t\int_\Omega |h_1|. \tag{26}$$

Now, we proceed to **Step** $2^{(b)}$. Let us denote

$$z_i = \frac{\partial v}{\partial x_i}, \quad z = \nabla v.$$

Then we have

$$\frac{\partial z_i}{\partial t} + \nabla\cdot\left(\frac{\partial f}{\partial u}(t,x,u)\ z_i\right) - \varepsilon\Delta z_i = -h_2^{(i)},$$

with

$$h_2^{(i)} = \frac{\partial^2 u^D}{\partial x_i\partial t} + \nabla\cdot\left(\frac{\partial f}{\partial u}(t,x,u)\ \frac{\partial u^D}{\partial x_i}\right) + \nabla\cdot\left(\frac{\partial f}{\partial x_i}(t,x,u)\right)$$
$$+ \frac{\partial}{\partial x_i}\Big(g(t,x,u)\Big) - \varepsilon\Delta\frac{\partial u^D}{\partial x_i}.$$

Multiplying the previous equation by $\partial\phi_\delta/\partial\xi_i(z)$, with $\phi_\delta(\xi) = (|\xi|^2+\delta^2)^{1/2}$, adding the terms ($i = 1, d$), we have, using the usual Einstein summation convention (i.e. whenever an index appears twice in one expression, the summation over this index is performed):

$$\int_0^t\int_\Omega \frac{\partial z_i}{\partial t}\frac{\partial\phi_\delta}{\partial\xi_i}(z) = \int_\Omega \phi_\delta(v(t,\cdot)) - \int_\Omega \phi_\delta(v(0,\cdot)),$$
$$\int_0^t\int_\Omega \Delta z_i\frac{\partial\phi_\delta}{\partial\xi_i}(z) = -\int_0^t\int_\Omega \frac{\partial z_i}{\partial x_j}\frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j} + \int_0^t\int_{\partial\Omega}\frac{\partial z_i}{\partial x_j}n_j\frac{\partial\phi_\delta}{\partial\xi_i}(z),$$
$$\int_0^t\int_\Omega \nabla\cdot\left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\ z_i\right)\frac{\partial\phi_\delta}{\partial\xi_i}(z)$$
$$= -\int_0^t\int_\Omega \frac{\partial f_j}{\partial u}(\cdot,\cdot,u)\ z_i\ \frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j} + \int_0^t\int_{\partial\Omega}\frac{\partial f_j}{\partial u}(\cdot,\cdot,u)n_j\ z_i\ \frac{\partial\phi_\delta}{\partial\xi_i}(z).$$

Due to the estimate (obtained exactly as in [6])

$$\begin{aligned}
&\varepsilon\frac{\partial z_i}{\partial x_j}\frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j}-\frac{\partial f_j}{\partial u}(\cdot,\cdot,u)\ z_i\ \frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j}\\
&\qquad=\frac{\delta^2}{(|z|^2+\delta^2)^{3/2}}\Big[\varepsilon|\nabla z|^2-\frac{\partial f_j}{\partial u}(\cdot,\cdot,u)\ z_i\ \frac{\partial z_i}{\partial x_j}\Big]\\
&\qquad\geq-\frac{1}{4\varepsilon}\Big|\frac{\partial f}{\partial u}(\cdot,\cdot,u)\Big|^2\frac{\delta^2|z|^2}{(|z|^2+\delta^2)^{3/2}}\geq-\frac{\mathcal{L}_{[f]}\ \delta}{4\varepsilon},
\end{aligned}$$

we obtain for $\delta\to 0$

$$\begin{aligned}
&\int_\Omega\Big|z(t,\cdot)\Big|-\Big|z(0,\cdot)\Big|\\
&\qquad\leq\int_0^t\int_\Omega|h_2|+\limsup_{\delta\to 0}\int_0^t\int_{\partial\Omega}\Big|\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot n\ z_j\ \frac{\partial\phi_\delta}{\partial\xi_j}(z)-\varepsilon\nabla z_i\cdot n\ \frac{\partial\phi_\delta}{\partial\xi_i}(z)\Big|.
\end{aligned}$$

Due to $z=0$ on Σ_T, we have on Σ_T

$$z=\nabla v=(\nabla v\cdot n)\ n,\quad \Delta v=D^2v(n,n)+\Delta s\nabla v\cdot n,$$

where D^2v is the bilinear form of the second differential of v. Therefore, the integrand can be rewritten as

$$\begin{aligned}
&\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot n\ z_j\ \frac{\partial\phi_\delta}{\partial\xi_j}(z)-\varepsilon\nabla z_i\cdot n\ \frac{\partial\phi_\delta}{\partial\xi_i}(z)\\
&\qquad=\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot n\ \frac{|\nabla v|^2}{(|\nabla v|^2+\delta^2)^{1/2}}-\varepsilon D^2v\left(n,\frac{\nabla v}{(|\nabla v|^2+\delta^2)^{1/2}}\right)\\
&\qquad=\left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot\nabla v-\varepsilon D^2v(n,n)\right)\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\nabla\cdot\Big(f(t,x,u)-f(t,x,u^D)\Big)\\
&\qquad=(\nabla\cdot f)(t,x,u)-(\nabla\cdot f)(t,x,u^D)+\frac{\partial f}{\partial u}\cdot\nabla u-\frac{\partial f}{\partial u}\cdot\nabla u^D\\
&\qquad=(\nabla\cdot f)(t,x,u)-(\nabla\cdot f)(t,x,u^D)+\frac{\partial f}{\partial u}(t,x,u)\cdot\nabla v\\
&\qquad\qquad-\left(\frac{\partial f}{\partial u}(t,x,u^D)-\frac{\partial f}{\partial u}(t,x,u)\right)\cdot\nabla u^D.
\end{aligned}$$

Thus, for $(t,x)\in\Sigma_T$, $\left(\frac{\partial f}{\partial u}(t,x,u^D)-\frac{\partial f}{\partial u}(t,x,u)\right)=0$, and we obtain

$$\begin{aligned}
&\frac{\partial f}{\partial u}(t,x,u)\cdot\nabla v\\
&\qquad=\nabla\cdot\Big(f(t,x,u)-f(t,x,u^D)\Big)-\Big\{(\nabla\cdot f)(t,x,u)-(\nabla\cdot f)(t,x,u^D)\Big\},
\end{aligned}$$

in which a non classical contribution appears, due to the fact that f may depend on x. Since $\partial v/\partial t = 0$ on Σ_T, we have for $(t,x) \in \Sigma_T$

$$\begin{aligned}
&\frac{\partial f}{\partial u}(t,x,u)\cdot n\ z_j\ \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n\ \frac{\partial \phi_\delta}{\partial \xi_i}(z) \\
&= \left(\frac{\partial v}{\partial t} + \nabla\cdot\Big(f(t,x,u) - f(t,x,u^D)\Big) - \varepsilon\Delta v + \varepsilon \Delta s \nabla v\cdot n\right)\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}} \\
&\quad -\Big\{\nabla\cdot f(t,x,u) - \nabla\cdot f(t,x,u^D)\Big\}\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}} \\
&= (-h_1 + \varepsilon\Delta s\nabla v\cdot n)\,\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}} \\
&\quad -\Big\{\nabla\cdot f(t,x,u) - \nabla\cdot f(t,x,u^D) + g(t,x,u) - g(t,x,u^D)\Big\}\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}} \\
&\quad -\Big\{g(t,x,u) - g(t,x,u^D)\Big\}\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}.
\end{aligned}$$

Since $(t,x) \in \Sigma_T$, we obtain ($u = u^D$):

$$\frac{\partial f}{\partial u}(t,x,u)\cdot n\ z_j\ \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n\ \frac{\partial \phi_\delta}{\partial \xi_i}(z) = (-h_1 + \varepsilon\Delta s\nabla v\cdot n)\,\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}.$$

Putting this in the last inequality gives:

$$\int_\Omega \left|z(t,\cdot)\right| \le \int_\Omega \left|z(0,\cdot)\right| + \int_0^t\int_\Omega |h_2| + \int_0^t\int_{\partial\Omega}\left\{\left|h_1\right| + \varepsilon\mathcal{R}\left|\nabla v\cdot n\right|\right\}$$

which, together with Inequality (26) implies

$$\int_\Omega \left\{\left|\nabla v(t,\cdot)\right| + \mathcal{R}\left|v(t,\cdot)\right|\right\} \le \int_\Omega \left\{\left|\nabla v(0,\cdot)\right| + \mathcal{R}\left|v(0,\cdot)\right|\right\} + \int_0^t\int_\Omega |h_2| + \int_0^t\int_{\partial\Omega}\left|h_1\right|$$

and, as a consequence,

$$\begin{aligned}
\int_\Omega \left\{\left|\nabla u(t,\cdot)\right| + \mathcal{R}\left|u(t,\cdot)\right|\right\} \le & \int_\Omega \left\{\left|\nabla u^D(t,\cdot)\right| + \mathcal{R}\left|u^D(t,\cdot)\right|\right\} \\
& + \int_\Omega \left\{\left|\nabla v(0,\cdot)\right| + \mathcal{R}\left|v(0,\cdot)\right|\right\} \\
& + \int_0^t\int_\Omega |h_2| + \int_0^t\int_{\partial\Omega}\left|h_1\right|. \qquad (27)
\end{aligned}$$

Let us analyse each term of Inequality (27).

▶ **(step $2^{(b)}$)** *Analysis of* $\int_\Omega \left\{\left|\nabla u^D(t,\cdot)\right| + \mathcal{R}\left|u^D(t,\cdot)\right|\right\}$. We easily state that

$$\int_\Omega \left\{\left|\nabla u^D(t,\cdot)\right| + \mathcal{R}\left|u^D(t,\cdot)\right|\right\} \leq c_6 := \left(\sup_{Q_T}\left|\nabla u^D\right| + \mathcal{R}\sup_{Q_T}\left|u^D\right|\right) |\Omega|\ T.$$

▶ **(step $2^{(b)}$)** *Analysis of* $\int_\Omega \left\{\left|\nabla v(0,\cdot)\right| + \mathcal{R}\left|v(0,\cdot)\right|\right\}$. Clearly, one has:

$$\begin{aligned}
&\int_\Omega \left\{\left|\nabla v(0,\cdot)\right| + \mathcal{R}\left|v(0,\cdot)\right|\right\} \\
&\qquad \leq \int_\Omega \left\{\left|\nabla u^0\right| + \left|\nabla u^D(0,\cdot)\right| + \mathcal{R}\left(\left|u^0\right| + \left|u^D(0,\cdot)\right)\right|\right\} \\
&\qquad \leq c_7 := \int_\Omega \left(\left|\nabla u^0\right| + \mathcal{R}\left|u^0\right|\right) + \left(\sup_{Q_T}\left|\nabla u^D\right| + \mathcal{R}\sup_{Q_T}\left|u^D\right|\right).
\end{aligned}$$

▶ **(step $2^{(b)}$)** *Analysis of* $\int_0^t\int_\Omega |h_1|$. Recalling the expression of h_1,

$$\begin{aligned}
\int_0^t\int_\Omega |h_1| &= \int_0^t\int_\Omega \left|\frac{\partial u^D}{\partial t} + \nabla\cdot(f(t,x,u^D)) + g(t,x,u^D) - \varepsilon\Delta u^D\right| \\
&\leq \left(\sup_{Q_T}\left|\frac{\partial u^D}{\partial t} + \Delta u^D\right| + \sup_{Q_T\times[a,b]}|g|\right)\ |\Omega|\ T + \int_0^t\int_\Omega \nabla\cdot(f(t,x,u^D)).
\end{aligned}$$

Since $\nabla\cdot(f(t,x,u^D)) = \nabla\cdot f(t,x,u^D) + \dfrac{\partial f}{\partial u}(t,x,u^D)\nabla u^D$, we have

$$\int_0^t\int_\Omega |h_1| \leq c_8,$$

with $c_8 = \left(\sup_{Q_T}\left\{\left|\dfrac{\partial u^D}{\partial t}\right| + \left|\Delta u^D\right| + \mathcal{L}_{[f]}\left|\nabla u^D\right|\right\} + \sup_{Q_T\times[a,b]}|g + \nabla\cdot f|\right)\ |\Omega|\ T.$

► **(step $2^{(b)}$)** *Analysis of* $\int_0^t \int_\Omega |h_2|$. First, let us develop the expression of h_2:

$$\begin{aligned} h_2^{(i)} = \frac{\partial^2 u^D}{\partial x_i \partial t} & + \nabla \cdot \frac{\partial f}{\partial u}(t,x,u) \frac{\partial u^D}{\partial x_i} + \frac{\partial^2 f}{\partial u^2}(t,x,u)\, \nabla u\, \frac{\partial u^D}{\partial x_i} \\ + \frac{\partial f}{\partial u}(t,x,u)\, \nabla \frac{\partial u^D}{\partial x_i} & + \nabla \cdot \frac{\partial f}{\partial x_i}(t,x,u) + \frac{\partial^2 f}{\partial x_i\, \partial u}(t,x,u)\, \nabla u \\ + \frac{\partial g}{\partial x_i}(t,x,u) & + \frac{\partial g}{\partial u}(t,x,u) \frac{\partial u}{\partial x_i} - \varepsilon \Delta \frac{\partial u^D}{\partial x_i}. \end{aligned}$$

in which non classical contributions appear (in comparison with [6]), due to the fact that f (resp. g) depends on (t,x) (resp. (t,x) and u). From this equality, we deduce the following estimate:

$$\int_0^t \int_\Omega |h_2| \le c_9^{(1)} + c_9^{(2)} \int_0^t \int_\Omega \left|\nabla u\right| \le c_9 \left(1 + \int_0^t \int_\Omega \left|\nabla u\right|\right)$$

with $c_9 = \max\left(c_9^{(1)}, c_9^{(2)}\right)$ and the following constants:

$$\begin{aligned} c_9^{(1)} = & \left(\sup_{Q_T \times [a,b]} \left|\nabla \cdot \frac{\partial f}{\partial u}\right| \sup_{Q_T} \left|\nabla u^D\right| + \sup_{Q_T \times [a,b]} \left|\nabla^2 f\right| + \sup_{Q_T \times [a,b]} \left|\nabla g\right|\right) |\Omega|\, T \\ & + \mathcal{L}_{[f]} \int_{Q_T} \left|\nabla^2 u^D\right| + \int_{Q_T} \left|\nabla^3 u^D\right|, \\ c_9^{(2)} = & \sup_{Q_T \times [a,b]} \left|\frac{\partial^2 f}{\partial u^2}\right| \sup_{Q_T} \left|\nabla u^D\right| + \mathcal{L}_{[g]} + \sup_{Q_T \times [a,b]} \left|\nabla \cdot \frac{\partial f}{\partial u}\right|. \end{aligned}$$

To conclude **Step 2**, we gather Inequality (27) with all the previous bounds:

$$\int_\Omega \left\{\left|\nabla u(t,\cdot)\right| + \mathcal{R}\left|u(t,\cdot)\right|\right\} \le c_{10} \left(1 + \int_0^t \int_\Omega \left|\nabla u\right|\right)$$

where $c_{10} = c_6 + c_7 + c_8 + c_9$ does not depend on ε. Finally, since u is a function with values in $[a,b]$, from the previous inequality, we infer that there exists c_{11} which does not depend on ε such that

$$\int_\Omega \left|\nabla u(t,\cdot)\right| \le c_{11} \left(1 + \int_0^t \int_\Omega \left|\nabla u\right|\right) \tag{28}$$

Now, we gather results obtained in **Step 1** and **Step 2**: using Inequalities (22) and (28) gives

$$\int_\Omega \left\{\left|\frac{\partial u}{\partial t}(t,\cdot)\right| + \left|\nabla u(t,\cdot)\right|\right\} \le c_{12} \left(1 + \int_0^t \int_\Omega \left\{\left|\frac{\partial u}{\partial t}\right| + \left|\nabla u\right|\right\}\right) \tag{29}$$

where c_{12} ($= c_5 + c_{11}$, for instance) does not depend on ε. Applying Gronwall's lemma concludes the proof of Lemma 13.

Theorem 14 (Existence) *Let us suppose that Assumption 1 holds. Let u_ε be the unique solution of Eq. (11)–(13) corresponding to the data $(u_\varepsilon^0, u_\varepsilon^D)$ satisfying Assumption 2 and let*

$$\lim_{\varepsilon \to 0} u_\varepsilon^D = u^D \quad \text{in } L^1(\Sigma_T), \qquad \lim_{\varepsilon \to 0} u_\varepsilon^0 = u^0 \quad \text{in } L^1(\Omega)$$

where $u^D \in L^\infty(\Sigma_T; [a,b])$ and $u^0 \in L^\infty(\Omega; [a,b])$. Then, the sequence $\{u_\varepsilon\}_\varepsilon$ converges to some function $u \in L^\infty(Q_T; [a,b])$ in $C^0([0,T], L^1(\Omega))$. Moreover u is a weak entropy solution of Eq. (1)–(3).

PROOF. Before entering into technical details, let us give the sketch of this proof. Our goal is to let ε tend to 0. Nevertheless, we cannot apply estimates stated in Lemma 13 on u_ε because $u_\varepsilon^D, u_\varepsilon^0$ satisfy compatibility conditions but do not necessarily have an extension over $\overline{Q}_T$ with sufficient regularity. Thus, we introduce, by means of construction, $(u_{\varepsilon,h}^D, u_{\varepsilon,h}^0)$ which both satisfy compatibility conditions and have an extension over $\overline{Q}_T$ with sufficient regularity. Moreover, $(u_{\varepsilon,h}^D, u_{\varepsilon,h}^0)$ are uniformly "close" to $(u_\varepsilon^D, u_\varepsilon^0)$ (as $h \to 0$, uniformly w.r.t. ε), which implies that $u_{\varepsilon,h}$ is "close" to u_ε (in a sense which will be precised further). Then, we apply Arzelà-Ascoli theorem on the sequence $\{u_\varepsilon\}$ in order to prove that it is relatively compact in $C^0([0,T]; L^1(\Omega))$. Of course, we have to verify that the sequence satisfies the hypotheses of the theorem (equicontinuity and pointwise relative compactness): for this, we use the properties of $u_{\varepsilon,h}$ and the fact that u_ε is "close" to $u_{\varepsilon,h}$. In order to use Lemma 13, we need some extension of u_ε^D and u_ε^0 to $\overline{Q}_T$, with sufficient regularity. Let us define the function $u_\varepsilon^{D,0}$ by

$$\begin{aligned} u_\varepsilon^{D,0}(t, r + s\, n(r)) &= u_\varepsilon^D(t,r), && t \in (0,T),\ r \in \partial\Omega,\ |s| \le \min(t,\delta) \\ u_\varepsilon^{D,0}(t,x) &= u_\varepsilon^0(x), && -\delta < t < \min(\operatorname{dist}(x,\partial\Omega),\delta),\ x \in \Omega \\ u_\varepsilon^{D,0}(t,x) &= 0, && \text{elsewhere.} \end{aligned}$$

Moreover, we mollify the above function (with a usual mollifier) which provides regularity on $\overline{Q}_T$:

$$u_{\varepsilon,h}^{D,0}(t,x) = \int_{\mathbb{R}^{d+1}} u_\varepsilon^{D,0}(t',x')\ \phi_h(t-t', x-x')\ dt'\ dx'$$

Now we denote by $u_{\varepsilon,h}^D$ (resp. $u_{\varepsilon,h}^0$) the restriction of $u_{\varepsilon,h}^{D,0}$ to Σ_T (resp. $\{0\} \times \Omega$). Let $u_{\varepsilon,h}$ be the solution of Eq. (11)–(13) corresponding to the boundary and initial conditions $u_{\varepsilon,h}^D$ and $u_{\varepsilon,h}^0$. On one hand, the uniform boundedness of u_ε^D, u_ε^0 implies the uniform boundedness of $u_{\varepsilon,h}^D$, $u_{\varepsilon,h}^0$ which provides (see Lemma 11) the uniform boundedness of u_ε, $u_{\varepsilon,h}$. Obviously, the following (strong) convergences hold:

$$\lim_{h\to 0} u^D_{\varepsilon,h} = u^D_\varepsilon \quad \text{in } L^1(\Sigma_T), \qquad \lim_{h\to 0} u^0_{\varepsilon,h} = u^0_\varepsilon \quad \text{in } L^1(\Omega)$$

uniformly w.r.t. ε. This and Inequality (16) (see Lemma 12) imply

$$\lim_{h\to 0} u_{\varepsilon,h} = u_\varepsilon \quad \text{in } C^0([0,T], L^1(\Omega)),$$

uniformly w.r.t. ε. On the other hand, it follows from the boundedness of $u^D_\varepsilon \in L^1(\Sigma_T)$ and $u^0_\varepsilon \in L^1(\Omega)$ that

$$\left\|u^D_{\varepsilon,h}\right\|_{\Sigma_T} \le \frac{c}{h^3}, \quad \left\|u^0_{\varepsilon,h}\right\|_{\Omega} \le \frac{c}{h^2}.$$

For fixed $h > 0$, it follows from Inequality (18) that the sequences

$$\left\{\frac{\partial u_{\varepsilon,h}}{\partial t}\right\}, \quad \{\nabla u_{\varepsilon,h}\}$$

are bounded in $C^0([0,T], L^1(\Omega))$. Now we propose to state that $\{u_\varepsilon\}_\varepsilon$ is precompact in $C^0([0,T], L^1(\Omega))$ with the Arzelà-Ascoli theorem:

(*i*) **Equicontinuity of $\{u_\varepsilon\}_\varepsilon$:** Let $\alpha > 0$. There exists some $h > 0$ such that

$$2\int_\Omega \left|u_{\varepsilon,h}(t,\cdot) - u_\varepsilon(t,\cdot)\right| \, dx < \alpha/2, \quad \forall t \in [0,T],\ \forall \varepsilon > 0$$

and, from the uniform boundedness of $\partial u_{\varepsilon,h}/\partial t$, there exists $\delta > 0$ such that

$$\delta \int_\Omega \left|\frac{\partial u_{\varepsilon,h}}{\partial t}(t,\cdot)\right| \, dx < \alpha/2, \quad \forall t \in [0,T],\ \forall \varepsilon > 0 \tag{30}$$

Thus, for all $\varepsilon > 0$ and all t_1, $t_2 \in [0,T]$ such that $|t_1 - t_2| \le \delta$, we have

$$\begin{aligned} \int_\Omega \left|u_\varepsilon(t_1,\cdot) - u_\varepsilon(t_2,\cdot)\right| &\le \sum_{i=1}^2 \int_\Omega \left|u_{\varepsilon,h}(t_i,\cdot) - u_\varepsilon(t_i,\cdot)\right| + \int_\Omega \left|u_{\varepsilon,h}(t_1,\cdot) - u_{\varepsilon,h}(t_2,\cdot)\right| \\ &\le \sum_{i=1}^2 \int_\Omega \left|u_{\varepsilon,h}(t_i,\cdot) - u_\varepsilon(t_i,\cdot)\right| + \left|t_1 - t_2\right| \sup_{t\in[t_1,t_2]} \int_\Omega \left|\frac{\partial u_{\varepsilon,h}}{\partial t}(t,\cdot)\right| \le \alpha \end{aligned}$$

Thus, the sequence u_ε is equicontinuous in $C^0([0,T], L^1(\Omega))$.

(*ii*) **Pointwise relative compactness of $\{u_\varepsilon\}_\varepsilon$:**
For this, we use the Kolmogorov-Fréchet-Weil theorem:
▷ Since $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty(Q_T)$, $\{u_\varepsilon(t,\cdot)\}$ is also bounded in $L^1(\Omega)$ (uniformly w.r.t. $t \in [0,T]$ and ε).

- ▷ Let $\eta > 0$. Let us consider $K_\eta \subset \Omega$, defined by $K_\eta = \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) \geq \eta\}$. Obviously, K_η is compact and

$$\sup_{u_\varepsilon(t,\cdot)} \int_{\Omega \setminus K_\eta} \Big| u_\varepsilon(t,\cdot) \Big| \leq \max(|a|,|b|)\ \operatorname{meas}(\Omega \setminus K_\eta) = C(a,b,\partial\Omega)\ \eta.$$

- ▷ Recalling the existence of $\delta > 0$ such that Inequality (30) holds, we get uniformly in $t \in [0,T]$ and $\varepsilon > 0$,

$$\int_{\Omega^{\Delta x}} \Big| u_\varepsilon(t, \cdot + \Delta x) - u_\varepsilon(t,\cdot) \Big| \leq 2 \int_\Omega \Big| u_{\varepsilon,h}(t,\cdot) - u_\varepsilon(t,\cdot) \Big| + |\Delta x| \int_\Omega \Big| \nabla u_{\varepsilon,h}(t,\cdot) \Big|$$

which is smaller than α for $|\Delta x| \leq \delta$ and $\Omega^{\Delta x} = \{x \in \Omega,\ x + \Delta x \in \Omega\}$.

Thus, the sequence $\{u_\varepsilon(t,\cdot)\}_{t \in [0,T],\ \varepsilon > 0}$ is relatively compact in $L^1(\Omega)$.

Thus, by the Arzelà-Ascoli theorem, $\{u_\varepsilon\}_\varepsilon$ is precompact in $C^0([0,T], L^1(\Omega))$, and since $C^0([0,T], L^1(\Omega))$ is complete, we infer that, up to a subsequence,

$$\lim_{\varepsilon \to 0} u_\varepsilon = u \quad \text{in } C^0([0,T], L^1(\Omega))$$

Moreover, $u \in L^\infty(Q_T; [a,b])$ (by passing to the limit on u_ε). Finally, u is a weak entropy solution of Eq. (1)–(3): recalling that

$$\lim_{\varepsilon \to 0} \int_\Omega \Big| 1 - \xi_\varepsilon \Big| = 0, \quad \lim_{\varepsilon \to 0} \varepsilon \int_\Omega \Big| \nabla \xi_\varepsilon \Big| = 0,$$

passing to the limit w.r.t. ε in Inequality (15) concludes the proof.

3 Uniqueness

Definition 15 *For any $k \in \mathbb{R}$, let us denote:*

$$\begin{aligned} (\widetilde{H}^k, \widetilde{Q}^k_{[f]}) :\ \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (z,w) &\longrightarrow \Big(\operatorname{dist}(z, \mathcal{I}(w,k)), \mathcal{F}_{[f]}(\cdot,\cdot,z,w,k) \Big) \end{aligned}$$

with $\mathcal{I}(w,k) = [\min(w,k), \max(w,k)]$ *and* $\mathcal{F}_{[f]} \in \mathcal{C}(\mathbb{R} \times \Omega \times \mathbb{R}^3)$ *defined as:*

$$\mathcal{F}_{[f]}(\cdot,\cdot,z,w,k) = \begin{cases} f(\cdot,\cdot,w) - f(\cdot,\cdot,z) & \text{for } \; z \leq w \leq k \\ 0 & \text{for } \; k \leq z \leq w \\ f(\cdot,\cdot,z) - f(\cdot,\cdot,k) & \text{for } \; w \leq k \leq z \\ f(\cdot,\cdot,k) - f(\cdot,\cdot,z) & \text{for } \; z \leq k \leq w \\ 0 & \text{for } \; w \leq z \leq k \\ f(\cdot,\cdot,z) - f(\cdot,\cdot,w) & \text{for } \; k \leq w \leq z \end{cases}$$

Lemma 16 *Let* $u \in L^\infty(Q_T)$ *satisfy* $(\mathcal{P}_{SK})$*; then one has:*

- *for all* $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$*, for all* $k \in \mathbb{R}$*,*

$$\int_{Q_T} \Big\{ |u-k| \; \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)\Big(f(t,x,u) - f(t,x,k)\Big) \; \nabla\varphi$$
$$- \operatorname{sgn}(u-k) \left(\nabla \cdot f(t,x,k) + g(t,x,u)\right) \; \varphi \Big\} \; dx \; dt$$
$$\geq \operatorname{ess} \lim_{\varrho \to 0^+} \int_{\Sigma_T} \Big\{ \operatorname{sgn}(u(t, r - \varrho \; n(r)) - k)\Big(f(t, r - \varrho \; n(r), u(t, r - \varrho \; n(r)))$$
$$- f(t, r - \varrho \; n(r), k)\Big) \Big\} \cdot n(r) \; \varphi(t,r) \; d\gamma(r) \; dt,$$

- *for all* $\beta \in L^1(\Sigma_T)$*,* $\beta \geq 0$ *a.e., and for all* $k \in \mathbb{R}$*,*

$$\operatorname{ess} \lim_{\varrho \to 0^+} \int_{\Sigma_T} \mathcal{F}_{[f]}(t, r, u(t, r - \varrho \; n(r)), u^D(t,r), k) \cdot n(r) \; \beta(t,r) \; d\gamma(r) \; dt \geq 0,$$

- *for all* $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$*, for all* $k \in \mathbb{R}$*,*

$$\int_{Q_T} \Big\{ |u-k| \; \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)\Big(f(t,x,u) - f(t,x,k)\Big) \; \nabla\varphi$$
$$- \operatorname{sgn}(u-k) \left(\nabla \cdot f(t,x,k) + g(t,x,u)\right) \; \varphi \Big\} \; dx \; dt$$
$$\geq \int_{\Sigma_T} \operatorname{sgn}(k - u^D)\Big(f(t,r,k) - f(t,r,u^D)\Big) \cdot n(r) \; \varphi(t,r) \; d\gamma(r) \; dt$$
$$- \operatorname{ess} \lim_{\varrho \to 0^+} \int_{\Sigma_T} \Big\{ \operatorname{sgn}(u(t, r - \varrho \; n(r)) - k)\Big(f(t, r, u(t, r - \varrho \; n(r)))$$
$$- f(t,r,k)\Big) \Big\} \cdot n(r) \; \varphi(t,r) \; d\gamma(r) \; dt.$$

PROOF.

▷ *1st inequality* - Adding the two inequalities defined by $(\mathcal{P}_{SK})$ with each "semi Kružkov entropy-flux pair" gives the following inequality

$$\int_{Q_T} \Big\{ |u-k| \, \frac{\partial \varphi}{\partial t} + \text{sgn}(u-k)\big(f(t,x,u) - f(t,x,k)\big) \, \nabla\varphi \\ - \text{sgn}(u-k) \, (\nabla \cdot f(t,x,k) + g(t,x,u)) \, \varphi \Big\} \, dx \, dt \geq 0, \tag{31}$$

for any $\varphi \in \mathcal{D}(Q_T)$. Thus, since u satisfies Inequality (31) along with the initial condition (6) (see Lemma 5), the result is obtained by following the same lines of the proof of Lemma 7.12. in [6].

▷ *2nd inequality* - The result is easily obtained by Lemma 4 applied to the particular "boundary fluxes" $\mathcal{F}_{[f]}$ (see Definition 15).

▷ *3rd inequality* - On the one hand, the function $2\mathcal{F}_{[f]}(\cdot,\cdot,z,w,k)$ is equal to

$$\text{sgn}(z-w) \, \big(f(\cdot,\cdot,z) - f(\cdot,\cdot,w)\big) - \text{sgn}(k-w) \, \big(f(\cdot,\cdot,k) - f(\cdot,\cdot,w)\big) \\ + \text{sgn}(z-k) \, \big(f(\cdot,\cdot,z) - f(\cdot,\cdot,k)\big).$$

On the second hand, terms of the form

$$\text{ess} \lim_{\varrho \to 0^-} \int_{\Sigma_T} \text{sgn}(u(t,r-\varrho\, n) - v^D(t,r)) \big\{ f(t,r,u(t,r-\varrho\, n)) \\ - f(t,r,v^D(t,r)) \big\} \cdot n \; \beta(t,r) \; d\gamma(r) \; dt$$

exist for all $\beta \in L^1(\Sigma_T)$, all $v^D \in L^\infty(\Sigma_T)$. Indeed, this term is obtained by using the proof of Lemma 4: it is sufficient to add the terms of (5) corresponding to each "semi Kružkov entropy-flux pair". Therefore, each term of the following inequality exists. Thus, from the 2nd inequality,

$$- \text{ess} \lim_{\varrho \to 0^+} \int_{\Sigma_T} F(t,r,u(t,r-\varrho\, n(r)),k) \cdot n(r) \; \beta(t,r) \; d\gamma(r) \; dt \\ \leq \text{ess} \lim_{\varrho \to 0^+} \int_{\Sigma_T} F(t,r,u(t,r-\varrho\, n(r)),u^D(t,r)) \cdot n(r) \; \beta(t,r) \; d\gamma(r) \; dt \\ - \int_{\Sigma_T} F(t,r,k,u^D(t,r)) \cdot n(r) \; \beta(t,r) \; d\gamma(r) \; dt$$

with the notation $F(t,r,u,k) = \operatorname{sgn}(u-k)\Big(f(t,r,u) - f(t,r,k)\Big)$, and the result is straightforward.

Lemma 17 *Let $u \in L^\infty(Q_T)$ (resp. $v \in L^\infty(Q_T)$) be a solution of $(\mathcal{P}_{SK})$ with data $(u^0, u^D) \in L^\infty(\Omega) \times L^\infty(\Sigma_T)$ (resp. $(v^0, v^D) \in L^\infty(\Omega) \times L^\infty(\Sigma_T)$); then*

$$- \int_{Q_T} \left\{ \Big|u - v\Big| \frac{\partial \beta}{\partial t} + \operatorname{sgn}(u-v)\Big(f(t,x,u) - f(t,x,v)\Big)\nabla\beta \right.$$
$$\left. - \operatorname{sgn}(u-v)\,(g(t,x,u) - g(t,x,v))\,\beta \right\} dx\ dt$$
$$\leq \int_{\Omega} \Big|u^0(x) - v^0(x)\Big|\ \beta(0,x)\ dx + \mathcal{L}_{[f]} \int_{\Sigma_T} \Big|u^D(t,r) - v^D(t,r)\Big|\ \beta(t,r)\ d\gamma(r)\ dt$$

for all $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$.

PROOF. As already pointed out, any term written under the form

$$\operatorname{ess}\lim_{\varrho \to 0^+} \int_{\Sigma_T} \left\{ \operatorname{sgn}(u(t, r - \varrho\, n(r)) - v^D(t,r))\Big(f(t,r,u(t, r - \varrho\, n(r)))\right.$$
$$\left. - f(t,r,v^D(t,r))\Big) \right\} \cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt$$

exists for all $\beta \in L^1(\Sigma_T)$, all $v^D \in L^\infty(\Sigma_T)$. Thus, we infer that there exists $\theta_{i,j} \in L^\infty(\Sigma_T)$ such that:

$$\int_{\Sigma_T} \theta_{1,1}(t,r)\beta(t,r)\, d\gamma(r)\, dt =$$
$$\operatorname{ess}\lim_{\varrho \to 0^+} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - \varrho\, n) - u^D)\Big(f(t,r,u(t,r-\varrho\, n)) - f(t,r,u^D)\Big) \cdot n\, \beta\, d\gamma(r)\, dt,$$

$$\int_{\Sigma_T} \theta_{1,2}(t,r)\, \beta(t,r)\, d\gamma(r)\, dt =$$
$$\operatorname{ess}\lim_{\varrho \to 0^+} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - \varrho\, n) - v^D)\Big(f(t,r,u(t,r-\varrho\, n)) - f(t,r,v^D)\Big) \cdot n\, \beta\, d\gamma(r)\, dt,$$

$$\int_{\Sigma_T} \theta_{2,2}(t,r)\, \beta(t,r)\, d\gamma(r)\, dt =$$
$$\operatorname{ess}\lim_{\varrho \to 0^+} \int_{\Sigma_T} \operatorname{sgn}(v(t, r - \varrho\, n) - v^D)\Big(f(t,r,v(t,r-\varrho\, n)) - f(t,r,v^D)\Big) \cdot n\, \beta\, d\gamma(r)\, dt,$$

$$\int_{\Sigma_T} \theta_{2,1}(t,r)\, \beta(t,r)\, d\gamma(r)\, dt =$$
$$\operatorname{ess}\lim_{\varrho \to 0^+} \int_{\Sigma_T} \operatorname{sgn}(v(t, r - \varrho\, n) - u^D)\Big(f(t,r,v(t,r-\varrho\, n)) - f(t,r,u^D)\Big) \cdot n\, \beta\, d\gamma(r)\, dt.$$

After this introduction of notations, we now apply the double variable method, initiated by Kružkov [3], to the 3rd inequality stated in Lemma 16. Let $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^{d+1})$ be a symmetric regularizing sequence. We will denote

$$
\begin{aligned}
p = (t,x) \in Q_T, \qquad & p' = (t',x') \in Q_T,\\
\gamma(p) = (t,r) \in \Sigma_T, \quad & \gamma(p') = (t',r') \in \Sigma_T,
\end{aligned}
$$

and let

$$
\beta_\varepsilon(p,p') = \beta\left(\frac{p+p'}{2}\right)\rho_\varepsilon\left(p-p'\right),
$$

for all $p,p' \in (Q_T)^2$, for a given $\beta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, $\beta \geq 0$. Hold $p' \in Q_T$ fixed and replace, in the 3rd inequality of Lemma 16, k by $v(p')$ and $\beta(p)$ by $\beta_\varepsilon(p,p')$. After integration over Q_T (with respect to the variable p'), and using the notation

$$
F(p,u(p),v(p')) = \operatorname{sgn}(u(p)-v(p'))\left(f(p,u(p)-f(p,v(p')))\right),
$$

we easily get $I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon + I_5^\varepsilon \leq I_6^\varepsilon + I_7^\varepsilon$, with

$$
\begin{aligned}
I_1^\varepsilon &= -\frac{1}{2}\int\limits_{Q_T}\int\limits_{Q_T} \left|u(p)-v(p')\right| \frac{\partial\beta}{\partial t}\left(\frac{p+p'}{2}\right)\rho_\varepsilon\left(p-p'\right)dp\,dp'\\
I_2^\varepsilon &= -\frac{1}{2}\int\limits_{Q_T}\int\limits_{Q_T} F(p,u(p),v(p'))\,\nabla\beta\left(\frac{p+p'}{2}\right)\rho_\varepsilon\left(p-p'\right)dp\,dp'\\
I_3^\varepsilon &= -\int\limits_{Q_T}\int\limits_{Q_T} \left|u(p)-v(p')\right| \frac{\partial\rho_\varepsilon}{\partial t}\left(p-p'\right)\,\beta\left(\frac{p+p'}{2}\right)dp\,dp'\\
I_4^\varepsilon &= -\int\limits_{Q_T}\int\limits_{Q_T} F(p,u(p),v(p'))\,\nabla\rho_\varepsilon\left(p-p'\right)\,\beta\left(\frac{p+p'}{2}\right)dp\,dp'\\
I_5^\varepsilon &= \int\limits_{Q_T}\int\limits_{Q_T} \operatorname{sgn}(u(p)-v(p'))\Big\{\nabla\cdot f\left(p,v(p')\right)+g(p,u(p))\Big\}\,\beta\left(\frac{p+p'}{2}\right)\rho_\varepsilon\left(p-p'\right)dp\,dp'\\
I_6^\varepsilon &= \int\limits_{Q_T}\int\limits_{\Sigma_T} \theta_{1,1}(\gamma(p))\,\beta\left(\frac{\gamma(p)+p'}{2}\right)\rho_\varepsilon\left(\gamma(p)-p'\right)d\gamma(p)\,dp'\\
I_7^\varepsilon &= -\int\limits_{Q_T}\int\limits_{\Sigma_T} F(p,v(p'),u^D(\gamma(p)))\cdot n\,\beta\left(\frac{\gamma(p)+p'}{2}\right)\rho_\varepsilon\left(\gamma(p)-p'\right)d\gamma(p)\,dp'.
\end{aligned}
$$

Now changing the role of $(u(p),p)$ and $(v(p'),p')$, we get in a similar way, the inequality $J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon + J_5^\varepsilon \leq J_6^\varepsilon + J_7^\varepsilon$, with

$$
\begin{aligned}
J_1^\varepsilon &= -\frac{1}{2}\int\limits_{Q_T}\int\limits_{Q_T} \left|v(p')-u(p)\right| \frac{\partial\beta}{\partial t}\left(\frac{p+p'}{2}\right)\rho_\varepsilon\left(p-p'\right)dp\,dp'\\
J_2^\varepsilon &= -\frac{1}{2}\int\limits_{Q_T}\int\limits_{Q_T} F(p',v(p'),u(p))\,\nabla\beta\left(\frac{p+p'}{2}\right)\rho_\varepsilon\left(p-p'\right)dp\,dp'
\end{aligned}
$$

$$J_3^\varepsilon = \int\limits_{Q_T}\int\limits_{Q_T} \left|v(p') - u(p)\right| \frac{\partial \rho_\varepsilon}{\partial t}\,(p-p')\;\beta\left(\frac{p+p'}{2}\right) dp\,dp'$$

$$J_4^\varepsilon = \int\limits_{Q_T}\int\limits_{Q_T} F(p', v(p'), u(p))\,\nabla\rho_\varepsilon\,(p-p')\;\beta\left(\frac{p+p'}{2}\right) dp\,dp'$$

$$J_5^\varepsilon = \int\limits_{Q_T}\int\limits_{Q_T} \mathrm{sgn}(v(p')-u(p))\Big\{\nabla\cdot f\,(p', u(p)) + g(p', v(p'))\Big\}\,\beta\left(\frac{p+p'}{2}\right)\,\rho_\varepsilon\,(p-p')\,dp\,dp'$$

$$J_6^\varepsilon = \int\limits_{Q_T}\int\limits_{\Sigma_T} \theta_{2,2}(\gamma(p'))\,\beta\left(\frac{p+\gamma(p')}{2}\right)\,\rho_\varepsilon\,(p-\gamma(p'))\,d\gamma(p')\,dp$$

$$J_7^\varepsilon = -\int\limits_{Q_T}\int\limits_{\Sigma_T} F(p', u(p), v^D(\gamma(p')))\cdot n\;\beta\left(\frac{p+\gamma(p')}{2}\right)\,\rho_\varepsilon\,(p-\gamma(p'))\,d\gamma(p')\,dp.$$

Adding the two inequalities, and noticing that $I_1^\varepsilon = J_1^\varepsilon$ and $I_3^\varepsilon = -J_3^\varepsilon$, we get

$$2I_1^\varepsilon + (I_2^\varepsilon + J_2^\varepsilon) + (I_4^\varepsilon + J_4^\varepsilon) + (I_5^\varepsilon + J_5^\varepsilon) \leq (I_6^\varepsilon + J_6^\varepsilon) + (I_7^\varepsilon + J_7^\varepsilon).$$

We are now ready to let ε tend to 0. Note that this method has been widely used in the works related to hyperbolic problems [3–6] but also parabolic problems [8] or elliptic-hyperbolic problems [13–16]. Thus, due to the convolution effect of ρ_ε, we obviously obtain

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = -\frac{1}{2}\int\limits_{Q_T} |u-v|\;\frac{\partial\beta}{\partial t},$$

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = \lim_{\varepsilon\to 0} J_2^\varepsilon = -\frac{1}{2}\int\limits_{Q_T} \mathrm{sgn}(u-v)\;\;(f(t,x,u) - f(t,x,v)))\;\;\nabla\beta,$$

$$\lim_{\varepsilon\to 0} I_6^\varepsilon = \frac{1}{2}\int\limits_{\Sigma_T}\theta_{1,1}\;\beta,\quad \lim_{\varepsilon\to 0} J_6^\varepsilon = \frac{1}{2}\int\limits_{\Sigma_T}\theta_{2,2}\;\beta,$$

$$\lim_{\varepsilon\to 0} I_7^\varepsilon = -\frac{1}{2}\int\limits_{\Sigma_T}\theta_{1,2}\;\beta,\quad \lim_{\varepsilon\to 0} J_7^\varepsilon = -\frac{1}{2}\int\limits_{\Sigma_T}\theta_{2,1}\;\beta.$$

Now, let us focus on $I_4^\varepsilon + J_4^\varepsilon + I_5^\varepsilon + J_5^\varepsilon$ which contains all the non classical contributions of the source term g and the non-autonomous property of the flux f. Interestingly, we will see that only the source term plays a role when passing to the limit on ε: this is because of the conservative form of the scalar conservation law. Let us reorganize the sum by rewriting it in the following form:

$$I_4^\varepsilon + J_4^\varepsilon + I_5^\varepsilon + J_5^\varepsilon = I_4^\varepsilon = K_1^{(1)} - K_1^{(1)} - (K_2^{(1)} - K_2^{(2)}) + K_3^{(1)} - K_3^{(2)}$$

with

$$K_1^{(1)} = \int\limits_{Q_T\times Q_T} F(p', u(p), v(p'))\, \beta\left(\frac{p+p'}{2}\right) \nabla\rho_\varepsilon\,(p-p')\;dp\,dp'$$
$$K_1^{(2)} = \int\limits_{Q_T\times Q_T} F(p, u(p), v(p'))\, \beta\left(\frac{p+p'}{2}\right) \nabla\rho_\varepsilon\,(p-p')\;dp\,dp'$$
$$K_2^{(1)} = \int\limits_{Q_T\times Q_T} \operatorname{sgn}(u(p)-v(p'))\, \nabla\cdot f(p', u(p))\, \beta\left(\frac{p+p'}{2}\right) \rho_\varepsilon\,(p-p')\;dp\,dp'$$
$$K_2^{(2)} = \int\limits_{Q_T\times Q_T} \operatorname{sgn}(u(p)-v(p'))\, \nabla\cdot f(p, v(p'))\, \beta\left(\frac{p+p'}{2}\right) \rho_\varepsilon\,(p-p')\;dp\,dp'$$
$$K_3^{(1)} = \int\limits_{Q_T\times Q_T} \operatorname{sgn}(u(p)-v(p'))\, g(p, u(p))\, \beta\left(\frac{p+p'}{2}\right) \rho_\varepsilon\,(p-p')\;dp\,dp'$$
$$K_3^{(2)} = \int\limits_{Q_T\times Q_T} \operatorname{sgn}(u(p)-v(p'))\, g(p', v(p'))\, \beta\left(\frac{p+p'}{2}\right) \rho_\varepsilon\,(p-p')\;dp\,dp'$$

Passing to the limit in $K_3^{(1)} - K_3^{(2)}$ obviously gives:

$$\lim_{\varepsilon\to 0} K_3^{(1)} - K_3^{(2)} = \int\limits_{Q_T} \operatorname{sgn}(u-v)\,(g(t,x,u)-g(t,x,v))\,\beta$$

Moreover, the limit of $K_1^{(1)} - K_1^{(2)} - (K_2^{(1)} - K_2^{(2)})$ may be analysed *exactly* in the same manner as in the paper of Kružkov (see p. 227 in [3]) and we have:

$$\lim_{\varepsilon\to 0} K_1^{(1)} - K_1^{(2)} - (K_2^{(1)} - K_2^{(2)}) = 0.$$

$$\lim_{\varepsilon\to 0}(I_4^\varepsilon + J_4^\varepsilon + I_5^\varepsilon + J_5^\varepsilon) = \int\limits_{Q_T} \operatorname{sgn}(u-v)\,(g(t,x,u)-g(t,x,v))\,\beta.$$

Finally we obtain:

$$-\int\limits_{Q_T} \left\{ \Big|u-v\Big|\,\frac{\partial\beta}{\partial t} + \operatorname{sgn}(u-v)\Big(f(t,x,u)-f(t,x,v)\Big)\nabla\beta \right.$$
$$\left. - \operatorname{sgn}(u-v)\,(g(t,x,u)-g(t,x,v))\,\beta \right\}$$
$$\leq \frac{1}{2}\int\limits_{\Sigma_T} (-\theta_{1,1}+\theta_{2,1}-\theta_{2,2}+\theta_{1,2})\,\beta,$$

for all $\beta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. As in [6], let us introduce the following definition:

$$\forall (t,r)\in\Sigma_T,\ \operatorname{diam}\Big(f(t,r,\cdot)\cdot n, \mathcal{I}(a,b)\Big) = \sup_{z_1,z_2\in\mathcal{I}(a,b)} \Big(\Big|f(t,r,z_1)\cdot n - f(t,r,z_2)\cdot n\Big|\Big)$$

Then, when discussing the cases, one sees that for all z_1, z_2, w_1, w_2,

$$\Big| \sum_{i,j=1}^{2} (-1)^{i+j} \mathrm{sgn}(z_i - w_j) \Big(f(t,r,z_i) - f(t,r,w_j) \Big) \cdot n \Big| \leq 2 \,\mathrm{diam}(f(t,r,\cdot)\cdot n, \mathcal{I}(w_1, w_2))$$

holds and using the property

$$\mathrm{diam}(f(t,r,\cdot)\cdot n, \mathcal{I}(u^D(t,r), v^D(t,r))) \leq \mathcal{L}_{[f]} \Big| u^D(t,r) - v^D(t,r) \Big|, \quad \forall (t,r) \in \Sigma_T,$$

one easily concludes that

$$\frac{1}{2} \Big| \int_{\Sigma_T} (-\theta_{1,1} + \theta_{2,1} - \theta_{2,2} + \theta_{1,2}) \ \beta \Big| \leq \mathcal{L}_{[f]} \int_{\Sigma_T} \Big| u^D - v^D \Big| \ \beta.$$

The initial term is obtained by slightly modifying the proof, with test functions in the appropriate space, namely $\mathcal{D}((-\infty, T) \times \mathbb{R}^d)$.

Theorem 18 (Uniqueness) *Under Assumption 1, problem* $(\mathcal{P}_{SK})$ *admits a unique weak entropy solution.*

PROOF. Considering the integral inequality of Lemma 17 with $v^D = u^D$ and $v^0 = u^0$ and a test function which only depends on time t, we get:

$$\int_{Q_T} \Big\{ \Big| u - v \Big| \ \alpha'(t) - \mathrm{sgn}(u - v) \Big(g(t,x,u) - g(t,x,v) \Big) \ \alpha(t) \Big\} \, dx \, dt \geq 0, \quad (32)$$

for all $\alpha \in \mathcal{D}(-\infty, T)$. Then, for an interval $[t_0, t_1] \subset]0, T[$, we can use in Inequality (32) the characteristic function of $[t_0, t_1]$, properly mollified, and pass to the limit on the mollifier parameter:

$$\int_{\Omega} \Big| u(t_1, \cdot) - v(t_1, \cdot) \Big| \leq \int_{\Omega} \Big| u(t_0, \cdot) - v(t_0, \cdot) \Big| + \int_{t_0}^{t_1} \int_{\Omega} \mathrm{sgn}(u - v) \left(g(t,x,v) - g(t,x,u) \right) \ dx \ dt.$$

Now, since we have, for all $(t,x) \in (0,T) \times \Omega$:

$$\mathrm{sgn}(u - v) \left(g(t,x,v) - g(t,x,u) \right) \leq \mathcal{L}_{[g]} \Big| u - v \Big|,$$

we obtain:

$$\int_{\Omega} \Big| u(t_1, \cdot) - v(t_1, \cdot) \Big| \leq \int_{\Omega} \Big| u(t_0, \cdot) - v(t_0, \cdot) \Big| + \mathcal{L}_{[g + \nabla \cdot f]} \int_{t_0}^{t_1} \Big| u(t, \cdot) - v(t, \cdot) \Big|_{L^1(\Omega)} dt.$$

From Gronwall's lemma, we conclude that:

$$\left|u(t_1,\cdot) - v(t_1,\cdot)\right|_{L^1(\Omega)} \leq \left|u(t_0,\cdot) - v(t_0,\cdot)\right|_{L^1(\Omega)} e^{\mathcal{L}_{[g]}(t_1-t_0)}.$$

As t_0 tends to 0, and using the fact that $v^0 = u^0$ along with the initial condition (6), the uniqueness is straightforward.

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