First-Order Theorem Proving and Vampire

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Outline

First-Order Logic: Exercises

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8. Having proofs is good.
9. Vampire is a first-order theorem prover.

Future and Our Motivation

1. Theorem proving will remain central in software verification and program analysis. The role of theorem proving in these areas will be growing.
2. Theorem provers will be used by a large number of users who do not understand theorem proving and by users with very elementary knowledge of logic.
3. Reasoning with both quantifiers and theories will remain the main challenge in practical applications of theorem proving (at least) for the next decade.
4. Theorem provers will be used in reasoning with very large theories. These theories will appear in knowledge mining and natural language processing.

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More formally: in a group "assuming that $x^2 = 1$ for all x prove that $x \cdot y = y \cdot x$ holds for all x, y."
What is implicit: axioms of the group theory.

$$\forall x(1 \cdot x = x)$$
$$\forall x(x^{-1} \cdot x = 1)$$
$$\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

Formulation in First-Order Logic

Axioms (of group theory):	$\forall x(1 \cdot x = x)$
	$\forall x(x^{-1} \cdot x = 1)$
	$\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
Assumptions:	$\forall x(x \cdot x = 1)$
Conjecture:	$\forall x \forall y(x \cdot y = y \cdot x)$

In the TPTP Syntax

The TPTP library (Thousands of Problems for Theorem Provers), `http://www.tptp.org` contains a large collection of first-order problems. For representing these problems it uses the TPTP syntax, which is understood by all modern theorem provers, including Vampire.

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In the TPTP syntax this group theory problem can be written down as follows:

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    ! [X] : mult(e,X) = X).
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Running Vampire of a TPTP file

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One can also run Vampire with various options, some of them will be explained later. For example, save the group theory problem in a file `group.tptp` and try

```
vampire --thanks ReRiSE group.tptp
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FOL	TPTP
$\bot, \top$	`$false, $true`
$\neg F$	`~F`
$F_1 \wedge \ldots \wedge F_n$	`F1 & ... & Fn`
$F_1 \vee \ldots \vee F_n$	`F1 | ... | Fn`
$F_1 \rightarrow F_n$	`F1 => Fn`
$(\forall x_1) \ldots (\forall x_n) F$	`! [X1,...,Xn] : F`
$(\exists x_1) \ldots (\exists x_n) F$	`? [X1,...,Xn] : F`

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Proof by Vampire (Slightliy Modified)

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202. sP1(mult(sK,sK0)) [backward demodulation 188,15]
188. mult(X8,X9) = mult(X9,X8) [superposition 22,87]
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Statistics

```
Version: Vampire 3 (revision 2038)
Termination reason: Refutation

Active clauses: 14
Passive clauses: 28
Generated clauses: 124
Final active clauses: 8
Final passive clauses: 6
Input formulas: 5
Initial clauses: 6

Splitted inequalities: 1

Fw subsumption resolutions: 1
Fw demodulations: 32
Bw demodulations: 12

Forward subsumptions: 53
Backward subsumptions: 1
Fw demodulations to eq. taut.: 6
Bw demodulations to eq. taut.: 1

Forward superposition: 41
Backward superposition: 28
Self superposition: 4

Memory used [KB]: 255
Time elapsed: 0.005 s
```

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- Completely automatic: once you started a proof attempt, it can only be interrupted by terminating the process.

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- Completely automatic: once you started a proof attempt, it can only be interrupted by terminating the process.
- Champion of the CASC world-cup in first-order theorem proving: won CASC 28 times.

Main applications

- Software and hardware verification;
- Static analysis of programs;
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- Program synthesis;
- Writing papers and giving talks at various conferences and schools ...

What an Automatic Theorem Prover is Expected to Do

Input:

- a set of axioms (first order formulas) or clauses;
- a conjecture (first-order formula or set of clauses).

Output:

- proof (hopefully).

Proof by Refutation

Given a problem with axioms and assumptions $F_1, \ldots, F_n$ and conjecture G,

1. negate the conjecture;
2. establish unsatisfiability of the set of formulas $F_1, \ldots, F_n, \neg G$.

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In this formulation the negation of the conjecture $\neg G$ is treated like any other formula. In fact, Vampire (and other provers) internally treat conjectures differently, to make proof search more goal-oriented.

General Scheme (simplified)

- Read a problem;
- Determine proof-search options to be used for this problem;
- Preprocess the problem;
- Convert it into CNF;
- Run a saturation algorithm on it, try to derive $\perp$.
- If $\perp$ is derived, report the result, maybe including a refutation.

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Trying to derive $\perp$ using a saturation algorithm is the hardest part, which in practice may not terminate or run out of memory.

Outline

Introduction

First-Order Logic and TPTP

Inference Systems

Saturation Algorithms

Redundancy Elimination

Equality

Inference System

- inference has the form

$$\frac{F_1 \quad \dots \quad F_n}{G},$$

  where $n \geq 0$ and $F_1, \dots, F_n, G$ are formulas.
- The formula G is called the conclusion of the inference;
- The formulas $F_1, \dots, F_n$ are called its premises.
- An inference rule R is a set of inferences.
- Every inference $I \in R$ is called an instance of R.
- An Inference system $\mathbb{I}$ is a set of inference rules.
- Axiom: inference rule with no premises.

Inference System: Example

Represent the natural number n by the string $\underbrace{|\ldots|}_{n \text{ times}}\varepsilon$.

The following inference system contains 6 inference rules for deriving equalities between expressions containing natural numbers, addition $+$ and multiplication $\cdot$.

$$\frac{}{\varepsilon = \varepsilon}\ (\varepsilon) \qquad \frac{x = y}{|x = |y}\ (|)$$

$$\frac{}{\varepsilon + x = x}\ (+_1) \qquad \frac{x + y = z}{|x + y = |z}\ (+_2)$$

$$\frac{}{\varepsilon \cdot x = \varepsilon}\ (\cdot_1) \qquad \frac{x \cdot y = u \quad y + u = z}{|x \cdot y = z}\ (\cdot_2)$$

Derivation, Proof

- Derivation in an inference system $\mathbb{I}$: a tree built from inferences in $\mathbb{I}$.
- If the root of this derivation is E, then we say it is a derivation of E.
- Proof of E: a finite derivation whose leaves are axioms.
- Derivation of E from $E_1, \ldots, E_m$: a finite derivation of E whose every leaf is either an axiom or one of the expressions $E_1, \ldots, E_m$.

Examples

For example,

$$\frac{||\varepsilon + |\varepsilon = |||\varepsilon}{|||\varepsilon + |\varepsilon = ||||\varepsilon}\ (+_2)$$

is an inference that is an instance (special case) of the inference rule

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The axiom

$$\frac{}{\varepsilon + |||\varepsilon = |||\varepsilon}\ (+_1)$$

is an instance of the rule

$$\frac{}{\varepsilon + x = x}\ (+_1)$$

Proof in this Inference System

Proof of $||\varepsilon \cdot ||\varepsilon = ||||\varepsilon$ (that is, $2 \cdot 2 = 4$).

$$
\dfrac{
  \dfrac{}{\varepsilon \cdot ||\varepsilon = \varepsilon}\,(\cdot_1)
  \qquad
  \dfrac{\dfrac{\dfrac{}{\varepsilon + \varepsilon = \varepsilon}\,(+_1)}{|\varepsilon + \varepsilon = |\varepsilon}\,(+_2)}{||\varepsilon + \varepsilon = ||\varepsilon}\,(+_2)
}{|\varepsilon \cdot ||\varepsilon = ||\varepsilon}\,(\cdot_2)
\qquad
\dfrac{\dfrac{\dfrac{}{\varepsilon + ||\varepsilon = ||\varepsilon}\,(+_1)}{|\varepsilon + ||\varepsilon = |||\varepsilon}\,(+_2)}{||\varepsilon + ||\varepsilon = ||||\varepsilon}\,(+_2)
$$

$$
\dfrac{|\varepsilon \cdot ||\varepsilon = ||\varepsilon \qquad ||\varepsilon + ||\varepsilon = ||||\varepsilon}{||\varepsilon \cdot ||\varepsilon = ||||\varepsilon}\,(\cdot_2).
$$

Derivation in this Inference System

Derivation of $||\varepsilon \cdot ||\varepsilon = |||||\varepsilon$ from $\varepsilon + ||\varepsilon = |||\varepsilon$ (that is, $2 + 2 = 5$ from $0 + 2 = 3$).

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$(\cdot_1)$}
\UnaryInfC{$\varepsilon \cdot ||\varepsilon = \varepsilon$}
\AxiomC{}
\RightLabel{$(+_1)$}
\UnaryInfC{$\varepsilon + \varepsilon = \varepsilon$}
\RightLabel{$(+_2)$}
\UnaryInfC{$|\varepsilon + \varepsilon = |\varepsilon$}
\RightLabel{$(+_2)$}
\UnaryInfC{$||\varepsilon + \varepsilon = ||\varepsilon$}
\RightLabel{$(\cdot_2)$}
\BinaryInfC{$|\varepsilon \cdot ||\varepsilon = ||\varepsilon$}
\AxiomC{$\varepsilon + ||\varepsilon = |||\varepsilon$}
\RightLabel{$(+_2)$}
\UnaryInfC{$|\varepsilon + ||\varepsilon = ||||\varepsilon$}
\RightLabel{$(+_2)$}
\UnaryInfC{$||\varepsilon + ||\varepsilon = |||||\varepsilon$}
\RightLabel{$(\cdot_2)$.}
\BinaryInfC{$||\varepsilon \cdot ||\varepsilon = ||||\varepsilon$}
\end{prooftree}
$$

Arbitrary First-Order Formulas

- A first-order signature (vocabulary): function symbols (including constants), predicate symbols. Equality is part of the language.
- A set of variables.
- Terms are built using variables and function symbols. For example, $f(x) + g(x)$.
- Atoms, or atomic formulas are obtained by applying a predicate symbol to a sequence of terms. For example, $p(a, x)$ or $f(x) + g(x) \geq 2$.
- Formulas: built from atoms using logical connectives $\neg$, $\land$, $\lor$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall$, $\exists$. For example, $(\forall x)x = 0 \lor (\exists y)y > x$.

Clauses

- Literal: either an atom A or its negation $\neg A$.
- Clause: a disjunction $L_1 \vee \ldots \vee L_n$ of literals, where $n \geq 0$.

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- Empty clause, denoted by $\square$: clause with 0 literals, that is, when $n = 0$.
- A formula in Clausal Normal Form (CNF): a conjunction of clauses.
- A clause is ground if it contains no variables.
- If a clause contains variables, we assume that it implicitly universally quantified. That is, we treat $p(x) \lor q(x)$ as $\forall x(p(x) \lor q(x))$.

Binary Resolution Inference System

The binary resolution inference system, denoted by $\mathbb{BR}$ is an inference system on propositional clauses (or ground clauses). It consists of two inference rules:

- Binary resolution, denoted by BR:

$$\frac{p \vee C_1 \quad \neg p \vee C_2}{C_1 \vee C_2} \text{ (BR)}.$$

- Factoring, denoted by Fact:

$$\frac{L \vee L \vee C}{L \vee C} \text{ (Fact)}.$$

Soundness

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

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$\mathbb{BR}$ is sound.

Consequence of soundness: let S be a set of clauses. If $\square$ can be derived from S in $\mathbb{BR}$, then S is unsatisfiable.

Example

Consider the following set of clauses

$$\{\neg p \vee \neg q,\ \neg p \vee q,\ p \vee \neg q,\ p \vee q\}.$$

The following derivation derives the empty clause from this set:

$$\dfrac{\dfrac{\dfrac{p \vee q \quad p \vee \neg q}{p \vee p}\,(\mathrm{BR})}{p}\,(\mathrm{Fact}) \qquad \dfrac{\dfrac{\neg p \vee q \quad \neg p \vee \neg q}{\neg p \vee \neg p}\,(\mathrm{BR})}{\neg p}\,(\mathrm{Fact})}{\Box}\,(\mathrm{BR})$$

Hence, this set of clauses is unsatisfiable.

Can this be used for checking (un)satisfiability

1. What happens when the empty clause cannot be derived from S?
2. How can one search for possible derivations of the empty clause?

Can this be used for checking (un)satisfiability

1. Completeness.

   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\Box$ from S in $\mathbb{BR}$.

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1. Completeness.

   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.

2. We have to formalize search for derivations.

However, before doing this we will introduce a slightly more refined inference system.

Selection Function

A literal selection function selects literals in a clause.

- If C is non-empty, then at least one literal is selected in C.

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Note: selection function does not have to be a function. It can be any oracle that selects literals.

Binary Resolution with Selection

We introduce a family of inference systems, parametrised by a literal selection function σ.
The binary resolution inference system, denoted by $\mathbb{BR}_\sigma$, consists of two inference rules:

- Binary resolution, denoted by BR

$$\frac{\underline{p} \vee C_1 \quad \underline{\neg p} \vee C_2}{C_1 \vee C_2} \text{ (BR).}$$

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- Positive factoring, denoted by Fact:

$$\frac{\underline{p} \vee \underline{p} \vee C}{p \vee C} \text{ (Fact).}$$

Completeness?

Binary resolution with selection may be incomplete, even when factoring is unrestricted (also applied to negative literals).

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Consider this set of clauses:

$$
\begin{array}{ll}
(1) & \neg q \vee \underline{r} \\
(2) & \neg p \vee \underline{q} \\
(3) & \neg r \vee \underline{\neg q} \\
(4) & \neg q \vee \underline{\neg p} \\
(5) & \neg p \vee \underline{\neg r} \\
(6) & \neg r \vee \underline{p} \\
(7) & r \vee q \vee \underline{p}
\end{array}
$$

Completeness?

Binary resolution with selection may be incomplete, even when factoring is unrestricted (also applied to negative literals).

Consider this set of clauses:

(1)	$\neg q \vee \underline{r}$
(2)	$\neg p \vee \underline{q}$
(3)	$\neg r \vee \underline{\neg q}$
(4)	$\neg q \vee \underline{\neg p}$
(5)	$\neg p \vee \underline{\neg r}$
(6)	$\neg r \vee \underline{p}$
(7)	$r \vee q \vee \underline{p}$

It is unsatisfiable:

(8)	$q \vee p$	$(6, 7)$
(9)	q	$(2, 8)$
(10)	r	$(1, 9)$
(11)	$\neg q$	$(3, 10)$
(12)	$\square$	$(9, 11)$

Note the linear representation of derivations (used by Vampire and many other provers).

However, any inference with selection applied to this set of clauses give either a clause in this set, or a clause containing a clause in this set.

Literal Orderings

Take any well-founded ordering $\succ$ on atoms, that is, an ordering such that there is no infinite decreasing chain of atoms:

$$A_0 \succ A_1 \succ A_2 \succ \cdots$$

In the sequel $\succ$ will always denote a well-founded ordering.

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- If $p \succ q$, then $p \succ \neg q$ and $\neg p \succ q$;
- $\neg p \succ p$.

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- If $p \succ q$, then $p \succ \neg q$ and $\neg p \succ q$;
- $\neg p \succ p$.

Exercise: prove that the induced ordering on literals is well-founded too.

Orderings and Well-Behaved Selections

Fix an ordering $\succ$. A literal selection function is well-behaved if

- If all selected literals are positive, then all maximal (w.r.t. $\succ$) literals in C are selected.

In other words, either a negative literal is selected, or all maximal literals must be selected.

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To be well-behaved, we sometimes must select more than one different literal in a clause. Example: $p \vee p$ or $p(x) \vee p(y)$.

Completeness of Binary Resolution with Selection

Binary resolution with selection is complete for every well-behaved selection function.

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Binary resolution with selection is complete for every well-behaved selection function.

Consider our previous example:

$$
\begin{array}{ll}
(1) & \neg q \vee \underline{r} \\
(2) & \neg p \vee \underline{q} \\
(3) & \neg r \vee \underline{\neg q} \\
(4) & \neg q \vee \underline{\neg p} \\
(5) & \neg p \vee \underline{\neg r} \\
(6) & \neg r \vee \underline{p} \\
(7) & r \vee q \vee \underline{p}
\end{array}
$$

A well-behave selection function must satisfy:

1. $r \succ q$, because of (1)
2. $q \succ p$, because of (2)
3. $p \succ r$, because of (6)

There is no ordering that satisfies these conditions.

End of Lecture 1

Slides for lecture 1 ended here . . .

Outline

How to Establish Unsatisfiability?

Completess is formulated in terms of derivability of the empty clause $\square$ from a set S_0 of clauses in an inference system $\mathbb{I}$. However, this formulations gives no hint on how to search for such a derivation.

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Idea:

- Take a set of clauses S (the search space), initially $S = S_0$. Repeatedly apply inferences in $\mathbb{I}$ to clauses in S and add their conclusions to S, unless these conclusions are already in S.
- If, at any stage, we obtain $\square$, we terminate and report unsatisfiability of S_0.

How to Establish Satisfiability?

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When we build a set S such that any inference applied to clauses in S is already a member of S. Any such set of clauses is called saturated (with respect to $\mathbb{I}$).

In first-order logic it is often the case that all saturated sets are infinite (due to undecidability), so in practice we can never build a saturated set.

The process of trying to build one is referred to as saturation.

Saturated Set of Clauses

Let $\mathbb{I}$ be an inference system on formulas and S be a set of formulas.

- S is called saturated with respect to $\mathbb{I}$, or simply $\mathbb{I}$-saturated, if for every inference of $\mathbb{I}$ with premises in S, the conclusion of this inference also belongs to S.
- The closure of S with respect to $\mathbb{I}$, or simply $\mathbb{I}$-closure, is the smallest set S' containing S and saturated with respect to $\mathbb{I}$.

Inference Process

Inference process: sequence of sets of formulas $S_0, S_1, \ldots$, denoted by

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

$(S_i \Rightarrow S_{i+1})$ is a step of this process.

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We say that this step is an $\mathbb{I}$-step if

1. there exists an inference

$$\frac{F_1 \quad \ldots \quad F_n}{F}$$

   in $\mathbb{I}$ such that $\{F_1, \ldots, F_n\} \subseteq S_i$;
2. $S_{i+1} = S_i \cup \{F\}$.

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2. $S_{i+1} = S_i \cup \{F\}$.

An $\mathbb{I}$-inference process is an inference process whose every step is an $\mathbb{I}$-step.

Property

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an $\mathbb{I}$-inference process and a formula F belongs to some S_i. Then S_i is derivable in $\mathbb{I}$ from S_0. In particular, every S_i is a subset of the $\mathbb{I}$-closure of S_0.

Limit of a Process

The limit of an inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$.

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In other words, the limit is the set of all derived formulas.

Suppose that we have an infinite inference process such that S_0 is unsatisfiable and we use a sound and complete inference system.

Question: does completeness imply that the limit of the process contains the empty clause?

Fairness

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an inference process with the limit S_∞.
The process is called fair if for every $\mathbb{I}$-inference

$$\frac{F_1 \quad \ldots \quad F_n}{F} \ ,$$

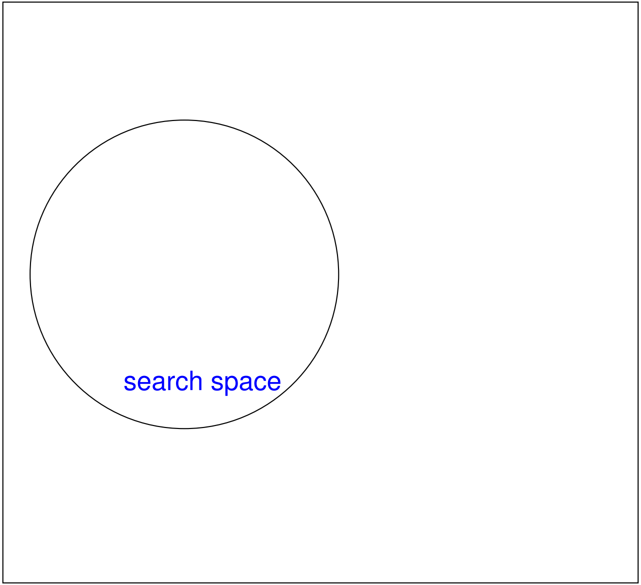
if $\{F_1, \ldots, F_n\} \subseteq S_\infty$, then there exists i such that $F \in S_i$.

Completeness, reformulated

Theorem Let $\mathbb{I}$ be an inference system. The following conditions are equivalent.

1. $\mathbb{I}$ is complete.
2. For every unsatisfiable set of formulas S_0 and any fair $\mathbb{I}$-inference process with the initial set S_0, the limit of this inference process contains $\square$.

Fair Saturation Algorithms: Inference Selection by Clause Selection



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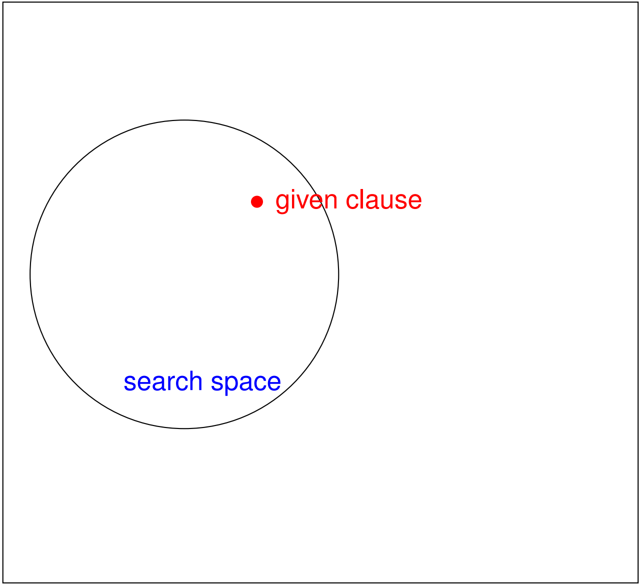

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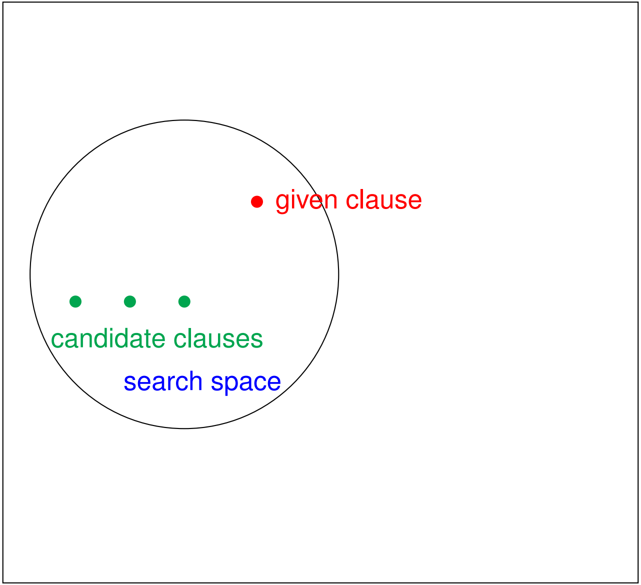

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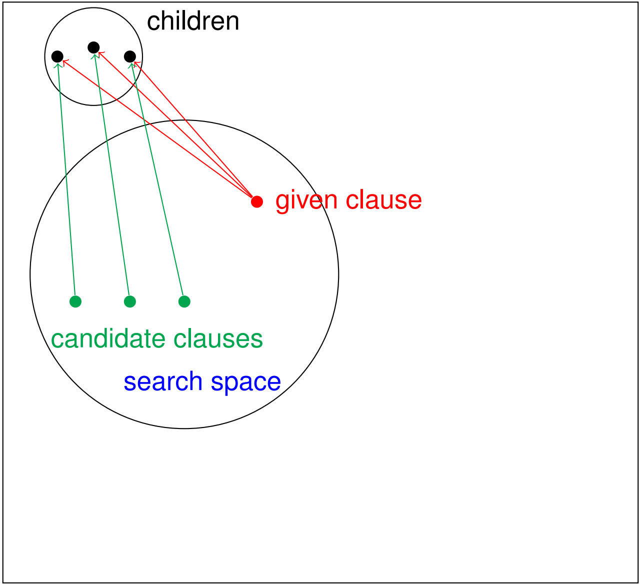

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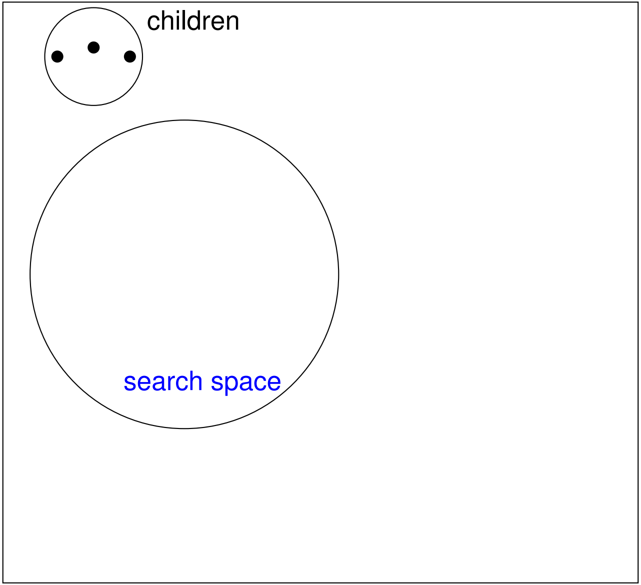

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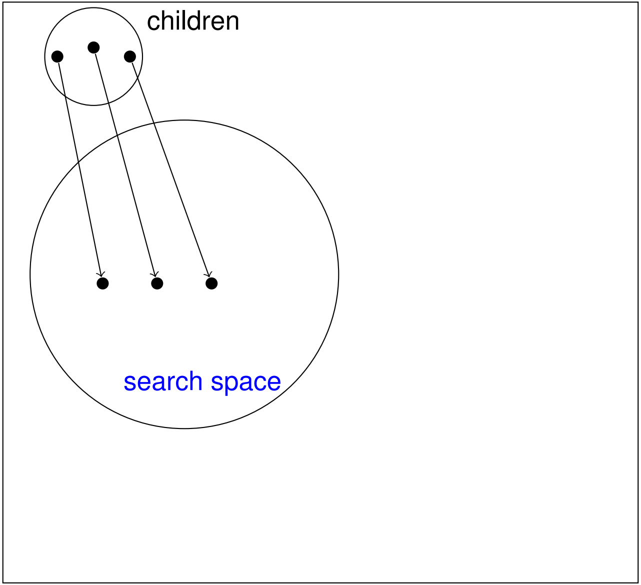

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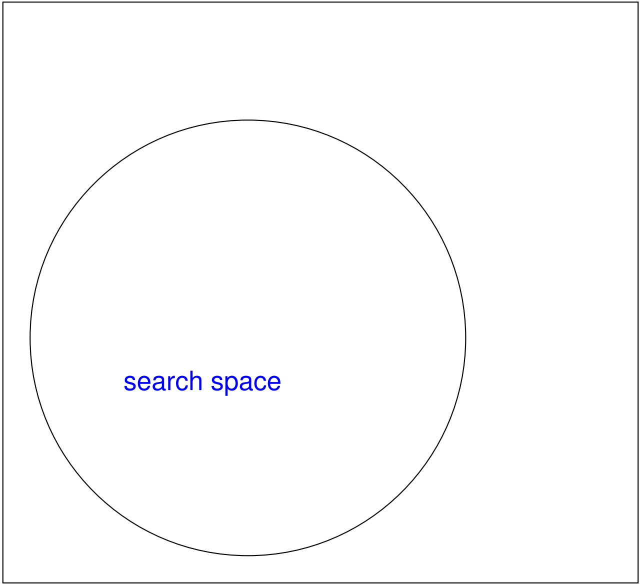

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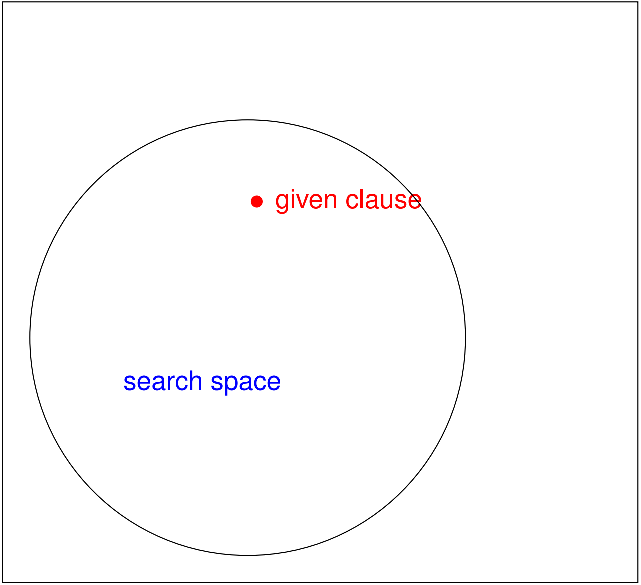

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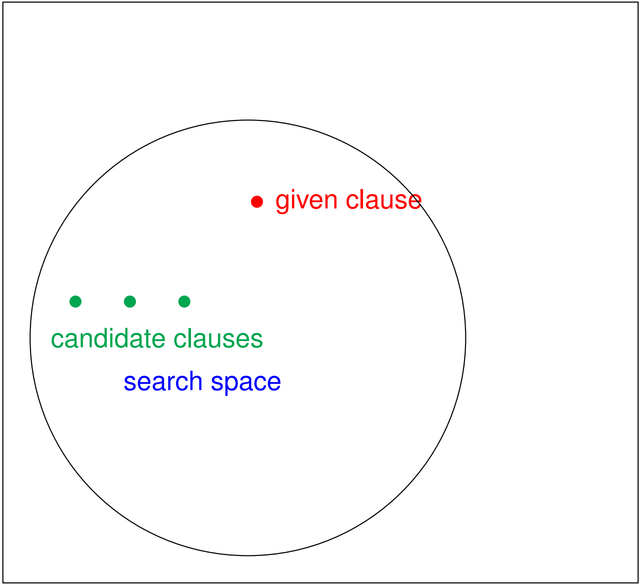

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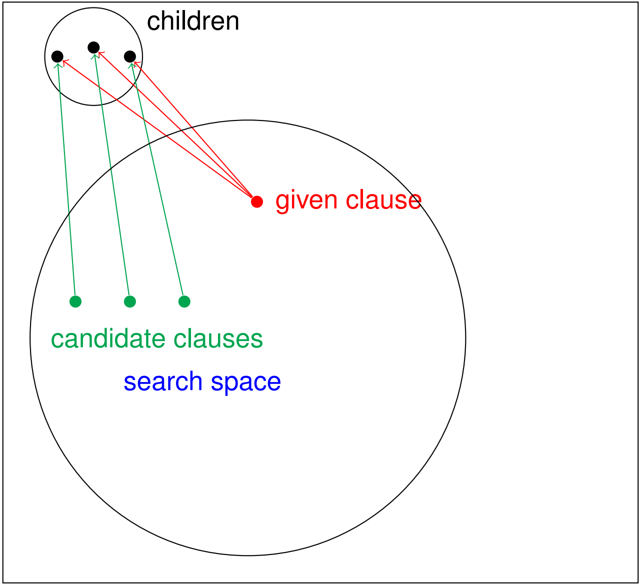

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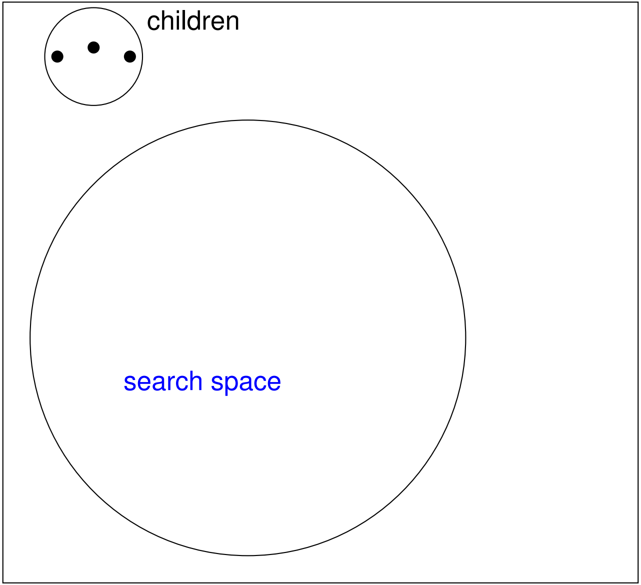

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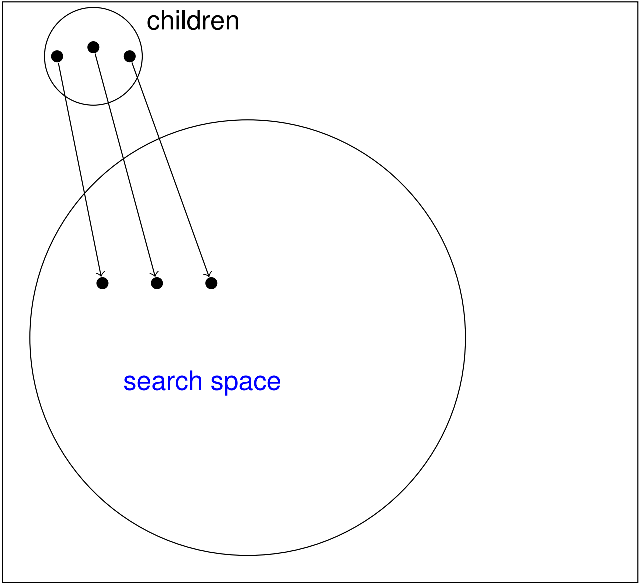

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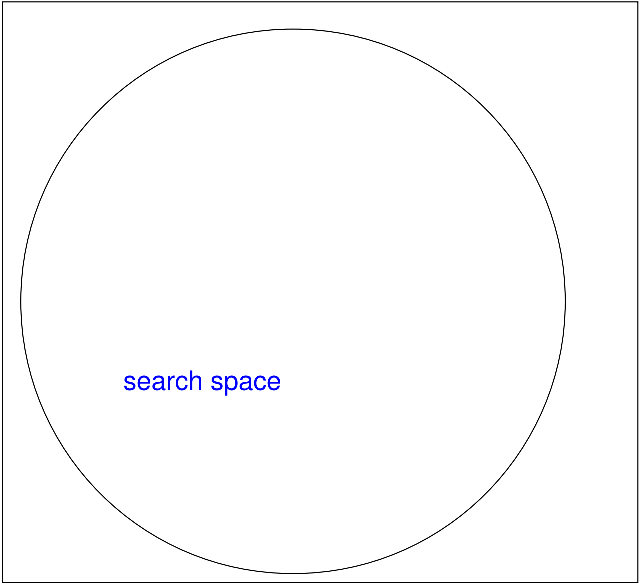

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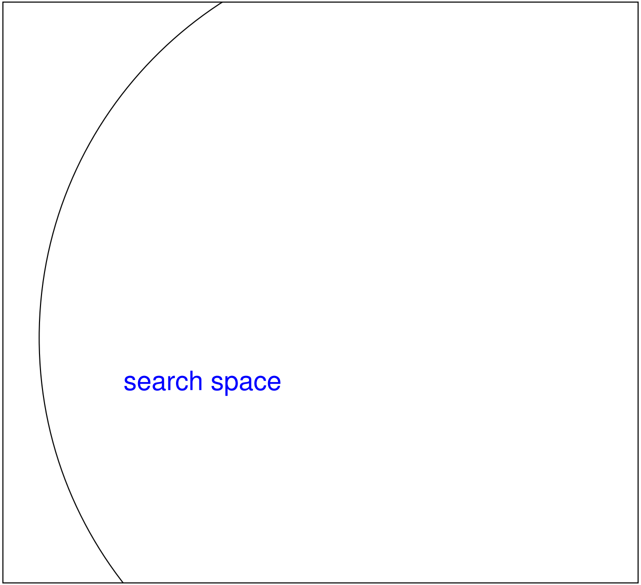

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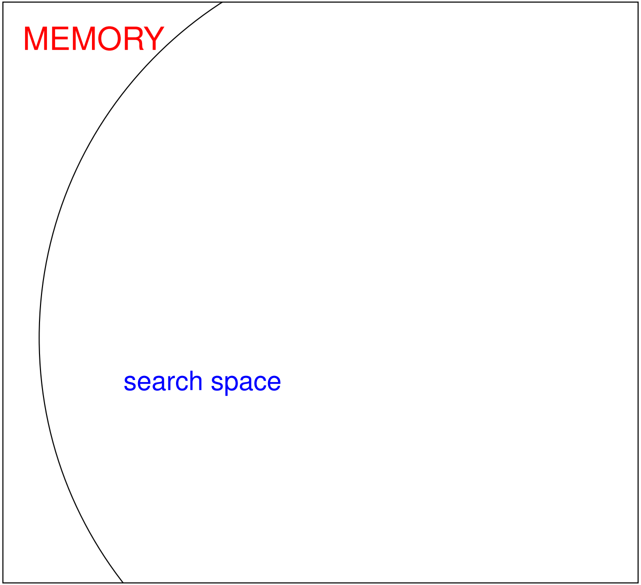

Saturation Algorithm

A saturation algorithm tries to saturate a set of clauses with respect to a given inference system.

In theory there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run forever, but without generating $\square$. In this case the input set of clauses is satisfiable.

Saturation Algorithm in Practice

In practice there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run until we run out of resources, but without generating $\square$. In this case it is unknown whether the input set is unsatisfiable.

Saturation Algorithm

Even when we implement inference selection by clause selection, there are too many inferences, especially when the search space grows.

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Solution: only apply inferences to the selected clause and the previously selected clauses.

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Thus, the search space is divided in two parts:

- active clauses, that participate in inferences;
- passive clauses, that do not participate in inferences.

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Thus, the search space is divided in two parts:

- active clauses, that participate in inferences;
- passive clauses, that do not participate in inferences.

Observation: the set of passive clauses is usually considerably larger than the set of active clauses, often by 2-4 orders of magnitude (depending on the saturation algorithm and the problem).

Outline

Subsumption and Tautology Deletion

A clause is a propositional tautology if it is of the form $A \lor \neg A \lor C$, that is, it contains a pair of complementary literals.
There are also equational tautologies, for example
$a \not\simeq b \lor b \not\simeq c \lor f(c, c) \simeq f(a, a)$.

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A clause C subsumes any clause $C \vee D$, where D is non-empty.

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A clause C subsumes any clause $C \lor D$, where D is non-empty.

It was known since 1965 that subsumed clauses and propositional tautologies can be removed from the search space.

Problem

How can we prove that completeness is preserved if we remove subsumed clauses and tautologies from the search space?

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Solution: general theory of redundancy.

Bag Extension of an Ordering

Bag = finite multiset.
Let $>$ be any ordering on a set X. The bag extension of $>$ is a binary relation $>^{bag}$, on bags over X, defined as the smallest transitive relation on bags such that

$$\{x, y_1, \ldots, y_n\} >^{bag} \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$
$$\text{if } x > x_i \text{ for all } i \in \{1 \ldots m\},$$

where $m \geq 0$.

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$$\text{if } x > x_i \text{ for all } i \in \{1 \ldots m\},$$

where $m \geq 0$.
Idea: a bag becomes smaller if we replace an element by any finite number of smaller elements.
The following results are known about the bag extensions of orderings:

1. $>^{bag}$ is an ordering;
2. If $>$ is total, then so is $>^{bag}$;
3. If $>$ is well-founded, then so is $>^{bag}$.

Clause Orderings

From now on consider clauses also as bags of literals. Note:

- we have an ordering $\succ$ for comparing literals;
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Hence

- we can compare clauses using the bag extension $\succ^{bag}$ of $\succ$.

For simpicity we denote the multiset ordering also by $\succ$.

Redundancy

A clause $C \in S$ is called redundant in S if it is a logical consequence of clauses in S strictly smaller than C.

Examples

A tautology $A \vee \neg A \vee C$ is a logical consequence of the empty set of formulas:

$$\models A \vee \neg A \vee C,$$

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We know that C subsumes $C \vee D$. Note

$$C \vee D \succ C$$
$$C \models C \vee D$$

therefore subsumed clauses are redundant.

If $\Box \in S$, then all non-empty other clauses in S are redundant.

Redundant Clauses Can be Removed

In $\mathbb{BR}_\sigma$ (and in all calculi we will consider later) redundant clauses can be removed from the search space.

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In $\mathbb{BR}_\sigma$ (and in all calculi we will consider later) redundant clauses can be removed from the search space.

Inference Process with Redundancy

Let $\mathbb{I}$ be an inference system. Consider an inference process with two kinds of step $S_i \Rightarrow S_{i+1}$:

1. Adding the conclusion of an $\mathbb{I}$-inference with premises in S_i.
2. Deletion of a clause redundant in S_i, that is

$$S_{i+1} = S_i - \{C\},$$

   where C is redundant in S_i.

Fairness: Persistent Clauses and Limit

Consider an inference process

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

A clause C is called persistent if

$$\exists i \forall j \geq i (C \in S_j).$$

The limit S_ω of the inference process is the set of all persistent clauses:

$$S_\omega = \bigcup_{i=0,1,\ldots} \bigcap_{j \geq i} S_j.$$

Fairness

The process is called $\mathbb{I}$-fair if every inference with persistent premises in S_ω has been applied, that is, if

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

is an inference in $\mathbb{I}$ and $\{C_1, \ldots, C_n\} \subseteq S_\omega$, then $C \in S_i$ for some i.

Completeness of $\mathbb{BR}_{\succ,\sigma}$

Completeness Theorem. Let $\succ$ be a simplification ordering and σ a well-behaved selection function. Let also

1. S_0 be a set of clauses;
2. $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be a fair $\mathbb{BR}_{\succ,\sigma}$-inference process.

Then S_0 is unsatisfiable if and only if $\square \in S_i$ for some i.

Saturation up to Redundancy

A set S of clauses is called saturated up to redundancy if for every $\mathbb{I}$-inference

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

with premises in S, either

1. $C \in S$; or
2. C is redundant w.r.t. S, that is, $S_{\prec C} \models C$.

End of Lecture 2

Slides for lecture 2 ended here . . .

Proof of Completeness

A trace of a clause C: a set of clauses $\{C_1, \ldots, C_n\} \subseteq S_\omega$ such that

1. $C \succ C_i$ for all $i = 1, \ldots, n$;
2. $C_1, \ldots, C_n \models C$.

Lemma 1. Every removed clause has a trace.
Lemma 2. The limit S_ω is saturated up to redundancy.
Lemma 3. The limit S_ω is logically equivalent to the initial set S_0.
Lemma 4. A set S of clauses saturated up to redundancy in $\mathbb{BR}_{\succ,\sigma}$ is unsatisfiable if and only if $\Box \in S$.

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Lemma 4. A set S of clauses saturated up to redundancy in $\mathbb{BR}_{\succ,\sigma}$ is unsatisfiable if and only if $\square \in S$.

Interestingly, only the last lemma uses rules of $\mathbb{BR}_{\succ,\sigma}$.

Binary Resolution with Selection

One of the key properties to satisfy this lemma is the following: the conclusion of every rule is strictly smaller that the rightmost premise of this rule.

- Binary resolution,

$$\frac{\underline{p} \vee C_1 \quad \underline{\neg p} \vee C_2}{C_1 \vee C_2} \text{ (BR)}.$$

- Positive factoring,

$$\frac{\underline{p} \vee \underline{p} \vee C}{p \vee C} \text{ (Fact)}.$$

Saturation up to Redundancy and Satisfiability Checking

Lemma 4. A set S of clauses saturated up to redundancy in $\mathbb{BR}_{\succ,\sigma}$ is unsatisfiable if and only if $\square \in S$.

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Therefore, if we built a set saturated up to redundancy, then the initial set S_0 is satisfiable. This is a powerful way of checking redundancy: one can even check satisfiability of formulas having only infinite models.

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Lemma 4. A set S of clauses saturated up to redundancy in $\mathbb{BR}_{\succ,\sigma}$ is unsatisfiable if and only if $\square \in S$.

Therefore, if we built a set saturated up to redundancy, then the initial set S_0 is satisfiable. This is a powerful way of checking redundancy: one can even check satisfiability of formulas having only infinite models.

The only problem with this characterisation is that there is no obvious way to build a model of S_0 out of a saturated set.

Outline

First-order logic with equality

- Equality predicate: $=$.
- Equality: $l = r$.

The order of literals in equalities does not matter, that is, we consider an equality $l = r$ as a multiset consisting of two terms l, r, and so consider $l = r$ and $r = l$ equal.

Equality. An Axiomatisation

- reflexivity axiom: $x = x$;
- symmetry axiom: $x = y \to y = x$;
- transitivity axiom: $x = y \wedge y = z \to x = z$;
- function substitution axioms:
  $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \to f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$, for every function symbol f;
- predicate substitution axioms:
  $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge P(x_1, \ldots, x_n) \to P(y_1, \ldots, y_n)$ for every predicate symbol P.

Inference systems for logic with equality

We will define a resolution and superposition inference system. This system is complete. One can eliminate redundancy (but the literal ordering needs to satisfy additional properties).

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Moreover, we will first define it only for ground clauses. On the theoretical side,

- Completeness is first proved for ground clauses only.
- It is then “lifted” to arbitrary clauses using a technique called lifting.
- Moreover, this way some notions (ordering, selection function) can first be defined for ground clauses only and then it is relatively easy to see how to generalise them for non-ground clauses.

Simple Ground Superposition Inference System

Superposition: (right and left)

$$\frac{l = r \vee C \quad s[l] = t \vee D}{s[r] = t \vee C \vee D}\ \text{(Sup)}, \quad \frac{l = r \vee C \quad s[l] \not\approx t \vee D}{s[r] \not\approx t \vee C \vee D}\ \text{(Sup)},$$

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Example

$$f(a) = a \vee g(a) = a$$
$$f(f(a)) = a \vee g(g(a)) \not\approx a$$
$$f(f(a)) \not\approx a$$

Can this system be used for efficient theorem proving?

Not really. It has too many inferences. For example, from the clause $f(a) = a$ we can derive any clause of the form

$$f^m(a) = f^n(a)$$

where $m, n \geq 0$.

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Worst of all, the derived clauses can be much larger than the original clause $f(a) = a$.
The recipe is to use the previously introduced ingredients:

1. Ordering;
2. Literal selection;
3. Redundancy elimination.

Atom and literal orderings on equalities

Equality atom comparison treats an equality $s = t$ as the multiset $\dot{\{}s, t\dot{\}}$.

- $(s' = t') \succ_{lit} (s = t)$ if $\dot{\{}s', t'\dot{\}} \succ \dot{\{}s, t\dot{\}}$.
- $(s' \not\approx t') \succ_{lit} (s \not\approx t)$ if $\dot{\{}s', t'\dot{\}} \succ \dot{\{}s, t\dot{\}}$.

Finally, we assert that all non-equality literals be greater than all equality literals.

Ground Superposition Inference System $\mathbb{S}\mathrm{up}_{\succ,\sigma}$

Let σ be a literal selection function.
Superposition: (right and left)

$$\frac{\underline{l = r} \lor C \quad \underline{s[l] = t} \lor D}{s[r] = t \lor C \lor D} \text{ (Sup)}, \quad \frac{\underline{l = r} \lor C \quad \underline{s[l] \not\approx t} \lor D}{s[r] \not\approx t \lor C \lor D} \text{ (Sup)},$$

where (i) $l \succ r$, (ii) $s[l] \succ t$, (iii) $l = r$ is strictly greater than any literal in C, (iv) $s[l] = t$ is greater than or equal to any literal in D.

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where (i) $l \succ r$, (ii) $s[l] \succ t$, (iii) $l = r$ is strictly greater than any literal in C, (iv) $s[l] = t$ is greater than or equal to any literal in D.
Equality Resolution:

$$\frac{\underline{s \not\simeq s} \vee C}{C} \text{ (ER)},$$

Equality Factoring:

$$\frac{\underline{s = t} \vee s = t' \vee C}{s = t \vee t \not\simeq t' \vee C} \text{ (EF)},$$

where (i) $s \succ t \succeq t'$; (ii) $s = t$ is greater than or equal to any literal in C.

Extension to arbitrary (non-equality) literals

- Consider a two-sorted logic in which equality is the only predicate symbol.
- Interpret terms as terms of the first sort and non-equality atoms as terms of the second sort.
- Add a constant $\top$ of the second sort.
- Replace non-equality atoms $p(t_1, \ldots, t_n)$ by equalities of the second sort $p(t_1, \ldots, t_n) = \top$.

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For example, the clause

$$p(a, b) \vee \neg q(a) \vee a \neq b$$

becomes

$$p(a, b) = \top \vee q(a) \not\simeq \top \vee a \neq b.$$

Binary resolution inferences can be represented by inferences in the superposition system

We ignore selection functions.

$$\frac{A \vee C_1 \quad \neg A \vee C_2}{C_1 \vee C_2} \text{ (BR)}$$

$$\cfrac{\cfrac{A = \top \vee C_1 \quad A \not\simeq \top \vee C_2}{\top \not\simeq \top \vee C_1 \vee C_2} \text{ (Sup)}}{C_1 \vee C_2} \text{ (ER)}$$

Exercise

Positive factoring can also be represented by inferences in the superposition system.

Simplification Ordering

The only restriction we imposed on term orderings was well-foundedness and stability under substitutions. When we deal with equality, these two properties are insufficient. We need a third property, called monotonicity.

An ordering $\succ$ on terms is called a simplification ordering if

1. $\succ$ is well-founded;
2. $\succ$ is monotonic: if $l \succ r$, then $s[l] \succ s[r]$;
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One can combine the last two properties into one:

2a. If $l \succ r$, then $s[l\theta] \succ s[r\theta]$.

End of Lecture 3

Slides for lecture 3 ended here . . .