

Fundamentals of Quaternionic Clifford Analysis III: Fischer Decomposition in Symplectic Harmonic Analysis

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Abstract

In the framework of quaternionic Clifford analysis in Euclidean space \mathbb{R}^{4p} , which constitutes a refinement of Euclidean and Hermitian Clifford analysis, the Fischer decomposition of the space of complex valued polynomials is obtained in terms of spaces of so-called (adjoint) symplectic spherical harmonics, which are irreducible modules for the symplectic group $\mathrm{Sp}(p)$. Its Howe dual partner is determined to be $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{so}(4, \mathbb{C})$.

1 Introduction

In 1917 Ernst Fischer proved (see [15]) that, given a homogeneous polynomial $q(X)$, $X \in \mathbb{R}^m$, every homogeneous polynomial $P_k(X)$ of degree k can be uniquely decomposed as

$$P_k(X) = Q_k(X) + q(X)R(X)$$

where $Q_k(X)$ is a homogeneous polynomial of degree k satisfying the partial differential equation

$$q(D)Q_k = 0$$

D being the differential operator corresponding to X through Fourier identification ($X_j \leftrightarrow \partial_{x_j}$, $j = 1, \dots, m$) and $R(X)$ is a homogeneous polynomial of suitable degree. If, in particular, $q(X) = |X|^2 = \sum_{j=1}^m X_j^2 = r^2$, then $q(D) = \sum_{j=1}^m \partial_{X_j}^2 = \Delta_m$, the Laplace operator in \mathbb{R}^m , and Q_k is harmonic, leading to the well-known decomposition

$$\mathcal{P}(\mathbb{R}^m; \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{p=0}^{\infty} r^{2p} \mathcal{H}_k(\mathbb{R}^m; \mathbb{C}) \quad (1)$$

of the space $\mathcal{P}(\mathbb{R}^m; \mathbb{C})$ of complex valued polynomials, in terms of the spaces $\mathcal{H}_k(\mathbb{R}^m; \mathbb{C})$ of complex valued harmonic homogeneous polynomials of degree k . This space $\mathcal{P}(\mathbb{R}^m; \mathbb{C})$ is a module over the special orthogonal group $\mathrm{SO}(m)$, its action being the regular representation

$$[g \cdot P](X) = P(g^{-1} \cdot X), \quad g \in \mathrm{SO}(m), \quad P \in \mathcal{P}(\mathbb{R}^m; \mathbb{C}), \quad X \in \mathbb{R}^m \quad (2)$$

Each of the constituents of the decomposition (1)

$$r^{2p} \mathcal{H}_k, \quad p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad k \in \mathbb{N}_0$$

is a subspace of $\mathcal{P}(\mathbb{R}^m; \mathbb{C})$ which is invariant under the $\mathrm{SO}(m)$ -action and all $\mathrm{SO}(m)$ -modules $\mathcal{H}_k(\mathbb{R}^m; \mathbb{C})$ are irreducible and mutually inequivalent. In particular, the space $\mathcal{P}_k(\mathbb{R}^m; \mathbb{C})$ of homogeneous polynomials of degree k , decomposes into $\mathrm{SO}(m)$ -irreducibles as

$$\mathcal{P}_k(\mathbb{R}^m; \mathbb{C}) = \bigoplus_{p=0}^{\lfloor \frac{k}{2} \rfloor} r^{2p} \mathcal{H}_{k-2p}(\mathbb{R}^m; \mathbb{C}) \quad (3)$$

The Fischer decomposition (1) may be rewritten in the triangular diagram

$$\begin{array}{ccccccc} \mathcal{H}_0 & & r^2 \mathcal{H}_0 & & r^4 \mathcal{H}_0 & & \cdots \\ & \mathcal{H}_1 & & r^2 \mathcal{H}_1 & & \cdots & \\ & & \mathcal{H}_2 & & r^2 \mathcal{H}_2 & & \cdots \\ & & & \mathcal{H}_3 & & \cdots & \\ & & & & \mathcal{H}_4 & & \cdots \end{array} \quad (4)$$

the vertical columns then reflecting the decomposition (3) of the spaces $\mathcal{P}_k(\mathbb{R}^m; \mathbb{C})$, $k = 0, 1, 2, \dots$

It is clear that in the Fischer decompositions (1) and (3) the operators $X := \frac{1}{2} r^2$ and $Y := -\frac{1}{2} \Delta_m$ play a key role. Note that they correspond to each other under natural or Fourier duality, also known as Fischer duality. They both commute with the action (2) of $\mathrm{SO}(m)$ on functions and on polynomials in particular, and their mutual commutator is

$$[X, Y] = \left[\frac{1}{2} r^2, -\frac{1}{2} \Delta_m \right] = \mathbb{E} + \frac{m}{2}$$

where $\mathbb{E} = r \partial_r = \sum_{j=1}^m X_j \partial_{X_j}$ is the Euler operator in \mathbb{R}^m . We then put

$$H := \mathbb{E} + \frac{m}{2}$$

and find that $[H, X] = 2X$ and $[H, Y] = -2Y$. This means that $\{H, X, Y\}$ generates a three-dimensional Lie algebra isomorphic with the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The action of $\mathfrak{sl}(2, \mathbb{C})$ on the decompositions (1) and (3) is:

$$\begin{aligned} X : r^{2p} \mathcal{H}_k &\longrightarrow r^{2p+2} \mathcal{H}_k \\ Y : r^{2p} \mathcal{H}_k &\longrightarrow r^{2p-2} \mathcal{H}_k \\ H : r^{2p} \mathcal{H}_k &\longrightarrow r^{2p} \mathcal{H}_k \end{aligned} \quad (5)$$

Taking the dimension m to be even: $m = 2n$, the standard complex structure \mathbb{I}_{2n} on \mathbb{R}^{2n} is introduced as follows. Let E_n denote the identity matrix in $M_n(\mathbb{C})$, the space of square $n \times n$ matrices with complex entries. Let

$$\varphi_n : M_n(\mathbb{C}) \longrightarrow M_{2n}(\mathbb{R})$$

stand for the injective homomorphism embedding $M_n(\mathbb{C})$ into the space $M_{2n}(\mathbb{R})$ of square $2n \times 2n$ real matrices. This embedding may be realized by substituting for each complex entry $a + bi$, the 2×2 real matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. In \mathbb{C}^n multiplication by the imaginary unit i is the \mathbb{C} -linear transformation associated to the matrix iE_n . The standard complex structure \mathbb{I}_{2n} then is the complex linear real matrix

$$\mathbb{I}_{2n} = \varphi_n(iE_n) = \mathrm{diag} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

As expected, there holds $\mathbb{I}_{2n}^2 = -E_{2n}$, E_{2n} being the identity matrix in $M_{2n}(\mathbb{R})$. Moreover \mathbb{I}_{2n} belongs to $\mathrm{SO}(2n)$, and a matrix $B \in M_{2n}(\mathbb{R})$ is complex linear, i.e. belongs to $\varphi_n(M_n(\mathbb{C}))$, if and only if B commutes with the complex structure \mathbb{I}_{2n} on \mathbb{R}^{2n} . We then have the following result (see also [5]).

Proposition 1. *The $\mathrm{SO}(2n)$ -matrices commuting with the complex structure \mathbb{I}_{2n} on \mathbb{R}^{2n} form a subgroup of $\mathrm{SO}(2n)$, denoted by $\mathrm{SO}_{\mathbb{I}}(2n)$, which is isomorphic with the unitary group $\mathrm{U}(n)$.*

The introduction of the complex structure \mathbb{I}_{2n} allows for considering the space $\mathcal{P}(\mathbb{R}^{2n}; \mathbb{C})$ of complex valued polynomials defined on Euclidean space of *even* dimension, as an $\mathrm{SO}_{\mathbb{I}}(2n) \cong \mathrm{U}(n)$ -module, the action of $\mathrm{SO}_{\mathbb{I}}(2n)$ being

$$[u \cdot P](X) = P(u^{-1} \cdot X), \quad u \in \mathrm{SO}_{\mathbb{I}}(2n), \quad P \in \mathcal{P}(\mathbb{R}^{2n}; \mathbb{C}), \quad X \in \mathbb{R}^{2n}$$

Since each complex valued polynomial in the real variables $(X_1, \dots, X_{2n}) = (x_1, \dots, x_n, y_1, \dots, y_n)$ may be written as a polynomial in the complex variables $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$, with $z_j = x_j + i y_j$, $\bar{z}_j = x_j - i y_j$, $j = 1, \dots, n$, i.e.

$$P(X) = P(x_1, \dots, x_n, y_1, \dots, y_n) = \tilde{P}(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$$

we have to determine the polynomials \tilde{P} which are invariant under the action of $\mathrm{SO}_{\mathbb{I}}(2n) \cong \mathrm{U}(n)$. As is well-known, the space of $\mathrm{U}(n)$ -invariant polynomials in $\mathcal{P}(\mathbb{R}^{2n}; \mathbb{C})$ is the space with basis

$$(1, r^2, r^4, \dots, r^{2p}, \dots)$$

where r^2 can be written as:

$$r^2 = |X|^2 = \sum_{j=1}^{2n} X_j^2 = \sum_{j=1}^n x_j^2 + y_j^2 = \sum_{j=1}^n z_j \bar{z}_j = \sum_{j=1}^n |z_j|^2$$

The differential operator corresponding, under Fourier duality, to the generator r^2 is the Laplace operator

$$\Delta_{2n} = \sum_{j=1}^n \partial_{x_j x_j}^2 + \partial_{y_j y_j}^2 = 4 \sum_{j=1}^n \partial_{z_j} \partial_{\bar{z}_j}$$

whence we are led to consider the space of harmonic polynomials in $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$. Its subspace $\mathcal{H}_k(\mathbb{R}^{2n}; \mathbb{C})$ of complex valued k -homogeneous harmonic polynomials may be decomposed as

$$\mathcal{H}_k(\mathbb{R}^{2n}; \mathbb{C}) = \bigoplus_{a+b=k} \mathcal{H}_{a,b}(\mathbb{R}^{2n}; \mathbb{C})$$

where $\mathcal{H}_{a,b}(\mathbb{R}^{2n}; \mathbb{C})$ is the space of the complex valued harmonic polynomials which are a -homogeneous in the variables z_j and at the same time b -homogeneous in the variables \bar{z}_j , i.e.

$$H_{a,b}(\lambda z_1, \dots, \lambda z_n, \mu \bar{z}_1, \dots, \mu \bar{z}_n) = \lambda^a \mu^b H_{a,b}(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$$

This leads to the Fischer decomposition

$$\mathcal{P}(\mathbb{R}^{2n}; \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{p=0}^{\infty} \bigoplus_{a=0}^k r^{2p} \mathcal{H}_{a,k-a}(\mathbb{R}^{2n}; \mathbb{C}) \quad (6)$$

where the constituents

$$r^{2p} \mathcal{H}_{a,k-a}, \quad p \in \mathbb{N}_0, \quad k \in \mathbb{N}_0, \quad a = 0, \dots, k$$

are irreducible invariant subspaces under the action of $\mathrm{U}(n)$. In particular, the space $\mathcal{P}_k(\mathbb{R}^{2n}; \mathbb{C})$ of k -homogeneous polynomials decomposes as

$$\mathcal{P}_k(\mathbb{R}^{2n}; \mathbb{C}) = \bigoplus_{p=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{a=0}^{k-2p} r^{2p} \mathcal{H}_{a,k-2p-a}(\mathbb{R}^{2n}; \mathbb{C}) \quad (7)$$

The corresponding diagram, similar to (4), looks like

$$\begin{array}{ccccccc}
\mathcal{H}_{0,0} & & r^2 \mathcal{H}_{0,0} & & r^4 \mathcal{H}_{0,0} & & \cdots \\
& & \mathcal{H}_{1,0} & & r^2 \mathcal{H}_{1,0} & & r^4 \mathcal{H}_{1,0} & \cdots \\
& & \mathcal{H}_{0,1} & & r^2 \mathcal{H}_{0,1} & & r^4 \mathcal{H}_{0,1} & \cdots \\
& & & & \mathcal{H}_{2,0} & & r^2 \mathcal{H}_{2,0} & \cdots \\
& & & & \mathcal{H}_{1,1} & & r^2 \mathcal{H}_{1,1} & \cdots \\
& & & & \mathcal{H}_{0,2} & & r^2 \mathcal{H}_{0,2} & \cdots \\
& & & & & & \mathcal{H}_{3,0} & & r^2 \mathcal{H}_{3,0} & \cdots \\
& & & & & & \mathcal{H}_{2,1} & & r^2 \mathcal{H}_{2,1} & \cdots \\
& & & & & & \mathcal{H}_{1,2} & & r^2 \mathcal{H}_{1,2} & \cdots \\
& & & & & & \mathcal{H}_{0,3} & & r^2 \mathcal{H}_{0,3} & \cdots \\
& & & & & & & & \mathcal{H}_{4,0} & \cdots \\
& & & & & & & & \mathcal{H}_{3,1} & \cdots \\
& & & & & & & & \mathcal{H}_{2,2} & \cdots \\
& & & & & & & & \mathcal{H}_{1,3} & \cdots \\
& & & & & & & & \mathcal{H}_{0,4} & \cdots \\
& & & & & & & & & \mathcal{H}_{5,0} & \cdots \\
& & & & & & & & & \mathcal{H}_{4,1} & \cdots \\
& & & & & & & & & \mathcal{H}_{3,2} & \cdots \\
& & & & & & & & & \mathcal{H}_{2,3} & \cdots \\
& & & & & & & & & \mathcal{H}_{1,4} & \cdots \\
& & & & & & & & & \mathcal{H}_{0,5} & \cdots
\end{array} \tag{8}$$

The smallest Lie algebra of complex polynomial differential operators generated by the polynomial r^2 and its dual operator Δ_{2n} again is $\mathfrak{sl}(2, \mathbb{C})$, since

$$[X, Y] = \left[\frac{1}{2} r^2, -\frac{1}{2} \Delta_{2n} \right] = \mathbb{E} + n = H$$

However, there is an additional natural invariant differential operator coming into play. Indeed, the Euler operator \mathbb{E} decomposes as

$$\mathbb{E} = \mathbb{E}_z + \mathbb{E}_z^\dagger$$

with

$$\mathbb{E}_z = \sum_{j=1}^n z_j \partial_{z_j} \quad \text{and} \quad \mathbb{E}_z^\dagger = \sum_{j=1}^n \bar{z}_j \partial_{\bar{z}_j}$$

Both these Euler operators in the complex variables are $U(n)$ -invariant, and so is their difference, up to a chosen constant,

$$\mathbb{E}_z^\dagger - \mathbb{E}_z + n$$

which commutes with $X = \frac{1}{2} r^2$, $Y = -\frac{1}{2} \Delta_{2n}$ and $H = \mathbb{E}_z + \mathbb{E}_z^\dagger + n$, since

$$[r^2, \mathbb{E}_z] = -r^2, \quad [r^2, \mathbb{E}_z^\dagger] = -r^2$$

and

$$[\Delta_{2n}, \mathbb{E}_z] = \Delta_{2n}, \quad [\Delta_{2n}, \mathbb{E}_z^\dagger] = \Delta_{2n}$$

In this way we end up with a reductive Lie algebra which is the direct sum of the three-dimensional Lie algebra generated by $\{H, X, Y\}$, isomorphic with $\mathfrak{sl}(2, \mathbb{C})$, and the one-dimensional abelian Lie algebra \mathbb{C} generated by $\{\mathbb{E}_z^\dagger - \mathbb{E}_z + n\}$. This is nothing else but the four dimensional general linear algebra $\mathfrak{gl}(2, \mathbb{C})$ with action

$$\begin{aligned} X &: r^{2p} \mathcal{H}_{a,b} &\longrightarrow & r^{2p+2} \mathcal{H}_{a,b} \\ Y &: r^{2p} \mathcal{H}_{a,b} &\longrightarrow & r^{2p-2} \mathcal{H}_{a,b} \\ H &: r^{2p} \mathcal{H}_{a,b} &\longrightarrow & r^{2p} \mathcal{H}_{a,b} \\ \mathbb{E}_z^\dagger - \mathbb{E}_z + n &: r^{2p} \mathcal{H}_{a,b} &\longrightarrow & r^{2p} \mathcal{H}_{a,b} \end{aligned} \tag{9}$$

since also

$$\begin{aligned} \mathbb{E}_z &: r^{2p} \mathcal{H}_{a,b} &\longrightarrow & r^{2p} \mathcal{H}_{a,b} \\ \mathbb{E}_z^\dagger &: r^{2p} \mathcal{H}_{a,b} &\longrightarrow & r^{2p} \mathcal{H}_{a,b} \end{aligned}$$

When comparing the Fischer decompositions (1) and (6), it becomes clear that refining the symmetry group from $\mathrm{SO}(2n)$ to its subgroup $\mathrm{SO}_{\mathbb{I}}(2n) \cong \mathrm{U}(n)$, results into the splitting of the space $\mathcal{H}_k(\mathbb{R}^{2n}; \mathbb{C})$ of homogeneous harmonic polynomials, now considered as functions in the complex variables $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$, according to the bidegrees of homogeneity:

$$\mathcal{H}_k(\mathbb{R}^{2n}; \mathbb{C}) = \bigoplus_{a+b=k} \mathcal{H}_{a,b}(\mathbb{R}^{2n}; \mathbb{C})$$

In [5] we have established in detail the fundamentals of a function theory called *quaternionic Clifford analysis* (see also [1, 2, 10, 11, 13, 21]), which is a refinement of Hermitian Clifford analysis (see e.g. [3, 4, 7, 14, 22, 23]), in its turn a refinement of Euclidean Clifford analysis. Clifford analysis (see e.g. [9, 12, 17, 18, 19]) is, in its most basic form, a generalization to higher dimension of holomorphic function theory in the complex plane. The fundamental group of Euclidean Clifford analysis in \mathbb{R}^m is the $\mathrm{Spin}(m)$ group, which doubly covers the $\mathrm{SO}(m)$ group. The fundamental group of Hermitian Clifford analysis in \mathbb{R}^{2n} is the $\mathrm{U}(n)$ group. The corresponding Fischer decompositions in terms of monogenic or Hermitian monogenic homogeneous polynomials respectively, are refinements of the Fischer decompositions (1) and (6) (see also [8]). As shown in [5], the fundamental group underlying quaternionic Clifford analysis in \mathbb{R}^{4p} (where the dimension now is a fourfold: $m = 2n = 4p$), is the symplectic group $\mathrm{Sp}(p)$. In order to obtain the corresponding Fischer decomposition it is crucial to know how to further decompose the space $\mathcal{H}_{a,b}(\mathbb{R}^{2n}; \mathbb{C})$ as a module for $\mathrm{Sp}(p)$. This is the problem we tackle in the present paper.

2 The symplectic Lie group and Lie algebra

The symplectic group $\mathrm{Sp}(p)$ is the real Lie group of square $p \times p$ matrices with quaternion entries, preserving the symplectic inner product

$$\langle \xi, \eta \rangle_{\mathbb{H}} = \xi_1 \bar{\eta}_1 + \xi_2 \bar{\eta}_2 + \dots + \xi_p \bar{\eta}_p \quad \xi, \eta \in \mathbb{H}^p$$

where $\bar{\cdot}$ stands for quaternionic conjugation. Equivalently, we can describe $\mathrm{Sp}(p)$ as

$$\mathrm{Sp}(p) = \{A \in \mathrm{GL}_p(\mathbb{H}) : AA^* = E_p\}$$

Square matrices in $M_p(\mathbb{H})$ may be embedded in $M_{2p}(\mathbb{C})$ by the injective homomorphism

$$\psi_p : M_p(\mathbb{H}) \longrightarrow M_{2p}(\mathbb{C})$$

where for each quaternion entry

$$z + wj = (x + yi) + (u + vi)j = x + yi + uj + vk$$

the 2×2 complex matrix $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ is substituted. In this way it turns out that $\psi_p(\mathrm{Sp}(p))$ is a subgroup of $\mathrm{SU}(2p)$.

At the level of Lie algebra we have the following picture. The real symplectic Lie algebra $\mathfrak{sp}(p)$ of skew-symplectic $M_p(\mathbb{H})$ matrices

$$\mathfrak{sp}(p) = \{A \in \mathrm{GL}_p(\mathbb{H}) : A + A^* = 0\}$$

is isomorphic with the subalgebra $\psi(\mathfrak{sp}(p))$ of the Lie algebra $\mathfrak{u}(2p)$ of skew-hermitian $M_{2p}(\mathbb{C})$ matrices. Moreover, for $A \in \mathfrak{sp}(p)$, the complex matrix $\psi(A)$ satisfies the relation

$$\psi(A)^T \mathbb{I}_{2p} + \mathbb{I}_{2p} \psi(A) = 0 \quad (10)$$

where \cdot^T stands for the transpose and \mathbb{I}_{2p} is the complex structure introduced in Section 1.

On the other hand, there is the complex symplectic Lie group $\mathrm{Sp}_{2p}(\mathbb{C})$ of complex linear matrices preserving the standard skew-hermitian form on \mathbb{C}^{2p} :

$$\mathrm{Sp}_{2p}(\mathbb{C}) = \{A \in \mathrm{GL}_{2p}(\mathbb{C}) : A^T \mathbb{I}_{2p} A = \mathbb{I}_{2p}\}$$

and its corresponding complex symplectic Lie algebra $\mathfrak{sp}_{2p}(\mathbb{C})$ given by

$$\mathfrak{sp}_{2p}(\mathbb{C}) = \{A \in \mathrm{GL}_{2p}(\mathbb{C}) : A^T \mathbb{I}_{2p} + \mathbb{I}_{2p} A = 0\}$$

This Lie algebra $\mathfrak{sp}_{2p}(\mathbb{C})$ is a subalgebra of $\mathfrak{sl}_{2p}(\mathbb{C})$; it can be decomposed into the direct sum of its Hermitian subspace and its skew-hermitian subalgebra, both spaces being isomorphic through multiplication by the imaginary unit i :

$$\mathfrak{sp}_{2p}(\mathbb{C}) = (\mathfrak{sp}_{2p}(\mathbb{C}) \cap \mathfrak{u}(2p)) \oplus i (\mathfrak{sp}_{2p}(\mathbb{C}) \cap \mathfrak{u}(2p))$$

In view of (10) this leads to the following result (see also [5]).

Proposition 2. *The real symplectic Lie algebra $\mathfrak{sp}(p)$ of skew-symplectic $M_p(\mathbb{H})$ -matrices is isomorphic with the compact form $\mathfrak{sp}_{2p}(\mathbb{C}) \cap \mathfrak{u}(2p)$ of the complex symplectic Lie algebra $\mathfrak{sp}_{2p}(\mathbb{C})$:*

$$\psi(\mathfrak{sp}(p)) = \mathfrak{sp}_{2p}(\mathbb{C}) \cap \mathfrak{u}(2p)$$

Henceforth we will use the Lie algebra $\mathfrak{sp}_{2p}(\mathbb{C})$.

Now, let us consider the space $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ of complex valued (a, b) -homogeneous harmonic polynomials in the variables $(z_1, z_2, \dots, z_{2p}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_{2p})$. Seen the surjectivity of the Laplace operator

$$\Delta_{4p} : \mathcal{P}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \longrightarrow \mathcal{P}_{a-1,b-1}(\mathbb{R}^{4p}; \mathbb{C})$$

we have

$$\begin{aligned} \dim \mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) &= \dim \mathcal{P}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) - \dim \mathcal{P}_{a-1,b-1}(\mathbb{R}^{4p}; \mathbb{C}) \\ &= \binom{2p+a-1}{a} \binom{2p+b-1}{b} - \binom{2p+a-2}{a-1} \binom{2p+b-2}{b-1} \quad (11) \\ &= \binom{2p+a-1}{2p-1} \binom{2p+b-1}{2p-1} - \binom{2p+a-2}{2p-1} \binom{2p+b-2}{2p-1} \end{aligned}$$

In order to decompose the space $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ into $\mathfrak{sp}_{2p}(\mathbb{C})$ -irreducibles, use could be made of existing branching rules when restricting $\mathfrak{gl}_{2p}(\mathbb{C})$ to $\mathfrak{sp}_{2p}(\mathbb{C})$. To that end we have to know the behaviour of $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ as a module for $\mathfrak{gl}_{2p}(\mathbb{C})$. Let us recall that in [14] the spaces $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ have been identified as irreducible modules for its simple Lie subalgebra $\mathfrak{sl}_{2p}(\mathbb{C})$, with highest weight vector

$$\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cong (a + b, b, \dots, b)$$

of length $2p - 1$. Interpreted as a representation space for $\mathfrak{gl}_{2p}(\mathbb{C})$ we have

$$\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cong (a, 0, \dots, 0, -b)$$

instead, where the highest weight vector now has length $2p$. In fact this is telling us that

$$\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cong (V^{\boxtimes a}) \boxtimes (\bar{V}^{\boxtimes b})$$

where \boxtimes stands for the Cartan product, with $V \cong \mathbb{C}^{2p}$ the fundamental representation and \bar{V} its dual. The branching rules when restricting $\mathfrak{gl}_{2p}(\mathbb{C})$ to $\mathfrak{sp}_{2p}(\mathbb{C})$ could be found in full generality in [20], the branching multiplicities being expressed in terms of Littlewood–Richardson coefficients. However, due to the rather simple highest weight to start with, the actual situation is not that complicated and one obtains, for $a > b$

$$\mathcal{H}_{a,b} \left| \begin{array}{l} \mathfrak{gl}_{2p}(\mathbb{C}) \\ \mathfrak{sp}_{2p}(\mathbb{C}) \end{array} \right. = (a, b)_s \oplus (a + 1, b - 1)_s \oplus \dots \oplus (a + b - 1, 1)_s \oplus (a + b)_s \quad (12)$$

where the shorthand notation $(\lambda)_s$ refers to an irreducible representation for $\mathfrak{sp}_{2p}(\mathbb{C})$, and stands for a *symplectic* highest weight $(\lambda, 0, \dots, 0)$ of length p . Also note that if a or b equals zero then no branching occurs, meaning that in that case $\mathcal{H}_{a,b}$ is symplectically irreducible.

In order to characterize the spaces of the form $(a, b)_s$ in terms of homogeneous polynomials on $\mathbb{R}^{4p} \cong \mathbb{C}^{2p}$, we will establish, in the next section, an alternative realization for $\mathfrak{sl}(2, \mathbb{C}) \cong \text{Alg}_{\mathbb{C}}(X, Y, H)$. To that end we introduce new differential operators appearing in quaternionic Clifford analysis.

3 Quaternionic Clifford analysis: the basics

As is well-known, when establishing Hermitian Clifford analysis (see e.g. [3]) use is made of the projection operators

$$\frac{1}{2} (1 \pm i \mathbb{I}_{2n})$$

where \mathbb{I}_{2n} is the standard complex structure on \mathbb{R}^{2n} (see Section 1), leading to the standard Hermitian vector variables

$$\underline{z} = \sum_{k=1}^n z_k \mathfrak{f}_k \quad \text{and} \quad \underline{z}^\dagger = \sum_{k=1}^n \bar{z}_k \mathfrak{f}_k^\dagger$$

and the Hermitian Dirac operators

$$\partial_{\underline{z}}^\dagger = \sum_{k=1}^n \partial_{\bar{z}_k} \mathfrak{f}_k \quad \text{and} \quad \partial_{\underline{z}} = \sum_{k=1}^n \partial_{z_k} \mathfrak{f}_k^\dagger$$

where the Witt basis vectors $(\mathfrak{f}_k, \mathfrak{f}_k^\dagger : k = 1, \dots, n)$ are given by

$$\mathfrak{f}_k = -\frac{1}{2}(\mathbf{1} - i\mathbb{I}_{2n})[e_{2k-1}] \quad \text{and} \quad \mathfrak{f}_k^\dagger = \frac{1}{2}(\mathbf{1} + i\mathbb{I}_{2n})[e_{2k-1}]$$

(e_1, \dots, e_{2n}) being an orthonormal basis in \mathbb{R}^{2n} .

Lemma 1. (see [6]) *The Hermitian variables and Dirac operators enjoy the anti-commutation relations*

$$\begin{aligned} \{\underline{z}, \underline{z}^\dagger\} &= |\underline{z}|^2 \\ \{\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger\} &= \frac{1}{4}\Delta_{2n} \\ \{\partial_{\underline{z}}, \underline{z}\} &= \mathbb{E}_z + \beta \\ \{\partial_{\underline{z}}^\dagger, \underline{z}^\dagger\} &= \mathbb{E}_z^\dagger + n - \beta \\ \{\partial_{\underline{z}}, \underline{z}^\dagger\} &= 0 = \{\partial_{\underline{z}}^\dagger, \underline{z}\} \end{aligned}$$

where β is the so-called spin-Euler operator given by $\beta = \sum_{k=1}^n \mathfrak{f}_k^\dagger \mathfrak{f}_k = n - \sum_{k=1}^n \mathfrak{f}_k \mathfrak{f}_k^\dagger$. They span the odd part of the Lie super algebra $\mathfrak{sl}(1|2) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{gl}(2, \mathbb{C}) = \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{C}) = \text{span}_{\mathbb{C}}(\mathbb{E}_z^\dagger - \mathbb{E}_z + n - 2\beta) \oplus \text{Alg}_{\mathbb{C}}(\mathbb{E}_z^\dagger + \mathbb{E}_z + n, \frac{1}{2}|\underline{z}|^2, -\frac{1}{2}\Delta_{2n}) \\ \mathfrak{g}_1 &= \text{span}_{\mathbb{C}}(\underline{z}, \underline{z}^\dagger, \partial_{\underline{z}}, \partial_{\underline{z}}^\dagger) \end{aligned}$$

The central notion in Hermitian Clifford analysis is that of a Hermitian monogenic function, the definition of which is as follows.

Definition 1. *A differentiable function F defined in a domain Ω of \mathbb{R}^{2n} and taking its values in the complex Clifford algebra \mathbb{C}_{2n} or in spinor space \mathbb{S} , is called Hermitian monogenic in Ω if it satisfies the system $\{\partial_{\underline{z}} F = 0, \partial_{\underline{z}}^\dagger F = 0\}$.*

Now, taking the dimension to be a fourfold: $m = 2n = 4p$, a quaternionic structure on \mathbb{R}^{4p} is established by introducing, next to the standard complex structure \mathbb{I}_{4p} , a second complex structure $\mathbb{J}_{4p} \in \text{SO}(4p)$ such that $\mathbb{J}_{4p}^2 = -E_{4p}$ and \mathbb{I}_{4p} and \mathbb{J}_{4p} are anti-commuting. This second complex structure \mathbb{J}_{4p} may be realized as

$$\mathbb{J}_{4p} = \text{diag} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

New twisted Hermitian variables and Dirac operators then are defined by

$$\begin{aligned} \underline{z}^J &= \mathbb{J}_{4p}[\underline{z}] = \sum_{k=1}^p \mathfrak{f}_{2k-1}^\dagger z_{2k} - \mathfrak{f}_{2k}^\dagger z_{2k-1} \\ \underline{z}^{\dagger J} &= \mathbb{J}_{4p}[\underline{z}^\dagger] = \sum_{k=1}^p \mathfrak{f}_{2k-1} \bar{z}_{2k} - \mathfrak{f}_{2k} \bar{z}_{2k-1} \\ \partial_{\underline{z}}^J &= \mathbb{J}_{4p}[\partial_{\underline{z}}] = \sum_{k=1}^p \mathfrak{f}_{2k-1} \partial_{z_{2k}} - \mathfrak{f}_{2k} \partial_{z_{2k-1}} \\ \partial_{\underline{z}}^{\dagger J} &= \mathbb{J}_{4p}[\partial_{\underline{z}}^\dagger] = \sum_{k=1}^p \mathfrak{f}_{2k-1}^\dagger \partial_{\bar{z}_{2k}} - \mathfrak{f}_{2k}^\dagger \partial_{\bar{z}_{2k-1}} \end{aligned}$$

Lemma 2. *The twisted Hermitian variables and Dirac operators enjoy the anti-commutation relations*

$$\begin{aligned}\{\underline{z}^J, \underline{z}^{\dagger J}\} &= |\underline{z}|^2 \\ \{\partial_{\underline{z}}^J, \partial_{\underline{z}}^{\dagger J}\} &= \frac{1}{4} \Delta_{4p} \\ \{\partial_{\underline{z}}^J, \underline{z}^J\} &= \mathbb{E}_z + 2p - \beta \\ \{\partial_{\underline{z}}^{\dagger J}, \underline{z}^{\dagger J}\} &= \mathbb{E}_z^\dagger + \beta \\ \{\partial_{\underline{z}}^J, \underline{z}^{\dagger J}\} &= 0 = \{\partial_{\underline{z}}^{\dagger J}, \underline{z}^J\}\end{aligned}$$

Remark 1. *Note the similarity of the anti-commutation relations of the twisted Hermitian variables and Dirac operators with those for the standard ones, which, quite naturally, follows from the fact that $\mathbb{J}_{4p} \in \text{SO}(4p)$.*

Remark 2. *While the operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ are invariant under $\text{U}(2p)$, the four Dirac operators $\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger, \partial_{\underline{z}}^J, \partial_{\underline{z}}^{\dagger J}$, taken together, are invariant under the action of the symplectic group $\text{Sp}(p)$.*

Definition 2. *A differentiable function $F : \mathbb{R}^{4p} \rightarrow \mathbb{S}$ is called quaternionic monogenic in the domain $\Omega \subset \mathbb{R}^{4p}$ if it is a simultaneous null-solution for the four operators $\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger, \partial_{\underline{z}}^J, \partial_{\underline{z}}^{\dagger J}$.*

New operators are now arising by considering the mixed anti-commutator relations of the standard and twisted Hermitian variables and Dirac operators. Indeed we can define

$$\begin{aligned}\mathcal{E} &:= \{\partial_{\underline{z}}^{\dagger J}, \underline{z}\} = \sum_{k=1}^p z_{2k-1} \partial_{\bar{z}_{2k}} - z_{2k} \partial_{\bar{z}_{2k-1}} \\ \mathcal{E}^\dagger &:= -\{\partial_{\underline{z}}^J, \underline{z}^\dagger\} = -\sum_{k=1}^p \bar{z}_{2k-1} \partial_{z_{2k}} - \bar{z}_{2k} \partial_{z_{2k-1}}\end{aligned}$$

and there also holds

$$\begin{aligned}\{\partial_{\underline{z}}^\dagger, \underline{z}^J\} &= \sum_{k=1}^p z_{2k} \partial_{\bar{z}_{2k-1}} - z_{2k-1} \partial_{\bar{z}_{2k}} = -\mathcal{E} \\ \{\partial_{\underline{z}}, \underline{z}^{\dagger J}\} &= \sum_{k=1}^p \bar{z}_{2k} \partial_{z_{2k-1}} - \bar{z}_{2k-1} \partial_{z_{2k}} = \mathcal{E}^\dagger\end{aligned}$$

These new operators enjoy the following properties.

Lemma 3. *The operators \mathcal{E} and \mathcal{E}^\dagger are invariant under the symplectic action.*

Lemma 4. *One has*

$$\mathfrak{sl}(2, \mathbb{C}) \cong \text{Alg}_{\mathbb{C}}(\mathbb{E}_z^\dagger - \mathbb{E}_z, \mathcal{E}^\dagger, \mathcal{E})$$

these three generating operators commuting with the harmonic triplet (H, X, Y) introduced in Section 1.

Proof

Direct computation shows that indeed:

- (i) $[\mathbb{E}_z^\dagger - \mathbb{E}_z, \mathcal{E}^\dagger] = 2\mathcal{E}^\dagger$
- (ii) $[\mathbb{E}_z^\dagger - \mathbb{E}_z, \mathcal{E}] = -2\mathcal{E}$

$$(iii) [\mathcal{E}^\dagger, \mathcal{E}] = \mathbb{E}_z^\dagger - \mathbb{E}_z$$

$$(iv) [\mathcal{E}^\dagger, \frac{1}{2}|\underline{z}|^2] = [\mathcal{E}, \frac{1}{2}|\underline{z}|^2] = 0$$

$$(v) [\mathcal{E}^\dagger, -\frac{1}{2}\Delta_{4p}] = [\mathcal{E}, -\frac{1}{2}\Delta_{4p}] = 0$$

As it was already shown that $\mathbb{E}_z^\dagger - \mathbb{E}_z$ commutes with $\frac{1}{2}|\underline{z}|^2$, $-\frac{1}{2}\Delta_{4p}$ and $\mathbb{E}_z + \mathbb{E}_z^\dagger + 2p$, the proof is complete. \square

Corollary 1. *One has*

$$\text{Alg}_{\mathbb{C}} \left(\mathbb{E}_z + \mathbb{E}_z^\dagger + 2p, \frac{1}{2}|\underline{z}|^2, -\frac{1}{2}\Delta_{4p} \right) \oplus \text{Alg}_{\mathbb{C}} (\mathbb{E}_z^\dagger - \mathbb{E}_z, \mathcal{E}^\dagger, \mathcal{E}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{so}(4, \mathbb{C})$$

4 Symplectic harmonics

If we consider the operators \mathcal{E} and \mathcal{E}^\dagger as acting between the spaces $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ of complex valued bi-homogeneous harmonic polynomials, we obtain

$$\mathcal{H}_{0,b+a} \xleftrightarrow{\mathcal{E}^\dagger} \cdots \xleftrightarrow{\mathcal{E}^\dagger} \mathcal{H}_{a-1,b+1} \xleftrightarrow{\mathcal{E}^\dagger} \mathcal{H}_{a,b} \xleftrightarrow{\mathcal{E}^\dagger} \mathcal{H}_{a+1,b-1} \xleftrightarrow{\mathcal{E}^\dagger} \cdots \xleftrightarrow{\mathcal{E}^\dagger} \mathcal{H}_{a+b,0}$$

and we define the kernel spaces

$$\mathcal{H}_{a,b}^S = \mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cap \text{Ker } \mathcal{E} = \mathcal{P}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cap \text{Ker}(\Delta_{4p}, \mathcal{E}) \quad (a \geq b)$$

and

$$\mathcal{H}_{a,b}^{S^\dagger} = \mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cap \text{Ker } \mathcal{E}^\dagger = \mathcal{P}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cap \text{Ker}(\Delta_{4p}, \mathcal{E}^\dagger) \quad (a \leq b)$$

These kernel spaces will show to be crucial in the decomposition of $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ in terms of $\text{Sp}(p)$ -irreducibles. We call their elements (*adjoint*) *symplectic harmonics*. It will be shown further on (see Corollary 4 and Proposition 4) that $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cap \text{Ker } \mathcal{E}^\dagger = \{0\}$ for $a > b$ and $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) \cap \text{Ker } \mathcal{E} = \{0\}$ for $a < b$.

Remark 3. *For all k , $\mathcal{P}_{k,0}(\mathbb{R}^{4p}; \mathbb{C}) = \mathcal{H}_{k,0} = \mathcal{H}_{k,0}^S$ and $\mathcal{P}_{0,k}(\mathbb{R}^{4p}; \mathbb{C}) = \mathcal{H}_{0,k} = \mathcal{H}_{0,k}^{S^\dagger}$, since the homogeneous polynomials in $\mathcal{H}_{k,0}$ (respectively $\mathcal{H}_{0,k}$) do not contain the variables $(\bar{z}_1, \dots, \bar{z}_{2p})$ (respectively (z_1, \dots, z_{2p})).*

With respect to the traditional Fischer inner product, given by

$$\langle f, g \rangle = f(\partial_{\underline{z}}^\dagger, \partial_{\underline{z}}) \bar{g} \Big|_{\underline{z}=0}$$

where $f(\partial_{\underline{z}}^\dagger, \partial_{\underline{z}})$ is obtained by substituting $\partial_{\bar{z}_j}$ for z_j and ∂_{z_j} for \bar{z}_j in $f(z_1, \dots, z_{2p}, \bar{z}_1, \dots, \bar{z}_{2p})$, each of the spaces $\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ can be decomposed as the direct sum

$$\begin{aligned} \mathcal{H}_{a,b} &= \mathcal{H}_{a,b}^S \oplus (\mathcal{H}_{a,b}^S)^\perp \quad a \geq b \\ \mathcal{H}_{a,b} &= \mathcal{H}_{a,b}^{S^\dagger} \oplus (\mathcal{H}_{a,b}^{S^\dagger})^\perp \quad a \leq b \end{aligned}$$

where the orthogonal complements $(\mathcal{H}_{a,b}^S)^\perp$ and $(\mathcal{H}_{a,b}^{S^\dagger})^\perp$ are isomorphic with $\text{Im}_{\mathcal{E}}(\mathcal{H}_{a,b})$ and $\text{Im}_{\mathcal{E}^\dagger}(\mathcal{H}_{a,b})$ respectively. We will now determine those orthogonal complements explicitly.

Lemma 5. *With respect to the Fischer inner product, the operators \mathcal{E} and \mathcal{E}^\dagger are adjoint operators, i.e. for polynomials $P \in \mathcal{P}_{a,b}$ and $Q \in \mathcal{P}_{a+1,b-1}$ there holds*

$$\langle \mathcal{E}P, Q \rangle = \langle P, \mathcal{E}^\dagger Q \rangle$$

Proof

It is clear that the Fischer inner product of two monomials is zero unless both monomials are equal up to a constant. This observation and a straightforward calculation lead to the desired result. \square

Proposition 3. *For $a \geq b$, the space $\mathcal{H}_{a,b}$ may be decomposed as*

$$\mathcal{H}_{a,b} = \mathcal{H}_{a,b}^S \oplus \mathcal{E}^\dagger \mathcal{H}_{a+1,b-1} \quad (13)$$

Proof

In fact we prove that, with respect to the Fischer inner product, $(\mathcal{E}^\dagger \mathcal{H}_{a+1,b-1})^\perp = \mathcal{H}_{a,b}^S$. It is important to note that if a function F is harmonic, then also $\mathcal{E}F$ and $\mathcal{E}^\dagger F$ are harmonic, since the Laplace operator commutes with both operators \mathcal{E} and \mathcal{E}^\dagger . Let $P \in \mathcal{H}_{a,b}^S$, then $\mathcal{E}P = 0$ and so $\langle P, \mathcal{E}^\dagger Q \rangle = 0$ for all $Q \in \mathcal{H}_{a+1,b-1}$, which means that P is orthogonal to $\mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}$ or $P \in (\mathcal{E}^\dagger \mathcal{H}_{a+1,b-1})^\perp$. Conversely, let $P \in (\mathcal{E}^\dagger \mathcal{H}_{a+1,b-1})^\perp$. Then $\mathcal{E}P \in \mathcal{H}_{a+1,b-1}$ and $\langle \mathcal{E}P, Q \rangle = \langle P, \mathcal{E}^\dagger Q \rangle = 0$ for all $Q \in \mathcal{H}_{a+1,b-1}$. In particular, for $Q = \mathcal{E}P$ we find $\langle \mathcal{E}P, \mathcal{E}P \rangle = 0$ whence $\mathcal{E}P = 0$ or $P \in \mathcal{H}_{a,b}^S$. \square

Corollary 2. *For $a \geq b$, the space $\mathcal{H}_{a,b}$ may be decomposed as*

$$\mathcal{H}_{a,b} = \mathcal{H}_{a,b}^S \oplus \mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}^S \oplus \mathcal{E}^{\dagger 2} \mathcal{H}_{a+2,b-2}^S \oplus \cdots \oplus \mathcal{E}^{\dagger b} \mathcal{H}_{a+b,0}^S \quad (14)$$

Proof

Consecutive application of the decomposition (13) leads to the desired result. \square

Lemma 6. *One has for $H_{\alpha,\beta} \in \mathcal{H}_{\alpha,\beta}$*

$$\mathcal{E} \mathcal{E}^{\dagger k} H_{\alpha,\beta} = k(\alpha - \beta - k + 1) \mathcal{E}^{\dagger(k-1)} H_{\alpha,\beta} + \mathcal{E}^{\dagger k} \mathcal{E} H_{\alpha,\beta}$$

and in particular for $H_{\alpha,\beta}^S \in \mathcal{H}_{\alpha,\beta}^S$

$$\mathcal{E} \mathcal{E}^{\dagger k} H_{\alpha,\beta}^S = k(\alpha - \beta - k + 1) \mathcal{E}^{\dagger(k-1)} H_{\alpha,\beta}^S$$

and

$$\mathcal{E}^\ell \mathcal{E}^{\dagger k} H_{\alpha,\beta}^S = k(k-1) \cdots (k-\ell+1)(\alpha - \beta - k + 1) \cdots (\alpha - \beta - k + \ell) \mathcal{E}^{\dagger(k-\ell)} H_{\alpha,\beta}^S$$

Proof

Straightforward computation based on the commutator $[\mathcal{E}, \mathcal{E}^\dagger] = \mathbb{E}_z - \mathbb{E}_z^\dagger$ (see Lemma 4). \square

Corollary 3. *For $a \geq b$, the mappings*

$$\begin{aligned} \mathcal{E} & : \mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}^S & \longrightarrow & \mathcal{H}_{a+1,b-1}^S \\ \mathcal{E} & : \mathcal{E}^{\dagger 2} \mathcal{H}_{a+2,b-2}^S & \longrightarrow & \mathcal{E}^\dagger \mathcal{H}_{a+2,b-2}^S \\ & \vdots & & \\ \mathcal{E} & : \mathcal{E}^{\dagger b} \mathcal{H}_{a+b,0}^S & \longrightarrow & \mathcal{E}^{\dagger(b-1)} \mathcal{H}_{a+b,0}^S \end{aligned}$$

are isomorphisms, their inverses being, up to constants, restrictions of the operator \mathcal{E}^\dagger to the corresponding spaces.

Proof

We prove that for $j = 1, \dots, b$

$$\mathcal{E} : \mathcal{E}^{\dagger j} \mathcal{H}_{a+j,b-j}^S \longrightarrow \mathcal{E}^{\dagger(j-1)} \mathcal{H}_{a+j,b-j}^S$$

is an isomorphism. First take $g \in \mathcal{E}^{\dagger(j-1)}\mathcal{H}_{a+j,b-j}^S$, meaning that $g = \mathcal{E}^{\dagger(j-1)}h$ with $h \in \mathcal{H}_{a+j,b-j}^S$, and consider

$$f = \frac{1}{j(a-b+j+1)} \mathcal{E}^{\dagger}g = \frac{1}{j(a-b+j+1)} \mathcal{E}^{\dagger j}h \in \mathcal{E}^{\dagger j}\mathcal{H}_{a+j,b-j}^S$$

Then, using the formulae of Lemma 6, it follows that $\mathcal{E}f = \mathcal{E}^{\dagger(j-1)}h = g$, and so the considered mapping is surjective. Moreover $(\mathcal{E}^{\dagger j}\mathcal{H}_{a+j,b-j}^S)^{\perp} = \mathcal{H}_{a,b}^S \oplus \mathcal{E}^{\dagger}\mathcal{H}_{a+1,b-1}^S \oplus \dots \oplus \mathcal{E}^{\dagger(j-1)}\mathcal{H}_{a+j-1,b-j+1}^S$ implying that this mapping is also injective. Clearly

$$\left(\mathcal{E} \Big|_{(\mathcal{E}^{\dagger j}\mathcal{H}_{a+j,b-j}^S)} \right)^{-1} = \frac{1}{j(a-b+j+1)} \mathcal{E}^{\dagger}$$

□

Corollary 4. For $a \geq b$ and $j = 1, \dots, b$ one has

$$\mathcal{H}_{a+j,b-j} \cap \text{Ker } \mathcal{E}^{\dagger} = \{0\}$$

Proof

Take $f \in \mathcal{H}_{a+j,b-j}$ with $\mathcal{E}^{\dagger}f = 0$ and hence also $\mathcal{E}\mathcal{E}^{\dagger}f = 0$. In view of Corollary 2, the function f can be decomposed as $f = \sum_{k=0}^{b-j} f_k$ with $f_k \in \mathcal{E}^{\dagger k}\mathcal{H}_{a+j+k,b-j-k}^S$. It then follows, in view of Corollary 3, that $\mathcal{E}\mathcal{E}^{\dagger}f = 0$ implies $f_0 = f_1 = \dots = f_{b-j} = 0$, and hence also $f = 0$. □

In a similar way as for the case where $a \geq b$, the following results hold for the case where $a \leq b$.

Proposition 4. For $a \leq b$ one has

(i) the space $\mathcal{H}_{a,b}$ may be decomposed as

$$\begin{aligned} \mathcal{H}_{a,b} &= \mathcal{H}_{a,b}^{S\dagger} \oplus \mathcal{E}\mathcal{H}_{a-1,b+1} \\ &= \mathcal{H}_{a,b}^{S\dagger} \oplus \mathcal{E}\mathcal{H}_{a-1,b+1}^{S\dagger} \oplus \mathcal{E}^2\mathcal{H}_{a-2,b+2}^{S\dagger} \oplus \dots \oplus \mathcal{E}^a\mathcal{H}_{0,b+a}^{S\dagger} \end{aligned} \quad (15)$$

(ii) the mappings

$$\begin{aligned} \mathcal{E}^{\dagger} &: \mathcal{E}\mathcal{H}_{a-1,b+1}^{S\dagger} &\longrightarrow & \mathcal{H}_{a-1,b+1}^{S\dagger} \\ \mathcal{E}^{\dagger} &: \mathcal{E}^2\mathcal{H}_{a-2,b+2}^{S\dagger} &\longrightarrow & \mathcal{E}\mathcal{H}_{a-2,b+2}^{S\dagger} \\ &\vdots & & \\ \mathcal{E}^{\dagger} &: \mathcal{E}^a\mathcal{H}_{0,b+a}^{S\dagger} &\longrightarrow & \mathcal{E}^{(a-1)}\mathcal{H}_{0,b+a}^{S\dagger} \end{aligned}$$

are isomorphisms, their inverses being, up to constants, restrictions of the operator \mathcal{E} to the corresponding spaces.

(iii) $\mathcal{H}_{a-j,b+j} \cap \text{Ker } \mathcal{E} = \{0\}$ for $j = 1, \dots, a$.

Corollary 5. For $a \geq b$ there holds

$$\begin{aligned} \dim \mathcal{H}_{a,b}^S(\mathbb{R}^{4p}; \mathbb{C}) &= \dim \mathcal{H}_{a,b} - \dim \mathcal{H}_{a+1,b-1} \\ &= \dim \mathcal{P}_{a,b} - \dim \mathcal{P}_{a-1,b-1} - \dim \mathcal{P}_{a+1,b-1} + \dim \mathcal{P}_{a,b-2} \\ &= \frac{(2p-1)(2p-2)(a-b+1)(a+b+2p-1)(a+2p-2)!(b+2p-3)!}{((2p-1)!)^2(a+1)!b!} \end{aligned}$$

For $a \leq b$ there holds

$$\begin{aligned}
\dim \mathcal{H}_{a,b}^{S^\dagger}(\mathbb{R}^{4p}; \mathbb{C}) &= \dim \mathcal{H}_{a,b} - \dim \mathcal{H}_{a-1,b+1} \\
&= \dim \mathcal{P}_{a,b} - \dim \mathcal{P}_{a-1,b-1} - \dim \mathcal{P}_{a-1,b+1} + \dim \mathcal{P}_{a-2,b} \\
&= \dim \mathcal{H}_{b,a}^S(\mathbb{R}^{4p}; \mathbb{C})
\end{aligned}$$

As $\dim \mathcal{H}_{a,b}^{S^\dagger} = \dim \mathcal{H}_{b,a}^S$, the spaces $\mathcal{H}_{a,b}^{S^\dagger}$ and $\mathcal{H}_{b,a}^S$ are isomorphic. Obviously this isomorphism is realized by complex conjugation which, indeed, maps $\mathcal{H}_{a,b}$ and $\mathcal{H}_{b,a}$ onto each other, since the Laplace operator is invariant under complex conjugation, and moreover the operators \mathcal{E} and \mathcal{E}^\dagger are complex conjugated up to a minus sign.

There is, however, another -nice- way to express this isomorphism, which is closely related to the quaternionic structure $(\mathbb{I}, \mathbb{J}, \mathbb{K})$ introduced in [5] to study the fundamentals of quaternionic Clifford analysis (see Section 3). For a function $F(z_1, \dots, z_{2p}, \bar{z}_1, \dots, \bar{z}_{2p})$ consider the transformation T , mapping F onto the function $T[F]$ by substituting for the variables $z_{2k-1}, z_{2k}, \bar{z}_{2k-1}, \bar{z}_{2k}$ the variables $-\bar{z}_{2k}, \bar{z}_{2k-1}, -z_{2k}, z_{2k-1}$ ($k = 1, \dots, p$) respectively. In fact this is the transformation associated to the second complex structure $\mathbb{J}_{4p} \in \text{SO}(4p)$ in the quaternionic structure. If $h_{a,b} \in \mathcal{H}_{a,b}$, then $T[h_{a,b}] \in \mathcal{H}_{b,a}$ since T commutes with the Laplace operator. Let us now compute the commutation relations of T with the operators \mathcal{E} and \mathcal{E}^\dagger .

Lemma 7. *For the transformation T introduced above, it holds*

$$\mathcal{E}^\dagger T = -T \mathcal{E} \quad \text{and} \quad \mathcal{E} T = -T \mathcal{E}^\dagger$$

Proof

We consecutively have

$$\begin{aligned}
\mathcal{E}^\dagger T[F] &= \sum_{k=1}^p (\bar{z}_{2k} \partial_{z_{2k-1}} - \bar{z}_{2k-1} \partial_{z_{2k}}) T[F] \\
&= \sum_{k=1}^p \bar{z}_{2k} T[\partial_{\bar{z}_{2k}} F] - \bar{z}_{2k-1} T[-\partial_{\bar{z}_{2k-1}} F] \\
&= T \left[\sum_{k=1}^p -z_{2k-1} \partial_{\bar{z}_{2k}} F + z_{2k} \partial_{\bar{z}_{2k-1}} F \right] \\
&= T[-\mathcal{E}F]
\end{aligned}$$

Next, taking into account that $T^2 = -\mathbf{1}$, we also have

$$T \mathcal{E}^\dagger T T = -T T \mathcal{E} T \quad \text{or} \quad T \mathcal{E}^\dagger = -\mathcal{E} T$$

□

Corollary 6. *If F is in $\text{Ker } \mathcal{E}$, then $T[F]$ is in $\text{Ker } \mathcal{E}^\dagger$ and vice versa, and, consequently*

$$T : \mathcal{H}_{a,b}^S \longleftrightarrow \mathcal{H}_{b,a}^{S^\dagger}$$

is an isomorphism.

Remark 4. *Taking for the operator T the operator associated with the third complex structure \mathbb{K}_{4p} , which corresponds to the change of variables $z_{2k-1} \mapsto i\bar{z}_{2k}, z_{2k} \mapsto -i\bar{z}_{2k-1}, \bar{z}_{2k-1} \mapsto -iz_{2k}, \bar{z}_{2k} \mapsto iz_{2k-1}$, also leads to an isomorphism between the spaces $\mathcal{H}_{a,b}^S$ and $\mathcal{H}_{b,a}^{S^\dagger}$. The operator associated to the first complex structure \mathbb{I}_{4p} is an automorphism of both spaces $\mathcal{H}_{a,b}^S$ and $\mathcal{H}_{a,b}^{S^\dagger}$.*

In the special case where $a = b = k$, the isomorphism between $\mathcal{H}_{k,k}^S$ and $\mathcal{H}_{k,k}^{S^\dagger}$ becomes the identity. This is a special case (for $j = 0$) of the following lemma, the proof of which invokes the Fischer decomposition established in the next section (see Theorems 1–2).

Lemma 8. *For all $j = 0, 1, \dots, k$ one has*

$$(\mathcal{E}^\dagger)^j \mathcal{H}_{k+j,k-j}^S = (\mathcal{E})^j \mathcal{H}_{k-j,k+j}^{S^\dagger}$$

Proof

Since $(\mathcal{E}^\dagger)^j \mathcal{H}_{k+j,k-j}^S$ and $(\mathcal{E})^j \mathcal{H}_{k-j,k+j}^{S^\dagger}$ both are non-trivial, irreducible \mathfrak{sp} -submodules of $\mathcal{H}_{k,k}$ with the same highest weight $(k+j, k-j)_s$, they coincide seen the Fischer decomposition of $\mathcal{H}_{k,k}$. \square

Also the case where $a - b = 1$ is interesting, and is obtained (by taking $j = 0$) from the following lemma, which also leans upon the Fischer decomposition.

Lemma 9. *For all $j = 0, 1, \dots, k$ one has*

$$(\mathcal{E}^\dagger)^{j+1} \mathcal{H}_{k+1+j,k-j}^S = (\mathcal{E})^j \mathcal{H}_{k-j,k+1+j}^{S^\dagger}$$

Proof

First note that $(\mathcal{E}^\dagger)^{j+1} \mathcal{H}_{k+1+j,k-j}^S$ is not the null-space, since $(\mathcal{E}^\dagger)^j \mathcal{H}_{k+1+j,k-j}^S \neq 0$ and $\text{Ker } \mathcal{E}^\dagger \cap \mathcal{H}_{k+1,k} = 0$. Similarly, also $(\mathcal{E})^j \mathcal{H}_{k-j,k+1+j}^{S^\dagger}$ is not the null-space. Since both spaces $(\mathcal{E}^\dagger)^{j+1} \mathcal{H}_{k+1+j,k-j}^S$ and $(\mathcal{E})^j \mathcal{H}_{k-j,k+1+j}^{S^\dagger}$ are, non-trivial, irreducible \mathfrak{sp} -submodules of $\mathcal{H}_{k,k+1}$ with the same highest weight $(k+1+j, k-j)$, they coincide seen the Fischer decomposition of $\mathcal{H}_{k,k+1}$. \square

Corollary 7. *For $a > b$, the mappings*

$$\mathcal{E}^{a-b} : \mathcal{H}_{b,a}^{S^\dagger} \longrightarrow \mathcal{H}_{a,b}^S$$

and

$$\mathcal{E}^{\dagger(a-b)} : \mathcal{H}_{a,b}^S \longrightarrow \mathcal{H}_{b,a}^{S^\dagger}$$

are isomorphisms.

5 Fischer decompositions

First assume that $a > b$ and compare the decomposition (14) for $\mathcal{H}_{a,b}$ in terms of symplectic harmonics, viz.

$$\mathcal{H}_{a,b} = \mathcal{H}_{a,b}^S \oplus \mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}^S \oplus \mathcal{E}^{\dagger 2} \mathcal{H}_{a+2,b-2}^S \oplus \dots \oplus \mathcal{E}^{\dagger b} \mathcal{H}_{a+b,0}^S$$

with the branching (12):

$$\mathcal{H}_{a,b} \left. \begin{array}{l} \mathfrak{gl}_{2p}(\mathbb{C}) \\ \mathfrak{sp}_{2p}(\mathbb{C}) \end{array} \right| = (a, b)_s \oplus (a+1, b-1)_s \oplus \dots \oplus (a+b-1, 1)_s \oplus (a+b)_s$$

It is then rather straightforward to conjecture that for $a > b$

$$(a, b)_s \cong \mathcal{H}_{a,b}^S(\mathbb{R}^{4p}; \mathbb{C})$$

That this indeed is the case is shown in the next theorem.

Theorem 1. *One has, with $a \geq b$,*

$$(a, b)_s \cong \mathcal{H}_{a,b}^S(\mathbb{R}^{4p}; \mathbb{C})$$

where $(a, b)_s = (a, b, 0, \dots, 0)_s$ stands for an irreducible $\mathfrak{sp}_{2p}(\mathbb{C})$ -representation, the highest weight being of length p , and

$$\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) = \mathcal{H}_{a,b}^S \oplus \mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}^S \oplus \mathcal{E}^{\dagger 2} \mathcal{H}_{a+2,b-2}^S \oplus \dots \oplus \mathcal{E}^{\dagger b} \mathcal{H}_{a+b,0}^S$$

is the Fischer decomposition of the space of complex valued bi-homogeneous harmonic polynomials in terms of $\mathfrak{sp}_{2p}(\mathbb{C})$ -irreducibles of complex valued bi-homogeneous symplectic harmonic polynomials.

Proof

We proceed by induction on b .

For $b = 0$ the result is trivial; indeed, as was already noticed in Section 2, in this case no branching occurs and $\mathcal{H}_{a,0}$ is symplectically irreducible.

Assuming that the theorem is true for $b - 1$ means that

$$(a + 1, b - 1)_s \cong \mathcal{H}_{a+1,b-1}^S(\mathbb{R}^{4p}; \mathbb{C})$$

and that

$$\mathcal{H}_{a+1,b-1}(\mathbb{R}^{4p}; \mathbb{C}) = \mathcal{H}_{a+1,b-1}^S \oplus \mathcal{E}^\dagger \mathcal{H}_{a+2,b-2}^S \oplus \mathcal{E}^{\dagger 2} \mathcal{H}_{a+3,b-3}^S \oplus \dots \oplus \mathcal{E}^{\dagger(b-1)} \mathcal{H}_{a+b,0}^S$$

is an $\mathfrak{sp}_{2p}(\mathbb{C})$ -irreducible decomposition, as then is also the case for

$$\mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}(\mathbb{R}^{4p}; \mathbb{C}) = \mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}^S \oplus \mathcal{E}^{\dagger 2} \mathcal{H}_{a+2,b-2}^S \oplus \mathcal{E}^{\dagger 3} \mathcal{H}_{a+3,b-3}^S \oplus \dots \oplus \mathcal{E}^{\dagger b} \mathcal{H}_{a+b,0}^S$$

which in fact also reads

$$\mathcal{E}^\dagger \mathcal{H}_{a+1,b-1} \left| \begin{array}{l} \mathfrak{gl}_{2p}(\mathbb{C}) \\ \mathfrak{sp}_{2p}(\mathbb{C}) \end{array} \right. = (a + 1, b - 1)_s \oplus (a + 2, b - 2)_s \oplus \dots \oplus (a + b - 1, 1)_s \oplus (a + b)_s$$

In view of the branching (12) and the decomposition (13) it follows that

$$(a, b)_s \cong \mathcal{H}_{a,b}^S(\mathbb{R}^{4p}; \mathbb{C})$$

which finishes the proof. □

If $a < b$ then we have to compare the decomposition (15)

$$\mathcal{H}_{a,b} = \mathcal{H}_{a,b}^{S^\dagger} \oplus \mathcal{E} \mathcal{H}_{a-1,b+1}^{S^\dagger} \oplus \mathcal{E}^2 \mathcal{H}_{a-2,b+2}^{S^\dagger} \oplus \dots \oplus \mathcal{E}^a \mathcal{H}_{0,b+a}^{S^\dagger}$$

with the branching rule

$$\mathcal{H}_{a,b} \left| \begin{array}{l} \mathfrak{gl}_{2p}(\mathbb{C}) \\ \mathfrak{sp}_{2p}(\mathbb{C}) \end{array} \right. = (b + a)_s \oplus (b + a - 1, 1)_s \oplus \dots \oplus (b, a)_s$$

leading to the complementary conjecture for $a < b$:

$$(b, a)_s \cong \mathcal{H}_{a,b}^{S^\dagger}(\mathbb{R}^{4p}; \mathbb{C})$$

which is proven in a similar way.

Theorem 2. *One has, with $a \leq b$,*

$$(b, a)_s \cong \mathcal{H}_{a,b}^{S^\dagger}(\mathbb{R}^{4p}; \mathbb{C})$$

and

$$\mathcal{H}_{a,b}(\mathbb{R}^{4p}; \mathbb{C}) = \mathcal{H}_{a,b}^{S^\dagger} \oplus \mathcal{E}\mathcal{H}_{a-1,b+1}^{S^\dagger} \oplus \mathcal{E}^2\mathcal{H}_{a-2,b+2}^{S^\dagger} \oplus \cdots \oplus \mathcal{E}^a\mathcal{H}_{0,b+a}^{S^\dagger}$$

is the Fischer decompositions of the space of complex valued bi-homogeneous harmonic polynomials in terms of $\mathfrak{sp}_{2p}(\mathbb{C})$ -irreducibles of complex valued bi-homogeneous adjoint symplectic harmonic polynomials.

Corollary 8. *With $a > b$, the spaces $\mathcal{H}_{a,b}^S(\mathbb{R}^{4p}; \mathbb{C})$ and $\mathcal{H}_{b,a}^{S^\dagger}(\mathbb{R}^{4p}; \mathbb{C})$ are isomorphic irreducible representations for $\mathfrak{sp}_{2p}(\mathbb{C})$.*

We already know the dimension of the spaces $\mathcal{H}_{a,b}^S$ and $\mathcal{H}_{b,a}^{S^\dagger}$ to be (see Corollary 5)

$$\dim \mathcal{H}_{a,b}^S = \dim \mathcal{H}_{b,a}^{S^\dagger} = \frac{(2p-1)(2p-2)(a-b+1)(a+b+2p-1)(a+2p-2)!(b+2p-3)!}{((2p-1)!)^2(a+1)!b!}$$

Now we are able to calculate this dimension in the following alternative way. If Γ_λ denotes an irreducible representation for $\mathfrak{sp}_{2p}(\mathbb{C})$ with highest weight $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, then (see [16], p. 406)

$$\dim \Gamma_\lambda = \prod_{i < j} \frac{\ell_i^2 - \ell_j^2}{m_i^2 - m_j^2} \prod_i \frac{\ell_i}{m_i}$$

with $\ell_i = \lambda_i + m_i$ and $m_i = p - i + 1, i = 1, \dots, p$. For the highest weight $(a, b)_s$ we have

$$m(m_1, \dots, m_p) = (p, p-1, \dots, 1)$$

and

$$\ell(\ell_1, \dots, \ell_p) = (a+p, b+p-1, p-2, p-3, \dots, 1)$$

A straightforward calculation then leads to

$$\dim (a, b)_s = \frac{(a-b+1)(a+b+2p-1)(a+2p-2)!(b+2p-3)!}{(2p-3)!(a+1)!b!}$$

which is, quite naturally, the dimension of $\mathcal{H}_{a,b}^S$ and $\mathcal{H}_{b,a}^{S^\dagger}$.

Let us give an illustrative example of the Fischer decompositions above. Take $p = 2$, and consider the decompositions

$$\begin{aligned} \mathcal{H}_{2,2}(\mathbb{R}^8; \mathbb{C}) &= \mathcal{H}_{2,2}^S \oplus \mathcal{E}^\dagger \mathcal{H}_{3,1}^S \oplus \mathcal{E}^{\dagger 2} \mathcal{H}_{4,0}^S \\ \mathcal{H}_{2,2}(\mathbb{R}^8; \mathbb{C}) &= \mathcal{H}_{2,2}^{S^\dagger} \oplus \mathcal{E} \mathcal{H}_{1,3}^{S^\dagger} \oplus \mathcal{E}^2 \mathcal{H}_{0,4}^{S^\dagger} \end{aligned}$$

The harmonic polynomial $z_3^2 \bar{z}_1^2 \in \mathcal{H}_{2,2}$ is decomposed as

$$z_3^2 \bar{z}_1^2 = P_1 + P_2 + P_3$$

with

$$\begin{aligned} P_1 &= \frac{1}{3} z_3^2 \bar{z}_1^2 + \frac{1}{3} z_2^2 \bar{z}_4^2 + \frac{2}{3} z_2 z_3 \bar{z}_1 \bar{z}_4 \\ P_2 &= \frac{1}{2} z_3^2 \bar{z}_1^2 - \frac{1}{2} z_2^2 \bar{z}_4^2 \\ P_3 &= \frac{1}{6} z_3^2 \bar{z}_1^2 + \frac{1}{6} z_2^2 \bar{z}_4^2 - \frac{2}{3} z_2 z_3 \bar{z}_1 \bar{z}_4 \end{aligned}$$

The polynomial P_1 belongs to $\mathcal{H}_{2,2}^S \equiv \mathcal{H}_{2,2}^{S^\dagger}$. The polynomial P_2 can be written as either

$$P_2 = \mathcal{E}^\dagger Q_2 \quad \text{with} \quad Q_2 = \frac{1}{2}(-z_2 z_3^2 \bar{z}_1 - z_2^2 z_3 \bar{z}_4)$$

or

$$P_2 = \mathcal{E} \widetilde{Q}_2 \quad \text{with} \quad \widetilde{Q}_2 = \frac{1}{2}(z_3 \bar{z}_1^2 \bar{z}_4 + z_2 \bar{z}_1 \bar{z}_4^2)$$

The polynomial Q_2 belongs to $\mathcal{H}_{3,1}^S$, while the polynomial \widetilde{Q}_2 belongs to $\mathcal{H}_{1,3}^{S^\dagger}$. The polynomial P_3 can be written as either

$$P_3 = \mathcal{E}^{\dagger 2} Q_3 \quad \text{with} \quad Q_3 = \frac{1}{12} z_2^2 z_3^2$$

or

$$P_3 = \mathcal{E}^2 \widetilde{Q}_3 \quad \text{with} \quad \widetilde{Q}_3 = \frac{1}{12} \bar{z}_1^2 \bar{z}_4^2$$

The polynomial Q_3 belongs to $\mathcal{H}_{4,0}^S \equiv \mathcal{H}_{4,0}$, while the polynomial \widetilde{Q}_3 belongs to $\mathcal{H}_{0,4}^{S^\dagger} \equiv \mathcal{H}_{0,4}$.

Corollary 9. *The Fischer decomposition of the space $\mathcal{P}_{a,b}(\mathbb{R}^{4p}; \mathbb{C})$ of complex valued bi-homogeneous polynomials in terms of irreducible symplectic modules, is given by*

$$\mathcal{P}_{a,b} = \bigoplus_{j=0}^b |\underline{z}|^{2j} \mathcal{H}_{a-j,b-j} = \bigoplus_{j=0}^b \bigoplus_{t=0}^{b-j} |\underline{z}|^{2j} \mathcal{E}^{\dagger t} \mathcal{H}_{a-j+t,b-j-t}^S \quad (a \geq b) \quad (16)$$

or

$$\mathcal{P}_{a,b} = \bigoplus_{j=0}^a |\underline{z}|^{2j} \mathcal{H}_{a-j,b-j} = \bigoplus_{j=0}^a \bigoplus_{t=0}^{a-j} |\underline{z}|^{2j} \mathcal{E}^t \mathcal{H}_{a-j-t,b-j+t}^{S^\dagger} \quad (a \leq b) \quad (17)$$

Corollary 10. *The space $\mathcal{P}(\mathbb{R}^{4p}; \mathbb{C})$ may be decomposed in terms of irreducible symplectic modules according to the following diagrams.*

For $\mathcal{P}_0(\mathbb{R}^{4p}; \mathbb{C})$:

$$\mathcal{H}_{0,0}^S$$

For $\mathcal{P}_2(\mathbb{R}^{4p}; \mathbb{C})$:

$$r^2 \mathcal{H}_{0,0}^S$$

$$\begin{array}{ccc} \mathcal{H}_{0,2}^{S^\dagger} & \mathcal{E}^\dagger \mathcal{H}_{2,0}^S & \mathcal{H}_{2,0}^S \\ & \mathcal{H}_{1,1}^{S^\dagger} & \end{array}$$

For $\mathcal{P}_4(\mathbb{R}^{4p}; \mathbb{C})$:

$$r^4 \mathcal{H}_{0,0}^S$$

$$\begin{array}{ccc} r^2 \mathcal{H}_{0,2}^{S^\dagger} & r^2 \mathcal{E}^\dagger \mathcal{H}_{2,0}^S & r^2 \mathcal{H}_{2,0}^S \\ & r^2 \mathcal{H}_{1,1}^{S^\dagger} & \end{array}$$

$$\begin{array}{ccccc} \mathcal{H}_{0,4}^{S^\dagger} & \mathcal{E} \mathcal{H}_{0,4}^{S^\dagger} & \mathcal{E}^{\dagger 2} \mathcal{H}_{4,0}^S & \mathcal{E}^\dagger \mathcal{H}_{4,0}^S & \mathcal{H}_{4,0}^S \\ & \mathcal{H}_{1,3}^{S^\dagger} & \mathcal{E}^\dagger \mathcal{H}_{3,1}^S & \mathcal{H}_{3,1}^S & \\ & & \mathcal{H}_{2,2}^S & & \end{array}$$

etc. for even degree polynomials.

For $\mathcal{P}_1(\mathbb{R}^{4p}; \mathbb{C})$:

$$\mathcal{H}_{0,1}^{S^\dagger} \quad \mathcal{H}_{1,0}^S$$

For $\mathcal{P}_3(\mathbb{R}^{4p}; \mathbb{C})$:

$$\begin{array}{cccc} r^2 \mathcal{H}_{0,1}^{S^\dagger} & r^2 \mathcal{H}_{1,0}^S & & \\ \mathcal{H}_{0,3}^{S^\dagger} & \mathcal{E} \mathcal{H}_{0,3}^{S^\dagger} & \mathcal{E}^\dagger \mathcal{H}_{3,0}^S & \mathcal{H}_{3,0}^S \\ & \mathcal{H}_{1,2}^{S^\dagger} & \mathcal{H}_{2,1}^S & \end{array}$$

For $\mathcal{P}_5(\mathbb{R}^{4p}; \mathbb{C})$:

$$\begin{array}{cccccc} r^4 \mathcal{H}_{0,1}^{S^\dagger} & r^4 \mathcal{H}_{1,0}^S & & & & \\ r^2 \mathcal{H}_{0,3}^{S^\dagger} & r^2 \mathcal{E} \mathcal{H}_{0,3}^{S^\dagger} & r^2 \mathcal{E}^\dagger \mathcal{H}_{3,0}^S & r^2 \mathcal{H}_{3,0}^S & & \\ & r^2 \mathcal{H}_{1,2}^{S^\dagger} & r^2 \mathcal{H}_{2,1}^S & & & \\ \mathcal{H}_{0,5}^{S^\dagger} & \mathcal{E} \mathcal{H}_{0,5}^{S^\dagger} & \mathcal{E}^2 \mathcal{H}_{0,5}^{S^\dagger} & \mathcal{E}^{\dagger 2} \mathcal{H}_{5,0}^S & \mathcal{E}^\dagger \mathcal{H}_{5,0}^S & \mathcal{H}_{5,0}^S \\ & \mathcal{H}_{1,4}^{S^\dagger} & \mathcal{E} \mathcal{H}_{1,4}^{S^\dagger} & \mathcal{E}^\dagger \mathcal{H}_{4,1}^S & \mathcal{H}_{4,1}^S & \\ & & \mathcal{H}_{2,3}^{S^\dagger} & \mathcal{H}_{3,2}^S & & \end{array}$$

etc. for odd degree polynomials.

6 Howe dual pair

The Fischer decomposition (1) of the space $\mathcal{P}(\mathbb{R}^m; \mathbb{C})$ of complex valued polynomials in terms of spherical harmonics, viz.

$$\mathcal{P}(\mathbb{R}^m; \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{p=0}^{\infty} r^{2p} \mathcal{H}_k(\mathbb{R}^m; \mathbb{C})$$

shows the drawback that it is not multiplicity-free: each of the $\mathrm{SO}(m)$ -irreducible invariant subspaces $\mathcal{H}_k(\mathbb{R}^m; \mathbb{C})$ appears with an infinite multiplicity, since all of

$$r^{2p} \mathcal{H}_k \quad , \quad p \in \mathbb{N}_0$$

$k \in \mathbb{N}_0$ being fixed, are isomorphic as $\mathrm{SO}(m)$ -modules. Expressing irreducibility with respect to $\mathfrak{g} \times \mathrm{SO}(m)$, \mathfrak{g} being an appropriate Lie algebra, aims at collecting the infinitely many copies of \mathcal{H}_k into one single irreducible representation. The so-called Howe dual pair $(\mathrm{SO}(m), \mathfrak{g})$ is to be found with respect to a bigger algebra in which $\mathfrak{so}(m)$ and \mathfrak{g} are commuting. Seen the action (5) of the operators $X := \frac{1}{2} r^2$, $Y := -\frac{1}{2} \Delta_m$ and $H := \mathbb{E} + \frac{m}{2}$, the Lie algebra \mathfrak{g} in this case is $\mathfrak{sl}(2, \mathbb{C})$. More background information is to be found in [6].

Similarly, the Fischer decomposition (6) of the space $\mathcal{P}(\mathbb{R}^{2n}; \mathbb{C})$ in terms of Hermitian spherical harmonics:

$$\mathcal{P}(\mathbb{R}^{2n}; \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{p=0}^{\infty} \bigoplus_{a=0}^k r^{2p} \mathcal{H}_{a,k-a}(\mathbb{R}^{2n}; \mathbb{C}) \quad (18)$$

is not multiplicity free since, for all a and b ,

$$r^{2p} \mathcal{H}_{a,b}, \quad p = 0, 1, 2, \dots$$

are isomorphic as $U(n)$ -modules. It turns out that the Howe dual pair here is $(U(n), \mathfrak{gl}(2, \mathbb{C}))$ (see also [6]), with

$$\begin{aligned}\mathfrak{gl}(2, \mathbb{C}) &= \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C} \\ &= \text{Alg}_{\mathbb{C}}(H, X, Y) \oplus \text{span}_{\mathbb{C}}(\mathbb{E}_z^\dagger - \mathbb{E}_z + n)\end{aligned}$$

Now let us have a look at the symplectic Fischer decomposition of the space $\mathcal{P}(\mathbb{R}^{4p}; \mathbb{C})$. By means of (16) and (17) we obtain

$$\begin{aligned}\mathcal{P}(\mathbb{R}^{4p}; \mathbb{C}) &= \bigoplus_{t=0}^{\infty} \bigoplus_{a < b} |\underline{z}|^{2t} \left(\mathcal{H}_{a,b}^{S^\dagger} \oplus \mathcal{E} \mathcal{H}_{a-1,b+1}^{S^\dagger} \oplus \dots \oplus \mathcal{E}^a \mathcal{H}_{0,b+a}^{S^\dagger} \right) \\ &\oplus \bigoplus_{a \geq b} |\underline{z}|^{2t} \left(\mathcal{H}_{a,b}^S \oplus \mathcal{E}^\dagger \mathcal{H}_{a+1,b-1}^S \oplus \dots \oplus \mathcal{E}^{\dagger b} \mathcal{H}_{a+b,0}^S \right)\end{aligned}\quad (19)$$

or, alternatively

$$\mathcal{P}(\mathbb{R}^{4p}; \mathbb{C}) = \bigoplus_{t=0}^{\infty} \bigoplus_{a \geq b} \bigoplus_{s=0}^{a-b} |\underline{z}|^{2t} \mathcal{E}^{\dagger s} \mathcal{H}_{a,b}^S \quad (20)$$

or still, interchanging the role of the operators \mathcal{E} and \mathcal{E}^\dagger ,

$$\mathcal{P}(\mathbb{R}^{4p}; \mathbb{C}) = \bigoplus_{t=0}^{\infty} \bigoplus_{a \leq b} \bigoplus_{s=0}^{b-a} |\underline{z}|^{2t} \mathcal{E}^s \mathcal{H}_{a,b}^{S^\dagger} \quad (21)$$

Clearly these decompositions are not multiplicity free. Assuming $a > b$, the isomorphic $\text{Sp}(p)$ -modules may be gathered in the following way

$$\begin{array}{cccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ |\underline{z}|^4 \mathcal{H}_{a,b}^S & \rightarrow & |\underline{z}|^4 \mathcal{E}^\dagger \mathcal{H}_{a,b}^S & \rightarrow \dots \rightarrow & |\underline{z}|^4 \mathcal{E}^{\dagger \alpha} \mathcal{H}_{a,b}^S \xrightarrow{iso} & |\underline{z}|^4 \mathcal{E}^\alpha \mathcal{H}_{b,a}^{S^\dagger} & \rightarrow \dots \rightarrow & |\underline{z}|^4 \mathcal{E} \mathcal{H}_{b,a}^{S^\dagger} & \rightarrow & |\underline{z}|^4 \mathcal{H}_{b,a}^{S^\dagger} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ |\underline{z}|^2 \mathcal{H}_{a,b}^S & \rightarrow & |\underline{z}|^2 \mathcal{E}^\dagger \mathcal{H}_{a,b}^S & \rightarrow \dots \rightarrow & |\underline{z}|^2 \mathcal{E}^{\dagger \alpha} \mathcal{H}_{a,b}^S \xrightarrow{iso} & |\underline{z}|^2 \mathcal{E}^\alpha \mathcal{H}_{b,a}^{S^\dagger} & \rightarrow \dots \rightarrow & |\underline{z}|^2 \mathcal{E} \mathcal{H}_{b,a}^{S^\dagger} & \rightarrow & |\underline{z}|^2 \mathcal{H}_{b,a}^{S^\dagger} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{H}_{a,b}^S & \rightarrow & \mathcal{E}^\dagger \mathcal{H}_{a,b}^S & \rightarrow \dots \rightarrow & \mathcal{E}^{\dagger \alpha} \mathcal{H}_{a,b}^S \xrightarrow{iso} & \mathcal{E}^\alpha \mathcal{H}_{b,a}^{S^\dagger} & \rightarrow \dots \rightarrow & \mathcal{E} \mathcal{H}_{b,a}^{S^\dagger} & \rightarrow & \mathcal{H}_{b,a}^{S^\dagger} \end{array}$$

where $\alpha = \lfloor \frac{a-b}{2} \rfloor$, and *iso* is either the identity if $a - b$ is even (see Lemma 8), or the mapping \mathcal{E}^\dagger if $a - b$ is odd (see Lemma 9). In this scheme the horizontal arrows represent the action of the operator \mathcal{E}^\dagger , while the vertical arrows correspond to multiplication by $|\underline{z}|^2$. If $a < b$ this scheme has to be reinterpreted "from right to left", the horizontal arrows, now oriented from right to left, then corresponding to the action of the operator \mathcal{E} . In the special case where $a = b$, the scheme reduces to

$$\begin{array}{ccc} \vdots & & \vdots \\ |\underline{z}|^4 \mathcal{H}_{a,a}^S & = & |\underline{z}|^4 \mathcal{H}_{a,a}^{S^\dagger} \\ \uparrow & & \uparrow \\ |\underline{z}|^2 \mathcal{H}_{a,a}^S & = & |\underline{z}|^2 \mathcal{H}_{a,a}^{S^\dagger} \\ \uparrow & & \uparrow \\ \mathcal{H}_{a,a}^S & = & \mathcal{H}_{a,a}^{S^\dagger} \end{array}$$

Apparently the Howe dual \mathfrak{g} is generated by the operators

$$\mathcal{E}, \mathcal{E}^\dagger, \mathbb{E}_z^\dagger - \mathbb{E}_z, |\underline{z}|^2, \Delta_{4p}, \mathbb{E}_z + \mathbb{E}_z^\dagger + 2p$$

which leads to (see also Corollary 1)

$$\begin{aligned} \mathfrak{g} &= \text{Alg}_{\mathbb{C}} \left(\mathbb{E}_z + \mathbb{E}_z^\dagger + 2p, \frac{1}{2} |\underline{z}|^2, -\frac{1}{2} \Delta_{4p} \right) \oplus \text{Alg}_{\mathbb{C}} (\mathbb{E}_z^\dagger - \mathbb{E}_z, \mathcal{E}^\dagger, \mathcal{E}) \\ &= \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \\ &= \mathfrak{so}(4, \mathbb{C}) \end{aligned}$$

So let us decompose the $\text{Sp}(p)$ -module $\mathcal{P}(\mathbb{R}^{4p}; \mathbb{C})$ under the combined action of the Howe dual pair $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})) \times \text{Sp}(p)$. For each irreducible $\text{Sp}(p)$ -module $\mathcal{H}_{a,b}^S$ we choose a basis $\{H_{a,b;j}^S : j = 1, 2, \dots, \dim \mathcal{H}_{a,b}^S\}$; this is a set of singular vectors, labeled by three parameters a, b and j . The repeated action of $X = \frac{1}{2} |\underline{z}|^2$ then generates the module $\mathbb{V}_{a,b;j}$ given by

$$\mathbb{V}_{a,b;j} = \text{span}_{\mathbb{C}} \{X^t H_{a,b;j}^S : t = 0, 1, 2, \dots\}$$

Each of the spaces $\mathbb{V}_{a,b;j}$ is a realization of a so-called Verma module, an infinite dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module, which we denote by $\mathbb{I}_{a,b}^\infty$. On the other hand, repeated action of \mathcal{E}^\dagger generates the module $\mathbb{W}_{a,b;j}$ given by

$$\mathbb{W}_{a,b;j} = \text{span}_{\mathbb{C}} \{\mathcal{E}^{\dagger s} H_{a,b;j}^S : s = 0, 1, 2, \dots, a - b\}$$

Each of the spaces $\mathbb{W}_{a,b;j}$ is a realization of a finite dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module, which we denote by $\mathbb{I}_{a,b}$. Finally, the space of (a, b) -homogeneous symplectic harmonic polynomials $\mathcal{H}_{a,b}^S$ is a realization of the irreducible $\text{Sp}(p)$ -module with highest weight $(a, b)_s = (a + b, 0, \dots, 0)_s$ of length p , which we denote by $\mathbb{H}_{a,b}$. For all (a, b) with $a > b$ the tensor product

$$(\mathbb{I}_{a,b}^\infty \otimes \mathbb{I}_{a,b}) \otimes \mathbb{H}_{a,b}$$

then is an irreducible $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})) \times \mathfrak{sp}_{2p}(\mathbb{C})$ -module. When regarded as a $\mathfrak{so}(4, \mathbb{C})$ -module it contains as many copies of $\mathbb{I}_{a,b}^\infty \otimes \mathbb{I}_{a,b}$ as the dimension of $\mathbb{H}_{a,b}$; when regarded as an $\mathfrak{sp}_{2p}(\mathbb{C})$ -module it contains infinitely many copies of $\mathbb{H}_{a,b}$. The symplectic Fischer decompositions (19)(20)(21) may thus be reformulated as follows.

Theorem 3. *Under the joint action of $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})) \times \text{Sp}(p)$, the space of complex valued polynomials $\mathcal{P}(\mathbb{R}^{4p}; \mathbb{C})$ is isomorphic with the multiplicity free irreducible direct sum decomposition*

$$\bigoplus_{a \geq b=0}^{\infty} (\mathbb{I}_{a,b}^\infty \otimes \mathbb{I}_{a,b}) \otimes \mathbb{H}_{a,b}$$

where $\mathbb{I}_{a,b}^\infty$ is a Verma $\mathfrak{sl}(2, \mathbb{C})$ -module with lowest weight $a + b + 2p$, $\mathbb{I}_{a,b}$ is an irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module with highest weight $a - b$ and $\mathbb{H}_{a,b}$ is an irreducible $\mathfrak{sp}_{2p}(\mathbb{C})$ -module with highest weight $(a + b, 0, \dots, 0)$.

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