Notre Dame Journal of Formal Logic
Volume XVIII, Number 4, October 1977
NDJFAM

FITCH-STYLE RULES FOR MANY MODAL LOGICS

DAVID F. SIEMENS, Jr.

Fitch's original natural deduction rules for modal logic [1] give a calculus similar to S4, provided one uses strong negation introduction (**SNI**) and strong negation elimination (**SNE**). If **SNI** and weak negation elimination (**WNE**) are used, an intuitionistic modal calculus results. And Fitch presents an even weaker calculus using weak negation introduction (**WNI**) and **WNE**. The combination of **WNI** and **SNE** merely allows alternation to the doubly negated proposition. To all these combinations of rules may be added the rules for quantification, giving a variety of modal functional calculi. *Cf.* [2].

Fitch [3] has shown some of the flexibility inherent in his rules by presenting formulations for the calculi known as B, M, S4, and S5, with quantification added, also for deontic versions of each of these, and for combinations of the alethic and deontic logics. Thomason [4] presented a Fitch-style calculus for the special calculus developed by Stalnaker and him, *cf.* [5] and [6]. However, the technique is even more flexible than has been indicated. Combining the following rules in various ways with one of the sets giving one of the propositional or lower functional calculi, watching for potential paradoxes with the latter, gives calculi equivalent to those known in the literature, and to some that do not match. The rules are:

For necessity:

1. Strong necessity introduction (**SLI**)

$$\left|\begin{array}{l} L\left|\begin{array}{l} \vdots \\ p \end{array}\right. \\ Lp \end{array}\right.$$

2. Weak necessity introduction (**WLI**)

$$\left|\begin{array}{l} \vdots \\ p \\ Lp \end{array}\right.$$

Received June 14, 1975

3. Strong necessity elimination (**SLE**)

$$\begin{array}{|l} \vdots \\ Lp \\ p \end{array}$$

4. Weak necessity elimination (**WLE**)

$$\begin{array}{|l} \vdots \\ Lp \\ Mp \end{array}$$

For possibility:

5. Strong possibility introduction (**SMI**)

$$\begin{array}{|l} \vdots \\ p \\ Mp \end{array}$$

6. Weak possibility introduction (**WMI**)

$$\begin{array}{|l} L \begin{array}{|l} \quad \vdots \\ \quad p \end{array} \\ Mp \end{array}$$

7. Strong possibility elimination (**SME**)

$$\begin{array}{|l} Mp \\ L \begin{array}{|l} \quad \underline{p} \\ \quad \vdots \\ \quad q \end{array} \\ Mq \end{array}$$

8. Weak possibility elimination (**WME**)

$$\begin{array}{|l} \vdots \\ MLp \\ Lp \end{array}$$

For combinations

9. Negated possibility introduction (**NMI**)

$$\begin{array}{|l} \vdots \\ LNp \\ NMp \end{array}$$

10. Negated possibility elimination (**NME**)

$$\begin{array}{|l} \vdots \\ NMp \\ LNp \end{array}$$

11. Possible necessity elimination (**MLE**)

$$\begin{array}{|l} \vdots \\ MLp \\ p \end{array}$$

For strict reiteration

12. Strong necessity reiteration (**SLR**)

$$\begin{array}{|l} Lp \\ L\begin{array}{|l} \\ \quad Lp \end{array} \end{array}$$

13. Weak necessity reiteration (**WLR**)

$$\begin{array}{|l} Lp \\ L\begin{array}{|l} \\ \quad p \end{array} \end{array}$$

14. Possibility reiteration (**MR** = **NLR**)

$$\begin{array}{|l} Mp \\ L\begin{array}{|l} \\ \quad Mp \end{array} \end{array} \qquad \text{or} \qquad \begin{array}{|l} NLp \\ L\begin{array}{|l} \\ \quad NLp \end{array} \end{array}$$

Of these, Fitch does not mention 2, 6, 8, 11, and 12. To these may be coupled one or two of the rules for the modal block, only the first of which Fitch adopts.

BR1 Allow any number of modal blocks.
BR2 Allow only one modal block.
BR2′ Allow only two modal blocks.

BR3 Allow no proof line inside a modal block.
BR4 Allow no hypotheses inside a modal block. (Disjunction and existence elimination rules do not involve true hypotheses.)

Fitch's technique of combining two different types of modalities, with or without interaction, may be added, as may the special reiteration rules used by Thomason [4] to give his calculus with an intransitive conditional connective. And finally, (15) the interdefinition of necessity and possibility may be added or excluded. It may first be noted that some combinations are equivalent to others. For example, with the usual rules for the propositional calculus, including **SNI** and **SNE**, plus 12. **SLR**, 1. **SLI** is equivalent to 7. **SME**; and 3. **SLE** to 5. **SMI**, and the set gives the interdefinition, 15. **D**. Alternatively, 1, 3, 13, and 15 give also 5 and 7. But the relationship changes if the intuitionistic propositional calculus or Fitch's weaker calculus is used. Other relationships are too obvious to be belabored. Combining these various rules with the usual propositional calculus gives rise to some of the well-known calculi.[1]

For K	1. **SLI**, 13. **WLR**, 15. **D**, and **BR1**
KB	1. **SLI**, 11. **MLE**, 13. **WLR**, 15. **D**, and **BR1**
KE	1. **SLI**, 8. **WME**, 13. **WLR**, 15. **D**, and **BR1**
D	1. **SLI**, 4. **WLE**, 13. **WLR**, 15. **D**, and **BR1**
T = M	1. **SLI**, 3. **SLE**, 13. **WLR**, 15. **D**, and **BR1**
K4	1. **SLI**, 12. **SLR**, 15. **D**, and **BR1** or **BR2**[2]
KB4	1. **SLI**, 11. **MLE**, 12. **SLR**, 15. **D**, and **BR1** or **BR2**
KE4	1. **SLI**, 8. **WME**, 12. **SLR**, 15. **D**, and **BR1** or **BR2**
D4	1. **SLI**, 4. **WLE**, 12. **SLR**, 15. **D**, and **BR1** or **BR2**
DB	1. **SLI**, 4. **WLE**, 11. **MLE**, 13. **WLR**, 15. **D**, and **BR1** or **BR2**
DE	1. **SLI**, 4. **WLE**, 8. **WME**, 13. **WLR**, 15. **D**, and **BR1** or **BR2**
T4 = S4	1. **SLI**, 3. **SLE**, 12. **SLR**, 15. **D**, and **BR1** or **BR2**
TB	1. **SLI**, 3. **SLE**, 11. **MLE**, 13. **WLR**, 15. **D**, and **BR1** or **BR2**
TE	1. **SLI**, 3. **SLE**, 8. **WME**, 13. **WLR**, 15. **D**, and **BR1** or **BR2**
T5 = S4	1. **SLI**, 3. **SLE**, 12. **SLR**, 14. **MR**, 15. **D**, and **BR1** or **BR2**
B	1. **SLI**, 3. **SLE**, 13. **WLR**, 14. **MR**, 15. **D**, and **BR1**

Fitch's [3] deontic systems, DB,[3] DM (= D), DS4 (= D4), and DS5 result from replacing 3. **SLE** in the corresponding calculus with 4. **WLE**. There apparently is not an exact equivalence, for D5 is T5 in the axiomatic systems, but DS5 does not seem to allow the proof of 3. **SLE** so as to be equivalent to S5.

1. The terminology of Lemmon [7] is used, except for B and M, and the deontic series (see footnote 3). M in [3] is not to be equated with $\underline{\underline{M}}$ in [2].

2. Here, and in the next calculi, **BR2** has no effect on the number of modal operators because of the effect of 12. **SLR**.

3. Note that 'D' means 'deontic' and relates to Lemmon's D-series. But the names are formed differently.

For a couple of weaker systems, replacing **BR1** with **BR2** in K gives a system slightly weaker than K; and 3. **SLE**, 13. **WLR**, 15. **D**, and **BR4** gives a calculus in which $LCpp$ is not a theorem, but $CLpLp$ and $CLLpLp$ are. Adding **BR2** to the weaker combinations gives results that do not seem to be paralleled by the axiomatic calculi. 2. **WLI** is, of course, the trivializing rule, corresponding to Lemmon's **C**.

If one wishes to combine two systems as Fitch [3] suggests, say D and S4, making the former dependent on the latter, one needs only use two necessities, with the rules for S4 applied to L and M, and the rules for D applied to L' and M' or, to use Fitch's operators, O and P, adding a new rule for reiteration:

12′. Alethic-deontic reiteration

$$\begin{array}{|l} \quad Lp \\ L' \begin{array}{|l} \\ \qquad Lp \end{array} \end{array} \qquad \text{or} \qquad \begin{array}{|l} \quad Lp \\ O \begin{array}{|l} \\ \qquad Lp \end{array} \end{array}$$

If the second system were subordinate to S5 instead of to S4, one might add, although he probably would not:

14′. Alethic-deontic possibility reiteration

$$\begin{array}{|l} \quad Mp \\ L' \begin{array}{|l} \\ \qquad Mp \end{array} \end{array} \qquad \text{or} \qquad \begin{array}{|l} \quad Mp \\ O \begin{array}{|l} \\ \qquad Mp \end{array} \end{array}$$

Triple systems are as easily constructed, with interactions of all systems or only of some.

With either the intuitionistic or Fitch's weaker calculus, different results may be had by the addition of the rules for possibility rather than 15. **D**. And Thomason's rules [4] add further possibilities. So the number of possible systems is great. Yet only a few have been investigated.

Another technique for modifying the results of Fitch's rules involves metalinguistic restrictions on the application of some of the rules of the propositional or functional calculi. But this is a topic for another study.

REFERENCES

[1] Fitch, F. B., *Symbolic Logic*, Ronald, New York (1952).

[2] Fitch, F. B., "Intuitionistic modal logic with quantifiers," *Portugaliae Mathematica*, vol. 7 (1948), pp. 113-118.

[3] Fitch, F. B., "Natural deduction rules for obligation," *American Philosophical Quarterly*, vol. 3 (1966), pp. 27-38.

[4] Thomason, R. H., "A Fitch-style formulation of conditional logic," *Logique et Analyse*, 13e an. (1970), pp. 397-412.

[5] Stalnaker, R. C., "A theory of conditionals," *American Philosophical Quarterly*, Monograph Series, vol. 2 (1968), pp. 98-112.

[6] Stalnaker, R. C., and R. H. Thomason, "A semantic analysis of conditional logic," *Theoria* (Lund), vol. 36 (1970), pp. 23-42.

[7] Lemmon, E. J., "Algebraic semantics for modal logics," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 46-65, 191-218.

Los Angeles Pierce College
Woodland Hills, California