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Five-dimensional gauge theories and the local B-model

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Abstract
We propose an effective framework for computing the prepotential of the topological B-model on a class of local Calabi–Yau geometries related to the circle compactification of five-dimensional $\mathcal{N} = 1$ super Yang–Mills theory with simple gauge group. In the simply laced case, we construct Picard–Fuchs operators from the Dubrovin connection on the Frobenius manifolds associated with the extended affine Weyl groups of type ADE. In general, we propose a purely algebraic construction of Picard–Fuchs ideals from a canonical subring of the space of regular functions on the ramification locus of the Seiberg–Witten curve, encompassing non-simply laced cases as well. We offer several precision tests of our proposal for simply laced cases by comparing with the gauge theory prepotentials obtained from the K-theoretic blow-up equations, finding perfect agreement. Whenever there is more than one candidate Seiberg-Witten curve for non-simply laced gauge groups in the literature, we employ our framework to verify which one agrees with the K-theoretic blow-up equations.

Keywords Seiberg-Witten · Picard-Fuchs · Instantons · Geometric engineering · Integrable systems

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1 Introduction

Five-dimensional supersymmetric gauge theories have been the subject of long-standing interest, in light of their intertwined roles both as low-energy descriptions of SCFTs in 5d [83], and as non-trivial quantum field theories engineered by M-theory compactification on a Calabi–Yau threefold [45, 58]. In particular, their stringy origin situates their Kaluza–Klein (KK) reduction on $\mathbb{R}^4 \times S^1$ at the centre of a web of correspondences relating instanton counting to, *inter alia*, the topological A-model on resolutions of local CY3 singularities [52, 58], a class of relativistic integrable models [36, 74, 82], a q-deformed version of the AGT correspondence [7, 79, 80], and in special instances, the large N limit of Chern–Simons theory on non-trivial 3-manifolds and related matrix models [2, 13, 14, 60].

As for any four-dimensional theory with eight supercharges, the infrared physics on the Coulomb branch $\mathcal{M}_C$ of the $\mathcal{N} = 2$ KK theory on $\mathbb{R}^4 \times S^1$ is encoded into its prepotential, which governs the exact Wilsonian effective action of the gauge theory. At weak coupling and for classical gauge groups, one approach to compute this micro-

scopically in the full Ω-background is by using localisation, asymptotically in the Coulomb moduli [72, 73, 75, 76]. Alternatively, and equivalently, the Ω-background prepotential coincides with the free energy of the refined A-model on the associated engineering CY3 geometry [5, 6, 46].

The spectacular results arising from direct instanton calculations come at a price, however, as explicit instanton partition functions become unwieldy in general, the treatment of exceptional gauge groups requires some degree of guesswork and/or extrapolation from the classical cases, and one is *a fortiori* stuck in the S-duality frame corresponding to the instanton expansion/large volume limit of the engineering geometry. By the work of Seiberg and Witten, one way around this is to consider the realisation of the gauge theory prepotential from the special geometry of a family of spectral curves fibred over the Coulomb branch $\mathcal{M}_{\mathsf{C}}$ [84]. Restricting for simplicity to the setting of the pure gauge theory with gauge group $\mathcal{G}$, the affine part $\mathcal{C}_{\mathcal{G};u} = \{\mathsf{P}_{\mathcal{G};u}(\mu, \lambda; u) = 0\}$ of the fibres of the family over a Coulomb moduli point $u \in \mathcal{M}_{\mathsf{C}}$ is given by the vanishing of a certain characteristic polynomial $\mathsf{P}_{\mathcal{G};u}(\mu, \lambda) \in \mathbb{C}[\mu, \lambda]$, and the gauge theory effective action is recovered from the special geometry relations

$$\frac{\partial \mathcal{F}_{\mathcal{G}}}{\partial a_i} = \frac{1}{2} \oint_{B_i} \log \mu \, \mathrm{d} \log \lambda, \qquad a_i = \frac{1}{2\pi \mathrm{i}} \oint_{A_i} \log \mu \, \mathrm{d} \log \lambda, \tag{1}$$

where $\{A_i, B_i\}_{i=1}^r$ is a duality-frame-dependent choice of a symplectic basis of integral homology 1-cycles on $\mathcal{C}_{\mathcal{G};u}$.

In principle, knowing $\mathsf{P}_{\mathcal{G};u}$ determines the prepotential of the gauge theory via the period integrals in (1), but translating this into practice, especially when $\mathcal{G} \neq \mathrm{SU}(N)$, is a much different story. Even being able to write down the integrals (let alone compute them) in (1) requires a delicate analysis in the choice of a symmetric combination of homology 1-cycles $\{A_i, B_i\}_{i=1}^r$, which realises a projection to a distinguished Prym–Tyurin subvariety[1] of $\mathrm{Jac}(\overline{\mathcal{C}_{\mathcal{G};u}})$, see [16, 66]. On top of this, the curves $\overline{\mathcal{C}_{\mathcal{G};u}}$ are usually non-hyperelliptic and of high genus on a dense open subset of $\mathcal{M}_{\mathsf{C}}$, *de facto* hampering a direct calculation of the period integrals. One possible perspective to overcome this impasse is offered by string theory engineering, wherein for $\mathcal{G} = \mathrm{SU}(N)$ the SW geometry gets identified with the Hori–Iqbal–Vafa mirror of a toric CY3 [15, 55]. The periods (1) are then solutions of a generalised hypergeometric system of coupled PDEs given by the Gelfand–Kapranov–Zelevinsky (GKZ) system for the corresponding toric variety. These solutions can be effectively computed asymptotically around the large complex structure/weak gauge coupling point using Frobenius' method: for $N = 2$ and fundamental matter, this method was brought to fruit in [33] (see also [24]). The toric condition on the engineering geometry, however, confines this approach to unitary gauge groups, and the case $\mathcal{G} \neq \mathrm{SU}(N)$ requires fundamentally different ideas to be treated.

In the present paper, we address this problem by providing a systematic construction of Picard–Fuchs operators annihilating the periods (1) for a general simple gauge group $\mathcal{G}$. We will offer two constructions of such global D-modules over the Coulomb branch

[1] We denote $\overline{\mathcal{C}_{\mathcal{G};u}}$ the smooth completion (normalisation of the Zariski closure in $\mathbb{P}^2$) of the affine curve $\mathcal{C}_{\mathcal{G};u}$. In particular $\overline{\mathcal{C}_{\mathcal{G};u}}$ is a compact Riemann surface.

and use them to test proposals for 5d SW curves arising from geometric engineering, brane constructions, and/or the theory of relativistic integrable systems.

Picard–Fuchs ideals from Frobenius manifolds

Suppose first that $\mathcal{G}$ is of type ADE. We propose that SW periods are, in the terminology of [30], the *odd* periods of the canonical Frobenius manifold structure on the orbits of the Dubrovin–Zhang extension of the affine Weyl group of type $\mathcal{G}$ in the reflection representation [31], which was recently explicitly constructed in [16, 20]. This is a natural generalisation of an idea of Dubrovin [30] (see also [34, 48] for previous work on this), where the polynomial Frobenius manifold of type ADE played a similar role in the reconstruction of the SW periods for four-dimensional $\mathcal{N} = 2$ super Yang–Mills with simply laced gauge symmetry. In particular, we propose that the Picard–Fuchs ideal annihilating the physical periods of the SW curve can be read off from a suitable specialisation of the quantum differential equations for the associated Dubrovin–Zhang Frobenius manifold. Schematically, and in close analogy to [34, 47, 48, 50], we propose that there exists a distinguished holomorphic chart $\{t_i(u)\}_{0\leq i\leq r}$ on[2] $\mathcal{M}_{\mathsf{C}} \times \mathbb{P}^1$ such that the periods (1) satisfy a holonomic system of Fuchsian PDEs in the form

$$\begin{aligned} &\left(\partial_{t_0} + \sum_i \mathfrak{q}_i t_i \partial_{t_i}\right)^2 \Pi = 4h_{\mathfrak{g}}^2 \partial_{t_r}^2 \Pi, \\ &\partial_{t_i t_j}^2 \Pi = \mathsf{C}_{ij}^k \partial_{t_k t_r}^2 \Pi \end{aligned} \tag{2}$$

In (2), $\{\mathfrak{q}_i\}_{i=1}^r$ are the α-basis coefficients of the highest element of the root system of type $\mathcal{G}$, $h_{\mathfrak{g}}$ is the Coxeter number, the coordinates t_i are a homogeneous choice of flat coordinates for the associated affine Weyl Frobenius manifold [20, 31], and C_{ij}^k are the structure constants of the Frobenius product in those coordinates. From the engineering point of view, and as an application to local mirror symmetry/(orbifold) Gromov–Witten theory, the Picard–Fuchs ideals (2) specialise to the GKZ system for the associated $Y^{N,0}$ toric singularities considered in [19, 33] for $\mathcal{G} = \mathrm{SU}(N)$. When $\mathcal{G} \neq \mathrm{SU}(N)$, they further generalise them to the non-toric singularities given by orbifolds of the singular conifold by a finite group action $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ which is McKay-dual to $\mathcal{G}$.

Picard–Fuchs ideals from Jacobi rings

When $\mathcal{G}$ is non-simply laced, a naive application of the above approach fails. The reasons are already well known from the study of the 4d set-up, where the auxiliary Frobenius manifold would arise from the spectral curves of the non-relativistic periodic Toda lattice. It was found shortly after the work of Seiberg–Witten [66] that the integrable system relevant for $\mathcal{N} = 2$ super Yang–Mills theory with gauge group $\mathcal{G}$

[2] The second factor accounts for the complexified radius of the fifth-dimensional circle, or rather its product with the holomorphic energy scale in the gauge theory.

is the Toda lattice associated with the *twisted* Kac–Moody algebra $(\mathcal{G}^{(1)})^\vee$, whereas the construction in [20, 31] of Frobenius manifolds on orbits of extended affine Weyl groups pertains to its untwisted counterpart. Unfortunately there turns out to be no strict analogue of the Frobenius manifolds of [31] in the twisted Kac–Moody world, as the would-be Frobenius metric constructed from the spectral curve becomes either degenerate, or curved, in that setting.

Motivated by work on associativity equations for 4d-prepotentials in [12, 63, 64], we propose an alternative method to construct Picard–Fuchs systems in the form (2) for the pure five-dimensional gauge theory on $\mathbb{R}^4 \times S^1$ in a completely algebraic fashion. Our procedure takes as its input datum the polynomial $\mathsf{P}_{\mathcal{G}}$ defining the five-dimensional spectral curve: the tensor C is identified in this context with the structure constants of a canonical subring of the algebra of regular functions on the zero-dimensional scheme corresponding to the ramification locus of the curve. Its (non-trivial) existence and explicit construction are reduced to a problem in commutative algebra, which we solve for all $\mathcal{G}$ and for all different realisations of Seiberg–Witten geometries when more than one is available (such as for non-simply laced classical groups). By the mirror theorem of [20], for simply laced $\mathcal{G}$ this restricts to the structure constants of the affine-Weyl Frobenius algebras mentioned previously.

Owing to the absence of an underlying Frobenius metric for non-simply laced $\mathcal{G}$, there is now no natural notion of "flat" coordinates $t_i(u)$, and it is a priori unclear how to fix a privileged chart for writing down something like (2). We claim that the constraints of rigid special Kähler geometry, and in particular the existence of a prepotential, are in fact sufficient to uniquely pin down an analogous coordinate frame $t_i(u)$ for general $\mathcal{G}$, without reference to an underlying flat metric.

Matching with the gauge theory, and an integrable systems surprise

Using Frobenius' method, the above constructions provide an efficient way of computing the prepotential given the SW/B-model mirror curve, and therefore the putative five-dimensional prepotentials for the corresponding pure gauge theory, around any boundary point in moduli space, and in particular at infinity in the Coulomb branch.

We put this to the test in a wide array of cases. When the SW geometry arises from geometric engineering [13, 52] or brane constructions in M-theory [42, 55, 59], we show that our constructions recover the microscopic gauge theory results with flying colours. We match the perturbative and instanton parts of the prepotential with the expressions arising from the K-theoretic version of the Nakajima–Yoshioka blow-up equations [73], as well as with their extrapolation to general gauge groups [54]. We also apply our proposal to the SW curves for classical non-simply laced groups obtained from string theory constructions with orientifold planes [15, 42, 59]. As a by-product of our construction we show, in type B_n, that the curves proposed by Brandhuber et al. [15] from an M-theory uplift of the Hanany–Witten construction with orientifold planes correctly reproduce the instanton prepotential for $\mathrm{SO}(2n+1)$ gauge groups, and in type C_n, we confirm a tension between the results of Brandhuber et al. [15] for $n = 1$ and more recent proposals of $\mathrm{Sp}(1)$ curves with discrete θ-angle at $\theta = \pi$ in [42, 59]. Although the two curves are closely related, and their periods are shown to be annihilated by the differential ideal generated by the second line of (2) for the same

choice of structure constants C_{ij}^k, in this rank-1 case determining the full expression for the periods requires fixing a finite-dimensional ambiguity akin to imposing a quasi-homogeneity condition as in the first line of (2): our calculations show that this is done differently for the curves from Brandhuber et al. [15] on one hand and of Hayashi et al. [42] and Li and Yagi [59] on the other, ruling out the former and confirming the correctness of the latter with a direct prepotential calculation. We also match our B-model construction applied to the $\theta = 0$ version of the brane web geometries with an O5-plane of Hayashi et al. [42] and Li and Yagi [59] against the corresponding instanton calculation of the prepotential—again finding perfect agreement.

We furthermore apply our construction to the spectral curves of the periodic relativistic Toda chain on the Langlands dual group of the affine Poisson–Lie group $\mathcal{G}^{(1)}$ [86], which are natural relativistic deformations of the well-known four-dimensional SW curves for the $\mathcal{N} = 2$ theory with gauge group $\mathcal{G}$ [66], and which have already implicitly appeared by the usual procedure of Dynkin folding in the context of geometric engineering on local CY singularities [13]. For non-simply laced cases, these are formally different from the curves obtained from string engineering—and for exceptional $\mathcal{G}$, they are to our knowledge the only candidates available so far for a B-model/SW description of the low-energy theory.

We show that our construction is (non-trivially) well defined in this context as well: there is an $(r + 1)$-dimensional vector subspace of the ring of regular functions of the branch locus of the SW curve, non-trivially closing under multiplication to a subring with structure constants $\mathsf{C}_{ij}^k, i, j, k = 0, \ldots, r$, and once again a distinguished chart in the sense of (2) is shown to exist. We then proceed to solving the resulting Picard–Fuchs system at large complex structure: for non-simply laced groups, the comparison with the gauge theory result surprisingly *fails* in this case, away from the divisor in the Coulomb branch corresponding to the non-relativistic/4d limit, already for the perturbative prepotential. We corroborate our findings with an explicit residue calculation of the triple derivatives of the 1-loop prepotential from the SW curve, which we find in agreement with the calculation from our proposed Picard–Fuchs system. It thus remains an open problem to identify the correct integrable system counterpart of non-simply laced gauge theories on $\mathbb{R}^4 \times S^1$, and, for exceptional non-simply laced groups, to determine their SW geometry.

Organisation of the paper

The paper is structured as follows. In Sect. 2, we give a review of instanton counting and blow-up equations in four and five dimensions on the one hand, and five-dimensional Seiberg–Witten curves from string theory engineering and integrable systems on the other. Then, in Sect. 3, we formulate our B-model approach to compute the gauge theory prepotential for simply laced Lie algebras from the odd periods of the corresponding extended affine Frobenius manifold and present detailed tests of our proposal in low rank. In Sect. 4, we generalise our construction to non-simply laced examples and provide an extended set of examples supporting its validity. A summary and prospects for future work are discussed in Sect. 5. We collect the notation used throughout the text in Table 1 for the reader's convenience.

2 Gauge theories and instanton counting in five dimensions

The field content of the minimally supersymmetric Yang–Mills theory with gauge group $\mathcal{G}$ in five dimensions is given by a gauge field A_μ with a Dirac spinor λ and a real scalar ϕ, both in the adjoint representation of $\mathcal{G}$. Upon compactification on a circle of radius R_5 to $\mathbb{R}^4 \times S^1_{R_5}$, the (classical/quantum) moduli space of the theory is parametrised by the (classical/quantum) vev of the complexified scalar $\varphi := \phi + \mathrm{i}A_5$. Fixing a set of linear generators $\{h_i\}_{i=1}^r$ for the Lie algebra of the maximal torus of $\mathcal{G}$ and in a diagonal gauge for φ, we shall write these, respectively, as

$$\langle\varphi\rangle_{\mathrm{cl}} = \sum_{i=1}^r a_i^{\mathrm{cl}} h_i, \quad \langle\varphi\rangle = \sum_{i=1}^r a_i h_i, \tag{3}$$

and write $q_i^{\mathrm{cl}} := \mathrm{e}^{2\pi\mathrm{i}R_5 a_i^{\mathrm{cl}}}$, $q_i := \mathrm{e}^{2\pi\mathrm{i}R_5 a_i}$ for the corresponding exponentiated linear coordinates on the Cartan torus. An alternative set of coordinates which arises naturally in B-model approaches to $\mathcal{N} = 2$ theories is given by the classical vev of (the conjugacy class of) the complexified Wilson loop $g = P\exp\int_{S^1_{R_5}}(-\mathrm{i}\varphi)$. A choice of r-independent Ad-invariant holomorphic functions on $\mathcal{G}$ fixes then a chart on the classical Coulomb branch, a natural one being given by the fundamental traces

$$u_i := \left\langle \mathrm{Tr}_{\rho_i} P\exp\int_{S^1_{R_5}} (A_5 - \mathrm{i}\phi)\right\rangle_{\mathrm{cl}}, \quad i = 1, \ldots, r, \tag{4}$$

where ρ_i is the irreducible representation having the ith fundamental weight ω_i of $\mathcal{G}$ as its highest weight. By (3) and their definition in (4), the u-coordinates are Weyl-invariant integral Laurent polynomials in $(q_1^{\mathrm{cl}}, \ldots, q_r^{\mathrm{cl}})$:

$$u_i \in \mathbb{Z}[q_1^{\mathrm{cl}}, \ldots, q_r^{\mathrm{cl}}]. \tag{5}$$

Treating further the compactification radius R_5 as a free parameter, there is an additional dimensionless modulus compared to the usual four-dimensional theory given by

$$u_0 := \left(\frac{1}{R_5\Lambda_{\mathrm{QCD}}}\right)^{2h_{\mathfrak{g}}}, \tag{6}$$

where $\Lambda_{\mathrm{QCD}} = \Lambda_{\mathrm{UV}}\mathrm{e}^{-\frac{1}{4g^2_{\mathrm{YM}}(\Lambda_{\mathrm{UV}})}}$ is the dynamical gauge theory scale, and $h_{\mathfrak{g}}$ is the dual Coxeter number. We will often write $q_0 := u_0^{-1}$ for its inverse, so that q_0 corresponds to the perturbative limit of the gauge theory.

When $R_5 = \infty$, the prepotential of the theory was shown by Intriligator–Morrison–Seiberg (IMS) to be exact at one loop [45], and it takes the form of a cubic polynomial

Table 1 Notation employed throughout the text

Symbol	Meaning	First instance
$\mathcal{G}$; $\mathcal{G}^{(1)}$	The compact form of a simple, simply connected Lie group; the corresponding affine Lie group.	Section 1
$\mathfrak{g}/\mathfrak{g}^{(n)}$	The simple complex Lie algebra $\mathrm{Lie}_{\mathbb{C}}(\mathcal{G})$; the corresponding affine version in Kac notation with twisting order n.	Sections 1 and 2.2.2
$\mathcal{K}(\mathfrak{g})/\mathcal{K}(\mathfrak{g}^{(n)})$	The Cartan matrix of the simple/(twisted)-affine Lie algebra $\mathfrak{g})/\mathfrak{g}^{(n)}$	(31)
$\mathcal{A}(\mathfrak{g})$; $\mathcal{A}(\mathfrak{g}^{(1)})$	A simply laced simple (resp. affine) Lie algebra, from which $\mathfrak{g}$ (resp. $(\mathfrak{g}^{(1)})^{\vee}$) is obtained as a quotient by outer automorphisms.	Section 2.2.2
$\Delta/\Delta^{+}/\Delta_{\ell}/\Delta_{\ell}^{+}$	The set of all/positive/long/long positive roots of $\mathfrak{g}$	Section 2
$\rho_{[i_1 \ldots i_r]}$	The irreducible representation of $\mathcal{G}$ with highest weight $\omega = [i_1 \ldots i_r]$ in Dynkin notation	Section 2.2.2
ω_i (resp. ρ_i)	The ith fundamental weight (resp. representation) of $\mathcal{G}$	Section 2
$\mathfrak{q}_i$, $\mathfrak{q}_i^{\vee}$	The Coxeter (resp. dual Coxeter) coefficients of $\mathfrak{g}$	(2)
$\mathcal{F}_{\mathcal{G}}$, $\mathcal{F}_{\mathcal{G}}^{[n]}$	The prepotential (resp. the nth instanton contribution to it) of the $\mathcal{N}=2$ pure theory with gauge group $\mathcal{G}$ on $\mathbb{R}^{1,3} \times S^1_{R_5}$.	(7)–(9)
$\mathsf{R}_{\mathcal{G},\rho;u}$, $\widetilde{\mathsf{R}}_{\mathcal{G};u}$	The characteristic (resp. reduced) polynomial of the Lax operator for the untwisted affine relativistic Toda chain in the representation ρ	(33)
$\mathsf{T}_{\mathcal{G},\rho;u}$, $\widetilde{\mathsf{T}}_{\mathcal{G},\rho;u}$	The characteristic (resp. reduced) polynomial of the Lax operator for the twisted affine relativistic Toda chain in the representation ρ	(38)
$\mathsf{Q}_{\mathcal{G};u}$, $\widetilde{\mathsf{Q}}_{\mathcal{G};u}$	The defining (resp. reduced) polynomial of the Hanany–Witten M-theory curve for type $\mathcal{G}$	(25)–(30)
$\mathsf{P}_{\mathcal{G};u}$; $\widetilde{\mathsf{P}}_{\mathcal{G};u}$	One of Q, R, T; resp. $\widetilde{\mathsf{Q}}$, $\widetilde{\mathsf{R}}$, $\widetilde{\mathsf{T}}$ above	Section 1
$(u_0, \ldots, u_r)$	Ad-invariant classical coordinates on the Coulomb branch	Section 2
$(q_0, \ldots, q_r)$	Exponentiated A-model Coulomb moduli	Section 2
$(z_0, \ldots, z_r)$	B-model coordinates around the large complex structure point	(60)
$(t_0, \ldots, t_r)$	Flat coordinates of $M_{\mathfrak{g}}^{\text{trig}}$; the distinguished coordinates of (95)	(2)
$\mathcal{V}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$	The canonical subring of the space of regular functions on the ramification locus spanned by $\{\partial_{t_i} \widetilde{\mathsf{P}}_{\mathcal{G};u}/\partial_{t_r} \widetilde{\mathsf{P}}_{\mathcal{G};u}\}$	Section 4.1
$\mathrm{GB}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$	A reduced Gröbner basis for $\mathcal{D}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$	Section 4.1
C_{ij}^{k}	Structure constants of $\mathcal{V}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$ in the t- chart	(2)
$M_{\mathfrak{g}}^{\text{trig}}$; $F_{\mathfrak{g}}^{\text{trig}}$	The extended affine Frobenius manifold of type $\mathfrak{g}$; its prepotential	Section 3.1

in the real scalars φ,

$$\mathcal{F}_{\mathcal{G}}\Big|_{R_5=\infty} = \frac{1}{2g_{\mathrm{YM}}^2}\mathsf{g}_{ij}\varphi^i\varphi^j + \frac{k_{\mathrm{CS}}}{6}\mathsf{d}_{ijk}\varphi^i\varphi^j\varphi^k + \frac{1}{6}\sum_{\alpha\in\Delta}\theta(\alpha(\varphi))\alpha(\varphi)^3 \tag{7}$$

where g_{ij} and d_{ijk} are, respectively, the Killing pairing and the cubic Casimir form for $\mathfrak{g}$, $k_{\mathrm{CS}} \in \mathbb{Z}$ is the five-dimensional Chern–Simons level, and $\theta(x)$ denotes Heaviside's step function. Upon compactification on $S^1_{R_5}$, the prepotential receives finite u_0 perturbative corrections from an infinite tower of excited Kaluza–Klein states, as well non-perturbative instanton corrections. The former are resummed as [74]

$$\mathcal{F}_{\mathcal{G}}^{[0]} = -\frac{\mathrm{i}}{8\pi^3}\sum_{\alpha\in\Delta}\left[(2\pi\mathrm{i}R_5\alpha(a))^2\,\frac{\log q_0}{2} + \mathrm{Li}_3\left(\mathrm{e}^{2\pi\mathrm{i}R_5\alpha(a)}\right)\right] \tag{8}$$

, while the latter gives rise to an infinite sum of the form

$$\mathcal{F}_{\mathcal{G}}^{\mathrm{np}} = \sum_{n>0} q_0^n\mathcal{F}_{\mathcal{G}}^{[n]}(q_1,\ldots,q_r), \tag{9}$$

with $\mathcal{F}_{\mathcal{G}}^{[n]} \in (\mathrm{i}/2\pi)^3\mathbb{Q}[[q_1,\ldots,q_r]]$.

2.1 A-model: K-theoretic instanton counting and blow-up equations

We start off by giving a quick summary of instanton partition functions in four and five dimensions, and of the generalised blow-up equations [54, 73] that recursively determine them.

2.1.1 Instanton counting à la Nekrasov: the SU($r+1$) case

Let $R_5 = 0$. By $\mathcal{N} = 2$ supersymmetry, the path integral calculation of non-perturbative contributions to the prepotential localises on instantons—anti-self-dual connections on a principal $\mathcal{G}$-bundle on $S^4 \simeq \mathbb{R}^4 \cup \{\mathrm{pt}\}$ with fixed second Chern character, modulo gauge equivalence. For $\mathcal{G} = \mathrm{SU}(r+1)$ [75], this space has an algebraic compactification to the framed moduli space $\mathcal{M}(r,n)$ of torsion-free sheaves on $\mathbb{P}^2$ of rank $r+1$ and $\langle c_2(E), [\mathbb{P}^2]\rangle = n$: given a pair of integers (r,n), $\mathcal{M}(r,n)$ parametrises isomorphism classes of a pair (E,Φ) such that [72]:

1. E is a torsion-free sheaf of rank$E = r+1$ and $\langle c_2(E), [\mathbb{P}^2]\rangle = n$ which is locally free in a neighbourhood of $\ell_\infty = \mathbb{P}^2\backslash\mathbb{C}^2$,
2. $\Phi : E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}^{\oplus r+1}$ is an isomorphism, which implies $c_1(E) = 0$

$\mathcal{M}(r,n)$ is a $2n(r+1)$-dimensional smooth quasi-projective complex variety, with the open subset $\mathcal{M}_0^{\mathrm{reg}}(r,n)$ of locally free sheaves coinciding with the moduli space of instantons on $\mathbb{S}^4$ of rank $r+1$ and $c_2 = n$ [26]. It also carries a $\tilde{T} := (\mathbb{C}^\star)^2 \times (\mathbb{C}^\star)^r$ action coming from the scaling actions on $\mathbb{P}^2$ and the maximal torus of SU$(r+1)$,

with zero-dimensional fixed loci $\mathcal{M}^{\tilde{T}}(r,n)$. Let $\mathbb{Q}(\epsilon_1,\epsilon_2;a_1,\dots,a_r)$ be the field of fractions of $H_{\tilde{T}}(\mathrm{pt}) = H(B\tilde{T},\mathbb{Q}) \simeq \mathbb{Q}[\epsilon_1,\epsilon_2;a_1,\dots,a_r]$. The instanton partition function is defined by the Atiyah–Bott formula as the equivariant volume

$$\mathcal{Z}_{\mathcal{G}}^{\mathrm{np},4d} := \sum_{n\geq 0} q_0^n \int_{\mathcal{M}^{\tilde{T}}(r,n)} \frac{1}{\mathrm{e}_{\tilde{T}}\left(N_{\mathcal{M}^{\tilde{T}}(r,n)/\mathcal{M}(r,n)}\right)} \in \mathbb{Q}(\epsilon_1,\epsilon_2,a_1,\dots,a_r). \quad (10)$$

The localisation formula reduces the computation of $\mathcal{Z}_{\mathcal{G}}^{\mathrm{np},4d}$ to a combinatorial question involving sums of 2D-partitions, see [71, 72] for explicit formulas. It was conjectured in [75] and proved in [72, 76] that the 4d prepotential computed from the Seiberg–Witten curve coincides with the logarithm of the instanton partition function in the non-equivariant/flat Ω-background limit, $\mathcal{F}_{\mathcal{G}}^{4\mathrm{d}} = \lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2 \log \mathcal{Z}_{\mathcal{G}}^{\mathrm{np},4\mathrm{d}}$. A natural generalisation to five compactified dimensions was also addressed by Nekrasov [75] and later proven by Nakajima–Yoshioka [73], by uplifting (10) to equivariant K-theory:

$$\mathcal{Z}_{\mathcal{G}}^{\mathrm{np},5d} = \sum_{n\geq 0}\left(q_0 \mathrm{e}^{-rR_5(\epsilon_1+\epsilon_2)/2}\right)^n Z_{\mathcal{G},n}^{\mathrm{np},5d}, \quad (11)$$

$$Z_{\mathcal{G},n}^{\mathrm{np},5d} = \sum_i (-1)^i \,\mathrm{ch}\, H^i(\mathcal{M}(r,n),\mathcal{O}), \quad (12)$$

where ch denotes the Hilbert series [72,Section 4.1]. Accordingly, the non-perturbative part of the prepotential (9) is given by

$$\mathcal{F}_{\mathcal{G}}^{\mathrm{np}} = \lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2 \log \mathcal{Z}_{\mathcal{G}}^{\mathrm{np},5\mathrm{d}}. \quad (13)$$

2.1.2 Blow-up equations

An efficient tool to evaluate (12) was put forward by [72], in the form of a comparison formula between the instanton partition function on $\mathbb{P}^2$ and the one on successive blow-ups at points. The upshot of that is a recursive relation for $\mathcal{Z}_{\mathcal{G}}^{\mathrm{np},5d}$ in terms of instanton numbers n. Let $\mathbb{F}_1$ be the first Hirzebruch surface,

$$\mathbb{F}_1 = \{[z_0,z_1,z_2],[x,y] \in \mathbb{P}^2\times\mathbb{P}^1 | z_1 y = z_2 x\}. \quad (14)$$

We denote by C the exceptional divisor defined by $z_0 = 0$, by $\mathcal{O}(C)$ the line bundle associated with the divisor C, and by $\mathcal{O}(mC)$ the m-th tensor power for some $m \in \mathbb{Z}_{\geq 0}$, and consider the framed moduli space $\widehat{\mathcal{M}}(r,n,k)$ of torsion-free sheaves (E,Φ) on $\mathbb{F}_1$, where

$$\langle c_1(E),[C]\rangle = -k, \quad \left\langle c_2(E) - \frac{r}{2(r+1)}c_1(E)^2, [\mathbb{F}_1]\right\rangle = n. \quad (15)$$

Although n is not an integer in general, $\widehat{\mathcal{M}}(r,n,k)$ is non-singular of dimension $2n(r+1)$, and it carries a $\mathbb{C}^2\times\mathbb{C}^r$-torus action as in the previous section. The corresponding K-theoretic instanton partition function for $\mathbb{F}_1$ is

$$\hat{Z}^{\mathrm{np},5d}_{\mathrm{SU}(r+1),m,k}=\sum_{n\geq 0}\left(q_0\mathrm{e}^{-rR_5(\epsilon_1+\epsilon_2)/2}\right)^n \mathrm{ch}\sum_i(-1)^iH^i(\widehat{\mathcal{M}}(r,n,k),\mathcal{O}(mC)). \tag{16}$$

and is defined for each tensor power $\mathcal{O}(mC)$. In terms of $\hat{Z}_{\mathrm{SU}(r+1),m,k}$, Nakajima–Yoshioka [73] establish a finite difference equation in the equivariant parameters and Coulomb moduli, as follows. Let $\mathcal{G}=\mathrm{SU}(r+1)$ and $\vec{a}\in\mathbb{R}^{r+1}$ with $\sum_i a_i=0$ be an element of the Cartan subalgebra of $\mathfrak{g}=\mathfrak{sl}_{r+1}$, and further write $K^\vee$ for the special linear coroot lattice, $K^\vee=\{\vec{k}\in\mathbb{Z}^{r+1}|\sum_i k_i=0\}$. Then for $k=0$, $\hat{Z}^{\mathrm{np},5d}_{\mathcal{G},m,0}$ satisfies

$$\begin{aligned}\hat{Z}^{\mathrm{np},5d}_{\mathcal{G},m,0}=\sum_{\vec{k}\in K^\vee}&\frac{(q_0\mathrm{e}^{R_5(\epsilon_1+\epsilon_2)(m-h^\vee_{\mathfrak{g}}/2)})^{(\vec{k},\vec{k})/2}e^{(\vec{k},\vec{a})m}}{\prod_{\alpha\in\Delta}l^{\vec{k}}_\alpha(\epsilon_1,\epsilon_2,\vec{a})}\\&\times Z^{\mathrm{np},5d}_{\mathcal{G}}(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\epsilon_1\vec{k};e^{R_5\epsilon_1(m-h^\vee_{\mathfrak{g}}/2)}q_0,R_5)\\&\times Z^{\mathrm{np},5d}_{\mathcal{G}}(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\epsilon_2\vec{k};e^{R_5\epsilon_2(m-h^\vee_{\mathfrak{g}}/2)}q_0,R_5),\end{aligned} \tag{17}$$

where

$$l^{\vec{k}}_\alpha(\epsilon_1,\epsilon_2,\vec{a})=\begin{cases}\prod_{\substack{i,j>0\\ i+j\leq-(\vec{k},\vec{a})-1}}\left(1-e^{R(i\epsilon_1+j\epsilon_2-(\vec{a},\alpha))}\right) & \text{if } (\vec{k},\alpha)<0,\\ \prod_{\substack{i,j>0\\ i+j\leq(\vec{k},\vec{a})-2}}\left(1-e^{R(-(i+1)\epsilon_1-(j+1)\epsilon_2-(\vec{a},\alpha))}\right) & \text{if } (\vec{k},\alpha)>1,\\ 1 & \text{otherwise}\end{cases} \tag{18}$$

Furthermore, for all $m\in\{0,1,...,r+1\}$, we have that

$$\hat{Z}^{\mathrm{np},5d}_{\mathcal{G},m,0}=Z^{\mathrm{np},5d}_{\mathcal{G}}, \tag{19}$$

Note that $h^\vee_{\mathfrak{sl}_{r+1}}=r+1$ for $\mathrm{SU}(r+1)$. From (17), it is possible to obtain a recursion for $Z^{\mathrm{np},5d}_{\mathcal{G},n}$, i.e. the expansion coefficients in terms of q_0 (16): see [73].

2.1.3 General gauge groups

The discussion above is restricted to the case of $\mathcal{G}=\mathrm{SU}(r+1)$, and except for the classical gauge groups where an ADHM construction is known [37, 61, 77], it is a priori unclear how to deduce an immediate generalisation of (17) for other Lie types. However, it is not hard to extrapolate (17) to general simple, simply connected $\mathcal{G}$ as the r.h.s. is entirely written in terms of purely root-theoretic data [54]: indeed, replacing the type A root system of (17) with the one of an arbitrary simple Lie algebra $\mathfrak{g}$ leads

to formulas for 1-instanton partition which are consistent with supersymmetric index calculations [8, 38, 53]. By taking these results into consideration, the authors of [54] derived another recursion for $Z^{\mathrm{np},5d}_{\mathcal{G},n}$ from (17) in the following form:

$$Z^{\mathrm{np},5d}_{\mathcal{G},n} = \frac{\mathrm{e}^{R_5 n(\epsilon_1+\epsilon_2)} I_n^{(0)} - \left(\mathrm{e}^{R_5 n\epsilon_1} + \mathrm{e}^{R_5 n\epsilon_2}\right) I_n^{(1)} + I_n^{(2)}}{(1-\mathrm{e}^{R_5 n\epsilon_1})(1-\mathrm{e}^{R_5 n\epsilon_2})}, \tag{20}$$

where

$$I_n^{(m)} = \sum_{\substack{\vec{k}\in K^\vee \\ i,j<n}}^{\frac{1}{2}(\vec{k},\vec{k})+i+j=n} \frac{\exp\left(R_5 m\left(i\epsilon_1 + j\epsilon_2 + (\vec{k},\vec{a}) + (\vec{k},\vec{k})\frac{\epsilon_1+\epsilon_2}{2}\right)\right)}{\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\epsilon_1,\epsilon_2,\vec{a})}$$
$$\times\ Z^{\mathrm{np},5d}_{\mathcal{G},i}(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\epsilon_1\vec{k},R_5)\, Z^{\mathrm{np},5d}_{\mathcal{G},j}(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\epsilon_2\vec{k},R_5). \tag{21}$$

Note that $I_n^{(m)}$ is determined by $Z^{\mathrm{np},5d}_{\mathcal{G},n'}$ with $n' < n$, and $Z^{\mathrm{np},5d}_{\mathcal{G},0} = 1$. In particular, denoting Δ_ℓ the set of long roots, we get

$$Z^{\mathrm{np},5d}_{\mathcal{G},1}(\epsilon_1,\epsilon_2,\vec{a}) = \frac{1}{(1-\mathrm{e}^{-\epsilon_1})(1-\mathrm{e}^{-\epsilon_2})}$$
$$\sum_{\gamma\in\Delta_\ell} \frac{1}{(1-\mathrm{e}^{-\epsilon_1-\epsilon_2+\gamma\cdot a})(1-\mathrm{e}^{-\gamma\cdot a})\prod_{\alpha\cdot\gamma=1}(1-\mathrm{e}^{-\alpha\cdot a})}, \tag{22}$$

from which we find

$$\mathcal{F}^{[1]}_{\mathcal{G}} = \sum_{\gamma\in\Delta_\ell} \frac{1}{(1-\mathrm{e}^{\gamma\cdot a})(1-\mathrm{e}^{-\gamma\cdot a})\prod_{\alpha\cdot\gamma=1}(1-\mathrm{e}^{-\alpha\cdot a})}, \tag{23}$$

where we used (13).

2.2 B-model: Seiberg–Witten geometries, old and new

We move on to review the B-model counterpart of instanton counting, given by special geometry on families of spectral curves fibred over the Coulomb branch. We discuss in turn the two main sources of these for five-dimensional gauge theories, namely M-theory engineering and the spectral curves from relativistic integrable models. The two approaches lead to a priori inequivalent B-model geometries for non-simply laced groups, as we now review.

2.2.1 Spectral curves from M-theory engineering

String theory embeddings are an extremely helpful tool to analyse the infrared behaviour of supersymmetric quantum field theories. For the case of eight supercharges, it is well known that one can realise SUSY gauge theories with gauge

symmetry given by products of classical Lie groups and bifundamental matter as the low-energy theory on systems of D-branes suspended between NS5-branes [41, 55, 89]. In the strong type II string coupling limit, the resulting M-theory picture is given by a single smooth five-brane set-up, containing the Seiberg–Witten curve as a factor [15, 89].

For 5d theories with a single $\mathcal{G} = \mathrm{SU}(N)$ factor and no hypermultiplets, one considers a IIB configuration on $\mathbb{R}^{1,3} \times S^1_{R_5} \times \mathbb{R} \times \mathbb{R}^4$ with N light D5-branes located at $x^i = 0$, $i = 5, 7, 8, 9, 10$, sweeping a full $\mathbb{R}^{1,3} \times S^1_{R_5}$ in the $x^0, \ldots x^4$ directions and suspended between two non-dynamical solitonic five branes classically situated at finite distance in the x^6 coordinate; the latter are extended in the $x^0, \ldots, x^5$ coordinates. The theory on the D5-branes is macroscopically five-dimensional, owing to the finite stretch in the x^6 direction. This brane set-up is T-dual to the constructions considered for theories with eight supercharges in 3d and 4d, respectively, by Hanany and Witten [41] and Witten [89], and it admits an M-theory description on $\mathbb{R}^{1,3} \times S^1_{R_5} \times \mathbb{R} \times \mathbb{R}^4 \times S^1_{R_{11}}$ in terms of an M5 brane wrapping a supersymmetric cycle $\mathbb{R}^{1,3} \times \mathcal{C}$, for a non-compact Riemann surface $\mathcal{C}$. In terms of the $\mathbb{C}^\star$-coordinates $\mu = \mathrm{e}^{-\mathrm{i}(x^4+\mathrm{i}x^5)/R_5}$ and $\lambda = \mathrm{e}^{-(x^6+\mathrm{i}x^{10})/R_{11}}$, $\mathcal{C}$ satisfies an algebraic equation of the form

$$\mathsf{Q}_{\mathrm{SU}(N);u}(\mu, \lambda) = \lambda^2 + Q_1(\mu)\lambda + Q_2(\mu) = 0 \tag{24}$$

where roots of $Q_1(\mu)$ (resp. $Q_2(\mu)$) correspond, in the type IIB limit, to the location of D5-branes between NS5s (resp. at $x^6 = \infty$, corresponding to fundamental hypermultiplets in the gauge theory on the D5 worldvolume). For the case of no hypermultiplets and Chern–Simons level $k_{\mathrm{CS}} \in \{0, \ldots, N\}$, one finds [15]

$$\mathsf{Q}_{\mathrm{SU}(N);u}(\mu, \lambda) = \lambda^2 + \lambda \left[1 + \sum_{k=1}^{N-1} u_k \mu^k + \mu^N \right] + q_0 \mu^{N-k_{\mathrm{CS}}}. \tag{25}$$

Other classical gauge groups can be analysed in the same manner through the addition of orientifold O5-planes in the type IIB picture. This again can be subsumed into an M-theory description in terms of a fivebrane with worldvolume containing the affine part $\{(\mu, \lambda) \in \mathbb{C}^\star \times \mathbb{C}^\star | \mathsf{Q}_{\mathcal{G};u}(\mu, \lambda) = 0\}$ of a hyperelliptic Riemann surface. In the special orthogonal case, this yields [15]

$$\begin{aligned} \mathsf{Q}_{\mathrm{Spin}(2N+1);u}(\mu, \lambda) &= \mu^{2N-3} \left(\mu^2 - 1\right) \left(\mu^2 + 1\right)^2 (\lambda^2 + q_0) \\ &\quad + \lambda \left[1 + \sum_{k=1}^{2N-1} u_k \mu^{2k} + \mu^{4N} \right], \end{aligned} \tag{26}$$

$$\begin{aligned} \mathsf{Q}_{\mathrm{Spin}(2N);u}(\mu, \lambda) &= \mu^{N-2} (\mu^2 - 1)^2 \left(\lambda^2 + q_0\right) \\ &\quad + \lambda \left[1 + \sum_{k=1}^{2N-1} u_k \mu^k + \mu^{2N} \right]. \end{aligned} \tag{27}$$

where $u_k = u_{2N-k}$. For symplectic gauge groups, since $\pi_4(\mathrm{Sp}(N)) = \mathbb{Z}/2$, the gauge theory path integral depends on an additional discrete ambiguity, which we can regard as a 5d analogue of a theta angle $\mathrm{e}^{\mathrm{i}\theta}$ where $\theta \in \{0, \pi\}$. The difference between the two choices can be reabsorbed in a rescaling of the masses of the fundamental hypermultiplets when present, but it is physical for the pure gauge theory, leading to inequivalent prepotentials. Accordingly, two sets of curves $\{\mathsf{Q}_{\mathrm{Sp}(N);u,\theta} = 0\}$ were put forward in [42, 59] to account for this,

$$\begin{aligned}\mathsf{Q}_{\mathrm{Sp}(N);u,0}(\mu,\lambda) = \mu^{N+2}\left(\lambda^2+q_0\right) + \lambda\Bigg[\sum_{k=2}^{N+2} u_k\left(\mu^{N+2+k}+\mu^{N+2-k}\right) \\ + \left(q_0 - 1 - \sum_{k=1}^{\frac{N-1}{2}} u_{2k+1}\right)\left(\mu^{N+3}+\mu^{N+1}\right) - 2\sum_{k=1}^{\frac{N+1}{2}} u_{2k}\Bigg],\end{aligned} \tag{28}$$

$$\begin{aligned}\mathsf{Q}_{\mathrm{Sp}(N);u,\pi}(\mu,\lambda) = \mu^{N+2}\left(\lambda^2+q_0\right) + \lambda\Bigg[\sum_{k=2}^{N+2} u_k\left(\mu^{N+2+k}+\mu^{N+2-k}\right) \\ - \left(1 + \sum_{k=1}^{\frac{N-1}{2}} u_{2k+1}\right)\left(\mu^{N+3}+\mu^{N+1}\right) - 2\left(q_0 + \sum_{k=1}^{\frac{N+1}{2}} u_{2k}\right)\Bigg].\end{aligned} \tag{29}$$

with $u_{N+2} = 1$. A pre-existing set of curves for the case with $N_f \leq N+2$ fundamental flavours was also proposed in [15], although the discussion of the θ-angle dependence was not carried out. Naively setting $N_f = 0$ leads to an alternative M-theory curve whose defining polynomial we denote by $\mathsf{Q}^{\flat}_{\mathrm{Sp}(N);u}$, where

$$\mathsf{Q}^{\flat}_{\mathrm{Sp}(N);u}(\mu,\lambda) = \mu^{N+2}\left(\lambda^2+q_0\right) + \lambda(\mu^2-1)^2\left[1 + \sum_{k=1}^{2N-1} u_k\mu^k + \mu^{2N}\right] \tag{30}$$

and $u_k = u_{2N-k}$. We shall in fact see in Sect. 4.5 in the example $N = 1$ that the prepotential computed from (30) disagrees with the Sp(1) gauge theory prepotential for either choice of discrete θ-angle.

2.2.2 Spectral curves from integrable systems

Another, and historically the first [74], main source of spectral curves for gauge theories in five compactified dimensions comes from the theory of relativistic integrable systems; and in particular, for the pure gauge theory, from the study of the affine relativistic Toda chain [21, 74, 82]. We refer to [13, 16, 36, 74, 86, 88][3] for background

[3] See also[39, 70] and references therein for spectral dualities between two different representations of the Lax operator.

and further details, condensing here the information relevant for the constructions of Sect. 4.

The phase space of the relativistic Toda chain associated with the (untwisted) affine Lie group $\mathcal{G}^{(1)}$ is the $(2r+2)$-dimensional Poisson algebraic torus $\big((\mathbb{C}_x^\star)^{r+1}\times(\mathbb{C}_y^\star)^{r+1},\{,\}_{\mathcal{G}}\big)$ with log-constant Poisson bracket

$$\{x_i,y_j\}_{\mathcal{G}} := \mathcal{K}(\mathfrak{g}^{(1)})_{ij}x_iy_j. \tag{31}$$

Since $\mathcal{K}(\mathfrak{g}^{(1)})$ has a one-dimensional kernel, (31) has a single Casimir function, given by

$$u_0 := \prod_{i=0}^{r} x_i^{-2\mathfrak{q}_i}, \tag{32}$$

with $\mathfrak{q}_i$ the Coxeter exponents of $\mathfrak{g}^{(1)}$. Dynamical commuting flows are specified by the spectral-parameter-dependent Lax operator [36, 57, 86],

$$L_{\mathcal{G}}(x,y;\lambda) = E_0(\lambda/y_0)E_{\bar{0}}(\lambda)\prod_{i=1}^{r} H_i(x_i)E_i(1)H_i(y_i)E_{-i}(1) \tag{33}$$

where $E_i(x)=\exp(xe_i)$, $H_i(x)=\exp(xh_i)$, and $\{e_{\pm i},h_i\}_{i=0,\dots,r}$ is a Cartan–Weyl basis for $\mathfrak{g}^{(1)}$.

Let ρ_i, $i=1,\dots,r$ be the ith fundamental representation of $\mathcal{G}$. Then, proceeding as in [16,Lem. 2.2], one finds that the λ dependence of the fundamental traces of the Lax operator (33) is given by

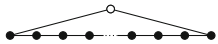
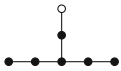
$$\mathrm{Tr}_{\rho_i} L_{\mathcal{G}}(x,y;\lambda) = u_i(x,y) + \begin{cases} \left(\lambda\delta_{iN}+\frac{\delta_{i,N+1}}{u_0\lambda}\right), & \mathcal{G}=\mathrm{SU}(2N+1) \\ \left(\lambda+\frac{1}{u_0\lambda}\right)\delta_{i\bar{k}}, & \text{else,} \end{cases} \tag{34}$$

where $\{u_i\}_{i=1}^r$ is a complete set of commuting first integrals w.r.t. the Poisson structure (31), and $\bar{k}$ labels the fundamental weight $\omega_{\bar{k}}$ corresponding to the largest irreducible Weyl orbit. Fixing a non-trivial representation $\rho'\in\mathrm{Rep}(\mathcal{G})$, these integrals determine, and can be retrieved from, the characteristic Laurent polynomial

$$\begin{aligned} \mathsf{R}_{\mathcal{G},\rho';u}(\mu,\lambda) &:= \det_{\rho'}\big[L_{\mathcal{G}}(x,y;\lambda)-\mu\big] \\ &= \sum_{i=0}^{\dim\rho'}(-\mu)^i\,\mathrm{Tr}_{\wedge^{\dim\rho'-i}\rho}L_{\mathcal{G}}(x,y;\lambda) \\ &\in \mathbb{Z}[\lambda^{\pm},\mu;u_0,\dots,u_r] \end{aligned} \tag{35}$$

where we have used (34) and the fact that $\mathrm{Rep}(\mathcal{G})$ is an integral polynomial ring in the fundamental characters. Note that, for $\mathcal{G}\neq\mathrm{SU}_{2N+1}$, (34) further ensures that there

Table 2 Dynkin diagrams and foldings for twisted affine Lie algebras

Type	Folding	Dynkin diagram
$B_l^\vee = A_{2l-1}^{(2)}$		
$C_l^\vee = D_{l+1}^{(2)}$		
$F_4^\vee = E_6^{(2)}$		
$G_2^\vee = D_4^{(3)}$		

exists $\widetilde{\mathsf{R}_{\mathcal{G},\rho';u}}(\mu,\nu) \in \mathbb{Z}[\mu,\nu;u_0,\ldots,u_r]$ such that

$$\widetilde{\mathsf{R}}_{\mathcal{G},\rho';u}(\mu,\nu)\Big|_{\nu=\lambda+1/(u_0\lambda)} = \mathsf{R}_{\mathcal{G},\rho';u}(\mu,\lambda). \tag{36}$$

As $(u_1,\ldots,u_r)$, from (34), are Weyl-invariant coordinates on the maximal torus of $\mathcal{G}$, the vanishing locus of $\mathsf{R}_{\mathcal{G};u}$ gives (after normalisation of the fibres) a family of spectral curves over the Coulomb branch. For classical simply laced groups[4] $\mathcal{G} = A_r$ or D_r one recovers [13, 57], in particular, the M-theory curves (25) and (27):

$$\mathsf{R}_{\mathrm{SU}(r+1),\rho_1;u} = \mathsf{Q}_{\mathrm{SU}(r+1);u}\Big|_{k_{\mathrm{CS}}=0}, \qquad \mathsf{R}_{\mathrm{Spin}(2r),\rho_1;u} = \mathsf{Q}_{\mathrm{Spin}(2r);u}. \tag{37}$$

The spectral polynomials $\mathsf{R}_{E_n,\rho_{n-1}}$ for $\mathcal{G} = E_n$ were computed[5] in [13, 16].

When $\mathcal{G}$ is of BCFG type, however, it is known that the prepotentials of the spectral curves $\{\mathsf{R}_{\mathcal{G},\rho';u} = 0\}$ are not expected to reproduce the low-energy effective action of the gauge theory. The reason for this was highlighted in [66]: in the $R_5 \to 0$ limit, corresponding to the limit of infinite speed of light of the relativistic Toda chain, the relevant dynamical system for $\mathcal{N} = 2$ super Yang–Mills is the affine non-relativistic Toda system associated with the *twisted* Kac–Moody algebras $(\mathfrak{g}^{(1)})^\vee$ [51, 81], rather than to $\mathfrak{g}^{(1)}$ itself. These are given by quotients of a "parent" untwisted Kac–Moody algebra $\mathcal{A}(\mathfrak{g}^{(1)})$ by their outer automorphism group $\mathrm{Out}(\mathcal{A}(\mathfrak{g}^{(1)}))$, associated with the folding of the Dynkin diagram of $\mathfrak{g}^{(1)}$: see Table 2.

[4] For the special unitary gauge group $\mathcal{G} = \mathrm{SU}(N)$, the equivalence is realised at Chern–Simons level $k_{\mathrm{CS}} = 0$. The effect of a shift in the Chern–Simons level on the integrable chain is discussed in [32, 65].

[5] Although the curves depend on the choice of the representation ρ' in (33), the prepotentials computed for different choices of ρ' only differ by an overall normalisation. The representation independence of the special geometry prepotential is non-trivial and follows from an isomorphism of the flows of the underlying integrable system when projected to a canonical Prym–Tyurin subvariety of the Jacobian: see [16, 67] for an extended discussion.

Accordingly, it is natural to cast the twisting constructions relevant for the affine Lie algebras/non-relativistic Toda chains/$\mathcal{N}=2$ $d=4$ theories at the level of the corresponding affine Lie groups/relativistic Toda chains/$\mathcal{N}=2$ KK theories, by considering the *twisted* Lax operator

$$L^{\vee}_{\mathcal{G}}(x,y;\lambda) := E^{\vee}_0(\lambda/y_0)E^{\vee}_{\bar{0}}(\lambda)\prod_{i=1}^{r} H^{\vee}_i(x_i)E^{\vee}_i(1)H^{\vee}_i(y_i)E^{\vee}_{-i}(1) \tag{38}$$

obtained from (33) by replacing all Chevalley operators with their twisted counterparts. Let $\mathcal{A}(\mathcal{G}) = \exp\mathcal{A}(\mathfrak{g})$): fixing $\mathbf{1} \neq \rho' \in \mathrm{Rep}(\mathcal{A}(\mathcal{G}))$, the associated characteristic polynomial is defined as in (35)

$$\begin{aligned}\mathsf{T}_{\mathcal{G},\rho';u}(\mu,\lambda) &:= \det_{\rho'}\left[L^{\vee}_{\mathcal{G}}(x,y;\lambda)-\mu\right]\\ &= \sum_{i=0}^{\dim\rho'}(-\mu)^i\,\mathrm{Tr}_{\wedge^{\dim\rho'-i}\rho}L^{\vee}_{\mathcal{G}}(x,y;\lambda)\end{aligned} \tag{39}$$

To determine the spectral dependence of $\mathsf{T}_{\mathcal{G},\rho';u}$ on the λ parameter, let $\mathrm{Eff}(\mathcal{A}(\mathfrak{g}))$ be the set of nodes in the Dynkin diagram of the "parent" simple Lie algebra $\mathcal{A}(\mathfrak{g})$ acted upon effectively by diagram automorphisms. This set of vertices splits in orbits $\mathrm{Eff}(\mathcal{A}(\mathfrak{g})) = \sqcup_j \mathsf{o}^{(2)}_j \sqcup_k \mathsf{o}^{(3)}_k$, where $\mathsf{o}^{(n)}_i$ is a cardinality-n set of Dynkin nodes upon which the outer automorphisms acts as the permutation group S_n in n-elements. Let $\sigma : \mathsf{o}^{(n)}_j \to \mathbb{Z}/n\mathbb{Z}$ be a bijection onto the set of nth roots of unity $\{\omega^l_n\}^{n-1}_{l=0}$, such that if p is the full cyclic permutation of order n in S_n, then $\sigma(p(\mathsf{v})) = \mathrm{e}^{2\pi\mathrm{i}/n}\sigma(\mathsf{v})$. We claim that, up to a rescaling and a shift of $\log\lambda$, the analogue of (34) in the twisted case is

$$\mathrm{Tr}_{\rho_i}L^{\vee}_{\mathcal{G}}(x,y;\lambda) = u_{\pi(i)}(x,y) + \begin{cases}\sigma(\mathsf{v}_i)\lambda + \frac{1}{u_0\lambda\sigma(\mathsf{v}_i)}, & \mathsf{v}_i \in \mathrm{Eff}(\mathcal{A}(\mathfrak{g})),\\ 0 & \text{else},\end{cases} \tag{40}$$

where now the Casimir function is expressed in terms of the *dual* Coxeter exponents

$$u_0 := \prod_{i=0}^{r} x_i^{-2\mathsf{q}^{\vee}_i}, \tag{41}$$

and $\pi : \{\mathsf{v}_i\} \to \{1,\ldots,r\}$ is a choice of label on the orbits, i.e. it is defined up to permutation by $\pi(\mathsf{v}_i) = \pi(\mathsf{v}_j)$ iff $\{\mathsf{v}_i,\mathsf{v}_j\} \subset \mathsf{o}^{(n)}_l$ for some n and l. From (40), we can define as we did for (36) the reduced characteristic polynomial

$$\widetilde{\mathsf{T}}_{\mathcal{G},\rho';u}(\mu,\nu)\Big|_{\nu=\lambda+1/(u_0\lambda)} = \mathsf{T}_{\mathcal{G},\rho';u}(\mu,\lambda). \tag{42}$$

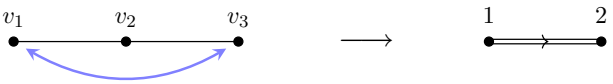


Fig. 1 The Dynkin diagram of $B_2 \simeq C_2$ via $A_3 \simeq D_3$-folding

Example 1 Let us consider the example of $\mathcal{G} = B_2$, so that $\left(B_2^{(1)}\right)^\vee = \left(C_2^{(1)}\right)^\vee = D_3^{(2)} = A_3^{(2)}$. In this case, the folding of the Dynkin diagram of the simple Lie algebra $\mathcal{A}(\mathfrak{b}_2) = \mathfrak{d}_3 \simeq \mathfrak{a}_3$ identifies the vertices $v_1 \leftrightarrow v_3$, corresponding to the highest weights [010] and [001] of the two Weyl spinor representations of D_3 (equivalently, the two complex-conjugate vector representations of A_3) leaving fixed v_2, which labels the six-dimensional defining representation $\rho_{[100]}$ of SO(6) (resp. the adjoint representation of SU(4)), so that $\mathrm{Eff}(\mathfrak{b}_2) = \mathsf{o}_1^{(2)} = \{v_1, v_3\}$: see Fig. 1.

For this example, let $\rho' = \rho_{[010]} = \mathbf{6_v}$ be the six-dimensional fundamental representation of A_3 (i.e. the vector representation of $D_3 = \mathrm{Spin}(6)$) and let $\varepsilon_{kl} \in \mathrm{Mat}(6, \mathbb{C})$ with $(\varepsilon_{kl})_{ij} = \delta_{ik}\delta_{lj}$. Then, the twisted Cartan–Weyl generators are given by Suris [86,Sec. 21.6]

$$e_1^\vee = \varepsilon_{21} - \varepsilon_{65}, \quad e_2^\vee = \varepsilon_{32} - \varepsilon_{53}, \quad e_0^\vee = \varepsilon_{14} - \varepsilon_{46}, \tag{43}$$

with $e_{\bar{i}}^\vee = (e_i^\vee)^T$ and $h_i = [e_i, e_{\bar{i}}]$, $i = 0, 1, 2$. Then, from (38), we find up to an affine transformation $\log\lambda \to a\log\lambda + b$ that

$$\partial_\lambda \mathrm{Tr}_{\wedge^i \rho'} L_{B_2}^\vee(x, y; \lambda) = \{0, 1, -2\}_i \left(2\lambda - \frac{2}{u_0\lambda^3}\right), \tag{44}$$

and $\mathrm{Tr}_{\wedge^i \rho'} L_{B_2}^\vee = \mathrm{Tr}_{\wedge^{6-i} \rho'} L_{B_2}^\vee$ by the reality of ρ'. Using the character relations in the exterior algebra

$$\wedge^2 \rho' \oplus \rho_{[000]} = \rho_{[100]} \otimes \rho_{[001]}, \quad \wedge^3 \rho' = \rho_{[200]} \oplus \rho_{[002]}, \tag{45}$$

plus the fact that $\rho_{[200]} \oplus \rho_{[010]} = \rho_{[100]} \otimes \rho_{[100]}$, $\rho_{[002]} \oplus \rho_{[010]} = \rho_{[001]} \otimes \rho_{[001]}$, and restricting to the invariant locus under the folding action, we retrieve the spectral parameter dependence claimed in (40):

$$\begin{aligned}
\mathrm{Tr}_{\rho_1} L_{B_2}^\vee(x, y; \lambda) &= u_1(x, y) + \left(\lambda + \frac{1}{u_0\lambda}\right), \\
\mathrm{Tr}_{\rho_3} L_{B_2}^\vee(x, y; \lambda) &= u_1(x, y) - \left(\lambda + \frac{1}{u_0\lambda}\right), \\
\mathrm{Tr}_{\rho_2} L_{B_2}^\vee(x, y; \lambda) &= u_2(x, y).
\end{aligned} \tag{46}$$

From (45) to (46), the reduced spectral polynomial in the six-dimensional vector representation $\rho' = \mathbf{6_v} \in \mathrm{Rep}(D_3)$ reads

$$\widetilde{\mathsf{T}}_{B_2, \mathbf{6_v}; u}(\mu, \nu) = 1 - u_2\mu + \left(u_1^2 - 1\right)\mu^2 - 2\left(u_1^2 - u_2\right)\mu^3$$

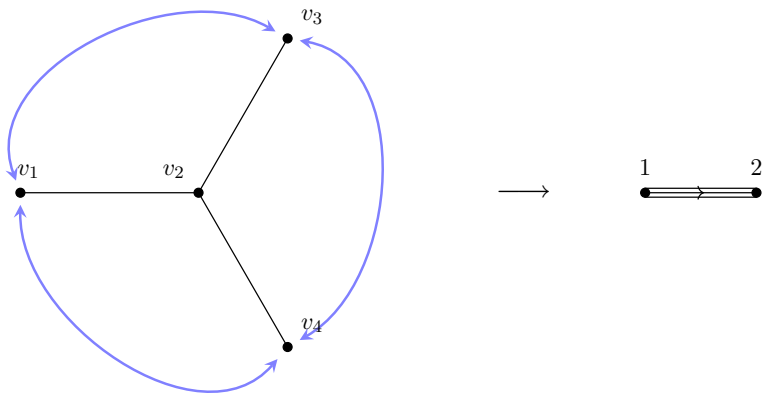


Fig. 2 The Dynkin diagram of G_2 via D_4-folding

$$+\left(u_1^2-1\right)\mu^4-u_2\mu^5+\mu^6-\nu^2(\mu+1)^2\mu^2. \tag{47}$$

Alternatively, using (46), we can get a more compact expression by letting $\rho'=\mathbf{4}_\pm$ to be either of the two $four$-dimensional complex conjugate irreducible representations of $\mathrm{Spin}(6)=\mathrm{SU}(4)$. Since $\wedge^i\rho'=\rho_i$ in this case, we obtain right off the bat from (46) that

$$\widetilde{\mathsf{T}}_{B_2,\mathbf{4}_\pm;u}(\mu,\nu)=1-(u_1+\nu)\mu+u_2\mu^2-(u_1-\nu)\mu^3+\mu^4. \tag{48}$$

Example 2 As a slightly more complicated example, consider $\left(G_2^{(1)}\right)^\vee=D_4^{(3)}$. In this case, the triality symmetry of the simple Lie algebra $\mathcal{A}(\mathfrak{g}_2)=\mathfrak{d}_4$ identifies the vertices $v_1\leftrightarrow v_3\leftrightarrow v_4$, corresponding to the highest weights [1000], [0010] and [0001] of the three eight-dimensional irreducible representations of Spin(8), whereas v_2, corresponding to the adjoint representation, is left fixed; see Fig. 2. In particular, we have that $\mathrm{Eff}(\mathfrak{g}_2)=\mathsf{o}_1^{(3)}=\{v_1,v_2,v_4\}$.

Fix $\rho'=\rho_{[1000]}$. Then, the twisted Cartan–Weyl generators are given in this representation by Suris [86,Sec. 21.13]

$$\begin{aligned}
e_1^\vee&=\varepsilon_{32}-\varepsilon_{76}\\
e_2^\vee&=\varepsilon_{21}+\varepsilon_{43}+\varepsilon_{53}-\varepsilon_{64}-\varepsilon_{65}-\varepsilon_{87}\\
e_0^\vee&=\varepsilon_{14}-\varepsilon_{58}+\omega_3\left(\varepsilon_{15}-\varepsilon_{48}\right)+\omega_3^2\left(\varepsilon_{26}-\varepsilon_{37}\right)
\end{aligned} \tag{49}$$

with again $e_{\bar{i}}^\vee=(e_i^\vee)^T$, $h_i=[e_i,e_{\bar{i}}]$, $i=0,1,2$, and where as before $\omega_3=\mathrm{e}^{2\pi\mathrm{i}/3}$. Plugging this into (38), it is straightforward to compute the spectral dependence of the exterior traces $\mathrm{Tr}_{\wedge^i\rho}L_{G_2}^\vee$, whose explicit expression we omit here. Using the following relations in the representation ring $\mathrm{Rep}(D_4)$,

$$\wedge^2\rho=\rho_{[0100]},\quad \wedge^3\rho=\rho_{[0010]}\otimes\rho_{[0001]},\quad \wedge^4\rho=\rho_{[020]}\oplus\rho_{[002]}, \tag{50}$$

and

$$\rho_{[0020]} = \rho_{[0010]} \otimes \rho_{[0010]} \oplus \rho_{[1000]},$$
$$\rho_{[0020]} = \rho_{[0010]} \otimes \rho_{[0001]} \oplus \rho_{[1000]} \tag{51}$$

it is immediate to verify that the λ-dependence of the exterior traces $\mathrm{Tr}_{\wedge^i \rho} L^{\vee}_{G_2}$ is induced by a shift of the fundamental traces as in (40). The reduced spectral polynomial then reads

$$\begin{aligned}\widetilde{T}_{G_2;\rho_1;u}(\mu,\nu) = (\mu-1)^2 &\left[u_2(\mu+1)^2\mu^2 - u_1^2\mu^3 - u_1\left(\mu^5+\mu\right) + \sum_{i=0}^{6}\mu^i\right] \\ &+ \nu(\mu-1)^2\mu\left(\mu^4+2\mu^3-u_1\mu^2+\mu^2+2\mu+1\right) \\ &- \nu^2\mu^3\left(\mu^2+\mu+1\right) + \frac{3(\mu+1)^2\mu^3}{u_0^2}.\end{aligned} \tag{52}$$

3 Picard–Fuchs equations: the simply laced case

3.1 Extended affine Weyl groups and Frobenius manifolds

We start by providing a very brief account of the construction of Dubrovin and Zhang [31] of a semisimple Frobenius manifold on the space of regular orbits of an extended affine Weyl group, referring the reader to Dubrovin and Zhang [31], Brini [16] and Brini and van Gemst [20] for a more extended treatment. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r, and write $\mathfrak{h}$ and $\mathcal{W}_{\mathfrak{g}}$ for, respectively, its Cartan subalgebra and the Weyl group. Let $\bar{k} \in \{1,\dots,r\}$ be a label marking the highest weight $\omega_{\bar{k}}$ corresponding to any of the highest-dimensional fundamental representations of $\mathfrak{g}$. The action of $\mathcal{W}_{\mathfrak{g}}$ on $\mathfrak{h}$ admits an affine extension $\widetilde{\mathcal{W}}_{\mathfrak{g},\omega_{\bar{k}}} \simeq \mathcal{W}_{\mathfrak{g}} \rtimes \Lambda^{\vee}_{\mathfrak{g}} \rtimes \mathbb{Z}$ on $\mathfrak{h}\times\mathbb{C}$,

$$\widetilde{\mathcal{W}}_{\mathfrak{g},\omega_{\bar{k}}} \times \mathfrak{h}\times\mathbb{C} \to \mathfrak{h}\times\mathbb{C} \tag{53}$$
$$((w,\alpha^{\vee},n),(h,\xi)) \mapsto (w(h)+\alpha^{\vee}+n\omega_{\bar{k}}, \xi-n). \tag{54}$$

The GIT quotient

$$M^{\mathrm{trig}}_{\mathfrak{g}} := (\mathfrak{h}^{\mathrm{reg}}\times\mathbb{C})//\widetilde{\mathcal{W}}_{\mathcal{R}} = \mathrm{Spec}(\mathcal{O}_{\mathfrak{h}\times\mathbb{C}}(\mathfrak{h}^{\mathrm{reg}}\times\mathbb{C}))^{\widetilde{\mathcal{W}}_{\mathfrak{g}}} \cong \mathcal{T}^{\mathrm{reg}}/\mathcal{W}_{\mathfrak{g}}\times\mathbb{C}^*, \tag{55}$$

is isomorphic as a complex manifold to the trivial one-dimensional family $\mathcal{M}_{\mathsf{C}}(\mathcal{G})\times\mathbb{C}^{\star}$, parametrised by $\xi := \log u_0 = -2h_{\mathfrak{g}}\log R_5\Lambda_{\mathrm{QCD}} \in \mathbb{P}^1$ of classical vacua with maximally broken gauge symmetry of the pure $\mathcal{N}=1$ gauge theory on $\mathbb{R}^4\times S^1_{R_5}$ with gauge group the real compact form $\mathcal{G}$ of $\exp(\mathfrak{g})$. In (55), $\mathfrak{h}^{\mathrm{reg}}$ is the set of regular orbits, and $\mathcal{T}^{\mathrm{reg}} = \exp(2\pi\mathrm{i}\mathfrak{h}^{\mathrm{reg}})$ is its image under the exponential map to the maximal torus $\mathcal{T}$.

In [31], the authors construct a canonical, semisimple Frobenius manifold structure on $M_{\mathfrak{g}}^{\text{trig}}$. Their construction asserts that there exist a chart $\{t_i(h,\xi)\}_{i=0}^r$ on $M_{\mathfrak{g}}^{\text{trig}}$ and a solution $F_{\mathfrak{g}}(t_0,\ldots,t_r)$ of the WDVV equation such that:

1. $F_{\mathfrak{g}} \in \mathbb{Q}[t_0,\ldots,t_r][\mathrm{e}^{t_0}]$;
2. $\eta_{ij} := \partial^3_{t_r t_i t_j} F_{\mathfrak{g}}$ is a constant and non-degenerate matrix;
3. $E(F) = 2F$, with

$$E := \sum_{j=1}^{r} \frac{\mathfrak{q}_j}{\mathfrak{q}_{\bar{k}}} t_j \partial_j + \frac{1}{\mathfrak{q}_{\bar{k}}} \partial_0, \tag{56}$$

up to quadratic terms.

Here, $\mathfrak{q}_i := \langle \omega_i, \omega_{\bar{k}} \rangle$ are the fundamental Coxeter degrees of $\mathfrak{g}$.

3.2 Seiberg–Witten periods as odd periods

The Frobenius manifolds $M_{\mathfrak{g}}^{\text{trig}}$ are a trigonometric version of the polynomial Frobenius manifolds $M_{\mathfrak{g}}^{\text{pol}}$ [28, 29] on quotients of the reflection representation of ordinary Weyl groups. For $\mathfrak{g} = \mathfrak{ade}$ these are isomorphic to the chiral rings[6] of the twisted, massively perturbed two-dimensional minimal $\mathcal{N} = 2$ SCFTs with central charge $d = c/3 < 1$ [25] of type ADE. It was proposed in [30, 34, 48, 49] that the SW periods of four-dimensional $\mathcal{N} = 2$ super-Yang–Mills theories with simply laced compact gauge group $\mathcal{G}$ are solutions of a Picard–Fuchs system determined by the Frobenius structure on the tensor product $M_{\mathfrak{g}}^{\text{pol}} \otimes QH(\mathbb{P}^1)$ ($\mathfrak{g} = \text{Lie}(\mathcal{G})_{\mathbb{C}}$) of the polynomial Frobenius manifold of type $\mathfrak{g}$ with the quantum cohomology of the projective line (the chiral ring of the topologically A-twisted σ-model). The claim is that there exists a change-of-variables $(u_1,\ldots,u_r,\Lambda_{\text{QCD}}) \to (t_1,\ldots,t_r,Q)$ such that the periods satisfy the holonomic system of PDEs [30,Prop. 5.19]

$$\left(\sum_{i=1}^{r} \mathfrak{q}_i t_i \partial_{t_i} - 1\right)^2 \Pi = 4h_{\mathfrak{g}}^2 Q \partial^2_{t_r} \Pi \tag{57}$$

$$\partial^2_{t_i t_j} \Pi = c_{ij}^k \partial^2_{t_r t_k} \Pi \tag{58}$$

In (57)–(58), the independent variables $\{t_i\}_{i=1}^r$ are flat coordinates for the Saito metric on $M_{\mathfrak{g}}^{\text{pol}}$ and are related polynomially to the classical Coulomb order parameters u_i [29]; the variable Q keeps track of the dependence on the holomorphic scale and is identified with the coordinate parametrising primary insertions of the Kähler class in the topological A-model on $\mathbb{P}^1$ [30], and finally, in (58), $c_{ij}^k(t)$ are the structure constants of the $\mathcal{N} = 2$ ADE chiral ring. In the language of [30], the gauge theory periods are identified with the *odd periods* of the Frobenius manifold $M_{\mathfrak{g}}^{\text{pol}} \otimes QH(\mathbb{P}^1)$ on the "coordinate cross" where all dependence on the primary insertions of the $\mathbb{P}^1$ theory are discarded.

[6] And for $\mathfrak{g} = \mathfrak{bcfg}$, the invariant subrings obtained by Dynkin folding [90].

Viewing the pure $\mathcal{N}=2$ KK theory on $\mathbb{R}^4 \times S^1_{R_5}$ with ADE gauge group as a 4d theory with $\widetilde{\mathrm{ADE}}$ loop group gauge symmetry, as in [74], we propose that the same tensor product construction of [30, 48] applies to the 5d setting upon replacing the type $\mathfrak{g}=\mathfrak{ade}$ polynomial Frobenius manifolds $M^{\mathrm{pol}}_{\mathfrak{g}}$ with their trigonometric, extended affine version $M^{\mathrm{trig}}_{\mathfrak{g}}$. This leads us to postulate that for ADE gauge groups the periods (1) satisfy the system of PDEs

$$\boxed{\begin{aligned} L_E^2\,\Pi &= 4\partial^2_{t_r}\Pi && \text{(59a)}\\ \partial^2_{t_i t_j}\Pi &= \mathsf{C}^k_{ij}\partial^2_{t_r t_k}\Pi && \text{(59b)}\end{aligned}}$$

where now $i=0,\dots,r$, the Euler vector field E is as in (56), and $\mathsf{C}^k_{ij}(t)$ are now the structure constants $\mathsf{C}^k_{ij}(t) := \eta^{kl}\partial^3_{t_l t_i t_j}F^{\mathrm{trig}}_{\mathfrak{g}}$ of $M^{\mathrm{trig}}_{\mathfrak{g}}$ in the flat chart $\{t^i\}_{i=0}^r$.[7]

It will be helpful to express (59a)–(59b) in a natural set of coordinates adapted to the weak coupling expansion in the gauge theory. These are the analogue of the locally monodromy-invariant coordinates around the large complex structure/maximally unipotent point in Hori–Vafa mirror symmetries, and they are related to the usual Coulomb branch u-coordinates as

$$z_0 = 1/u_0, \qquad z_i = \prod_{j=1}^{r} u_j^{-\mathcal{K}_{ij}(\mathfrak{g})} \tag{60}$$

where $\mathcal{K}(\mathfrak{g})$ is the Cartan matrix of $\mathfrak{g}$. In these coordinates, (59a)–(59b) read

$$\boxed{\begin{aligned} \left(z_0\partial_{z_0}\right)^2\Pi &= 4h^2_{\mathfrak{g}}\left(X^{ij}(z)\partial^2_{z_i z_j} + Y^i(z)\partial_{z_i}\right)\Pi, && \text{(61a)}\\ \partial^2_{z_i z_j}\Pi &= \left(A^{kl}_{ij}(z)\partial^2_{z_k z_l} + B^k_{ij}\partial_{z_k}\right)\Pi, && \text{(61b)}\end{aligned}}$$

where

$$\begin{aligned} X^{ij} &= \frac{\partial z_i}{\partial t_r}\frac{\partial z_j}{\partial t_r}, & Y^i &= \frac{\partial^2 z_i}{\partial t_r \partial t_k}\frac{\partial z_k}{\partial t_r},\\ A^{kl}_{ij} &= \frac{\partial t_m}{\partial z_i}\frac{\partial t_n}{\partial z_j}\mathsf{C}^p_{mn}\frac{\partial z_k}{\partial t_p}\frac{\partial z_l}{\partial t_r}, & B^k_{ij} &= \frac{\partial t_m}{\partial z_i}\frac{\partial t_n}{\partial z_j}\mathsf{C}^p_{mn}\frac{\partial^2 z_k}{\partial t_p \partial t_r} + \frac{\partial^2 t_m}{\partial z_i \partial z_j}\frac{\partial z_k}{\partial t_m}. \end{aligned} \tag{62}$$

By design, the large complex structure/semiclassical expansion point corresponds in these coordinates to $z=0$: from (6) and (60), sending Λ_{QCD} to infinity sets $z_0=0$, and from (4) sending a_i to infinity leads to a damping behaviour of the form

$$z_i \sim \mathrm{e}^{-\sum_j \mathcal{K}_{ij}(\mathfrak{g})a_j} \sim \mathrm{e}^{-\sum_j \mathcal{K}_{ij}(\mathfrak{g})a_j^{\mathrm{cl}}} \tag{63}$$

[7] A remark is in order regarding the type A series, since in that case we have an additional parametric dependence on the five-dimensional Chern–Simons term. By (34), the tensor product with the quantum cohomology of $\mathbb{P}^1$, whose LG superpotential has the form $\lambda + 1/(u_0\lambda)$, picks out a CS level such that the symmetry $\lambda \leftrightarrow (u_0\lambda)^{-1}$ is realised at the spectral curve level. This occurs at $k_{\mathrm{CS}}=0$ for $\mathcal{G}=\mathrm{SU}(2N)$ and $k_{\mathrm{CS}}=1$ for $\mathcal{G}=\mathrm{SU}(2N+1)$.

so that $z_i \to 0$, $i = 1, \ldots, r$ in that limit. The determination of the gauge theory prepotential from (59a) to (59b) proceeds then along the following steps.

1. We write down the $t-$chart Picard–Fuchs system (59a)–(59b) from the Frobenius manifold prepotential $F^{\text{trig}}_{\mathfrak{g}}(t)$, recently found for all $\mathfrak{g}$ in [16, 20].
2. We then derive from this the Picard–Fuchs system (61a)–(61b) in the z-coordinates, by using the expression of the flat coordinates $t^i(z)$ in terms of the basic invariants/classical gauge theory Casimirs found in [16, 20], and using (60).
3. We finally look for solutions to (61a)–(61b) in the form

$$\Pi = \sum_{j,k} a_{jk} \log z_j \log z_k + \sum_l \sum_{J \in \frac{1}{|Z(\mathcal{G})|}\mathbb{Z}^{r+1}} b_{l,J} \log z_l \prod_k z_k^{J_k} + \sum_{J \in \frac{1}{|Z(\mathcal{G})|}\mathbb{Z}^{r+1}} c_J \prod_k z_k^{J_k} \tag{64}$$

with at worst double-logarithmic singularities around $z = 0$. In (64), we also allow fractional exponents with denominators that divide the order of the centre of the group, as the latter coincides with the determinant of the Cartan matrix, and may arise as a consequence of the change-of-variables in (60).

We shall find that (61a)–(61b) admit a $2r+2$-dimensional solution space of the form (64): two of these are always $\log z_0$ and constant solution, and that the remaining $2r$ solutions satisfy the special geometry relations (1), from which the gauge theory prepotential can be computed.

We will put this strategy to the test in some of the lowest rank ADE examples in the next section.

3.3 Examples

3.3.1 A_1

We start off by illustrating in detail the simplest example of $\mathcal{G} = A_1$ at vanishing θ-angle. In this case, the Frobenius manifold $M^{\text{trig}}_{\mathfrak{g}}$ coincides with the quantum cohomology ring of $\mathbb{P}^1$, and its prepotential is simply

$$F^{\text{trig}}_{A_1}(t_0, t_1) = t_0 t_1^2 + \mathrm{e}^{2t_0}, \tag{65}$$

with Euler vector field $E = \partial_{t_0} + t_1 \partial t_1$ from (56), while the flat coordinates (t_0, t_1) are related to the classical Coulomb moduli as [20]

$$t_0 = \log u_0, \qquad t_1 = u_0 u_1. \tag{66}$$

From this, it is immediate to verify that the flat metric is anti-diagonal, $\mathsf{C}^j_{1i} = \delta^j_i$, and

$$\mathsf{C}^j_{0i} = \begin{pmatrix} 0 & \mathrm{e}^{t_0} \\ 1 & 0 \end{pmatrix}. \tag{67}$$

Finally, the semiclassical expansion coordinates (60) are:

$$z_0 = \frac{1}{u_0}, \qquad z_1 = \frac{1}{u_1^2} \tag{68}$$

We now have all the ingredients to write the Picard–Fuchs system (61a)–(61b), which reads

$$\begin{aligned} &\left(z_0\partial_{z_0}\right)^2 \Pi = 8z_0^2 z_1^2 (3\partial_{z_1}\Pi + 2z_1\partial_{z_1}^2)\Pi, \\ &(z_0\partial_{z_0}^2)\Pi = 4\left[z_1^2(4z_1-1)\partial_{z_1}^2 + z_1(6z_1-1)\partial_{z_1} + z_0 z_1 \partial_{z_0}\partial_{z_1}\Pi\right]. \end{aligned} \tag{69}$$

Inserting the ansatz (64) into (69), we find that the solution space of (69) is a four-dimensional complex vector space with coordinates $(b_{0,0}, c_{0,0}, c_{0,1}, c_{0,2})$:

$$\begin{aligned} \Pi = &\frac{2c_{0,2} - 3c_{0,1}}{14}\left[\log^2(z_1) - 2\log(z_0)\log(z_1)\right] \\ &+ \frac{\log z_0}{7}\Big[\left(7b_{0,0} + z_1\left(4c_{0,2} - 6c_{0,1}\right) + z_1^2\left(6c_{0,2} - 9c_{0,1}\right)\right) \\ &+ z_0^2\left(4c_{0,2} - 6c_{0,1}\right)z_1 + \cdots\Big] \\ &+ \frac{\log z_1}{14}\Big[\left(13c_{0,1} - 4c_{0,2}\right) + z_1\left(8c_{0,2} - 12c_{0,1}\right) + z_1^2\left(12c_{0,2} - 18c_{0,1}\right) \\ &+ z_0^2\left(8c_{0,2} - 12c_{0,1}\right)z_1 + \cdots\Big] \\ &+ \left(c_{0,0} + z_1 c_{0,1} + z_1^2 c_{0,2}\right) + z_0^2\left(c_{0,1}z_1 + \left(\frac{18c_{0,1}}{7} + \frac{16c_{0,2}}{7}\right)z_1^2\right) + \cdots \end{aligned} \tag{70}$$

Setting $c_{0,1} = 2/3c_{0,2}$ we find a constant holomorphic solution at $c_{0,2} = b_{0,0} = 0$, and two logarithmic solutions

$$\begin{aligned} \Pi_{A_0} &= \log z_0 \\ \Pi_{A_1} &= \log z_1 + 2z_1 + 3z_1^2 + 2z_1 z_0^2 + 12 z_1^2 z_0^2 + \cdots \end{aligned} \tag{71}$$

for $(b_{0,0}, c_{0,2}) = (1, 0)$ and $(0, 3)$, respectively. From (6) and (60), we identify $\mathrm{e}^{\Pi_{A_0}} = z_0 = q_0$, while Π_{A_1} equates to minus the scalar vev $a = \langle\varphi\rangle$ in (1), in terms of which

the inverse mirror map reads

$$z_1(q_0, q_1) = q_1 - 2\left(q_0^2+1\right)q_1^2 + 3\left(q_0^4+1\right)q_1^3 - 4\left(q_0^6+q_0^4+q_0^2+1\right)q_1^4 + 5\left(q_0^8-5q_0^4+1\right)q_1^5 + \cdots \tag{72}$$

Setting instead $c_{0,1} = 1$ and $c_{0,2} = 1/4$ gives the dual period

$$\begin{aligned}\Pi_{B_1} &= \frac{1}{4}\left(\log^2 z_1 + 2\log z_0 \log z_1 - 4\log z_1\right) \\ &\quad + \log(z_0 z_1)\left(z_1 + \frac{3}{2}z_1^2 + z_0^2 z_1 + 6z_0^2 z_1^2 + \cdots\right) \\ &\quad - z_1 z_0^2 + z_1^2\left(\frac{1}{4} + 2z_0^2\right) + \cdots \\ &= \frac{1}{4}\left(\log^2 q_1 + 2\log q_0 \log q_1 - 4\log q_1\right) \\ &\quad + \left(1+q_0^2\right)q_1 + \frac{1}{4}\left(1+16q_0^2+q_0^4\right)q_1^2 + \frac{1}{9}\left(1+81q_0^2+81q_0^4+q_0^6\right)q_1^3 + \\ &\quad + \frac{1}{16}\left(1+256q_0^2+1040q_0^4+256q_0^6+q_0^8\right)q_1^4 + O\left(q_1^5\right)\end{aligned} \tag{73}$$

Identifying Π_{B_1} with the gradient $\partial_a \mathcal{F}$ of the gauge theory prepotential, the above calculation retrieves the expression of the latter for the five-dimensional pure SU(2) theory at vanishing θ-angle [74].

3.3.2 A_2

As a slightly more difficult case, let us consider another special unitary case given by the SU(3) gauge theory with $k_{\mathrm{CS}} = 1$. The prepotential of $M^{\mathrm{trig}}_{\mathfrak{a}_2}$ is given by

$$F(t_0, t_1, t_2) = \mathrm{e}^{\frac{3}{2}t_0} + \frac{3}{4}t_0 t_2^2 - \frac{1}{96}t_1^4 + \frac{1}{4}t_1^2 t_2, \tag{74}$$

with flat coordinates related to the classical Coulomb moduli as

$$(t_0, t_1, t_2) = (\log u_0, u_0^{\frac{1}{2}} u_1, u_0 u_2). \tag{75}$$

As before the flat metric is anti-diagonal, with non-trivial structure constants

$$\mathsf{C}^j_{0i} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{9}{2}\mathrm{e}^{\frac{3t_0}{2}} & 0 & 0 \\ \frac{9}{4}\mathrm{e}^{\frac{3t_0}{2}} t(1) & \frac{3}{2}\mathrm{e}^{\frac{3t_0}{2}} & 0 \end{pmatrix}_{ij}, \quad \mathsf{C}^j_{1i} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & -\frac{t_1}{2} & 1 \\ \frac{3}{2}\mathrm{e}^{\frac{3t_0}{2}} & 0 & 0 \end{pmatrix}, \tag{76}$$

and the Euler vector field is $E = \partial_{t_0} + t_1/2\partial_{t_1} + t_2\partial_{t_2}$. Finally, the semiclassical coordinates (60) are here given by

$$z_0 = \frac{1}{u_0}, \quad z_1 = \frac{u_2}{u_1^2}, \quad z_2 = \frac{u_1}{u_2^2}, \tag{77}$$

The strategy in this case follows *verbatim* the discussion of the A_1 example. As expected, the solution space of (61a)–(61b) is six-dimensional, linearly generated by one constant, three logarithmic, and two doubly logarithmic solutions. By way of example, we obtain for the logarithmic solutions

$$\begin{aligned}
\Pi_{A_0} &= \log z_0, \\
\Pi_{A_1} &= \log(z_1) - z_2 - 8z_2z_1^3 + \frac{20z_1^3}{3} - 2z_2z_1^2 + 3z_1^2 \\
&\quad - \frac{3z_2^2}{2} + z_2^2z_1 + 2z_1 + 6z_0^2z_2^{4/3}z_1^{8/3} + \cdots \\
\Pi_{A_2} &= \log(z_2) - z_1 + 2z_2 - \frac{10z_1^3}{3} - 3z_0^2z_2^{4/3}z_1^{8/3} \\
&\quad + z_2z_1^2 - \frac{3z_1^2}{2} - 8z_2^3z_1 - 2z_2^2z_1 + \cdots
\end{aligned} \tag{78}$$

The final result for the prepotential then is

$$\begin{aligned}
-8\pi^3\mathrm{i}\mathcal{F}_{A_2} &= (2\log^2 q_1 + 2\log q_2 \log q_1 + 4\log^2 q_2)\log q_0 \\
&\quad - \frac{20\log^3 q_1}{27} - \frac{10}{9}\log q_2 \log^2 q_1 - \frac{8}{9}\log^2 q_2 \log q_1 - \frac{16\log^3 q_2}{27} \\
&\quad + 2(q_1 + 2q_1q_2 + 2q_2) + \frac{1}{4}\left(q_1^2 + q_1^2q_2^2 + q_2^2))\right) + \cdots
\end{aligned} \tag{79}$$

Once again we have matched this at the first few orders in q_i with the instanton calculation[8] of the gauge theory prepotential at Chern–Simons level 1. We have verified in the same manner that the same applies for higher rank cases as well, such as $\mathcal{G} = \mathrm{SU}(4)$ at vanishing CS level, and that moreover our expressions agree with the direct period integral calculations of [19].

3.3.3 D_4

Let us now move on to uncharted territory and test the Picard–Fuchs construction of (61a)–(61b) on the first non-trivial, non-unitary case corresponding to $\mathcal{G} = \mathrm{Spin}(8)$. While a direct analysis of period integrals is too hard to carry out in this case, our

[8] For the $\mathrm{SU}(N)$ theory at an arbitrary Chern–Simons level k_{CS}, see [87,Section 3] for the perturbative prepotential $\mathcal{F}^{[0]}_{\mathrm{SU}(N)_{k_{\mathrm{CS}}}}$ and [40] for the instanton corrections $\mathcal{F}^{[n]}_{\mathrm{SU}(N)_{k_{\mathrm{CS}}}}$ with $n > 0$.

method allows to compute their weak coupling expansion efficiently. The prepotential, Euler vector field, and flat coordinates for this case are, respectively, given by

$$F^{\text{trig}}_{\mathfrak{d}_4} = \frac{\mathrm{e}^{2t_0}}{2} + \mathrm{e}^{t_0} t_1^2 + \mathrm{e}^{t_0} t_3^2 + \mathrm{e}^{t_0} t_4^2 + 2\mathrm{e}^{\frac{t_0}{2}} t_1 t_3 t_4 + \frac{1}{2} t_2 t_1^2 + \frac{1}{2} t_2 t_3^2 + \frac{1}{2} t_2 t_4^2 - \frac{t_1^4}{48} - \frac{t_3^4}{48} - \frac{t_4^4}{48} + \frac{t_2^3}{6} + \frac{1}{2} t_0 t_2^2, \tag{80}$$

$$E = \partial_{t_0} + \frac{t_1}{2}\partial_{t_1} + t_2 \partial_{t_2} + \frac{t_3}{2}\partial_{t_3} + \frac{t_4}{2}\partial_{t_4}, \tag{81}$$

$$t_0 = \log u_0, \quad t_1 = u_0^{\frac{1}{2}} u_1, \quad t_2 = u_0(u_2 + 2), \quad t_3 = u_0^{\frac{1}{2}} u_3, \quad t_4 = u_0^{\frac{1}{2}} u_4. \tag{82}$$

The expansion coordinates around the large complex structure point in the moduli space read, from (60),

$$z_0 = \frac{1}{u_0}, \quad z_1 = \frac{u_2}{u_1^2}, \quad z_2 = \frac{u_1 u_3 u_4}{u_2^2}, \quad z_3 = \frac{u_2}{u_3^2}, \quad z_4 = \frac{u_2}{u_4^2}. \tag{83}$$

Explicit expressions for the differential system (61a)–(61b) and its solutions are omitted here as they are obtained in a straightforward manner from (80)–(83); they are available upon request. Once again our strategy can be shown to retrieve the gauge theory prepotential for this case as well, which we have verified up to one-instanton, and up to $\mathcal{O}(z_i^{12})$.[9]

3.3.4 E_6

Finally, we will briefly treat here the exceptional case of $\mathcal{G} = E_6$. From [20], the prepotential of $M^{\text{trig}}_{\mathfrak{e}_6}$ reads

$$\begin{aligned} F^{\text{trig}}_{\mathfrak{e}_6} = &-\frac{t_1^6}{3240} - \frac{1}{648} t_4 t_1^4 + \frac{1}{12}\mathrm{e}^{\frac{t_0}{3}} t_5 t_1^4 + \mathrm{e}^{t_0} t_1^3 + \mathrm{e}^{\frac{t_0}{2}} t_6 t_1^3 - \frac{1}{216} t_4^2 t_1^2 + \frac{5}{3}\mathrm{e}^{\frac{2t_0}{3}} t_5^2 t_1^2 \\ &+ \frac{1}{6}\mathrm{e}^{\frac{2t_0}{3}} t_2 t_1^2 - \frac{1}{6}\mathrm{e}^{\frac{t_0}{3}} t_4 t_5 t_1^2 + \frac{1}{6}\mathrm{e}^{\frac{t_0}{6}} t_5^2 t_6 t_1^2 + \frac{1}{6}\mathrm{e}^{\frac{t_0}{6}} t_2 t_6 t_1^2 + \frac{1}{12}\mathrm{e}^{\frac{t_0}{3}} t_5^4 t_1 + \frac{1}{12}\mathrm{e}^{\frac{t_0}{3}} t_2^2 t_1 \\ &+ \frac{1}{6}\mathrm{e}^{\frac{t_0}{3}} t_2 t_5^2 t_1 + 3\mathrm{e}^{\frac{t_0}{3}} t_5 t_6^2 t_1 - \frac{1}{3} t_3 t_4 t_1 + 3\mathrm{e}^{\frac{4t_0}{3}} t_5 t_1 - \mathrm{e}^{\frac{t_0}{2}} t_4 t_6 t_1 + 6\mathrm{e}^{\frac{5t_0}{6}} t_5 t_6 t_1 \\ &+ \frac{\mathrm{e}^{2t_0}}{2} - \frac{t_5^6}{3240} + \frac{1}{648} t_2 t_5^4 - \frac{t_6^4}{16} + \frac{t_2^3}{648} - \frac{t_4^3}{648} + \mathrm{e}^{t_0} t_5^3 + 2\mathrm{e}^{\frac{t_0}{2}} t_6^3 \end{aligned}$$

[9] As for such high rank the calculations become rapidly quite cumbersome, for our purposes it is quicker to match the special geometry and gauge theory expressions by a partial reverse engineering of the solutions of (61a)–(61b). We computed the single-logarithmic solutions of (61a)–(61b)—i.e. the mirror map—and then used these to find a conjectural expression for the dual periods Π_{B_i} in terms of the semiclassical variables B-model variables z_i by employing the known results for the 1-loop and 1-instanton contributions to the gauge theory prepotential. We then verified explicitly in an expansion around $z = 0$ that these are indeed solutions of our differential system (61a)–(61b).

$$+\frac{1}{2}t_0t_3^2-\frac{1}{216}t_2^2t_5^2-\frac{1}{6}\mathrm{e}^{\frac{2t_0}{3}}t_4t_5^2+3\mathrm{e}^{t_0}t_6^2+\frac{3}{2}t_3t_6^2-\frac{1}{6}\mathrm{e}^{\frac{2t_0}{3}}t_2t_4+\frac{1}{12}\mathrm{e}^{\frac{t_0}{3}}t_4^2t_5$$
$$+\frac{1}{3}t_2t_3t_5+\mathrm{e}^{\frac{t_0}{2}}t_5^3t_6-\frac{1}{6}\mathrm{e}^{\frac{t_0}{6}}t_4t_5^2t_6-\frac{1}{6}\mathrm{e}^{\frac{t_0}{6}}t_2t_4t_6+\mathrm{e}^{\frac{t_0}{2}}t_2t_5t_6, \tag{84}$$

with Euler vector field

$$E=\partial_{t_0}+\frac{t_1}{3}\partial_{t_1}+\frac{2t_2}{3}\partial_{t_2}+t_3\partial_{t_3}+\frac{2t_4}{4}\partial_{t_4}+\frac{t_5}{3}\partial_{t_5}+\frac{t_6}{2}\partial_{t_6} \tag{85}$$

and flat and z-coordinates given, respectively, by

$$\begin{aligned}
&t_0=\log u_0,\quad t_1=u_0^{1/3}u_1,\quad t_2=u_0^{2/3}\left(u_1^2-6u_2-12u_5\right),\\
&t_3=u_0\left(u_3+2u_1u_5+4u_6+3\right),\quad t_4=u_0^{2/3}\left(12u_1-u_5^2+6u_4\right),\\
&t_5=u_0^{1/3}u_5,\quad t_6=\sqrt{u_0}\left(u_6+2\right),
\end{aligned} \tag{86}$$

and

$$\begin{aligned}
&z_0=\frac{1}{u_0},\quad z_1=\frac{u_2}{u_1^2},\quad z_2=\frac{u_1u_3}{u_2^2},\quad z_3=\frac{u_2u_4u_6}{u_3^2},\\
&z_4=\frac{u_3u_5}{u_4^2},\quad z_5=\frac{u_4}{u_5^2},\quad z_6=\frac{u_3}{u_6^2}.
\end{aligned} \tag{87}$$

As in the previous case of $\mathcal{G}=D_4$, to deal with the complexity of the system (61a)–(61b) it is convenient to focus on the calculation of single-logarithmic solutions from which the mirror map can be constructed, with the dual periods being recovered as an *a posteriori* check. The z-chart differential system (61a)–(61b) contains now an unmanageably large number of terms spawned by the change-of-variables (62), thereby impeding a direct computational approach based on solving by brute force for the coefficients in the ansatz (64). We adopt a hybrid method to circumvent the issue by considering instead the t-chart Picard–Fuchs operators (59a)–(59b), whose expressions are significantly simpler, working out their action on a monomial $\prod_{i=0}^{r}z_i^{m_i}(t)$, finally expressing the result in z-coordinates to obtain recursion relations for the coefficients (a,b,c) in the z-chart ansatz (64). Restricting, for example, to the single-logarithmic solutions ($a_{jk}=0, b_{l,J}=\delta_{J0}\delta_{li}$ for $i=0,\dots,r$) we obtain from (59a)

$$\begin{aligned}
(J_0+2)^2\,S_0^{-2}\prod_{i=1}^{6}S_i^{-2\mathrm{q}_i}c_J=4\Big[&J_2^2+\left(-4J_3+2J_4+2J_6-1\right)J_2+J_4^2+4J_3^2\\
&+J_6^2-J_4+J_3\left(-4J_4-4J_6+2\right)+2J_4J_6-J_6\Big]c_J,
\end{aligned} \tag{88}$$

where $S_i c_{(J_0,\ldots,J_i,\ldots J_r)} := c_{(J_0,\ldots,J_i-1,\ldots J_r)}$ is the left-shift in the ith component of the multi-index J in (64). The recursions from (59b) are obtained similarly; the simplest one is obtained for $(i, j) = (4, 2)$, where, for example, we obtain

$$\begin{aligned}
0 = {} & (J_1 - 2J_2 + J_3)(J_3 - 2J_4 + J_5)\, c_J - 2 \\
& \times (J_3 - 2J_6 + 1)(J_2 - 2J_3 + J_4 + J_6 + 1)\, S_3 S_6 c_J \\
& + 2\Big[J_2^2 + (-4J_3 + 2J_4 + 2J_6 - 13)\, J_2 \\
& + 4J_3^2 + J_4^2 + J_6^2 - 13J_4 + J_3(-4J_4 - 4J_6 + 26) + 2J_4J_6 - 13J_6 + 42 \Big] \\
& \times S_1 S_2^2 S_3^4 S_4^2 S_5 S_6^2 c_J \\
& - \Big[J_2^2 + (-4J_3 + 2J_4 + 2J_6 + 3)\, J_2 + 4J_3^2 + J_4^2 + J_6^2 + 3J_4 + 2J_4J_6 + 3J_6 \\
& - 2J_3(2J_4 + 2J_6 + 3) + 2 \Big] S_3 c_J .
\end{aligned} \tag{89}$$

The initial data of the recursions are fixed by the coefficients $(b_{0,0}, \ldots, b_{r,0})$ in (64), leading to an $r + 1$-dimensional vector space of solutions of (61a)–(61b) with log-singularities as expected: for example, the above differential constraint (59b) for $(i, j) = (4, 2)$ sets

$$c_{(0,0,0,1,0,0,0)} = -1/2 c_{(0,1,2,4,2,1,2)} = -b_{0,2} + 2b_{0,3} - b_{0,4} - b_{0,6}, \tag{90}$$

while (59a) sets $c_{(2,4,8,12,8,4,6)} = -b_{0,2} + 2b_{0,3} - b_{0,4} - b_{0,6}$. The full set of recursions for all (i, j) in (59b) and the ensuing reconstruction of the solutions are omitted for reasons of space and are available upon request.

4 Picard–Fuchs equations: the non-simply laced case

The construction of the previous section is constrained to apply to ADE gauge groups: although there exists a definition of an analogous canonical Frobenius manifold structure on the quotient of the reflection representation of extended affine Weyl groups associated with any root system [30], including non-simply laced cases, these are not expected to retrieve the gauge theory periods as solutions of the differential system (61a)–(61b). The reason for this was already hinted in Sect. 2.2: the mirror theorem of [20] asserts that the Frobenius manifolds $M_{\mathfrak{g}}^{\text{trig}}$ are isomorphic to certain Hurwitz strata associated with the spectral curves of the relativistic Toda chain associated with the co-extended loop group of $\mathcal{G}$, which in the non-relativistic limit reduce to the usual spectral curves of the Toda chain associated with the *untwisted* affine Kac–Moody algebra $\mathfrak{g}^{(1)}$. For non-simply laced $\mathfrak{g}$, this is different from the *twisted* Toda chain relevant for the gauge theory: to account for this, it would in fact be natural to speculate that if one replaced, in the Landau–Ginzburg construction of[20], the affine relativistic Toda chain with its twisted (Langlands/Montonen–Olive dual) version, then this

would yield a "twisted" Frobenius manifold $(M_{\mathfrak{g}}^{\mathrm{trig}})^{\vee}$ from which the gauge theory periods would be retrieved by the same construction of the previous section.

Both expectations turn out to be false. As we shall see, the natural Frobenius metric associated with the spectral curves of Sect. 2.2.2 is either degenerate or not flat, and perhaps more surprisingly, the twisted relativistic Toda spectral curves will turn out to be the *wrong* integrable system for the pure gauge theory in five compactified dimensions, away from $R_5 \to 0$. As such, a new strategy is required to treat the non-simply laced cases along the same lines as the ADE setting. In the next section we will describe one such strategy, partly inspired by previous studies of associativity equations for the prepotentials of the 4d and the (perturbative) 5d theory, which also included non-simply laced cases [12, 43, 44, 63, 64].

4.1 The Jacobi algebra of a Seiberg–Witten curve

By the mirror theorem of [20], the structure constants C_{ij}^{k} in the previous section can be read off from the spectral curve of the type-$\mathcal{G}^{(1)}$ relativistic Toda chain, $\mathsf{R}_{\mathcal{G},\rho';u}(\mu,\lambda)$, as follows. Let $\nu : \overline{C}_{\mathcal{G};u}^{\mathrm{red}} \to \mathbb{P}^1$ be the Cartesian projection to the ν-axis from the perturbative spectral curve $\overline{C}_{\mathcal{G};u}^{\mathrm{red}} := \overline{\{\widetilde{\mathsf{R}}_{\mathcal{G},\rho';u}(\mu,\nu/u_0)=0\}}$ for a choice of non-trivial irreducible representation ρ': as the below is independent of the particular choice, we will henceforth suppress ρ' from our notation. Then, from [20,Thm. 3.1], we have that

$$\partial_{t_i}\nu(\mu)\ \partial_{t_j}\nu(\mu) = \mathsf{C}_{ij}^{k}\partial_{t_k}\nu(\mu)\partial_{t_r}\nu(\mu) + \mathsf{D}_{ij}(\mu)\partial_{\mu}\nu(\mu) \tag{91}$$

for some meromorphic function $\mathsf{D}_{ij}(\mu)$, and where the t_i-derivatives are taken at constant μ. This states that $M_{\mathfrak{g}}^{\mathrm{trig}}$ is isomorphic, as a holomorphic family of commutative rings, to the Jacobi algebra of the meromorphic projection ν. By the implicit function theorem, we can then rewrite (91) as

$$\partial_{t_i}\widetilde{\mathsf{R}}_{\mathcal{G};u}\ \partial_{t_j}\widetilde{\mathsf{R}}_{\mathcal{G};u} = \mathsf{C}_{ij}^{k}\ \partial_{t_k}\widetilde{\mathsf{R}}_{\mathcal{G};u}\ \partial_{t_r}\widetilde{\mathsf{R}}_{\mathcal{G};u} \quad \mod\ (\widetilde{\mathsf{R}}_{\mathcal{G};u},\partial_{\mu}\widetilde{\mathsf{R}}_{\mathcal{G};u}) \tag{92}$$

This can be phrased more poignantly as follows. Let $\widetilde{V}_{(i)}(\widetilde{\mathsf{R}}_{\mathcal{G};u}) := \partial_{t_i}\widetilde{\mathsf{R}}_{\mathcal{G};u}/\partial_{t_r}\widetilde{\mathsf{R}}_{\mathcal{G};u}$. Then for all $u \in \mathbb{C}^{r+1}$, the tangent space $T_{t(u)}M_{\mathfrak{g}}^{\mathrm{trig}}$ is isomorphic as a commutative, associative unital algebra to

$$\mathcal{V}(\widetilde{\mathsf{R}}_{\mathcal{G};u}) := \mathrm{span}_{\mathbb{C}}\{\widetilde{V}_{(i)}\}_{i=0}^{r}.$$

The latter is obviously a vector subspace of the quotient ring $\mathbb{C}[\mathrm{e}^{t_0},t_1\ldots,t_r][\lambda,\nu]/\mathcal{I}(\widetilde{\mathsf{R}}_{\mathcal{G};u})$, where

$$\mathcal{I}(\widetilde{\mathsf{R}}_{\mathcal{G};u}) := \left\langle \widetilde{\mathsf{R}}_{\mathcal{G};u}(\mu,\nu/u_0),\partial_{\mu}\widetilde{\mathsf{R}}_{\mathcal{G};u}(\mu,\nu/u_0)\right\rangle,$$

and (92) states that it is actually a subalgebra, closed under product. The structure constants C_{ij}^k (although not the Frobenius metric yet) of $M_{\mathfrak{g}}^{\text{trig}}$ can be obtained from

$$\widetilde{V}_{(i)}(\widetilde{\mathsf{R}}_{\mathcal{G};u})\widetilde{V}_{(j)}(\widetilde{\mathsf{R}}_{\mathcal{G};u}) = \mathsf{C}_{ij}^k(t)\widetilde{V}_{(k)}(\widetilde{\mathsf{R}}_{\mathcal{G};u}) \quad \text{mod} \quad \mathcal{I}(\widetilde{\mathsf{R}}_{\mathcal{G};u}), \tag{93}$$

in a purely algebraic way, given the knowledge of $\widetilde{\mathsf{R}}_{\mathcal{G};u}$ and the unit vector $e = \widetilde{V}_{(r)}(\widetilde{R}_{\mathcal{G};u})$.

How does this help in dealing with non-simply laced cases? In all instances considered in Sect. 2.2, including non-simply laced gauge groups, the construction of SW curves for the gauge theory leads to a spectral polynomial of the form $\mathsf{P}_{\mathcal{G};u}(\mu,\lambda) = \widetilde{\mathsf{P}}_{\mathcal{G};u}(\mu,\lambda+q_0/\lambda)$ for some $\widetilde{\mathsf{P}}_{\mathcal{G};u} \in \mathbb{Z}[\mu,\nu;u_0,\ldots,u_r]$, exactly as for the untwisted relativistic Toda polynomials relevant for simply laced $\mathcal{G}$ in the discussion above: we have $\mathsf{P}_{\mathcal{G};u} = \mathsf{Q}_{\mathcal{G};u}$ in the M-theory engineering curves (25)–(27), and $\mathsf{P}_{\mathcal{G};u} = \mathsf{R}_{\mathcal{G};u}$ (resp. $\mathsf{P}_{\mathcal{G};u} = \mathsf{T}_{\mathcal{G};u}$) for the untwisted (resp. twisted) relativistic Toda curves of (39). It is then natural to look at the possibility of replacing the input polynomial $\widetilde{\mathsf{R}}_{\mathcal{G};u}$ in (93) with the Hanany–Witten/relativistic Toda reduced characteristic polynomials $\widetilde{\mathsf{Q}}_{\mathcal{G};u}/\widetilde{\mathsf{T}}_{\mathcal{G};u}$, and construct an associated Frobenius manifold out of them.

Indeed, as we shall show in Claim 4.1, for all the curves of Sect. 2.2 we will be able to associate a holomorphic family of commutative and associative algebras via (93), and whose existence is itself non-trivial. But these will not come in general from a Frobenius manifold: when $\mathsf{P}_{\mathcal{G};u} \neq \mathsf{R}_{\mathcal{G};u}$: one can check that the construction of Dubrovin [27] and Brini and van Gemst [20] applied to the curves in Sects. 2.2.1 and 2.2.2 for non-simply laced $\mathcal{G}$ gives rise to a Frobenius metric that is either degenerate (for $\mathsf{Q}_{B_r;u}$ and $\mathsf{T}_{\mathcal{G};u}$) or curved (for $\mathsf{Q}_{C_r;u}$), so there is no clear notion of a privileged "flat" coordinate system where something like (59a)–(59b) could hold. These difficulties can, however, be side-stepped by imposing the constraints of rigid special Kähler geometry: we condense this in the following list of statements.

Claim 4.1 *Let $\widetilde{P}_{\mathcal{G};u}$ be equal to one of $\widetilde{R}_{\mathcal{G};u}$, $\widetilde{Q}_{\mathcal{G};u}$ or $\widetilde{T}_{\mathcal{G};u}$. Up to affine-linear transformations, there exists a unique chart $\{t^i(u)\}_{i=0}^r$ such that*

1. *there exist holomorphic functions $C_{ij}^k(t)$ s.t.*

$$\widetilde{V}_{(i)}(\widetilde{P}_{\mathcal{G};u})\widetilde{V}_{(j)}(\widetilde{P}_{\mathcal{G};u}) = C_{ij}^k(t)\widetilde{V}_{(k)}(\widetilde{P}_{\mathcal{G};u}) \quad \text{mod} \quad \mathcal{I}(\widetilde{P}_{\mathcal{G};u}) \tag{94}$$

2. *the periods associated with the SW curve $\{P_{\mathcal{G};u}(\mu,\lambda) = 0\}$ satisfy the system of 2nd order PDEs*

$$\boxed{\partial^2_{t_it_j}\Pi = C_{ij}^k(t)\partial^2_{t_rt_k}\Pi} \tag{95}$$

The coordinates $t_i(u)$ are uniquely fixed by imposing that the inverse $u_i(t)$ is a collection of trigonometric polynomials,

$$u_0 = \mathrm{e}^{t_0}, \quad u_i \in \mathbb{C}[\mathrm{e}^{-t_0};t_1,\ldots,t_r], \tag{96}$$

and that single-logarithmic singular solutions of (95) exist at the point of large complex structure.

Our claim therefore postulates the existence of a differential ideal annihilating the periods of the SW curve, which is specified entirely by the product structure on $\mathcal{V}(\widetilde{P}_{\mathcal{G};u})$ and by the choice of a canonical system of coordinates $t^i(u)$. The latter behaves as a surrogate of the flat coordinates of the previous section, in spite of the lack of a flat non-degenerate Frobenius metric.

As we did for the ADE gauge groups, we will subject Claim 4.1 to a variety of tests, including comparisons with the gauge theory calculations from Sect. 2.1 and direct period integral calculations from the spectral curve, giving in turn an effective method to compute the periods in (95)–(97) around the large complex structure limit. We will proceed, accordingly, in three steps.

Step 1: the canonical ring structure. In analogy with the ADE cases, we assume that some of the special coordinates t_i are determined by the same residue formula as Brini and van Gemst [20,Lemma. 4.1] which is a specialisation of the residue formulas of [28,Lecture 5] for the flat coordinates of a Hurwitz Frobenius manifold. For the examples below, we will indeed find that all t_i (but perhaps t_r) are determined by the residue computation. Again similarly to the simply laced setting, we further impose that $t_r = u_0(u_r + f(u_1, ...u_{r-1}))$ where f is a polynomial in $u_1, ..., u_{r-1}$. We then proceed to verify the first point in Claim 4.1 by an algebraic construction of the ring structure on $\mathcal{V}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$. To do this, we first compute a reduced Gröbner basis $\mathrm{GB}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$ for the ideal $\mathcal{I}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$ and then use multi-variate polynomial division to compute the reduction of (94) w.r.t. $\mathrm{GB}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$. Since the latter is a Gröbner basis, the reduction is zero iff the above expression is in the ideal $\mathcal{I}(\widetilde{\mathsf{P}}_{\mathcal{G};u})$, and imposing the vanishing of the remainder of the division gives an *a priori* highly overconstrained inhomogeneous linear system for the indeterminates C^k_{ij}. While for a general family of plane curves the system would admit no solutions, we will always find that, remarkably, for all the spectral polynomials $\widetilde{\mathsf{P}}_{\mathcal{G};u}$ in Sect. 2.2 a unique solution exists.

Step 2: the special coordinates. We are now in a position to write down the differential system (95) in a quasi-polynomial t-chart. At this point we still have a parametric dependence on the coefficients of the polynomial $f(u_1, ..., u_{r-1})$, which we fix as follows. We first write (95) in the z-chart defined by (60), and look for a condition such that solutions in the form (64) exist. Remarkably, all coefficients of $f(u_1, ..., u_{r-1})$ turn out to be fixed this way, by just looking at the classical piece of the solutions, i.e. their limit as $z_0 \to 0$.

Step 3: taming the solutions. We still lack the analogue of the quasi-homogeneity condition (59a) at this stage, which entails that the solution space of (95) will be *a priori* of much higher dimension than the expected $2r + 2$.It turns out that this freedom can be fixed entirely,[10] in either of two ways. For the spectral curves with $\mathcal{G} = B_r$, associated either with the twisted affine relativistic Toda chain or the M-theory engineering of Brandhuber et al. [15], we find that there exist complex numbers $\alpha, \beta \in \mathbb{C}$, and $l \in \{0, \ldots, r\}$ such that

[10] With the one exception of the Sp(1) curves, where a small remaining ambiguity will need additional input to be fixed.

$$\boxed{\left(t_0+\sum_{i=1}^{r}\mathfrak{q}_i^{\vee}t_i\partial_{t_i}\right)^2\Pi=\alpha\mathrm{e}^{\beta t_0}\partial^2_{t_lt_r}\Pi.} \tag{97}$$

In the quasi-homogeneity Eq. (97), the Euler vector field is specified in terms of the dual Coxeter exponents of $\mathcal{G}$ (i.e. the α-basis coefficients of the highest short root), and we slightly generalise (97) by admitting a more general second derivative term in the r.h.s., where possibly $t_l \neq t_r$.[11] For $\mathcal{G} \neq B_r$, we show that the imposition of the existence of a prepotential whose gradient returns the double-log solutions, $\partial_{\Pi_{A_i}}\mathcal{F}=\Pi_{B_i}$, and whose analytic structure at weak coupling has the form

$$\mathcal{F}_{\mathcal{G}}(q_0,\ldots,q_r)=C_3(\log q_0,\ldots,\log q_r)+\sum_{I\in(\mathbb{N})^{r+1}}a_I\prod_j q_j^{I_j}, \tag{98}$$

allows to fix uniquely all remaining ambiguities.

Having outlined our computational strategy, we will now verify the above proposal in several examples.

4.2 B_2

From (26), the perturbative spectral curve for $\mathcal{G}=B_2=\mathrm{Spin}(5)$ reads

$$\widetilde{\mathrm{Q}}_{B_2}(\mu,\nu;u)=\left(\mu^2-1\right)\left(\mu^2+1\right)^2\mu\,\nu+u_2\mu^4+u_1\left(\mu^6+\mu^2\right)+\mu^8+1. \tag{99}$$

As far as Claim 4.1 is concerned, we find that for this example, and indeed for all B_r cases we have checked (up to $r=4$), the privileged coordinate set $t_i(u)$ including t_r can be found by the same residue calculation of Brini and van Gemst [20,Lemma 4.1] applied to the perturbative SW curve (99). We thus get

$$t_0=\log u_0,\quad t_1=u_0^{1/2}\sqrt{2-2u_1+u_2},\quad t_2=u_0(2+2u_1+u_2). \tag{100}$$

with inverse

$$u_0=\mathrm{e}^{t_0},\quad u_1=-\frac{1}{4}\mathrm{e}^{-t_0}\left(t_1^2-t_2\right),\quad u_2=\frac{1}{2}\mathrm{e}^{-t_0}\left(t_1^2-4\mathrm{e}^{t_0}+t_2\right). \tag{101}$$

A reduced Gröbner basis for the discriminant ideal $\mathcal{I}(\widetilde{\mathrm{Q}}_{B_2;u})$ can be computed using Buchberger's algorithm with respect to the lexicographic monomial ordering $\mu\prec\nu$. We obtain $\mathrm{GB}(\widetilde{\mathrm{Q}}_{B_2;u})=\{P(\mu),Q(\mu,\nu)\}$, with

$$P(\mu)=\left(\mu^2+1\right)^4\left(4\left(\mu^2-1\right)^2e^{t_0}-\mu^2t_2\right)$$

[11] Although we believe this can be reduced to the form appearing in (97), i.e. with $t_l=t_r$, as an equation on cohomology (the periods), it will be easier for us to consider the generalised, and equivalent equation (97) by seeing this as following from a stronger constraint at the level of forms (the SW differential).

$$
\begin{aligned}
&+\mu^2\left(\mu^2-1\right)^2\left(\mu^4-6\mu^2+1\right)t_1^2,\\
Q(\mu,\nu)=\mu\Big[&\left(39\mu^8-313\mu^6+557\mu^4-347\mu^2+64\right)t_1^2\\
&+4(\mu^2-1)\left(39\mu^8+116\mu^6+78\mu^4-68\mu^2-101\right)e^{t_0}\\
&-\left(39\mu^8+155\mu^6+233\mu^4+165\mu^2+64\right)t_2\Big]+128\nu. \qquad (102)
\end{aligned}
$$

Reducing (94) to the ideal $\mathcal{I}(\widetilde{\mathsf{Q}}_{B_2;u}) = \left\langle \mathrm{GB}(\widetilde{\mathsf{Q}}_{B_2;u})\right\rangle$ w.r.t. to the reduced basis $\mathrm{GB}(\widetilde{\mathsf{Q}}_{B_2;u})$ and solving for C_{ij}^k, we obtain, for the structure constants of (95),

$$
\mathsf{C}_{0i}^j=\begin{pmatrix}-t_1^2-16\mathrm{e}^{t_0}+t_2 & -16\mathrm{e}^{t_0}t_1 & 16\mathrm{e}^{t_0}t_2\\ -2t_1 & -16\mathrm{e}^{t_0} & 0\\ 1 & 0 & 0\end{pmatrix}_i^j, \qquad (103)
$$

$$
\mathsf{C}_{1i}^j=\begin{pmatrix}-2t_1 & -16\mathrm{e}^{t_0} & 0\\ -2 & -t_1 & 2t_2\\ 0 & 1 & 0\end{pmatrix}_i^j. \qquad (104)
$$

Finally, let us show how to derive the quasi-homogeneity condition (97). We will directly impose that (97) holds as a constraint on the Seiberg–Witten differential $\log\mu\mathrm{d}\log\lambda$ for some α, β and l. The full SW curve $\overline{\mathcal{C}}_{B_r;u}$ is hyperelliptic, with a degree two map $\mu:\overline{\mathcal{C}}_{\mathcal{G};u}\to\mathbb{P}^1$ realising it as a branched cover of the complex line. In particular there is a subset of homology cycles $\{\gamma_1,\ldots,\gamma_{g(\overline{\mathcal{C}}_{B_r;u})}\}$ in $\overline{\mathcal{C}}_{B_r;u}$ such that

$$
\Pi_{\gamma_i}:=\oint_{\gamma_i}\log\mu\mathrm{d}\log\lambda=\int_{l_i}(\log\lambda_+-\log\lambda_-)\,\mathrm{d}\log\mu \qquad (105)
$$

for chains $l_i=\mu(\gamma_i)\subset\mathbb{P}^1$, where an integration by parts has been performed. Let now $\mathfrak{D}=\sum_{ij}\mathsf{a}_{ij}(u)\partial^2_{u_iu_j}+\sum_i\mathsf{b}_i\partial_{u_i}$ be a second order differential operator whose symbol $\sigma(\mathfrak{D})$ has vanishing constant term. Then it is easy to show that $\mathfrak{D}(\log\lambda_+(\mu)-\log\lambda_-(\mu))=P_1/P_2^{3/2}$ for $P_1,P_2\in\mathbb{C}[\mu;u_0,\ldots,u_r]$. Imposing that $P_1\equiv 0$ identically in μ gives an in principle overconstrained system of equations in the indeterminates a, b. Happily, this can be shown to non-trivially admit a one-dimensional space of solutions, parametrised by overall rescalings of $\mathfrak{D}$: in flat coordinates and at the level of periods, this gives

$$
\left(\partial_{t_0}+\frac{1}{2}t_1\partial_{t_1}+t_2\partial_{t_2}\right)^2\Pi=-256\mathrm{e}^{-t_0}\partial_{t_0}\partial_{t_2}\Pi. \qquad (106)
$$

completing the construction of the PF system in (95) and (97).

Armed with this, we proceed as in the analysis for the simply laced cases. We find three linearly independent single-log solutions in the z_i-coordinates (60), and two

double-log solutions, confirming our Claim 4.1. Up to 2-instanton corrections, we find that the perturbative level is unambiguously fixed by (95), while order by order in z_0, each solution comes with finitely many constants which are equivalently fixed by the quasi-homogeneity Eq. (97) or by imposing the existence of the prepotential. We find that the final result

$$\begin{aligned}-4\pi^3 \mathrm{i}\mathcal{F}_{B_2} = &\frac{5}{6}\log^3 q_2 + \log q_0 \log^2 q_2 + \frac{5}{4}\log q_1 \log^2 q_2 \\ &+ \frac{3}{4}\log^2 q_1 \log q_2 + \log q_0 \log q_1 \log q_2 \\ &+ q_2 + q_1 q_2 + q_1 q_2^2 + \frac{q_2^2}{8} + \frac{q_2^3}{27} + \frac{q_2^4}{64} + \frac{q_1^2 q_2^2}{8} + \cdots \\ &+ q_0\Big(q_1 q_2 + 3q_1^2 q_2 + 5q_1^3 q_2 + 4q_1^2 q_2^2 + 7q_1^4 q_2 \\ &+ 8q_1^3 q_2^2 + 3q_1^2 q_2^3 + 9q_1^5 q_2 + 12q_1^4 q_2^2 \\ &+ 9q_1^3 q_2^3 + 11q_1^6 q_2 + 16q_1^5 q_2^2 + 15q_1^4 q_2^3 + 8q_1^3 q_2^4 + \cdots\Big)\end{aligned} \tag{107}$$

is in precise agreement with the IMS, 1-loop, and 1-instanton prepotential, which we have checked up to $\mathcal{O}(q_i^8)$.

4.3 B_3

The reduced characteristic polynomial for the M-theory curve of $\mathcal{G} = B_3$ is given from (26) as

$$\begin{aligned}\widetilde{\mathrm{Q}}_{B_3;u}(\mu,\nu) = &(\mu^2-1)\mu^3\left(\mu^2+1\right)^2 \nu \\ &+ u_3\mu^6 + u_1\left(\mu^{10}+\mu^2\right) + u_2\left(\mu^8+\mu^4\right) + \mu^{12} + 1,\end{aligned} \tag{108}$$

We follow exactly the same procedure as for $\mathcal{G} = B_2$ to compute the structure constants of the Jacobi algebra (93) and the flat coordinates. We omit details and present the final result for both. For the coordinates, we get

$$\begin{aligned}t_0 &= \log u_0,\ t_1 = u_0^{\frac{2}{3}}(u_1-1),\ t_2 \\ &= u_0^{1/2}(2u_1 - 2u_2 + u_3 - 2)^{1/2},\ t_3 = u_0(2 + 2u_1 + 2u_2 + u_3),\end{aligned} \tag{109}$$

with polynomial inverses

$$\begin{aligned}u_0 &= \mathrm{e}^{t_0},\ u_1 = \mathrm{e}^{-\frac{2t_0}{3}}t_1 + 1,\ u_2 = \frac{1}{4}\mathrm{e}^{-t_0}t_2^2 - \frac{1}{4}\mathrm{e}^{-t_0}t_3 + 1,\ u_3 \\ &= \frac{1}{2}\mathrm{e}^{-t_0}t_2^2 - 2\mathrm{e}^{-\frac{2t_0}{3}}t_1 + \frac{1}{2}\mathrm{e}^{-t_0}t_3 - 2,\end{aligned} \tag{110}$$

while the structure constants in the t-chart read

$$\mathsf{C}^j_{00} = \begin{pmatrix} \frac{4e^{\frac{t_0}{3}} t_1 \left(t_1 - 24e^{\frac{t_0}{3}}\right) + 3t_3 - 3t_2^2}{9} \\ \frac{12e^{\frac{2t_0}{3}} t_1^2 + 288e^{t_0} t_1 + 3\left(t_3 - t_2^2\right) t_1 - 432e^{\frac{4t_0}{3}} + e^{\frac{t_0}{3}} \left(18\left(5t_2^2 + t_3\right) - 8t_1^3\right)}{27} \\ \left(16e^{t_0} - \frac{16}{3} e^{\frac{2t_0}{3}} t_1\right) t_2 \\ \frac{16e^{\frac{2t_0}{3}}}{3} \end{pmatrix}_j \tag{111}$$

$$\mathsf{C}^j_{1i} = \begin{pmatrix} \frac{4}{3} e^{\frac{t_0}{3}} \left(t_1 - 6e^{\frac{t_0}{3}}\right) & \frac{1}{9}\left(-8e^{\frac{t_0}{3}} t_1^2 - 36e^{\frac{2t_0}{3}} t_1 + 72e^{t_0} - 3t_2^2 + 3t_3\right) & -\frac{16}{3} e^{\frac{2t_0}{3}} t_2 & \frac{16}{3} e^{\frac{2t_0}{3}} t_3 \\ 4e^{\frac{t_0}{3}} & -\frac{4}{3} e^{\frac{t_0}{3}} \left(2t_1 + 9e^{\frac{t_0}{3}}\right) & 0 & 0 \\ 0 & -2t_2 & -16e^{\frac{2t_0}{3}} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^j_i \tag{112}$$

$$\mathsf{C}^j_{2i} = \begin{pmatrix} -2t_2 & 8e^{\frac{t_0}{3}} t_2 & 16e^{t_0} - \frac{32}{3} e^{\frac{2t_0}{3}} t_1 & 0 \\ 0 & -2t_2 & -16e^{\frac{2t_0}{3}} & 0 \\ -6 & 2\left(t_1 + 6e^{\frac{t_0}{3}}\right) & -t_2 & 2t_3 \\ 0 & 0 & 1 & 0 \end{pmatrix}^j_i, \tag{113}$$

and $\mathsf{C}^j_{3i} = \delta^j_i$. As for the B_2 case, we also look for a smaller set of differential equations satisfied directly by the SW differential at for fixed μ. It turns out that there is a four-dimensional space of differential operators annihilating (105) in this case, and one basis element is in the desired form:

$$\left(\partial_{t_0} + \frac{2}{3} t_1 \partial_{t_1} + \frac{1}{2} t_2 \partial_{t_2} + t_3 \partial_{t_3}\right)^2 \Pi = -4096 e^{-t_0/3} \partial_1 \partial_3 \Pi. \tag{114}$$

Once again the corresponding solutions are found to have the desired asymptotic behaviour at infinity in the Coulomb branch. We verified that the resulting prepotential agrees with the gauge theory calculation up to 1-instanton corrections and up to order $\mathcal{O}(q_i^7)$, $i = 1, 2, 3$.

4.4 C_1 at $\theta = 0$

For symplectic gauge groups, $\mathcal{G} = C_r$, we have the two different sets of curves (30) and (28)–(29), from Brandhuber et al. [15] and Hayashi et al. [42] and Brandhuber et al. [59], respectively. We consider here for the purpose of illustration the rank one case at different theta angles.

For $\theta = 0$, (28) gives

$$\widetilde{\mathsf{Q}}_{C_1;u,0}(\mu, \nu/u_0) = \mu^3\nu + \left(\mu^2 - 1\right)^2 \left(u_1\mu + \mu^2 + 1\right) + \frac{\mu^2\left(\mu^2+1\right)}{u_0}. \quad (115)$$

For the coordinates $t^i(u)$ and the $t-$chart structure constants, we find

$$t_0 = \log u_0, \quad t_1 = u_0 u_1, \quad (116)$$

$$\mathsf{C}^j_{0i} = \begin{pmatrix} -\frac{2t_1}{3} & \frac{1}{3}\mathrm{e}^{t_0}\left(4\mathrm{e}^{t_0} - 1\right) \\ 1 & 0 \end{pmatrix}, \quad (117)$$

and again $\mathsf{C}^j_{1i} = \delta^j_i$.
The resulting Picard–Fuchs system is superficially different from the one of A_1 in (67). Also the absence of an analogue of the quasi-homogeneity condition (97) leads to a space of solutions of dimension higher than $2r + 2 = 4$: while for higher rank the imposition of the existence of a prepotential is sufficient to project to a solution space of the correct dimension, the case of rank one is pathological and leads to a small finite ambiguity at every instanton level. In this case, perturbative parts and 1-instanton corrections are unambiguously fixed, but for higher instanton corrections we find two free parameters at each even order, and none at odd orders, which we can fix by matching with a direct period expansion from the curve. By doing so, we verify that the resulting prepotential is in agreement with the SU(2) calculation at vanishing θ-angle, which we confirmed in an expansion at weak coupling up to 5-instantons.

4.5 C_1 at $\theta = \pi$

We will jointly consider in this section the curves (29) and (30). We will show that, while formally different, these surprisingly give rise to the same algebra structure on the discriminant of the perturbative SW curve, and therefore to the same differential system (95). As in the previous section, however, in this rank one situation there will be a finite (two-dimensional) ambiguity at each order of the q_0 expansion of the solutions, which as for $\theta = 0$ we fix from a direct period calculation from the full SW curve. It turns out that the free parameters thus fixed will be *different* according to which curve one picks: the respective prepotential will also be different, with only the curve (29) returning the correct gauge theory prepotential.

The perturbative SW curve (30) of [15] reads

$$\widetilde{\mathsf{Q}}^{\flat}_{C_1;u}(\mu, \nu) = \mu^3\nu + \left(\mu^2 - 1\right)^2 \left(u_1\mu - \mu^2 - 1\right), \quad (118)$$

whereas the $C_1^{\theta=0}$ curve in [42] is given by

$$\widetilde{\mathsf{Q}}_{C_1;u,\pi}(\mu, \nu) = \mu^3\nu - \left(\mu^2 - 1\right)^2 \left(u_1\mu + \mu^2 + 1\right) + \frac{2\mu^3}{u_0} \quad (119)$$

Note that, up to rescalings of ν and μ, the curves differ only by the last term which depends on u_0 and is therefore proportional to the gauge theory scale: this implies that the 1-loop prepotentials agree for these two curves. It is, however, less trivial to see whether this equality survives instanton corrections. In the same special coordinates for $\mathcal{G} = A_1$ and $\mathcal{G} = C_1^{\theta=0}$,

$$t_0 = \log u_0, \qquad t_1 = u_0 u_1, \tag{120}$$

we find that the structure constants for both curves indeed surprisingly agree:

$$\mathsf{C}_{0i}^{j} = \begin{pmatrix} -\frac{2t_1}{3} & \frac{4e^{2t_0}}{3} \\ 1 & 0 \end{pmatrix}, \tag{121}$$

and $\mathsf{C}_{1i}^{j} = \delta_i^j$. We then solve (95) in an expansion in $z_0 = 1/u_0$, $z_1 = 1/\sqrt{u_1}$ around $(z_0, z_1) = (0, 0)$.

As before, in this rank-1 case the imposition of the existence of a prepotential does not give enough constraints, and the equations (95) leave a two-dimensional ambiguity at every instanton level which we fix, at any given order in q_0, by an expansion of the period integrals to leading order in $z_1 = 1/u_1$. It turns out that different values are obtained for these free parameters according to whether one picks (29) or (30): for example, for the non-trivial single-log solution of (95), we find

$$\begin{aligned} \Pi_A = \log z_1 + z_1^2 + C_1 z_0 z_1^3 + \frac{3z_1^4}{2} + 6C_1 z_0 z_1^5 + z_1^6 \left(C_2 z_0^2 + \frac{10}{3} \right) \\ + 30 C_1 z_0 z_1^7 + z_1^8 \left(14 C_2 z_0^2 + \frac{35}{4} \right) + \mathcal{O}\left(z_1^9 \right) \end{aligned} \tag{122}$$

where $(C_1, C_2, \dots) = (2, 15, \dots)$ for (29) and $(C_1, C_2, \dots) = (0, 5, \dots)$ for (30). These impact the prepotential directly: for the curve (29) we find

$$\begin{aligned} -4\pi^3 \mathrm{i} \mathcal{F}_{C_1, \theta=\pi} = & \frac{1}{8} \log^2 q_1 \log q_0 + \frac{1}{12} \log^3 q_1 + q_1 + \frac{q_1^2}{8} + \frac{q_1^3}{27} + \frac{q_1^4}{64} + \dots \\ & - \frac{q_0}{2} \left(1 + 3q_1 + 5q_1^2 + 7q_1^3 + 9q_1^4 + 11q_1^5 + \cdots \right) \\ & + q_0^2 \left(-\frac{q_1}{16} + \frac{45}{16} q_1^2 + 16 q_1^3 + \frac{875}{16} q_1^4 + 144 q_1^5 + \dots \right), \end{aligned} \tag{123}$$

in precise agreement with the genus zero topological string calculation from the engineering geometry $K_{\mathbb{F}_1}$ [23]. For the curve (30) instead we get the correct expression for the 1-loop term, but disagreement is already found at 1-instanton level for all terms, where, for example, the $[q_0^1 q_1^0]$ coefficient in the expansion of the prepotential is predicted to vanish. This confirms the tension between the results of Brandhuber et al. [15] on the one hand and Hayashi et al. [42] and Li and Yagi [59] on the other, ruling out the curve (30) of Brandhuber et al. [15] as a candidate Seiberg–Witten geometry for the Sp(1) theory with no flavours.

4.6 The relativistic Toda chain on a twisted loop group

4.6.1 B_2

From (48), the spectral curve of the twisted B_2 relativistic Toda chain can be obtained from the reduced characteristic polynomial

$$\widetilde{\mathsf{T}}_{B_2;u}(\mu,\nu) = u_2\mu^2 - u_1\left(\mu^3+\mu\right) + \mu^4 + 1 + \left(\mu^2-1\right)\mu\nu. \tag{124}$$

Note that (124) is different from (99), even after the natural redefinition $\mu \to \sqrt{\mu}$. In the non-relativistic limit, the full curve $\mathsf{T}_{B_2;u}(\mu,\lambda) = \widetilde{\mathsf{T}}_{B_2;u}(\mu,\lambda+q_0/\lambda) = 0$ reduces to the four-dimensional $(B_2^{(1)})^\vee = A_3^{(2)}$ curve of [66].

Even though t_2 is not determined by the residue formula of Brini and van Gemst [20,Lemma 4.1], we can still find it by applying the strategy outlined above. Omitting computational details, the special geometry condition on the dual periods gives for the t-chart the following expressions:

$$t_0 = \log u_0, \quad t_1 = u_0u_1, \quad t_2 = u_0(2+u_1+u_2). \tag{125}$$

The Gröbner basis calculation for $\mathcal{I}\left(\widetilde{\mathsf{T}}_{B_2;u}\right)$ gives then

$$\begin{aligned}\mathrm{GB}\left(\widetilde{\mathsf{T}}_{B_2;u}\right) = \Big\{&\mathrm{e}^{-t_0}\left(\left(\mu^2+4\mu+1\right)\mu^2t_1 + \left(\mu^2+1\right)\left(\left(\mu^2-1\right)^2\mathrm{e}^{t_0} - \mu^2t_2\right)\right),\\ &\mathrm{e}^{-t_0}\left(2\nu + 3\mu^5\mathrm{e}^{t_0} - 2\mu^3\mathrm{e}^{t_0} - \left(3\mu^2+4\right)\mu t_2\right.\\ &\left.+\left(3\mu^3+12\mu^2+4\mu+2\right)t_1 - \mu\mathrm{e}^{t_0}\right)\Big\},\end{aligned} \tag{126}$$

from which the structure constants of the Jacobi algebra are read off as

$$\mathsf{C}_{0i}^j = \begin{pmatrix} -t_1 - 4\mathrm{e}^{t_0} + t_2 & 4\mathrm{e}^{t_0}t_1 & 4\mathrm{e}^{t_0}t_2 \\ -1 & t_2 - t_1 & 3t_1 + t_2 \\ 1 & 0 & 0 \end{pmatrix}, \tag{127}$$

$$\mathsf{C}_{1i}^j = \begin{pmatrix} -1 & t_2 - t_1 & 3t_1 + t_2 \\ \mathrm{e}^{-t_0} & -2 & 3 \\ 0 & 1 & 0 \end{pmatrix}. \tag{128}$$

As for the B_2 curve of Brandhuber et al. [15] considered in Sect. 4.2, since $\overline{\{\mathsf{T}_{B_2;u}(\mu,\lambda+q_0/\lambda)=0\}}$ is hyperelliptic, we can directly derive a quasi-homogeneity equation satisfied by the Seiberg–Witten differential in the form (97):

$$L_E^2\Pi = 4\mathrm{e}^{-t_0}\partial_0\partial_2\Pi. \tag{129}$$

With the above in place, it is now straightforward to compute the prepotential. Surprisingly, this *disagrees* with the gauge theory result, away from the four-dimensional, $R_5 \to 0$ limit, already at the 1-loop level. We find:

$$\begin{aligned}
-4\pi^3 \mathrm{i}\mathcal{F}_{B_2} = {} & \frac{\log^3 q_1}{3} + \frac{1}{2}\log q_0 \log^2 q_1 + \frac{1}{2}\log^2 q_1 \log q_2 + \frac{3}{8}\log q_1 \log^2 q_2 \\
& + \frac{1}{2}\log q_0 \log q_1 \log q_2 + \frac{\log^3 q_2}{8} + \frac{1}{4}\log q_0 \log^2 q_2 \\
& + \frac{q_2}{4} + \frac{q_2^2}{32} + \frac{q_2^3}{108} + q_1 + q_1 q_2 + \frac{q_1^2}{8} + \frac{q_1^2 q_2}{4} + \frac{q_1^2 q_2^2}{8} + \frac{q_1^3}{27} + \frac{q_1^3 q_2^3}{27} + \cdots \\
& + q_0 \left(\frac{1}{2} q_1^2 q_2 + q_1^3 q_2 + \frac{3}{2} q_1^4 q_2 + q_1^3 q_2^2 + 2 q_1^5 q_2 + 2 q_1^4 q_2^2 + \frac{5}{2} q_1^6 q_2 \right. \\
& \left. + 3 q_1^5 q_2^2 + \frac{3}{2} q_1^4 q_2^3 + \cdots \right) \\
& + q_0^2 \left(\frac{1}{32} q_1^4 q_2^2 + \frac{3}{4} q_1^5 q_2^2 + \frac{1}{16} q_1^4 q_2^3 + \frac{65}{16} q_1^6 q_2^2 + q_1^5 q_2^3 \right. \\
& \left. + \frac{1}{8} q_1^4 q_2^4 + \frac{55}{4} q_1^7 q_2^2 + \frac{99}{16} q_1^6 q_2^3 + \cdots \right).
\end{aligned} \tag{130}$$

We see from (130) that the W-bosons multi-cover contributions to the prepotential take the form $\sum_{\alpha \in \Delta^+(B_2)} 4/(\alpha,\alpha)^2 \mathrm{Li}_3(q^\alpha)$, where contributions of long roots appear weighted with their inverse square length ($= 1/4$), instead of appearing with weight one. Note that this is not incompatible with the curve being correct in the four-dimensional limit, as such weighing factor can be reabsorbed when $R_5 \to 0$ by a rescaling of the Coulomb moduli and an immaterial quadratic shift of the prepotential. It should not come as much of a surprise that the higher order instanton terms also disagree with the gauge theory calculation from the blow-up formulas (23).

At this point one might wonder whether the mismatch with the instanton calculation is to be ascribed to a failure of our formalism in this case, rather than a genuine deviation of the twisted Toda spectral curve from the correct description of the low energy theory. A good litmus test would be to contrast (130) with a direct period integral calculation from the curve: as the discrepancy with the gauge theory prepotential affects the perturbative level as well, this can be probed already in the $q_0 \to 0$ limit, where by (40) the spectral curve reduces to the vanishing locus of (124). This is a rational curve, as the μ-projection is unramified and maps it isomorphically to a $\mathbb{P}^1$. As a result, the general expression for the triple derivatives of the prepotential boils down to a sum of residues at the divisors of zeroes and poles of μ and ν:

$$\begin{aligned}
& \partial^3_{\log q_i, \log q_j, \log q_k} \mathcal{F}_{B_2} \\
& = \sum_{\mathrm{d}\nu = 0} \frac{q_i q_j q_k \partial_{q_i}(\log\mu \, \mathrm{d}\log\nu) \partial_{q_j}(\log\mu \, \mathrm{d}\log\nu) \partial_{q_k}(\log\mu \, \mathrm{d}\log\nu)}{\mathrm{d}\log\mu \, \mathrm{d}\log\nu} \\
& = \sum_{\nu,\mu \in \{0,\infty\}} \frac{q_i q_j q_k \partial_{q_i}(\log\nu \, \mathrm{d}\log\mu) \partial_{q_j}(\log\nu \, \mathrm{d}\log\mu) \partial_{q_k}(\log\nu \, \mathrm{d}\log\mu)}{\mathrm{d}\log\mu \, \mathrm{d}\log\nu}
\end{aligned} \tag{131}$$

where in going from the first to the second line we have turned the contour around in the μ-plane and picked up poles of the integrand in the complement of $\{\mathrm{d}\nu(\mu) = 0\}$; we have further used the fact that $\{\mathrm{d}\mu(\nu) = 0\} = \emptyset$; and we have finally employed the "thermodynamic identity" $\partial_{q_i}\mu\mathrm{d}\nu = -\partial_{q_i}\nu\mathrm{d}\mu$ [28,Lecture 5]. The residues in (131) are then immediate to compute as, from (124), they are just residues of the rational function $\nu(\mu)$ at $\mu = 0, \infty$, and $\mathrm{e}^{\pi \mathrm{i} k/2}$, $k = 0, 1, 2, 3$. This returns the $q_0 = 0$ part of (130), confirming the correctness of the proposed Picard–Fuchs system, and the discrepancy with the gauge theory alongside it.

4.6.2 G_2

Our last example is the spectral curve geometry of the twisted relativistic Toda chain associated with the G_2 root lattice. The reduced characteristic polynomial is given by

$$\begin{aligned}\widetilde{T}_{G_2;u}(\mu,\nu) = (\mu-1)^2\left[u_2(\mu+1)^2\mu^2 - u_1^2\mu^3 - u_1\left(\mu^5+\mu\right) + \sum_{i=0}^{6}\mu^i\right]\\ + \nu(\mu-1)^2\mu\left(\mu^4+2\mu^3-u_1\mu^2+\mu^2+2\mu+1\right)\\ - \nu^2\mu^3\left(\mu^2+\mu+1\right) + \frac{3(\mu+1)^2\mu^3}{u_0^2}.\end{aligned} \tag{132}$$

The corresponding spectral curve $\{\widetilde{\mathsf{T}}_{G_2,u}(\mu,\nu+q_0/\nu) = \mathsf{T}_{G_2,u}(\mu,\lambda) = 0\}$ has already appeared in the literature as a factor of the spectral curve for $\mathrm{SU}(N)$ Chern–Simons theory on the Seifert manifold $\Sigma_{2,3,3} \simeq S^3/\mathbb{T}_{24}$, where $\mathbb{T}_{24}$ is the binary tetrahedral group: the spectral parameter dependence of (40) is indeed the same as that of [13,Eq.(6.33)–(6.35)]. This is large-N dual to the topological A-string on the quotient $[\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(-1)/\mathbb{T}_{24}]$ of the resolved conifold, which engineers the pure gauge theory with $\mathcal{G} = E_6$. The family of spectral curves for the twisted G_2 Toda chain curve then sit as a three-parameter subfamily of the E_6 five-dimensional SW curves. It is furthermore immediate to see that it returns the affine $G_2^\vee$ periodic Toda spectral curve of Martinec and Warner [66] in the $R_5 \to 0$ limit.

The quasi-polynomiality condition and the condition of existence of single-log solutions at large complex structure uniquely fixes the privileged coordinate t-frame as

$$t_0 = \log u_0, \quad t_1 = u_0^{\frac{1}{2}}(u_1+2), \quad t_2 = u_0\left(\frac{1}{2}u_1^2 + 7u_1 + 7 + u_2\right), \tag{133}$$

in terms of which the structure constants read

$$\mathsf{C}_{0i}^{j} = \begin{pmatrix} \frac{6t_2+18\mathrm{e}^{t_0}-t_1^2+18t_1\mathrm{e}^{\frac{t_0}{2}}}{4} & \frac{2t_2t_1+12\mathrm{e}^{\frac{t_0}{2}}(4t_1^2+t_2)-3t_1^3-54\mathrm{e}^{t_0}t_1}{8} & \frac{4t_2t_1^2+6\mathrm{e}^{\frac{t_0}{2}}(9t_1^2+14t_2)t_1-4t_2^2+18\mathrm{e}^{t_0}(3t_1^2-4t_2-3t_1^4)}{8} \\ -3\mathrm{e}^{\frac{t_0}{2}} & -t_1^2+\frac{9}{2}\mathrm{e}^{\frac{t_0}{2}}t_1+t_2 & \mathrm{e}^{\frac{t_0}{2}}\left(9t_1^2+6t_2\right)+\frac{1}{4}\left(2t_1t_2-3t_1^3\right) \\ 1 & 0 & 0 \end{pmatrix}_i^j, \tag{134}$$

$$\mathsf{C}^j_{1i} = \begin{pmatrix} -3\mathrm{e}^{\frac{t_0}{2}} & -t_1^2 + \frac{9}{2}\mathrm{e}^{\frac{t_0}{2}} t_1 + t_2 & \mathrm{e}^{\frac{t_0}{2}}\left(9t_1^2 + 6t_2\right) + \frac{1}{4}\left(2t_1t_2 - 3t_1^3\right) \\ 2 & -3t_1 & -\frac{3t_1^2}{2} + 9\mathrm{e}^{\frac{t_0}{2}} t_1 - t_2 \\ 0 & 1 & 0 \end{pmatrix}^j_i, \quad (135)$$

and $\mathsf{C}^j_{2i} = \delta^j_i$.

Unlike the twisted B_2 case, the spectral curve is not hyperelliptic here and it is more difficult to come up with a partial differential equation in the moduli which is already satisfied at the level of differentials. To fix the prepotential uniquely, we shall first consider the full solution space of (95), without imposing any quasi-homogeneity condition of the form (97). We find that the solutions thus obtained possess the following properties in the z-chart:

- all Taylor coefficients of $\mathcal{O}(z_0^0)$ in the expansion around $z = 0$ are unambiguously fixed only by (95),
- order by order in z_0, each solution comes with finitely many constant parameters that cannot be fixed purely by (61b) due to the lack of the quasi-homogeneity equation (61a), but imposing the existence of the prepotential in the form (98) *uniquely* fixes such constant parameters.

Exactly as for the previous case of $\mathcal{G} = B_2$, the perturbative prepotential contains as expected a sum of trilogarithms indexed by positive roots, but the long roots contributions are weighted by a relative factor of $4/(\alpha, \alpha)^2 = 1/9$ compared to the short ones, in disagreement with the gauge theory calculation already at the perturbative level (for $R_5 \neq 0$). The higher-order corrections in $z_0 = q_0$ are computed efficiently and in a similar manner: rather unsurprisingly at this point, they also differ from the blow-up formula calculation of Keller and Song [54]. We find

$$\begin{aligned} -4\pi^3 \mathrm{i}\mathcal{F}_{G_2} = {} & \frac{4}{3}\log^3 q_1 + \frac{1}{2}\log q_0 \log^2 q_1 + 2\log^2 q_1 \log q_2 \\ & + \frac{7}{6}\log q_1 \log^2 q_2 + \frac{7}{27}\log^3 q_2 \\ & + \frac{1}{6}\log q_0 \log^2 q_2 + \frac{1}{2}\log q_0 \log q_2 \log q_1 \\ & + q_1 + \frac{q_2}{9} + \frac{q_1^2}{8} + q_2 q_1 + \frac{q_2^2}{72} + \frac{q_1^3}{27} + q_1^2 q_2 + \frac{q_2^3}{243} + \frac{q_1^4}{64} \\ & + \frac{1}{9}q_1^3 q_2 + \frac{1}{8}q_1^2 q_2^2 + \frac{q_2^4}{576} + \frac{q_1^5}{125} + \cdots \\ & + q_0\Big(q_1^4 q_2^2 + 2q_1^5 q_2^2 + 3q_1^6 q_2^2 + 2q_1^5 q_2^3 + 4q_1^7 q_2^2 + 6q_1^6 q_2^3 + 5q_1^8 q_2^2 \\ & + 10q_1^7 q_2^3 + 3q_1^6 q_2^4 + \cdots\Big) \\ & + q_0^2\Big(\frac{1}{8}q_1^8 q_2^4 + 3q_1^9 q_2^4 + \frac{65}{4}q_1^{10} q_2^4 \end{aligned}$$

$$+3q_1^9q_2^5+55q_1^{11}q_2^4+25q_1^{10}q_2^5+\cdots\Big)+\mathcal{O}(q_0^3). \tag{136}$$

5 Conclusion

In this paper, we formulated an algebraic construction of Picard–Fuchs D-modules for the periods of a class of B-model geometries related to $\mathcal{N}=1$ super Yang–Mills theory on $\mathbb{R}^4\times S^1$, including non-unitary gauge groups/non-toric engineering geometries. Our proposal passes a host of fine checks involving explicit period integral calculations on the B-model, and/or indirect tests from multi-instanton calculations in the gauge theory/A-model. One first upshot of our study is a direct confirmation of the validity of various proposals in the literature arising from the theory of relativistic integrable systems (for simply laced gauge groups) and from five-brane web constructions (for classical groups): in doing so we also resolve a tension between two inequivalent proposals for symplectic gauge groups in [15] and [59], in favour of the latter. A second surprising consequence is a disproof of the relevance of the twisted version of the affine relativistic Toda chain for non-simply laced gauge symmetry: the corresponding spectral curves are shown *not* to reproduce the low-energy prepotential of the gauge theory for BCFG types, in sharp contrast with the well-established situation for pure $\mathcal{N}=2$ super Yang–Mills theory on $\mathbb{R}^4$.

Our results furthermore open several further avenues of investigation, which we defer to future work. We describe some of these below.

- One obvious strand of implications is for the possible generalisation to a non-trivial Ω-background, possibly including surface operators. In particular, in the Nekrasov–Shatashvili limit [78], it would be fascinating to explore to what extent the commutative algebra construction of Sect. 4 of Picard–Fuchs systems from the ring of regular functions on the discriminant admits a canonical non-commutative quantisation: this aligns well with both the spirit of the quantum curve constructions of Aganagic et al. [1] and the fact that the NS twisted effective superpotential is known to be computable from an $\hbar$-deformation of the usual special geometry relations [68, 69]. While it is clear how to promote the Seiberg–Witten curve to a quantum finite-difference operator, the notion of a non-commutative discriminant ideal relevant for the constructions of Sect. 4 is less obvious. Moreover, the fact that the system (95) is holonomic rests on the underlying ring structure on the discriminant being commutative, which guarantees that the structure constants C_{ij}^k can be viewed as Christoffel symbols of a torsion-free connection on the Coulomb branch: how this feature would persist in the non-commutative setting would need to be clarified.
- As far as the self-dual slice of the Ω-background is concerned, the construction proposed here gives an effective technique for finding the mirror maps in the context of local mirror symmetry, and it would be interesting to combine it with the Chekhov–Eynard–Orantin topological recursion [22, 35] to compute open and closed higher genus A-model amplitudes in the self-dual Ω background, as considered in [3, 4, 18, 56] in relation to the AGT correspondence with surface operators, and studied in detail in [2, 13, 14, 16, 17] for ADE gauge groups in relation to

the large N duality with Chern–Simons theory on Seifert manifolds and related matrix models. A fascinating relating development is the appearance of recursive structures for the self-dual theory arising from the non-autonomous Toda chain in four dimensions, as recently been studied in [11], raising the prospects of a possible uplift to the 5d set-up considered here.

- Our story took place wholly in five dimensions and without hypermultiplets. The absence of matter leads to a symmetry under the involution $\lambda \leftrightarrow q_0/\lambda$ which simplifies the task of determining the PF ideal to calculations on the reduced (perturbative) Seiberg–Witten curves $\{\widetilde{\mathrm{P}}_{\mathcal{G};u} = 0\}$, as in [48]. The lack of such symmetry may render the calculations more difficult in the case with non-trivial matter content, but it should possible to extend at least the treatment of Sect. 4 to these cases as well. Furthermore, the trigonometric Frobenius manifolds of Sect. 3 have canonical elliptic deformations in terms of Frobenius structures on orbits of Jacobi groups, as in [9, 10]: it is only natural to conjecture that these would play for a 6D uplift of the theory the same role that their rational (Weyl) and trigonometric (extended affine Weyl) had for the four and five-dimensional set-up, respectively. Note that for $\mathcal{G} = \mathrm{SU}(N)$ their almost-dual Frobenius manifolds [30] are indeed already known to closely reproduce the six-dimensional perturbative gauge theory prepotential [62] in terms of Beilinson's elliptic polylogarithm, in exactly the same way as the almost-duals of their rational [30] and trigonometric [85] limits Frobenius manifolds gave the 1-loop prepotential of the four and five-dimensional gauge theories, respectively.
- The surprising failure of the twisted affine relativistic Toda system to reproduce the low-energy effective action of the theory with BCFG gauge symmetry raises several interesting questions. The first, obvious one is what would then be the correct integrable model underlying the gauge theory. This should be a relativistic deformation of the twisted affine Toda chain which is qualitatively different from the natural one considered here by Dynkin folding. That would be similar to the way in which the integrable models underlying the $\mathrm{SU}(N)$ theory at different Chern–Simons levels provide inequivalent relativistic deformations of the periodic Toda chain, to which they, however, all reduce in the four-dimensional/non-relativistic limit [19, 32]. For classical groups, the dimer model strategy of [32] applied to the M-theory curves of Brandhuber et al. [15] and Li and Yagi [59] might be a promising avenue to make headway into this and possibly hint to a solution encompassing exceptional non-simply laced gauge groups as well.
- The reverse question is also of some non-trivial relevance, i.e. what is the physical and/or geometrical significance of the B-model on the local geometry associated with the spectral curves of the twisted affine relativistic Toda chain? Part of the answer is already known: these curves indeed arise via affine Dynkin folding on a sublocus of the Kähler moduli space of the topological A-model on ADE-orbifolds of the resolved conifold, where they have already been found *ante litteram* to describe (at small 't Hooft coupling/small Kähler volume) the large N limit of Chern–Simons theory on Seifert manifolds of positive curvature [13]. So even though our results in Sect. 4.6 disprove that the resulting special geometry prepotentials are the gauge theory prepotentials of the pure KK theory with $\mathcal{G} = \mathrm{BCFG}$, there is still, however, room for a gauge theory angle on them. For

example, although the G_2 twisted relativistic Toda curves do not describe the weak coupling expansion of the pure G_2 theory in 5d, they do sit as the Seiberg–Witten curves of the E_6 theory on a special three-dimensional locus of its Coulomb branch [13] determined by the folding procedure. It would nonetheless be interesting to give a more poignant characterisation of the large radius expansions of Sect. 4.6, whether intrinsically or in terms of a suitable deformation of the $\mathcal{N} = 2$ KK theory with non-simply laced gauge symmetry.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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References

1. Aganagic, M., Cheng, M.C.N., Dijkgraaf, R., Krefl, D., Vafa, C.: Quantum geometry of refined topological strings. JHEP **11**, 019 (2012). arXiv:1105.0630
2. Aganagic, M., Klemm, A., Marino, M., Vafa, C.: Matrix model as a mirror of Chern–Simons theory. JHEP **02**, 010 (2004). arXiv:hep-th/0211098
3. Alday, L.F., Gaiotto, D., Gukov, S., Tachikawa, Y., Verlinde, H.: Loop and surface operators in $\mathcal{N} = 2$ gauge theory and Liouville modular geometry. JHEP **01**, 113 (2010). arXiv:0909.0945
4. Awata, H., Fuji, H., Kanno, H., Manabe, M., Yamada, Y.: Localization with a surface operator, irregular conformal blocks and open topological string. Adv. Theor. Math. Phys. **16**, 725–804 (2012). arXiv:1008.0574
5. Awata, H., Kanno, H.: Instanton counting, Macdonald functions and the moduli space of D-branes. JHEP **05**, 039 (2005). arXiv:hep-th/0502061
6. Awata, H., Kanno, H.: Refined BPS state counting from Nekrasov's formula and Macdonald functions. Int. J. Mod. Phys. A **24**, 2253–2306 (2009). arXiv:0805.0191
7. Awata, H., Yamada, Y.: Five-dimensional AGT conjecture and the deformed Virasoro algebra. JHEP **01**, 125 (2010). arXiv:0910.4431
8. Benvenuti, S., Hanany, A., Mekareeya, N.: The Hilbert series of the one instanton moduli space. JHEP **06**, 100 (2010). arXiv:1005.3026
9. Bertola, M.: Frobenius manifold structure on orbit space of Jacobi groups. I. Difier. Geom. Appl. **13**, 19–41 (2000)

10. Bertola, M.: Frobenius manifold structure on orbit space of Jacobi groups. II. Difier. Geom. Appl. **13**, 213–233 (2000)
11. Bonelli, G., Globlek, F., Tanzini, A.: Counting Yang–Mills instantons by surface operator renormalization group flow. Phys. Rev. Lett. **126**, 231602 (2021). arXiv:2102.01627
12. Bonelli, G., Matone, M.: Nonperturbative relations in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory and the Witten–Dijkgraaf–Verlinde–Verlinde equation. Phys. Rev. Lett. **77**, 4712–4715 (1996). arXiv:hep-th/9605090
13. Borot, G., Brini, A.: Chern–Simons theory on spherical Seifert manifolds, topological strings and integrable systems. Adv. Theor. Math. Phys. **22**, 305–394 (2018). arXiv:1506.06887
14. Borot, G., Eynard, B., Weisse, A.: Root systems, spectral curves, and analysis of a Chern–Simons matrix model for Seifert fibered spaces. Sel. Math. New Ser. **23**, 915–1025 (2017). arXiv:1407.4500
15. Brandhuber, A., Itzhaki, N., Sonnenschein, J., Theisen, S., Yankielowicz, S.: On the M theory approach to (compactified) 5-D field theories. Phys. Lett. B **415**, 127–134 (1997). arXiv:hep-th/9709010
16. Brini, A.: E_8 spectral curves. Proc. Lond. Math. Soc. **3**(121), 954–1032 (2020). arXiv:1711.05958
17. Brini, A., Griguolo, L., Seminara, D., Tanzini, A.: Chern–Simons theory on L(p, q) lens spaces and Gopakumar–Vafa duality. J. Geom. Phys. **60**, 417–429 (2010). arXiv:0809.1610
18. Brini, A., Marino, M., Stevan, S.: The uses of the refined matrix model recursion. J. Math. Phys. **52**, 052305 (2011). arXiv:1010.1210
19. Brini, A., Tanzini, A.: Exact results for topological strings on resolved Y(p, q) singularities. Commun. Math. Phys. **289**, 205–252 (2009). arXiv:0804.2598
20. Brini, A., van Gemst, K.: Mirror symmetry for extended afine Weyl groups. arXiv:2103.12673
21. Bruschi, M., Ragnisco, O.: Lax representation and complete integrability for the periodic relativistic Toda lattice. Phys. Lett. A **134**, 365–370 (1989)
22. Chekhov, L., Eynard, B.: Hermitean matrix model free energy: Feynman graph technique for all genera. JHEP **03**, 014 (2006). arXiv:hep-th/0504116
23. Chiang, T.M., Klemm, A., Yau, S.-T., Zaslow, E.: Local mirror symmetry: calculations and interpretations. Adv. Theor. Math. Phys. **3**, 495–565 (1999). arXiv:hep-th/9903053
24. Closset, C., Magureanu, H.: The U-plane of rank-one 4d $\mathcal{N} = 2$ KK theories. arXiv:2107.03509
25. Dijkgraaf, R., Verlinde, H.L., Verlinde, E.P.: Topological strings in $d<1$. Nucl. Phys. B **352**, 59–86 (1991)
26. Donaldson, S.K.: Instantons and geometric invariant theory. Commun. Math. Phys. **93**, 453–460 (1984)
27. Dubrovin, B.: Integrable systems in topological field theory. Nucl. Phys. B **379**, 627–689 (1992)
28. Dubrovin, B.: Geometry of 2D topological field theories, in "Integrable systems and quantum groups" (Montecatini Terme, 1993). Lecture Notes in Math. **1620**, 120–348 (1994). arXiv:hep-th/9407018
29. Dubrovin, B.: Differential geometry of the space of orbits of a Coxeter group. Surv. Differ. Geom. **4**, 181–211 (1998). arXiv:hep-th/9303152
30. Dubrovin, B.: On almost duality for Frobenius manifolds, in "Geometry, topology, and mathematical physics". Am. Math. Soc. Transl. Ser. **2**(212), 75–132 (2004)
31. Dubrovin, B., Zhang, Y.: Extended afine Weyl groups and Frobenius manifolds. Compos. Math. **111**, 167–219 (1998)
32. Eager, R., Franco, S., Schaeffer, K.: Dimer models and integrable systems. JHEP **06**, 106 (2012). arXiv:1107.1244
33. Eguchi, T., Kanno, H.: Five-dimensional gauge theories and local mirror symmetry. Nucl. Phys. B **586**, 331–345 (2000). arXiv:hep-th/0005008
34. Eguchi, T., Yang, S.-K.: A new description of the E_6 singularity. Phys. Lett. B **394**, 315–322 (1997). arXiv:hep-th/9612086
35. Eynard, B., Orantin, N.: Invariants of algebraic curves and topological expansion. Commun. Number Theory Phys. **1**, 347–452 (2007). arXiv:math-ph/0702045
36. Fock, V.V., Marshakov, A.: Loop groups. Clusters, Dimers and Integrable systems. arXiv:1401.1606
37. Fucito, F., Morales, J.F., Poghossian, R.: Instantons on quivers and orientifolds. JHEP **10**, 037 (2004)
38. Gaiotto, D., Neitzke, A., Tachikawa, Y.: Argyres–Seiberg duality and the Higgs branch. Commun. Math. Phys. **294**, 389–410 (2010). arXiv:0810.4541
39. Gorsky, A., Mironov, A.: Solutions to the reaction equation and integrable systems for $\mathcal{N} = 2$ SQCD with classical groups. Nucl. Phys. B **550**, 513–530 (1999). arXiv:hep-th/9902030
40. Gottsche, L., Nakajima, H., Yoshioka, K.: K-theoretic Donaldson invariants via instanton counting. Pure Appl. Math. Q. **5**, 1029–1111 (2009). arXiv:math/0611945

41. Hanany, A., Witten, E.: Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. Nucl. Phys. B **492**, 152–190 (1997). arXiv:hep-th/9611230
42. Hayashi, H., Kim, S.-S., Lee, K., Yagi, F.: Discrete theta angle from an O5-plane. JHEP **11**, 041 (2017). arXiv:1707.07181
43. Hoevenaars, L.K.: The WDVV equations in pure Seiberg–Witten theory. Acta Appl. Math. **86**, 49–102 (2005)
44. Hoevenaars, L.K., Martini, R.: On the WDVV equations in five-dimensional gauge theories. Phys. Lett. B **557**, 94–104 (2003). arXiv:math-ph/0212016
45. Intriligator, K.A., Morrison, D.R., Seiberg, N.: Five-dimensional supersymmetric gauge theories and degenerations of Calabi–Yau spaces. Nucl. Phys. B **497**, 56–100 (1997). arXiv:hep-th/9702198
46. Iqbal, A., Kozcaz, C., Vafa, C.: The refined topological vertex. JHEP **10**, 069 (2009). arXiv:hep-th/0701156
47. Ito, K.: Picard-Fuchs equations and prepotential in $\mathcal{N} = 2$ supersymmetric G2 Yang–Mills theory. Phys. Lett. B **406**, 54–59 (1997). arXiv:hep-th/9703180
48. Ito, K., Yang, S.-K.: Flat coordinates, topological Landau–Ginzburg models and the Seiberg–Witten period integrals. Phys. Lett. B **415**, 45–53 (1997). arXiv:hep-th/9708017
49. Ito, K., Yang, S.-K.: A–D–E singularity and prepotentials in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. Int. J. Mod. Phys. A **13**, 5373–5390 (1998). arXiv:hep-th/9712018
50. Ito, K., Yang, S.-K.: The WDVV equations in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. Phys. Lett. B **433**, 56–62 (1998). arXiv:hep-th/9803126
51. Kac, V.G.: Infinite-Dimensional Lie Algebras. Cambridge University Press, Cambridge (1990)
52. Katz, S.H., Klemm, A., Vafa, C.: Geometric engineering of quantum field theories. Nucl. Phys. B **497**, 173–195 (1997). arXiv:hep-th/9609239
53. Keller, C.A., Mekareeya, N., Song, J., Tachikawa, Y.: The ABCDEFG of instantons and W-algebras. JHEP **03**, 045 (2012)
54. Keller, C.A., Song, J.: Counting exceptional instantons. JHEP **07**, 085 (2012). arXiv:1205.4722
55. Kol, B.: 5-D field theories and M theory. JHEP **11**, 026 (1999). arXiv:hep-th/9705031
56. Kozcaz, C., Pasquetti, S., Wyllard, N.: A & B model approaches to surface operators and Toda theories. JHEP **08**, 042 (2010). arXiv:1004.2025
57. Kruglinskaya, O., Marshakov, A.: On lie groups and Toda lattices. J. Phys. A **48**, 125201 (2015). arXiv:1404.6507
58. Lawrence, A.E., Nekrasov, N.: Instanton sums and five-dimensional gauge theories. Nucl. Phys. B **513**, 239–265 (1998). arXiv:hep-th/9706025
59. Li, X., Yagi, F.: Thermodynamic limit of Nekrasov partition function for 5-brane web with O5-plane. JHEP **06**, 004 (2021). arXiv:2102.09482
60. Marino, M.: Chern–Simons theory, matrix integrals, and perturbative three-manifold invariants. Commun. Math. Phys. **253**, 25–49 (2004). arXiv:hep-th/0207096
61. Marino, M., Wyllard, N.: A note on instanton counting for $\mathcal{N} = 2$ gauge theories with classical gauge groups. JHEP **05**, 021 (2004)
62. Marshakov, A., Mironov, A.: 5-d and 6-d supersymmetric gauge theories: prepotentials from integrable systems. Nucl. Phys. B **518**, 59–91 (1998). arXiv:hep-th/9711156
63. Marshakov, A., Mironov, A., Morozov, A.: WDVV - like equations in $\mathcal{N} = 2$ SUSY Yang–Mills theory. Phys. Lett. B **389**, 43–52 (1996). arXiv:hep-th/9607109
64. Marshakov, A., Mironov, A., Morozov, A.: WDVV equations from algebra of forms. Mod. Phys. Lett. A **12**, 773–788 (1997). arXiv:hep-th/9701014
65. Marshakov, A., Semenyakin, M.: Cluster integrable systems and spin chains. JHEP **10**, 100 (2019). arXiv:1905.09921
66. Martinec, E.J., Warner, N.P.: Integrable systems and supersymmetric gauge theory. Nucl. Phys. B **459**, 97–112 (1996). arXiv:hep-th/9509161
67. McDaniel, A., Smolinsky, L.: Lax equations, weight lattices, and Prym–Tjurin varieties. Acta Math. **181**, 283–305 (1998)
68. Mironov, A., Morozov, A.: Nekrasov functions and exact Bohr–Zommerfeld integrals. JHEP **04**, 040 (2010). arXiv:0910.5670
69. Mironov, A., Morozov, A.: Nekrasov functions from exact BS periods: the case of SU(N). J. Phys. A **43**, 195401 (2010). arXiv:0911.2396
70. Mironov, A., Morozov, A., Zenkevich, Y., Zotov, A.: Spectral duality in integrable systems from AGT conjecture. JETP Lett. **97**, 45–51 (2013). arXiv:1204.0913

71. Nakajima, H., Yoshioka, K.: Lectures on instanton counting. In: CRM Proc. Lecture Notes, vol. 38. American Mathematical Society, Providence, pp. 31–101 (2004)
72. Nakajima, H., Yoshioka, K.: Instanton counting on blowup. 1. Invent. Math. **162**, 313–355 (2005). arXiv:math/0306198
73. Nakajima, H., Yoshioka, K.: Instanton counting on blowup. II. K-theoretic partition function. Transform. Groups **10**, 489– 519 (2005). arXiv:math/0505553
74. Nekrasov, N.: Five dimensional gauge theories and relativistic integrable systems. Nucl. Phys. B **531**, 323–344 (1998). arXiv:hep-th/9609219
75. Nekrasov, N.: Seiberg–Witten prepotential from instanton counting. Adv. Theor. Math. Phys. **7**, 831–864 (2003). arXiv:hep-th/0206161
76. Nekrasov, N., Okounkov, A.: Seiberg–Witten theory and random partitions. Progr. Math. **244**, 525–596 (2006). arXiv:hep-th/0306238
77. Nekrasov, N., Shadchin, S.: ABCD of instantons. Commun. Math. Phys. **252**, 359–391 (2004)
78. Nekrasov, N., Shatashvili, S.: Quantization of integrable systems and four dimensional gauge theories. In: Contribution to 16th International Congress on Mathematical Physics (ICMP09), pp.265–289
79. Nieri, F., Pasquetti, S., Passerini, F.: 3d and 5d gauge theory partition functions as q-deformed CFT correlators. Lett. Math. Phys. **105**, 109–148 (2015). arXiv:1303.2626
80. Nieri, F., Pasquetti, S., Passerini, F., Torrielli, A.: 5D partition functions, q-Virasoro systems and integrable spin-chains. JHEP **12**, 040 (2014). arXiv:1312.1294
81. Olive, D.I., Turok, N.: The symmetries of Dynkin diagrams and the reduction of Toda field equations. Nucl. Phys. B **215**, 470–494 (1983)
82. Ruijsenaars, S.N.M.: Relativistic Toda systems. Commun. Math. Phys. **133**, 217–247 (1990)
83. Seiberg, N.: Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics. Phys. Lett. B **388**, 753–760 (1996). arXiv:hep-th/9608111
84. Seiberg, N., Witten, E.: Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. Nucl. Phys. B **426**, 19–52 (1994). arXiv:hep-th/9407087
85. Stedman, R., Strachan, I.A.B.: Extended $\bigvee$-systems and trigonometric solutions to the WDVV equations. J. Math. Phys. **62**, 022301 (2021)
86. Suris, Y.B.: The problem of integrable discretization: Hamiltonian approach. In: Progress in Mathematics, vol. 219. Birkhäuser (2003)
87. Tachikawa, Y.: Five-dimensional Chern-Simons terms and Nekrasov's instanton counting. JHEP **02**, 050 (2004). arXiv:hep-th/0401184
88. Williams, H.: Double Bruhat clls in Kac–Moody groups and integrable systems. Lett. Math. Phys. **103**, 389–419 (2013). arXiv:1204.0601
89. Witten, E.: Solutions of four-dimensional field theories via M-theory. Nucl. Phys. B **500**, 3–42 (1997). arXiv:hep-th/9703166
90. Zuber, J.B.: On Dubrovin topological field theories. Mod. Phys. Lett. A **9**, 749–760 (1994). arXiv:hep-th/9312209

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