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Fixed and periodic point theorems for T-contractions on cone metric spaces

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Abstract. Recently, Filipović et al. [M. Filipović, L. Paunović, S. Radenović, M. Rajović, Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings", Math. Comput. Modelling. 54 (2011) 1467-1472] proved several fixed and periodic point theorems for solid cones on cone metric spaces. In this paper several fixed and periodic point theorems for T-contraction of two maps on cone metric spaces with solid cone are proved. The results of this paper extend and generalize well-known comparable results in the literature.

1. Introduction and preliminaries

In 1922, Banach proved the following famous fixed point theorem [3]. Suppose that (X, d) is a complete metric space and a self-map T of X satisfies $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$; that is, T is a contractive mapping. Then T has a unique fixed point. Afterward, other people considered various definitions of contractive mappings and proved several fixed point theorems [4, 8, 11, 12, 17]. In 2007, Huang and Zhang [9] introduced cone metric space and proved some fixed point theorems. Several fixed and common fixed point results on cone metric spaces were introduced in [1, 15, 16, 18, 19].

Recently, Morales and Rajes [14] introduced T-Kannan and T-Chatterjea contractive mappings in cone metric spaces and proved some fixed point theorems. Later, Filipović et al. [6] defined T-Hardy-Rogers contraction in cone metric space and proved some fixed and periodic point theorems. In this work we prove several fixed and periodic point theorems for a T-contraction of two maps on cone metric spaces. Our results extend various comparable results of Abbas and Rhoades [2], Filipović et al. [6] and, Morales and Rajes [14].

We begin with some important definitions.

Definition 1.1. (*See [7, 9]*). *Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if*
(a) *P is closed, non-empty and $P \neq \{\theta\}$;*
(b) *$a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;*
(c) *if $x \in P$ and $-x \in P$, then $x = \theta$.*

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Given a cone $P \subset E$, a partial ordering $\preceq$ with respect to P is defined by
$x \preceq y \Longleftrightarrow y - x \in P$.

We shall write $x \prec y$ to mean $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). If $intP \neq \emptyset$, the cone P is called solid. A cone P is called normal if there exists a number $K > 0$ such that, for all $x, y \in E$,

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Example 1.2. (*See [16]*).
(*i*) *Let* $E = C_{\mathbf{R}}[0, 1]$ *with the supremum norm and* $P = \{f \in E : f \geq 0\}$. *Then* P *is a normal cone with normal constant* $K = 1$.
(*ii*) *Let* $E = C^2_{\mathbf{R}}[0, 1]$ *with the norm* $\|f\| = \|f\|_\infty + \|f'\|_\infty$ *and consider the cone* $P = \{f \in E : f \geq 0\}$ *for every* $K \geq 1$. *Then* P *is a non-normal cone.*

Definition 1.3. (*See [9]*). *Let* X *be a nonempty set. Suppose that the mapping* $d : X \times X \to E$ *satisfies*
($d1$) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;
($d2$) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
($d3$) $d(x, z) \preceq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.
Then, d *is called a cone metric on* X *and* (X, d) *is called a cone metric space.*

Example 1.4. (*See [9]*). *Let* $E = R^2$, $P = \{(x, y) \in E | x, y \geq 0\} \subset R^2$, $X = R$ *and* $d : X \times X \to E$ *is such that* $d(x, y) = (|x - y|, \alpha|x - y|)$, *where* $\alpha \geq 0$ *is a constant. Then* (X, d) *is a cone metric space.*

Definition 1.5. (*See [6]*). *Let* (X, d) *be a cone metric space,* $\{x_n\}$ *a sequence in* X *and* $x \in X$. *Then*
(*i*) $\{x_n\}$ *converges to* x *if, for every* $c \in E$ *with* $\theta \ll c$ *there exists an* $n_0 \in \mathbf{N}$ *such that* $d(x_n, x) \ll c$ *for all* $n > n_0$. *We denote this by* $\lim_{n\to\infty} d(x_n, x) = \theta$
(*ii*) $\{x_n\}$ *is called a Cauchy sequence if, for every* $c \in E$ *with* $\theta \ll c$ *there exists an* $n_0 \in \mathbf{N}$ *such that* $d(x_n, x_m) \ll c$ *for all* $m, n > n_0$. *We denote this by* $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

The notation $\theta \ll c$ for $c \in intP$ of a positive cone is used by Krein and Rutman [13]. Also, a cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. In the sequel we shall always suppose that E is a real Banach space, P is a solid cone in E, and $\preceq$ is a partial ordering with respect to P.

Lemma 1.6. (*See [6]*). *Let* (X, d) *be a cone metric space over an ordered real Banach space* E. *Then the following properties are often used, particularly when dealing with cone metric spaces in which the cone need not be normal.*
(P_1) *If* $x \preceq y$ *and* $y \ll z$, *then* $x \ll z$.
(P_2) *If* $\theta \preceq x \ll c$ *for each* $c \in intP$, *then* $x = \theta$.
(P_3) *If* $x \preceq \lambda x$ *where* $x \in P$ *and* $0 \leq \lambda < 1$, *then* $x = \theta$.
(P_4) *Let* $x_n \to \theta$ *in* E *and* $\theta \ll c$. *Then there exists a positive integer* n_0 *such that* $x_n \ll c$ *for each* $n > n_0$.

Definition 1.7. (*See [6]*). *Let* (X, d) *be a cone metric space,* P *a solid cone and* $S : X \to X$. *Then*
(*i*) S *is said to be sequentially convergent if we have, for every sequence* $\{x_n\}$, *if* $\{Sx_n\}$ *is convergent, then* $\{x_n\}$ *also is convergent.*
(*ii*) S *is said to be subsequentially convergent if, for every sequence* $\{x_n\}$ *that* $\{Sx_n\}$ *is convergent,* $\{x_n\}$ *has a convergent subsequence.*
(*iii*) S *is said to be continuous if* $\lim_{n\to\infty} x_n = x$ *implies that* $\lim_{n\to\infty} Sx_n = Sx$, *for all* $\{x_n\}$ *in* X.

Definition 1.8. (*See [6]*). *Let* (X, d) *be a cone metric space and* $T, f : X \to X$ *be two mappings. A mapping* f *is said to be a T-Hardy-Rogers contraction, if there exist* $\alpha_i \geq 0$, $i = 1, \cdots, 5$ *with* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ *such that for all* $x, y \in X$,

$$d(Tfx, Tfy) \preceq \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Tfx) + \alpha_3 d(Ty, Tfy) + \alpha_4 d(Tx, Tfy) + \alpha_5 d(Ty, Tfx). \tag{1}$$

In Definition 1.8 *if one assumes that* $\alpha_1 = \alpha_4 = \alpha_5 = 0$ *and* $\alpha_2 = \alpha_3 \neq 0$ (*resp.* $\alpha_1 = \alpha_2 = \alpha_3 = 0$ *and* $\alpha_4 = \alpha_5 \neq 0$), *then one obtains a T-Kannan* (*resp. T-Chatterjea*) *contraction.*(*See [14].*)

2. Fixed point results

The following is the cone metric space version of a contractive condition of Ćirić for an ordinary metric space.

Definition 2.1. *Let (X, d) be a cone metric space. A mapping $f : X \to X$ is said to be a λ-generalized contraction if and only if for every $x, y \in X$, there exist nonnegative functions $q(x, y)$, $r(x, y)$, $s(x, y)$ and $t(x, y)$ such that*

$$\sup_{x,y\in X}\{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \leq \lambda < 1$$

and

$$d(fx, fy) \leq q(x, y)d(fx, fy) + r(x, y)d(x, fx) + s(x, y)d(y, fy) + 2t(x, y)[d(x, fy) + d(y, fx)]$$

holds for all $x, y \in X$.

Theorem 2.2. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let f and g be two mappings of X satisfying*

$$\begin{aligned} d(Tfx, Tgy) \leq\ & q(x, y)d(Tx, Ty) + r(x, y)d(Tx, Tfx) + s(x, y)d(Ty, Tgy)] \\ & + t(x, y)[d(Tx, Tgy) + d(Ty, Tfx)], \end{aligned} \tag{2}$$

for all $x, y \in X$, where q, r, s, and t are nonnegative functions satisfying

$$\sup_{x,y\in X}\{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \leq \lambda < 1; \tag{3}$$

that is, f and g are T-contractions. Then
(1) There exists a $z_x \in X$ such that $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = z_x$.
(2) If T is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.
(3) There exists a unique $w_x \in X$ such that $fw_x = gw_x = w_x$; that is, f and g have a unique common fixed point.
(4) If T is sequentially convergent, then the sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to w_x.

Proof. Suppose that x_0 is an arbitrary point of X, and define $\{x_n\}$ by
$x_1 = fx_0 \ , \ x_2 = gx_1 \ , \ \cdots \ , \ x_{2n+1} = fx_{2n} \ , \ x_{2n+2} = gx_{2n+1} \ \ for \ \ n = 0, 1, 2, ...$
First we shall prove that $\{Tx_n\}$ is a Cauchy sequence. Applying the triangle inequality we get

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(Tfx_{2n}, Tgx_{2n+1}) \\ &\leq q(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) + r(x_{2n}, x_{2n+1})d(Tx_{2n}, Tfx_{2n}) \\ &\quad + s(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tgx_{2n+1}) + t(x_{2n}, x_{2n+1})[d(Tx_{2n}, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfx_{2n})] \\ &= q(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) + r(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) \\ &\quad + s(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tx_{2n+2}) + t(x_{2n}, x_{2n+1})[d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})] \\ &\leq (q + r + t)(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) + (s + t)(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tx_{2n+2}). \end{aligned}$$

Consequently

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \frac{q(x_{2n}, x_{2n+1}) + r(x_{2n}, x_{2n+1}) + t(x_{2n}, x_{2n+1})}{1 - s(x_{2n}, x_{2n+1}) - t(x_{2n}, x_{2n+1})}d(Tx_{2n}, Tx_{2n+1}). \tag{4}$$

Using (3), we have

$$\frac{q(x, y) + r(x, y) + t(x, y)}{1 - s(x, y) - t(x, y)} \leq \lambda$$

for all $x, y \in X$. Thus, from (4), it follows that

$$d(Tx_{2n+1}, Tx_{2n+2}) \preceq \lambda d(Tx_{2n}, Tx_{2n+1}),$$

which shows that a generalized contraction is a contraction for certain pairs of points. Following arguments similar to those given above, we obtain

$$d(Tx_{2n+3}, Tx_{2n+2}) \preceq \lambda d(Tx_{2n+2}, Tx_{2n+1}),$$

where

$$\frac{q(x,y) + s(x,y) + t(x,y)}{1 - r(x,y) - t(x,y)} \leq \lambda$$

for all $x, y \in X$. Therefore, for all n,

$$d(Tx_n, Tx_{n+1}) \preceq \lambda d(Tx_{n-1}, Tx_n) \preceq \lambda^2 d(Tx_{n-2}, Tx_{n-1}) \preceq \cdots \preceq \lambda^n d(Tx_0, Tx_1). \tag{5}$$

Now, for any $m > n$ and $\lambda < 1$,

$$\begin{aligned} d(Tx_n, Tx_m) &\preceq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m) \\ &\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(Tx_0, Tx_1) \\ &\preceq \frac{\lambda^n}{1-\lambda} d(Tx_0, Tx_1) \to \theta \qquad as \qquad n \to \infty. \end{aligned}$$

From (P_4) we have $(\lambda^n/(1-\lambda)) d(Tx_0, Tx_1) \ll c$ for all n sufficiently large and $\theta \ll c$. From (P_1), we have $d(Tx_n, Tx_m) \ll c$. It follows that $\{Tx_n\}$ is a Cauchy sequence by Definition 1.5.(ii). Since a cone metric space X is complete, there exists a $z_x \in X$ such that $Tx_n \to z_x$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} Tfx_{2n} = z_x, \qquad \lim_{n\to\infty} Tgx_{2n+1} = z_x. \tag{6}$$

Now, if T is subsequentially convergent, $\{fx_{2n}\}$ (resp. $\{gx_{2n+1}\}$) has a convergent subsequence. Thus, there exist $w_{x_1} \in X$ and $\{fx_{2n_i}\}$ (resp. $w_{x_2} \in X$ and $\{gx_{2n_i+1}\}$) such that

$$\lim_{n\to\infty} fx_{2n_i} = w_{x_1}, \qquad \lim_{n\to\infty} gx_{2n_i+1} = w_{x_2}. \tag{7}$$

Because of the continuity of T, we have

$$\lim_{n\to\infty} Tfx_{2n_i} = Tw_{x_1}, \qquad \lim_{n\to\infty} Tgx_{2n_i+1} = Tw_{x_2}. \tag{8}$$

From (6) and (8) and using the injectivity of T, there exists a $w_x \in X$ (set $w_x = w_{x_1} = w_{x_2}$) such that $Tw_x = z_x$. On the other hand, from (d_3) and (2) we have

$$\begin{aligned} d(Tw_x, Tgw_x) &\preceq d(Tw_x, Tgx_{2n_i+1}) + d(Tgx_{2n_i+1}, Tfx_{2n_i}) + d(Tfx_{2n_i}, Tgw_x) \\ &\preceq d(Tw_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + q(x_{2n_i}, w_x) d(Tx_{2n_i}, Tw_x) \\ &\quad + r(x_{2n_i}, w_x) d(Tx_{2n_i}, Tx_{2n_i+1}) + s(x_{2n_i}, w_x) d(Tw_x, Tgw_x) \\ &\quad + t(x_{2n_i}, w_x)[d(Tx_{2n_i}, Tgw_x) + d(Tw_x, Tx_{2n_i+1})] \\ &\preceq d(Tw_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + (q+t)(x_{2n_i}, w_x) d(Tx_{2n_i}, Tw_x) \\ &\quad + r(x_{2n_i}, w_x) d(Tx_{2n_i}, Tx_{2n_i+1}) + t(x_{2n_i}, w_x) d(Tw_x, Tx_{2n_i+1}) \\ &\quad + (s+t)(x_{2n_i}, w_x) d(Tw_x, Tgw_x). \end{aligned} \tag{9}$$

Now, by (3), (5) and (9) we have

$$\begin{aligned} d(Tw_x, Tgw_x) &\preceq \frac{1}{1-\lambda} d(Tw_x, Tx_{2n_i+2}) + \frac{1}{1-\lambda} d(Tx_{2n_i+2}, Tx_{2n_i+1}) + \frac{\lambda}{1-\lambda} d(Tx_{2n_i}, Tw_x) \\ &\quad + \frac{\lambda}{1-\lambda} d(Tx_{2n_i}, Tx_{2n_i+1}) + \frac{\lambda}{1-\lambda} d(Tw_x, Tx_{2n_i+1}) \\ &= B_1 d(Tw_x, Tx_{2n_i+2}) + B_2 \lambda^{2n_i+1} + B_3 d(Tx_{2n_i}, Tw_x) + B_4 d(Tw_x, Tx_{2n_i+1}), \end{aligned}$$

where

$$B_1 = \frac{1}{1-\lambda} \quad , \quad B_2 = \frac{1}{1-\lambda} d(Tx_0, Tx_1) \quad , \quad B_3 = \frac{\lambda}{1-\lambda} \quad , \quad B_4 = \frac{\lambda}{1-\lambda}.$$

Let $\theta \ll c$. Since $\lambda^{2n_i+1} \to \theta$ and $Tx_{n_i} \to Tw_x$ as $i \to \infty$, there exists a natural number n_0 such that, for each $i \geq n_0$, $\big($by Definition 1.5.$(i)\big)$ we have

$$d(Tw_x, Tx_{2n_i+2}) \ll \frac{c}{4B_1} \quad , \quad \lambda^{2n_i} \ll \frac{c}{4B_2} \quad , \quad d(Tx_{2n_i}, Tw_x) \ll \frac{c}{4B_3} \quad , \quad d(Tw_x, Tx_{2n_i+1}) \ll \frac{c}{4B_4}.$$

By (P_1), we obtain

$$d(Tw_x, Tgw_x) \ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c.$$

Thus, $d(Tw_x, Tgw_x) \ll c$ for each $c \in intP$. Using (P_2), we obtain $d(Tw_x, Tgw_x) = \theta$; that is, $Tw_x = Tgw_x$. Since T is one to one, $gw_x = w_x$. Now we shall show that $fw_x = w_x$.

$$\begin{aligned} d(Tfw_x, Tw_x) &= d(Tfw_x, Tgw_x) \\ &\leq q(w_x, w_x)d(Tw_x, Tw_x) + r(w_x, w_x)d(Tw_x, Tfw_x) + s(w_x, w_x)d(Tw_x, Tgw_x) \\ &\quad + t(w_x, w_x)[d(Tw_x, Tgw_x) + d(Tw_x, Tfw_x)] \\ &= (r + t)(w_x, w_x)d(Tw_x, Tfw_x) \leq \lambda d(Tw_x, Tfw_x). \end{aligned}$$

Using (P_3), it follows that $d(Tfw_x, Tw_x) = \theta$, which implies the equality $Tfw_x = Tw_x$. Since T is one to one, then $fw_x = w_x$. Thus $fw_x = gw_x = w_x$; that is, w_x is a common fixed point of f and g. Now we shall show that w_x is the unique common fixed point. Suppose that w'_x is another common fixed point of f and g. Then

$$\begin{aligned} d(Tw_x, Tw'_x) &= d(Tfw_x, Tgw'_x) \\ &\leq q(w_x, w'_x)d(Tw_x, Tw'_x) + r(w_x, w'_x)d(Tw_x, Tfw_x) + s(w_x, w'_x)d(Tw'_x, Tgw'_x) \\ &\quad + t(w_x, w'_x)[d(Tw_x, Tgw'_x) + d(Tw'_x, Tfw_x)] \\ &= (q + 2t)(w_x, w'_x)d(Tw_x, Tw'_x) \leq \lambda d(Tw_x, Tw'_x). \end{aligned}$$

Using (P_3), it follows that $d(Tw_x, Tw'_x) = \theta$, which implies the equality $Tw_x = Tw'_x$. Since T is one to one, $w_x = w'_x$. Thus f and g have a unique common fixed point.
Ultimately, if T is sequentially convergent, then we can replace n by n_i. Thus we have

$$\lim_{n\to\infty} fx_{2n} = w_x, \qquad \lim_{n\to\infty} gx_{2n+1} = w_x.$$

Therefore if T is sequentially convergent, then the sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to w_x. □

The following results is obtained from Theorem 2.2.

Corollary 2.3. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let f and g be two maps of X satisfying*

$$d(Tfx, Tgy) \leq \alpha d(Tx, Ty) + \beta[d(Tx, Tfx) + d(Ty, Tgy)] + \gamma[d(Tx, Tgy) + d(Ty, Tfx)], \tag{10}$$

for all $x, y \in X$, where

$$\alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + 2\gamma < 1; \tag{11}$$

that is, f and g are T-contractions. Then
(1) *There exists a $z_x \in X$ such that* $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = z_x$.
(2) *If T is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.*
(3) *There exists a unique $w_x \in X$ such that $fw_x = gw_x = w_x$; that is, f and g have a unique common fixed point.*
(4) *If T is sequentially convergent, then the sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to w_x.*

Proof. Corollary 2.3 follows from Theorem 2.2 by setting $q = \alpha$, $r = s = \beta$ and $t = \gamma$ □

Corollary 2.4. *Let (X, d) be a complete cone metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, let the mapping f be a map of X satisfying*

$$d(Tfx, Tfy) \leq q(x, y)d(Tx, Ty) + r(x, y)d(Tx, Tfx) + s(x, y)d(Ty, Tfy) + t(x, y)[d(Tx, Tfy) + d(Ty, Tfx)], \tag{12}$$

for all $x, y \in X$, where q, r, s and t are nonnegative functions satisfying

$$\sup_{x,y\in X}\{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \leq \lambda < 1; \tag{13}$$

that is, f is a T-contraction. Then

(1) *For each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence, (Define the iterate sequence $\{x_n\}$ by $x_{n+1} = f^{n+1}x_0$).*
(2) *There exists a $z_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = z_{x_0}$.*
(3) *If T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence.*
(4) *There exists a unique $w_{x_0} \in X$ such that $fw_{x_0} = w_{x_0}$; that is, f has a unique fixed point.*
(5) *If T is sequentially convergent, then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to w_{x_0}.*

Corollary 2.5. *Let (X, d) be a complete cone metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, let the mapping f be a map of X satisfying*

$$d(Tfx, Tfy) \leq \alpha d(Tx, Ty) + \beta[d(Tx, Tfx) + d(Ty, Tfy)] + \gamma[d(Tx, Tfy) + d(Ty, Tfx)], \tag{14}$$

for all $x, y \in X$, where

$$\alpha, \beta, \gamma \geq 0 \qquad \text{and} \qquad \alpha + 2\beta + 2\gamma < 1; \tag{15}$$

that is, f be a T-contraction. Then

(1) *For each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence, (Define the iterate sequence $\{x_n\}$ by $x_{n+1} = f^{n+1}x_0$).*
(2) *There exists a $z_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = z_{x_0}$.*
(3) *If T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence.*
(4) *There exists a unique $w_{x_0} \in X$ such that $fw_{x_0} = w_{x_0}$; that is, f has a unique fixed point.*
(5) *If T is sequentially convergent, then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to w_{x_0}.*

Example 2.6. *(See [14]). Let $X = [0, 1]$, $E = C^2_{\mathbf{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $P = \{f \in E | f \geq 0\}$ and $d(x, y) = |x - y|2^t$ where $2^t \in P \subset E$. Moreover, suppose that $Tx = x^2$ and $fx = x/2$, which map the set X into X. (X, d) is a cone metric space with non-normal solid cone [9, 16]. Also, T is a one to one, continuous mapping, and f is not a Kannan contraction [14]. All of the conditions of Corollary 2.5 are satisfied with $\alpha = \gamma = 0$ and $\beta = \frac{1}{3}$. Therefore, $x = 0$ is the unique fixed point of f.*

Corollary 2.7. *Let (X, d) be a complete cone metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, let the mapping f be a T-Hardy-Rogers contraction. Then, the results of the previous Corollary hold.*

Proof. See [6]. □

3. Periodic point results

Obviously, if f is a map which has a fixed point z, then z is also a fixed point of f^n for each $n \in \mathbf{N}$. However the converse need not be true [2]. If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbf{N}$, where $Fix(f)$ stands for the set of fixed points of f [10], then f is said to have property P. Recall also that two mappings $f, g : X \to X$ are siad to have property Q if $Fix(f) \bigcap Fix(g) = Fix(f^n) \bigcap Fix(g^n)$ for each $n \in N$. The following results extend some theorems of [2, 6].

Theorem 3.1. *Let (X, d) be a cone metric space, P be a solid cone and $T : X \to X$ be a one to one mapping. Moreover, let the mapping f be a map of X satisfyiing*
(i) $d(fx, f^2x) \preceq \lambda d(x, fx)$ for all $x \in X$, where $\lambda \in [0, 1)$, or (ii) with strict inequality, $\lambda = 1$ for all $x \in X$ with $x \neq fx$. If $Fix(f) \neq \emptyset$, then f has property P.

Proof. See [6]. □

Theorem 3.2. *Let (X, d) be a complete cone metric space, and P a solid cone. Suppose that mappings $f, g : X \to X$ satisfy all of the conditions of Corollary 2.3. Then f and g have property Q.*

Proof. From Corollary 2.3, $Fix(f) \bigcap Fix(g) = \{w\}$, where w is the unique common fixed point of f and g. Suppose that $z \in Fix(f^n) \bigcap Fix(g^n)$, where $n > 1$ is arbitrary. Then we have

$$\begin{aligned} d(Tw, Tz) = d(Tf^n w, Tg^n z) &= d(Tf(f^{n-1}w), Tg(g^{n-1}z)) \\ &\preceq \alpha d(Tf^{n-1}w, Tg^{n-1}z) + \beta[d(Tf^{n-1}w, Tf^n w) + d(Tg^{n-1}z, Tg^n z)] \\ &\quad + \gamma[d(Tf^{n-1}w, Tg^n z) + d(Tg^{n-1}z, Tf^n w)] \\ &= \alpha d(Tw, Tg^{n-1}z) + \beta[\theta + d(Tg^{n-1}z, Tz)] \\ &\quad + \gamma[d(Tw, Tz) + d(Tg^{n-1}z, Tw)] \\ &\preceq \alpha d(Tw, Tg^{n-1}z) + \beta[d(Tg^{n-1}z, Tw) + d(Tw, Tz)] \\ &\quad + \gamma[d(Tw, Tz) + d(Tg^{n-1}z, Tw)], \end{aligned}$$

which implies that

$$d(Tw, Tz) = d(Tw, Tg^n z) \preceq \lambda d(Tw, Tg^{n-1}z),$$

where $\lambda = (\alpha + \beta + \gamma)/(1 - \beta - \gamma) < 1$ (by relation (11)). Now, we have

$$d(Tw, Tz) = d(Tw, Tg^n z) \preceq \lambda d(Tw, Tg^{n-1}z) \preceq \lambda^2 d(Tw, Tg^{n-2}z) \cdots \preceq \lambda^n d(Tw, Tz).$$

Since $\lambda^n \in [0, 1)$, according to (P_3), we have $d(Tw, Tz) = \theta$; that is, $Tw = Tz$. Since T is one to one, then $w = z$, which implies that f and g have property Q. □

Theorem 3.3. *Let (X, d) be a complete cone metric space, and P a solid cone. Suppose that the mapping $f : X \to X$ satisfies all of the conditions of Corollary 2.5. Then f has property P.*

Proof. From Corollary 2.5, f has a unique fixed point in X. Suppose that $z \in Fix(f^n)$. Then we have

$$\begin{aligned} d(Tz, Tfz) &= d(Tf(f^{n-1}z), Tf(f^n z)) \\ &\preceq \alpha d(Tf^{n-1}z, Tf^n z) + \beta[d(Tf^{n-1}z, Tf^n z) + d(Tf^n z, Tf^{n+1}z)] \\ &\quad + \gamma[d(Tf^{n-1}z, Tf^{n+1}z) + d(Tf^n z, Tf^n z)] \\ &\preceq \alpha d(Tf^{n-1}z, Tz) + \beta[d(Tf^{n-1}z, Tz) + d(Tz, Tfz)] + \gamma[d(Tf^{n-1}z, Tz) + d(Tz, Tfz)] \\ &= (\alpha + \beta + \gamma) d(Tf^{n-1}z, Tz) + (\beta + \gamma) d(Tz, Tfz), \end{aligned}$$

which implies that
$d(Tz, Tfz) \preceq \lambda d(Tf^{n-1}z, Tz)$ where $\lambda = (\alpha + \beta + \gamma)/(1 - \beta - \gamma) < 1$, (by relation (15)). Hence, $d(Tz, Tfz) = d(Tf^n z, Tf^{n+1}z) \preceq \lambda d(Tf^{n-1}z, Tz) \preceq \cdots \preceq \lambda^n d(Tfz, Tz)$. Therefore we have $d(Tfz, Tz) = \theta$; that is, $Tfz = Tz$. Since T is one to one, $fz = z$. □

Corollary 3.4. *Let (X, d) be a complete cone metric space, and P be a solid cone. Suppose that the mapping $f : X \to X$ satisfies all of the conditions of Corollary 2.7. Then f has property P.*

Proof. See [6]. □

References

[1] M. Abbas, G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008) 416–420.
[2] M. Abbas, B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 22 (2009) 511–515.
[3] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, Fund. Math. J. 3 (1922) 133–181.
[4] L.B. Ćirić , A generalization of Banach contraction principle, Proc. Amer. Math. Soc. 45 (1974) 267–273.
[5] L.B. Ćirić , Generalized contractions and fixed point theorems, Publ. Inst. Math. (Beograd) 12 26 (1971) 19–26.
[6] M. Filipović, L. Paunović, S. Radenović, M. Rajović, Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings", Math. Comput. Modelling. 54 (2011) 1467–1472.
[7] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, 1985.
[8] G.E. Hardy, T.D. Rogers, A generalization of a fixed point theorem of Reich, Canad. Math. Bull.1 6 (1973) 201–206.
[9] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1467–1475.
[10] G.S. Jeong, B.E. Rhoades, Maps for which $F(T) = F(T^n)$, Fixed Point Theory Appl. 6 (2005) 87–131.
[11] G. Jungck, Commuting maps and fixed points, Amer. Math. Monthly 83 (1976) 261–263.
[12] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71–76.
[13] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach spaces, Uspekhi Mat. Nauk (NS) 3 (1) (1948) 3–95.
[14] J.R. Morales, E. Rojas, Cone metric spaces and fixed point theorems of T-Kannan contractive mappings, Int. J. Math. Anal. 4 (4) (2010) 175–184.
[15] S. Radojević, Lj. Paunović, S, Radenović, Abstract metric spaces and HardyRogers-type theorems, Appl. Math. Lett. 24 (2011) 553–558.
[16] S. Rezapour, R. Hamlbarani, Some note on the paper cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 345 (2008) 719–724.
[17] B.E. Rhoades, A comparison of various definition of contractive mappings, Trans. Amer. Math. Soc. 266 (1977) 257–290.
[18] G. Song, X. Sun, Y. Zhao, G. Wang, New common fixed point theorems for maps on cone metric spaces, Appl. Math. Lett. 23 (2010) 1033–1037.
[19] S. Wang, B. Guo, Distance in cone metric spaces and common fixed point theorems, Appl. Math. Lett. 24 (2011) 1735–1739.