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Fixed-disc results via simulation functions

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Abstract: In this paper, our aim is to obtain new fixed-disc results on metric spaces. To do this, we present a new approach using the set of simulation functions and some known fixed-point techniques. We do not need to have some strong conditions such as completeness or compactness of the metric space or continuity of the self-mapping in our results. Taking only one geometric condition, we ensure the existence of a fixed disc of a new type contractive mapping.

Key words: Fixed disc, fixed circle, simulation function, metric space

1. Introduction and preliminaries

Let (X, d) be a metric space and T a self-mapping on X. If T has more than one fixed point then the investigation of the geometric properties of fixed points appears a natural and interesting problem. For example, let $X = \mathbb{R}$ be the set of all real numbers with the usual metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. The self-mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x^2 - 2$ has two fixed points $x_1 = -1$ and $x_2 = 2$. Fixed points of T form the circle $C_{\frac{1}{2},\frac{3}{2}} = \left\{x \in \mathbb{R} : \left|x - \frac{1}{2}\right| = \frac{3}{2}\right\}$. In recent years, the fixed-circle problem and the fixed-disc problem have been studied with this perspective on metric and some generalized metric spaces (see [1, 9, 10, 12–16, 18–20, 23–29] for more details). As a consequence of some fixed-circle theorems, fixed-disc results have been also appeared. For example, the self-mapping S on $\mathbb{R}$ defined by

$$Sx = \begin{cases} x & ; \quad x \in [0, 2] \\ x + \sqrt{2} & ; \quad \text{otherwise} \end{cases}$$

fixes all points of the disc $D_{1,1} = \{x \in \mathbb{R} : |x - 1| \leq 1\}$. Clearly, S fixes all circles contained in the disc $D_{1,1}$. Therefore, it is an attractive problem to study new fixed-disc results and their consequences on metric spaces.

In this paper, our aim is to present new fixed-disc results. To do this, we provide a new technique using simulation functions defined in [8]. The function $\zeta : [0, \infty)^2 \to \mathbb{R}$ is said to be a simulation function, if it satisfies the following conditions :

(ζ_1) $\zeta(0, 0) = 0$,

(ζ_2) $\zeta(t, s) < s - t$ for all $s, t > 0$,

(ζ_3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

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then

$$\limsup_{n\to\infty}\zeta(t_n,s_n)<0.$$

The set of all simulation functions is denoted by $\mathcal{Z}$ [8]. In [8], the notion of a $\mathcal{Z}$-contraction was defined to generalize the Banach contraction as follows:

Definition 1.1 *[8] Let (X,d) be a metric space and $T:X\to X$ a mapping and $\zeta\in\mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied for all $x,y\in X$:*

$$\zeta\left(d\left(Tx,Ty\right),d\left(x,y\right)\right)\geq 0. \tag{1.1}$$

Every $\mathcal{Z}$-contraction mapping is contractive and hence continuous (see [3, 8, 21] for basic properties and some examples of a $\mathcal{Z}$-contraction). In [8], Khojasteh et al. used the notion of a simulation function to unify several existing fixed-point results in the literature.

We note that the notion of a simulation function has many interesting applications (see [3, 5, 7] and the references therein). In a very recent paper, a new solution is given to an open problem raised by Rhoades about the discontinuity problem at fixed point using the family of simulation functions (see [19] and [22]).

2. Main results

Let (X,d) be a metric space, $D_{x_0,r}=\{x\in X:d(x,x_0)\leq r\}$ $(r\in\mathbb{R}^+\cup\{0\})$ a disc and T a self-mapping on X. If $Tx=x$ for all $x\in D_{x_0,r}$ then the disc $D_{x_0,r}$ is called the fixed disc of T [29].

From now on we assume that (X,d) is a metric space and $T:X\to X$ a self-mapping. To obtain new fixed-disc results, we define several new contractive mappings. At first, we give the following definition.

Definition 2.1 *Let $\zeta\in\mathcal{Z}$ be any simulation function. T is said to be a $\mathcal{Z}_c$-contraction with respect to ζ if there exists an $x_0\in X$ such that the following condition holds for all $x\in X$:*

$$d(Tx,x)>0\Rightarrow\zeta\left(d(Tx,x),d(Tx,x_0)\right)\geq 0.$$

If T is a $\mathcal{Z}_c$-contraction with respect to ζ, then we have

$$d(Tx,x)<d(Tx,x_0), \tag{2.1}$$

for all $x\in X$ with $Tx\neq x_0$. Indeed, if $Tx=x$ then the inequality (2.1) is satisfied trivially. If $Tx\neq x$ then $d(Tx,x)>0$. By the definition of a $\mathcal{Z}_c$-contraction and the condition (ζ_2), we obtain

$$0\leq\zeta\left(d(Tx,x),d(Tx,x_0)\right)<d(Tx,x_0)-d(Tx,x)$$

and so Equation (2.1) is satisfied.

In all of our fixed disc results we use the number $\rho\in\mathbb{R}^+\cup\{0\}$ defined by

$$\rho=\inf_{x\in X}\{d(x,Tx)\mid Tx\neq x\}. \tag{2.2}$$

We begin with the following theorem.

Theorem 2.2 *If T is a $\mathcal{Z}_c$-contraction with respect to ζ with $x_0 \in X$ and the condition $0 < d(Tx, x_0) \leq \rho$ holds for all $x \in D_{x_0,\rho} - \{x_0\}$ then $D_{x_0,\rho}$ is a fixed disc of T.*

Proof Let $\rho = 0$. In this case we have $D_{x_0,\rho} = \{x_0\}$. If $Tx_0 \neq x_0$ then $d(x_0, Tx_0) > 0$ and using the definition of a $\mathcal{Z}_c$-contraction we get

$$\zeta\left(d(Tx_0, x_0), d(Tx_0, x_0)\right) \geq 0.$$

This is a contradiction by the condition (ζ_2). Hence, it should be $Tx_0 = x_0$.

Assume that $\rho \neq 0$. Let $x \in D_{x_0,\rho}$ be such that $Tx \neq x$. By the definition of ρ, we have $0 < \rho \leq d(x, Tx)$ and using the condition (ζ_2) we find

$$\begin{aligned} \zeta\left(d(Tx, x), d(Tx, x_0)\right) &< d(Tx, x_0) - d(Tx, x) \\ &\leq \rho - d(Tx, x) \leq \rho - \rho = 0, \end{aligned}$$

a contradiction with the $\mathcal{Z}_c$-contractive property of T. It should be $Tx = x$, so T fixes the disc $D_{x_0,\rho}$. □

In the following corollaries we obtain new fixed-disc results.

Corollary 2.3 *Let $x_0 \in X$. If T satisfies the following conditions then $D_{x_0,\rho}$ is a fixed disc of T:*

1) $d(Tx, x) \leq \lambda d(Tx, x_0)$ *for all* $x \in X$,

where $\lambda \in [0, 1)$.

2) $0 < d(Tx, x_0) \leq \rho$ *holds for all* $x \in D_{x_0,\rho} - \{x_0\}$.

Proof Let us consider the function $\zeta_1 : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\zeta_1(t, s) = \lambda s - t \text{ for all } s, t \in [0, \infty)$$

(see Corollary 2.10 given in [8]). Using the hypothesis, it is easy to see that the self-mapping T is a $\mathcal{Z}_c$-contraction with respect to ζ_1 with $x_0 \in X$. Hence, the proof follows by setting $\zeta = \zeta_1$ in Theorem 2.2. □

Corollary 2.4 *Let $x_0 \in X$. If T satisfies the following conditions then $D_{x_0,\rho}$ is a fixed disc of T:*

1) $d(Tx, x) \leq d(Tx, x_0) - \varphi\left(d(Tx, x_0)\right)$ *for all* $x \in X$,

where $\varphi : [0, \infty) \to [0, \infty)$ *is lower semicontinuous function and* $\varphi^{-1}(0) = 0$.

2) $0 < d(Tx, x_0) \leq \rho$ *holds for all* $x \in D_{x_0,\rho} - \{x_0\}$.

Proof Consider the function $\zeta_2 : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\zeta_2(t, s) = s - \varphi\left(s\right) - t,$$

for all $s, t \in [0, \infty)$ (see Corollary 2.11 given in [8]). Using the hypothesis, it is easy to verify that the self-mapping T is a $\mathcal{Z}_c$-contraction with respect to ζ_2 with $x_0 \in X$. Hence, the proof follows by setting $\zeta = \zeta_2$ in Theorem 2.2. □

Corollary 2.5 *Let $x_0 \in X$. If T satisfies the following conditions then $D_{x_0,\rho}$ is a fixed disc of T:*

1) $d(Tx,x) \leq \varphi\left(d(Tx,x_0)\right) d(Tx,x_0)$ *for all* $x \in X$,

where $\varphi : [0,\infty) \to [0,1)$ *be a mapping such that* $\limsup_{t\to r^+} \varphi(t) < 1$, *for all* $r > 0$.

2) $0 < d(Tx,x_0) \leq \rho$ *holds for all* $x \in D_{x_0,\rho} - \{x_0\}$.

Proof Consider the function $\zeta_3 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by

$$\zeta_3(t,s) = s\varphi(s) - t,$$

for all $s,t \in [0,\infty)$ (see Corollary 2.13 given in [8]). Using the hypothesis, it is easy to verify that the self-mapping T is a $\mathcal{Z}_c$-contraction with respect to ζ_3 with $x_0 \in X$. Therefore, the proof follows by setting $\zeta = \zeta_3$ in Theorem 2.2. □

Corollary 2.6 *Let* $x_0 \in X$. *If* T *satisfies the following conditions then* $D_{x_0,\rho}$ *is a fixed disc of* T:

1) $d(Tx,x) \leq \eta\left(d(Tx,x_0)\right)$ *for all* $x \in X$,

where $\eta : [0,\infty) \to [0,\infty)$ *be an upper semicontinuous mapping such that* $\eta(t) < t$ *for all* $t > 0$.

2) $0 < d(Tx,x_0) \leq \rho$ *holds for all* $x \in D_{x_0,\rho} - \{x_0\}$.

Proof Consider the function $\zeta_4 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by

$$\zeta_4(t,s) = \eta(s) - t,$$

for all $s,t \in [0,\infty)$ (see Corollary 2.14 given in [8]). Using the hypothesis, it is easy to verify that the self-mapping T is a $\mathcal{Z}_c$-contraction with respect to ζ_4 with $x_0 \in X$. Therefore, the proof follows by setting $\zeta = \zeta_4$ in Theorem 2.2. □

Corollary 2.7 *Let* $x_0 \in X$. *If* T *satisfies the following conditions then* $D_{x_0,\rho}$ *is a fixed disc of* T:

1) $\int_0^{d(Tx,x)} \phi(t)dt \leq d(Tx,x_0)$ *for all* $x \in X$,

where $\phi : [0,\infty) \to [0,\infty)$ *is a function such that* $\int_0^{\varepsilon} \phi(t)dt$ *exists and* $\int_0^{\varepsilon} \phi(t)dt > \varepsilon$, *for each* $\varepsilon > 0$.

2) $0 < d(Tx,x_0) \leq \rho$ *holds for all* $x \in D_{x_0,\rho} - \{x_0\}$.

Proof Consider the function $\zeta_5 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by

$$\zeta_5(t,s) = s - \int_0^t \phi(u)du,$$

for all $s,t \in [0,\infty)$ (see Corollary 2.15 given in [8]). Using the hypothesis, it is easy to verify that the self-mapping T is a $\mathcal{Z}_c$-contraction with respect to ζ_5 with $x_0 \in X$. Therefore, the proof follows by taking $\zeta = \zeta_4$ in Theorem 2.2. □

We give the following example.

Example 2.8 *Let* $X = \mathbb{R}$ *and* (X, d) *be the usual metric space with* $d(x, y) = |x - y|$. *Let us define the self-mapping* $T_1 : X \to X$ *as*

$$T_1x = \begin{cases} x & ; \quad x \in [-1, 1] \\ 2x & ; \quad x \in (-\infty, -1) \cup (1, \infty) \end{cases},$$

for all $x \in \mathbb{R}$. *Then* T_1 *is a* $\mathcal{Z}_c$*-contraction with* $\rho = 1$, $x_0 = 0$ *and the function* $\zeta_6 : [0, \infty)^2 \to \mathbb{R}$ *defined as* $\zeta_6(t, s) = \frac{3}{4}s - t$. *Indeed, it is clear that*

$$0 < d(T_1x, 0) = |x - 0| = |x| \leq 1,$$

for all $x \in D_{0,1} - \{0\}$ *and we have*

$$\zeta_6 \left(d(T_1x, x), d(T_1x, x_0)\right) = \zeta \left(|x|, |2x|\right) = \frac{1}{2} |x| > 0$$

for all $x \in \mathbb{R}$ *such that* $d(Tx, x) > 0$. *Consequently,* T_1 *fixes the disc* $D_{0,1} = [-1, 1]$.

Now we consider the self-mapping $T_2 : X \to X$ *defined by*

$$T_2x = \begin{cases} x & ; \quad |x - x_0| \leq \mu \\ 2x_0 & ; \quad |x - x_0| > \mu \end{cases},$$

for all $x \in \mathbb{R}$ *with* $0 < x_0$ *and* $\mu \geq 2x_0$. *The self-mapping* T_2 *is not a* $\mathcal{Z}_c$*-contraction with respect to any* $\zeta \in \mathcal{Z}$ *with* $x_0 \in X$. *However,* T_2 *fixes the disc* $D_{x_0,\mu}$. *Indeed, by the condition* (ζ_2), *for all* $x \in (-\infty, x_0 - \mu) \cup (x_0 + \mu, \infty)$ *we have*

$$\begin{aligned} \zeta \left(d(Tx, x), d(Tx, x_0)\right) &= \zeta \left(|2x_0 - x|, |2x_0 - x_0|\right) \\ &= \zeta \left(|2x_0 - x|, |x_0|\right) < |x_0| - |2x_0 - x| < 0. \end{aligned}$$

This example shows that the converse statement of Theorem 2.2 is not true everywhen.

Remark 2.9 1) *We note that the radius* ρ *of the fixed disc* $D_{x_0,\rho}$ *is not maximal in Theorem 2.2* (*resp. Corollary 2.3-Corollary 2.7*). *That is, if* D_{x_0,ρ_1} *is another fixed disc of the self-mapping* T *then it can be* $\rho \leq \rho_1$. *Indeed, if we consider the self mapping* $T_3 : \mathbb{R} \to \mathbb{R}$ *defined by*

$$T_3x = \begin{cases} x & ; \quad x \in [-3, 3] \\ x + 1 & ; \quad \textit{otherwise} \end{cases}$$

with the usual metric on $\mathbb{R}$, *then the self-mapping* T_3 *is a* $\mathcal{Z}_c$*-contraction with* $\rho = 1$, $x_0 = 0$ *and the function* $\zeta_7 : [0, \infty)^2 \to \mathbb{R}$ *defined as* $\zeta_7(t, s) = \frac{1}{2}s - t$. *Hence,* T_1 *fixes the disc* $D_{0,1} = [-1, 1]$ *by Theorem 2.2. However, the disc* $D_{0,2} = [-2, 2]$ *is another fixed disc of the self-mapping* T_3.

2) *The radius* ρ *of the fixed disc* $D_{x_0,\rho}$ *is independent from the center* x_0 *in Theorem 2.2* (*resp. Corollary 2.3-Corollary 2.7*). *Again, if we consider the self-mapping* T_3 *defined in* (1), *it is easy to verify that* T_3 *is also a* $\mathcal{Z}_c$*-contraction with* $\rho = 1$, $x_0 = 1$ *and the function* ζ_7. *Clearly, the disc* $D_{1,1} = [0, 2]$ *is another fixed disc of* T_3.

In [1], Aydi et al. introduced the notion of a α-x_0-admissible map as follows:

Definition 2.10 *[1] Let X be a nonempty set. Given a function $\alpha : X \times X \to (0,\infty)$ and $x_0 \in X$. T is said to be an α-x_0-admissible map if for every $x \in X$,*

$$\alpha(x_0, x) \geq 1 \Rightarrow \alpha(x_0, Tx) \geq 1.$$

Then using this notion it was given new fixed-disc results on a rectangular metric space in [1]. Now we give the following definition.

Definition 2.11 *Let T be a self-mapping defined on a metric space (X,d). If there exist $\zeta \in \mathcal{Z}$, $x_0 \in X$ and $\alpha : X \times X \to (0,\infty)$ such that*

$$d(Tx,x) > 0 \Rightarrow \zeta\left(\alpha(x_0,Tx)d(x,Tx), d(Tx,x_0)\right) \geq 0 \text{ for all } x \in X,$$

then T is called as an α-$\mathcal{Z}_c$-contraction with respect to ζ.

Remark 2.12 1) *If T is an α-$\mathcal{Z}_c$-contraction with respect to ζ, then we have*

$$\alpha(x_0,Tx)d(x,Tx) < d(Tx,x_0), \tag{2.3}$$

for all $x \in X$ such that $Tx \neq x_0$. If $Tx \neq x_0$ then we have $d(Tx,x_0) > 0$.

Case 1. If $Tx = x$, then $\alpha(x_0,Tx)d(x,Tx) = 0 < d(Tx,x_0)$.

Case 2. If $Tx \neq x$, then $d(Tx,x) > 0$. Since $\alpha(x_0,Tx) > 0$, then by the condition (ζ_2) and the definition of an α-$\mathcal{Z}_c$-contraction, we find

$$0 \leq \zeta\left(\alpha(x_0,Tx)d(x,Tx), d(Tx,x_0)\right) < d(Tx,x_0) - \alpha(x_0,Tx)d(x,Tx)$$

and hence

$$\alpha(x_0,Tx)d(x,Tx) < d(Tx,x_0).$$

2) *If $\alpha(x_0,Tx) = 1$ then an α-$\mathcal{Z}_c$-contraction T turns into a $\mathcal{Z}_c$-contraction with respect to ζ and the equation (2.3) turns into Equation (2.1).*

Now we give the following theorem.

Theorem 2.13 *Let T be an α-$\mathcal{Z}_c$-contraction with respect to ζ with $x_0 \in X$. Assume that T is α-x_0-admissible. If $\alpha(x_0,x) \geq 1$ for $x \in D_{x_0,\rho}$ and $0 < d(Tx,x_0) \leq \rho$ for $x \in D_{x_0,\rho} - \{x_0\}$, then $D_{x_0,\rho}$ is a fixed disc of T.*

Proof Let $\rho = 0$. In this case $D_{x_0,\rho} = \{x_0\}$ and the α-$\mathcal{Z}_c$-contractive hypothesis yields $Tx_0 = x_0$. Indeed, if $Tx_0 \neq x_0$ then $d(x_0,Tx_0) > 0$ and using the definition of an α-$\mathcal{Z}_c$-contraction we get

$$\zeta\left(\alpha(x_0,Tx_0)d(Tx_0,x_0), d(Tx_0,x_0)\right) \geq 0.$$

We have a contradiction by the condition (ζ_2). Hence, it should be $Tx_0 = x_0$.

Assume that $\rho \neq 0$. Let $x \in D_{x_0,\rho}$ be such that $Tx \neq x$. By the hypothesis, we have $\alpha(x_0,x) \geq 1$ and by the α-x_0-admissible property of T we get $\alpha(x_0,Tx) \geq 1$. Then using the condition (ζ_2) we find

$$\begin{aligned}\zeta\left(\alpha(x_0,Tx)d(Tx,x), d(Tx,x_0)\right) &< d(Tx,x_0) - \alpha(x_0,Tx)d(Tx,x) \\ &< \rho - d(Tx,x) \leq \rho - \rho = 0,\end{aligned}$$

a contradiction with the α-$\mathcal{Z}_c$-contractive property of T. It should be $Tx = x$, so T fixes the disc $D_{x_0,\rho}$. □

Let us consider the number $m^*(x,y)$ defined as follows:

$$m^*(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}. \tag{2.4}$$

Using simulation functions and the number $m^*(x,y)$, new fixed-point results were obtained in [17]. Moreover, using this number, some discontinuity results at fixed point was given in [2]. Now we obtain a new fixed-disc result using the number $m^*(x,y)$ and the set of simulation functions.

We give the following definition.

Definition 2.14 *Let (X,d) be a metric space, $T : X \to X$ a self-mapping and $\zeta \in \mathcal{Z}$. T is said to be a Ćirić type $\mathcal{Z}_c$-contraction with respect to ζ if there exist an $x_0 \in X$ such that the following condition holds for all $x \in X$:*

$$d(Tx,x) > 0 \Rightarrow \zeta\left(d(Tx,x), m^*(x,x_0)\right) \geq 0.$$

Now we give the following theorem.

Theorem 2.15 *Let (X,d) be a metric space and $T : X \to X$ a Ćirić type $\mathcal{Z}_c$- contraction with respect to ζ with $x_0 \in X$. If the condition $0 < d(Tx,x_0) \leq \rho$ holds for all $x \in D_{x_0,\rho} - \{x_0\}$ then $D_{x_0,\rho}$ is a fixed disc of T.*

Proof Let $\rho = 0$. In this case we have $D_{x_0,\rho} = \{x_0\}$ and the Ćirić type $\mathcal{Z}_c$-contractive hypothesis yields $Tx_0 = x_0$. Indeed, if $Tx_0 \neq x_0$ then we have $d(x_0,Tx_0) > 0$. By the definition of a Ćirić type $\mathcal{Z}_c$-contraction we have

$$\zeta\left(d(Tx_0,x_0), m^*(x_0,x_0)\right) \geq 0. \tag{2.5}$$

Since we have

$$\begin{aligned} m^*(x_0,x_0) &= \max\left\{d(x_0,x_0), d(x_0,Tx_0), d(x_0,Tx_0), \frac{d(x_0,Tx_0)+d(x_0,Tx_0)}{2}\right\} \\ &= d(x_0,Tx_0), \end{aligned}$$

we find

$$\zeta\left(d(Tx_0,x_0), m^*(x_0,x_0)\right) = \zeta\left(d(Tx_0,x_0), d(x_0,Tx_0)\right) < 0$$

by the condition (ζ_2). This is a contradiction to Equation (2.5). Hence, it should be $Tx_0 = x_0$.

Assume that $\rho \neq 0$. Let $x \in D_{x_0,\rho}$ be such that $Tx \neq x$. Then we have

$$\begin{aligned} m^*(x,x_0) &= \max\left\{d(x,x_0), d(x,Tx), d(x_0,Tx_0), \frac{d(x,Tx_0)+d(x_0,Tx)}{2}\right\} \\ &= \max\left\{d(x,x_0), d(x,Tx), \frac{d(x,x_0)+d(x_0,Tx)}{2}\right\}. \end{aligned}$$

By the hypothesis, we have

$$\zeta\left(d(Tx,x), m^*(x,x_0)\right) \geq 0$$

and so

$$\zeta\left(d(Tx,x),\max\left\{d(x,x_0),d(x,Tx),\frac{d(x,x_0)+d(x_0,Tx)}{2}\right\}\right)\geq 0. \tag{2.6}$$

We have the following cases:

Case 1. Let $\max\left\{d(x,x_0),d(x,Tx),\frac{d(x,Tx_0)+d(x_0,Tx)}{2}\right\}=d(x,x_0)$. From (2.6) we get

$$\zeta\left(d(Tx,x),d(x,x_0)\right)\geq 0.$$

Using the condition (ζ_2) and considering definition of ρ, we find

$$\zeta\left(d(Tx,x),d(x,x_0)\right)<d(x,x_0)-d(Tx,x)\leq\rho-d(Tx,x)\leq\rho-\rho=0.$$

This is a contradiction with the Ćirić type $\mathcal{Z}_c$-contractive property of T.

Case 2. Let $\max\left\{d(x,x_0),d(x,Tx),\frac{d(x,x_0)+d(x_0,Tx)}{2}\right\}=d(x,Tx)$. From (2.6) we get

$$\zeta\left(d(Tx,x),d(x,Tx)\right)\geq 0.$$

Using the condition (ζ_2), again we get a contradiction.

Case 3. Let $\max\left\{d(x,x_0),d(x,Tx),\frac{d(x,x_0)+d(x_0,Tx)}{2}\right\}=\frac{d(x,x_0)+d(x_0,Tx)}{2}$. From (2.6) we get

$$\zeta\left(d(Tx,x),\frac{d(x,x_0)+d(x_0,Tx)}{2}\right)\geq 0.$$

Using the condition (ζ_2), we get

$$\begin{aligned}\zeta\left(d(Tx,x),\frac{d(x,x_0)+d(x_0,Tx)}{2}\right) &< \frac{d(x,x_0)+d(x_0,Tx)}{2}-d(Tx,x)\\ &\leq \rho-d(Tx,x)\leq\rho-\rho=0.\end{aligned}$$

Again this is a contradiction with the Ćirić type $\mathcal{Z}_c$-contractive property of T.

In all of the above cases we have a contradiction. Hence, it should be $Tx=x$ and consequently, T fixes the disc $D_{x_0,\rho}$. □

3. A common fixed-disc theorem

In this section, we give a common fixed-disc result for a pair of self-mappings (T,S) of a metric space (X,d). If $Tx=Sx=x$ for all $x\in D_{x_0,r}$ then the disc $D_{x_0,r}$ is called the common fixed disc of the pair (T,S). At first, we modify the number defined in (2.4) for a pair of self-mappings as follows:

$$m^*_{S,T}(x,y)=\max\left\{d(Tx,Sy),d(Tx,Sx),d(Ty,Sy),\frac{d(Tx,Sy)+d(Ty,Sx)}{2}\right\}. \tag{3.1}$$

Then we give the following theorem using the numbers $m^*_{S,T}(x,y)$, $\rho_T=\inf_{x\in X}\{d(x,Tx)\mid Tx\neq x\}$, $\rho_S=\inf_{x\in X}\{d(x,Sx)\mid Sx\neq x\}$ and $r\in\mathbb{R}^+\cup\{0\}$ defined by

$$r=\inf_{x\in X}\{d(Tx,Sx)\mid Tx\neq Sx\}. \tag{3.2}$$

Let $\mu=\min\{\rho_T,\rho_S,r\}$.

Theorem 3.1 *Let $T, S: X \to X$ be two self-mappings on a metric space. Assume that there exists $\zeta \in \mathcal{Z}$ and $x_0 \in X$ such that*

$$d(Tx, Sx) > 0 \Rightarrow \zeta\left(d(Tx, Sx), m^*_{S,T}(x, x_0)\right) \geq 0 \text{ for all } x \in X$$

and

$$d(Tx, x_0) \leq \mu,\ d(Sx, x_0) \leq \mu \text{ for all } x \in D_{x_0,\mu}.$$

If T is a $\mathcal{Z}_c$-contraction with respect to ζ with x_0 such that $0 < d(Tx, x_0) \leq \rho_T$ for $x \in D_{x_0,\rho_T} - \{x_0\}$ (or S is a $\mathcal{Z}_c$-contraction with respect to ζ with x_0 such that $0 < d(Sx, x_0) \leq \rho_S$ for $x \in D_{x_0,\rho_S} - \{x_0\}$), then $D_{x_0,\mu}$ is a common fixed disc of T and S in X.

Proof At first, we show that x_0 is a coincidence point of T and S, that is, $Tx_0 = Sx_0$. Conversely, assume that $Tx_0 \neq Sx_0$, so $d(Tx_0, Sx_0) > 0$. Using the condition (ζ_2), we have

$$\zeta\left(d(Tx_0, Sx_0), m^*_{S,T}(x_0, x_0)\right) = \zeta\left(d(Tx_0, Sx_0), d(Tx_0, Sx_0)\right) < 0.$$

However, this is a contradiction by the hypothesis. Hence, we find $Tx_0 = Sx_0$, that is, x_0 is a coincidence point of T and S. If T is a $\mathcal{Z}_c$-contraction (or S is a $\mathcal{Z}_c$-contraction) then we have $Tx_0 = x_0$ (or $Sx_0 = x_0$) and $Tx_0 = Sx_0 = x_0$.

Let $\mu = 0$. In this case we have $D_{x_0,\mu} = \{x_0\}$ and clearly $D_{x_0,\mu}$ is a common fixed-disc of T and S.

Let $\mu > 0$ and $x \in D_{x_0,\mu}$ be an arbitrary point. Suppose $Tx \neq Sx$ and so $d(Tx, Sx) > 0$. Using the hypothesis $d(Tx, x_0) \leq \mu$, $d(Sx, x_0) \leq \mu$ for all $x \in D_{x_0,\mu}$ and considering the definition of μ we get

$$\begin{aligned}
\zeta\left(d(Tx, Sx), m^*_{S,T}(x, x_0)\right) &= \zeta\left(d(Tx, Sx), \max\left\{\begin{array}{c} d(Tx, Sx_0), d(Tx, Sx), \\ d(Tx_0, Sx_0), \frac{d(Tx, Sx_0) + d(Tx_0, Sx)}{2} \end{array}\right\}\right) \\
&= \zeta\left(d(Tx, Sx), \max\left\{ d(Tx, x_0), d(Tx, Sx), 0, \frac{d(Tx, x_0) + d(x_0, Sx)}{2} \right\}\right) \\
&= \zeta\left(d(Tx, Sx), d(Tx, Sx)\right).
\end{aligned}$$

This leads a contradiction by the condition (ζ_2). Therefore, x is a coincidence point of T and S.

Now, if $u \in D_{x_0,\mu}$ is a fixed point of T then clearly u is also a fixed point of S and vice versa. If T is a $\mathcal{Z}_c$-contraction (or S is a $\mathcal{Z}_c$-contraction) then by Theorem 2.2, we have $Tx = x$ (or $Sx = x$) and hence $Tx = Sx = x$ for all $x \in D_{x_0,\mu}$. That is, the disc $D_{x_0,\mu}$ is a common fixed-disc of T and S. □

Example 3.2 *Let us consider the usual metric space $X = \mathbb{R}$ and the self-mapping T_1 defined in Example 2.8. Define the self-mapping $T_4 : \mathbb{R} \to \mathbb{R}$ by*

$$T_4 x = \left\{\begin{array}{ccc} x & ; & x \in [-3, 3] \\ 3x & ; & x \in (-\infty, -3) \cup (3, \infty) \end{array}\right. .$$

Clearly, we have $\mu = 1$. Then the pair (T_1, T_4) satisfies the conditions of Theorem 3.1 for $\mu = 1$, $x_0 = 0$ and the function $\zeta_6 : [0, \infty)^2 \to \mathbb{R}$ defined as $\zeta_6(t, s) = \frac{3}{4}s - t$. Hence, the disc $D_{0,1} = [-1, 1]$ is the common fixed disc of the self-mappings T_1 and T_4.

4. Applications of fixed points in neural networks

In this section, we discuss some possible applications of our fixed-disc results in the study of neural networks. It is well known that some fixed point results have been extensively used in various types of neural networks and that the multistability analysis of neural networks depends on the type of used activation functions (see [11] and the references therein). For example, in [31], using the Brouwer's fixed point theorem, the multistability analysis was discussed for neural networks with a class of continuous Mexican-hat-type activation functions. In numerical simulations, the following Mexican-hat-type function was used:

$$g(x) = \begin{cases} -1 & , \quad -\infty < x < -1 \\ x & , \quad -1 \leq x \leq 1 \\ -x+2 & , \quad 1 < x \leq 3 \\ -1 & , \quad 3 < x < +\infty \end{cases} .$$

Notice that the disc $D_{0,1}$ is a fixed disc of the activation function $g(x)$. The graphic of $g(x)$ can be shown in the figure (this graphic is drawn using Mathematica [32]).

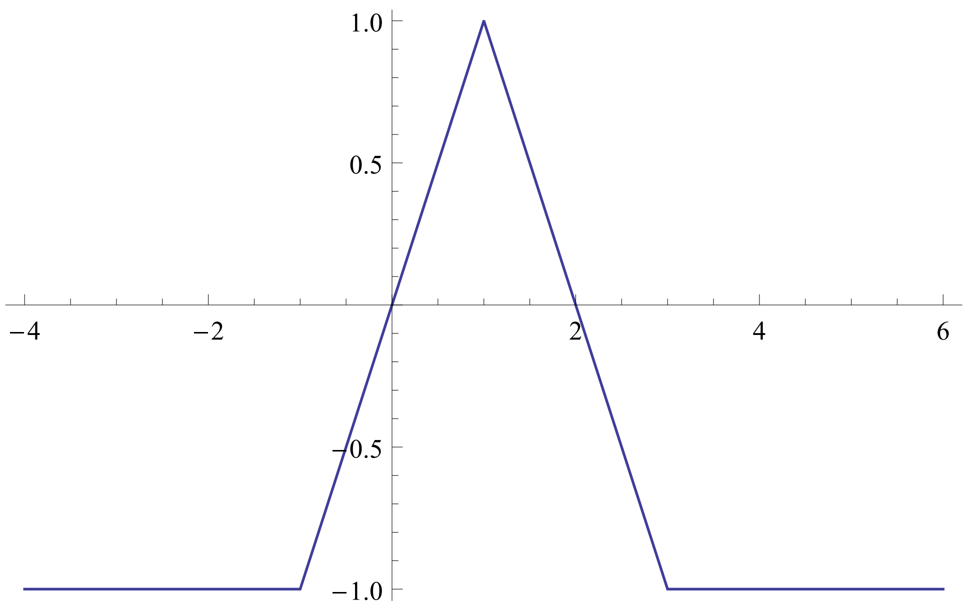

Figure 1. The graph of the Mexican-hat-type activation function $g(x)$.

On the other hand, it is worth to mention that most of the popular activation functions used in neural networks are those mappings having fixed-discs. For example, exponential linear unit (ELU) function defined by

$$f(x) = \begin{cases} x & ; \quad \text{if } x \geq 0 \\ \alpha(\exp(x) - 1) & ; \quad \text{if } x < 0 \end{cases} ,$$

where α is constant of ELUs, and S-shaped rectified linear unit function (SReLU) defined by

$$h(x_i) = \begin{cases} t_i^r + a_i^r(x - t_i^r) & ; \quad x_i \geq t_i^r \\ x_i & ; \quad t_i^r > x_i > t_i^l \\ t_i^l + a_i^l(x - t_i^l) & ; \quad x_i \leq t_i^l \end{cases} ,$$

where $\{t_i^r, a_i^r, a_i^l, t_i^l\}$ are four learnable parameters used to model an individual SReLU activation unit, are well-known activation functions (see [4] and [6] for more details).

Therefore, the study of features of mappings which have fixed-discs has significance in both theory and application.

5. Conclusion and future work

In this paper, we have obtained new fixed-disc results presenting a new approach via simulation functions. Using similar approaches, it can be studied new fixed-disc results on metric and some generalized metric spaces. As a future work, it is a meaningful problem to investigate some conditions to exclude the identity map of X from Theorem 2.2, Theorem 2.13, Theorem 2.15 and related results.

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