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Fixed point free equivariant homotopy classes

by

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Abstract. Let G be a compact Lie group. For an equivariant self-map f of a compact smooth G-manifold M, an equivariant homotopy invariant $L(f)$ is defined, and it is shown that, under given conditions on M, this invariant detects the equivariant homotopy classes of fixed point free maps. In this context, the question of the existence of a non-singular equivariant vector field extending a vector field given on the boundary is also discussed.

0. Introduction. The Lefschetz fixed point theorem states that, if $f\colon X \to X$ is homotopic to a fixed point free map, then the Lefschetz number of f vanishes. This theorem is valid for a wide class of spaces X and maps f, e.g., for all compact ANRs and continuous maps. The question arises whether the vanishing of the Lefschetz number is a sufficient condition for f to be fixed point free up to a homotopy. The answer is negative in general, even for polyhedra. If X is a Wecken space the answer is positive. The simplest and most important example of a Wecken space is a compact simply connected manifold, with or without boundary, of dimension at least three. A full, detailed exposition of the Lefschetz fixed point theorem, its converse and related topics can be found in [3].

In this paper the question of the existence of a fixed point free map homotopic to a given self-map of a compact smooth manifold is considered in a G-equivariant category, G being any compact Lie group. In Section 2 with each equivariant map f we associate a family of integers, denoted by $L(f)$, which depends only on the equivariant homotopy class of f and has properties analogous to that of the usual Lefschetz number. In particular, the Lefschetz theorem 2.6 is valid. This invariant detects the equivariant homotopy classes of fixed point free maps. Of course, some additional hypotheses, as in the non-equivariant case, are needed. In fact, we prove the following theorem:

THEOREM A. *Let M be a compact smooth G-manifold such that all connected components of M^H are simply connected and of dimension at least three for any isotropy subgroup H with a finite Weyl group in G. Then an equivariant map $f\colon M \to M$ is fixed point free up to an equivariant homotopy if and only if $L(f) = 0$.*

If f is the identity the same is true without any restrictions on the fundamental group and dimension.

THEOREM B. *A compact smooth G-manifold M admits a fixed point free equivariant deformation if and only if $\chi(M) = 0$, where $\chi(M) = L(\mathrm{id}_M)$.*

The proof of Theorem A is contained in Section 3. Theorem B follows from a more general result about equivariant vector fields proved in Section 4. Theorem 4.4 generalizes two theorems of Komiya — [10, 4.3] and [9, 1.1].

1. Preliminaries. Let G be a compact Lie group and let H be a closed subgroup. We denote by $N(H)$ the normalizer of H in G and by $W(H)$ the Weyl group $N(H)/H$. The conjugacy class of H we call the *isotropy type of* H and denote by (H).

If X is a left G-space and $x \in X$, then G_x denotes the isotropy subgroup of x and Gx denotes the orbit of x. The set of orbits, called the *orbit space*, will be denoted by X/G. For each subgroup H of G let X^H denote the fixed point set of H; it consists of all points x of X such that $H \subseteq G_x$. The set of points of X for which G_x is precisely H will be denoted by X_H and its complement in X^H by X^H_s. The union of orbits of points in H^H, X_H and X^H_s we denote by $X^{(H)}$, $X_{(H)}$ and $X^{(H)}_s$, respectively. If C is a subset of $X_{(H)}/G$ then $X_{(H),C}$ is used to denote $p^{-1}(C)$, $p\colon X_{(H)} \to X_{(H)}/G$ being the projection. We also use $X^{(H)}_C = X^{(H)} - X_{(H),C}$, $X_{H,C} = X_{(H),C} \cap X^H$ and $X^H_C = X^{(H)}_C \cap X^H$.

An equivariant map $f\colon X \to Y$, i.e., a map commuting with the actions of G on X and Y, induces for each H, in an obvious way, the following maps: $X^H \to Y^H$, $X^{(H)} \to Y^{(H)}$, $X^H_s \to Y^H_s$ and $X^{(H)}_s \to Y^{(H)}_s$, which are denoted by f^H, $f^{(H)}$, f^H_s and $f^{(H)}_s$, respectively. Passing to the orbit spaces, f induces $f/G\colon X/G \to Y/G$. An equivariant map which takes values in the unit segment $I = [0, 1]$, with trivial G-action on I, is called an *invariant function.*

A homotopy $F\colon X \times I \to Y$ into a metric space Y we call an *ε-homotopy* ($\varepsilon > 0$) if $\operatorname{diam} F(x, I) < \varepsilon$ for each $x \in X$.

For a map $f\colon A \to X$, $A \subseteq X$ we say that f is fixed point free if $f(x) \neq x$, $x \in A$.

Recall that a G-space X is said to be *equivariantly triangulable for G a finite group* if there is an equivariant triangulation $K \to X$, i.e., an equivariant homeomorphism of an equivariant simplicial complex K onto X. A simplicial complex (= a geometric simplicial complex considered as a topological space with the topology generated by the simplexes) on which G acts through simplicial maps is called an *equivariant simplicial complex* if each group element takes the interior of any n-simplex homeomorphically to the interior of some other n-simplex, and via identity whenever a simplex is mapped onto itself. We will make use of the fact that every compact smooth G-manifold (a smooth manifold with a smooth G-action), with or without boundary, is equivariantly triangulable. In fact, this is one of the results of [7].

If K and L are equivariant simplicial complexes, an equivariant simplicial map $K \to L$ is an equivariant map which is simplicial as a map between the ordinary complexes K and L.

There is also an equivariant version of the simplicial approximation theorem (cf. [2, Ex. I.6]).

Suppose $f\colon K \to L$ is an equivariant map between equivariant simplicial complexes K and L which is simplicial on an invariant simplicial subcomplex $K_1 \subset K$. There is an equivariant subdivision K' of K modulo K_1 and an equivariant simplicial map $g\colon K' \to L$ equivariantly homotopic $\operatorname{rel} K_1$ to f.

2. The Lefschetz fixed point theorem. Let X be a compact ENR and let A be a closed neighbourhood retract in X. Suppose that the complement $X-A$ is a disjoint union of a family $\{D\}$ of its open subsets. Let $i_D\colon (X, A) \to (X, X-D)$ be the inclusion. Then on singular cohomology groups (with integers as coefficients) we obtain the isomorphism

$$\{i_D^*\}\colon \oplus_D H^*(X, X-D) \to H^*(X, A).$$

Let $\{p_D^*\colon H^*(X, A) \to H^*(X, X-D)\}$ be the family of projections associated with the direct sum isomorphism.

For a map $f\colon (X, A) \to (X, A)$ define the number $L(f, X, A, D)$ as the Lefschetz trace of $p_D^* \circ f^* \circ i_D^*$.

Let U be a neighbourhood of A in X. A map $f\colon X \to Y$ is said to be *taut over A in U* if there is a retraction $r\colon U \to A$ such that $f_{|U} = f_{|A} \circ r$.

PROPOSITION 2.1. *If $f\colon (X, A) \to (X, A)$ is taut over A in U, then for each D*

$$L(f, X, A, D) = I_{f|V},$$

where V is any open subset of X such that $\overline{V} \subset D \subset U \cup V$ and $I_{f|V}$ is the fixed point index [5] of $f_{|V}$.

Proof. Since $X/X-D$ may be identified with D^+, the one-point compactification of D, we see that $L(f, X, A, D)+1$ is the same as the Lefschetz number of the map $g = i_D \circ f \circ p_D\colon D^+ \to D^+$, where $p_D\colon D^+ \to X/A$ is the inclusion. The conclusion follows from the additivity of the fixed point index, because $I_{g|U'} = 1$, U' being the image of U under the identification map $X \to D^+$. ■

Remark. In order to use the fixed point index, it should be noted that D^+ is an ENR. To see this, observe first that $X-D$ is an ENR. In fact, $X-D$ is a neighbourhood retract in X, and the retraction is

$$(r \cup \mathrm{id})\colon (U \cap D) \cup (X-D) \to X-D.$$

Thus the space $D^+ = X/X-D$, being locally contractible at each point (cf. [5, 4.8.7]), is an ENR according to [1, 5.10.3].

For the rest of this section X denotes a compact equivariant Euclidean neighbourhood retract (G-ENR), i.e., a compact G-space which can be equivariantly embedded as an equivariant neighbourhood retract in an orthogonal representation of G (we always assume X to be equipped with a metric d induced via this embedding by the metric defined by the inner product in the Euclidean space), and A denotes an equivariant closed neighbourhood retract in X. For an equivariant map $f\colon X \to Y$, we say that f is *taut over A in an invariant neighbourhood U* if there is an equivariant retraction $r\colon U \to A$ such that $f_{|U} = f_{|A} \circ r$.

PROPOSITION 2.2. *Given $\varepsilon>0$, there exists an invariant neighbourhood W of the diagonal $d(X)$ in $X\times X$ and an equivariant ε-homotopy $W\times I\to X$ from the first projection to the second, which is constant on $d(X)$.*

Proof. Embed X into the orthogonal representation of G and then proceed as in [3, 3.A.3]. ■

COROLLARY 2.3. *For any G-space Y and $\varepsilon>0$, there is a $\delta>0$ such that, if $f,g\colon Y\to X$ are equivariant maps and $d(f(y),g(y))<\delta$ for all $y\in Y$, then f and g are equivariantly ε-homotopic through homotopy constant on the coincidence set of f and g.* ■

PROPOSITION 2.4. *Let Z be a normal G-space and $B\subseteq Z$ an invariant closed subspace. Assume that $f\colon Z\to X$ is an equivariant map and $F'\colon B\times I\to X$ is an equivariant ε-homotopy of $f_{|B}$ for a given $\varepsilon>0$. There is an equivariant ε-homotopy $F\colon Z\times I\to X$ of f which extends F'.*

Proof. By the Tietze–Gleason theorem [2, 1.2.3] there is an extension of F' to an equivariant homotopy $F''\colon U\times I\to X$ of $f_{|U}$ for an invariant neighbourhood U of B. If U is sufficiently close to B, then F'' becomes an ε-homotopy. Let $h\colon Z\to I$ be an invariant function such that $h_{|B}=1$ and $h_{|Z-U}=0$.

Define F by

$$F(x,t)=\begin{cases} f(x) & \text{for } x\notin U,\\ F''(x,th(x)) & \text{for } x\in U.\end{cases}$$ ■

PROPOSITION 2.5. *Let Y be a compact G-ENR and let $f\colon X\to Y$ be an equivariant map. Then, for a given $\varepsilon>0$, there is an invariant neighbourhood U of A in X and an equivariant ε-homotopy* rel *A from f to a map which is taut over A in U.*

Proof. Choose an $\varepsilon'>0$ such that $d(f(x),f(y))<\varepsilon$ if $d(x,y)<\varepsilon'$, $x,y\in X$. For a $\delta>0$, denote by U the set $\{x\in V\colon d(x,r(x))<\delta\}$, where $r\colon V\to A$ is an equivariant neighbourhood retraction. If δ is small enough, then there is an equivariant ε-homotopy rel A from $f_{|U}$ to $f_{|A}\circ r_{|U}$, because for small δ the maps i_U and $i_A\circ r_{|U}$ are equivariantly ε'-homotopic rel A by 2.3, i_U and i_A denote the inclusions of U and A into X. Finally, extend the resulting ε-homotopy to an equivariant ε-homotopy of f by the use of 2.4. ■

Let $A(X)$ denote the free abelian group generated by the set of all pairs $((H),C)$, where (H) is an isotropy type on X such that the Weyl group $W(H)$ is finite and C is a connected component of $X_{(H)}/G$. By $A'(X)$ we denote the set of all elements z of $A(X)$, whose $((H),C)$-th coordinate (denoted by $z_{((H),C)}$ or shortly by z_C) is 0 or 1 for any pair $((H),C)$ such that $\dim C=0$.

For an equivariant map $f\colon X\to X$ let $L(f)$ and $L'(f)$ denote the unique elements of $A(X)$ and $A'(X)$, respectively, such that for each pair $((H),C)$

$$L(f)_{((H),C)}=L(f^H,X^H,X^H_s,X_{H,C}),$$

$$L'(f)_{((H),C)}=|W(H)|^{-1}L(f^H,X^H,X^H_s,X_{H,C}).$$

It is clear from [6, 8.18] and from 2.5 and 2.1 above that $L(f^H,X^H,X^H_s,X_{H,C})$ is divisible by $|W(H)|$ and hence $L'(f)$ is well defined. Since $L(f)$ and $L'(f)$ both depend only on the equivariant homotopy class of f, there are well defined maps $L\colon [X,X]_G\to A(X)$ and $L'\colon [X,X]_G\to A'(X)$, where $[X,X]_G$ stands for the set of equivariant homotopy classes of self-maps of X.

THEOREM 2.6. *If an equivariant map $f\colon X\to X$ is fixed point free, then $L(f)=0$.*

Proof. Let $((H),C)$ be a generator of $A(X)$; we will show $L(f)_C=0$. There is an $\varepsilon>0$ such that $d(f(x),x)>2\varepsilon$ for all $x\in X$. Since $X_s^{(H)}$ is a G-ENR by [8, 2.1], the application of 2.5 yields an equivariant map $g\colon X\to X$ which is taut over $X_s^{(H)}$ in an invariant neighbourhood and such that $L(g)_C=L(f)_C$, and $d(g(x),x)>\varepsilon$ for all $x\in X$. In particular, g is fixed point free and hence $L(g)_C=0$ by 2.1. ■

For equivariant self-maps of a closed smooth G-manifold S which is a semilinear G-sphere (this means that S^H is a homotopy sphere for any isotropy subgroup H), the following equivariant version of the Hopf theorem holds:

THEOREM 2.7. *The map $L'\colon [S,S]_G\to A'(S)$ is bijective. In particular, equivariant maps $f,g\colon S\to S$ are equivariantly homotopic if and only if $L(f)=L(g)$.*

Proof. Suppose $L'(f)=L'(g)$. Let (H) be an isotropy type on S. Suppose further that $f_s^{(H)}$ and $g_s^{(H)}$ are equivariantly homotopic. Then there is a map $\hat{g}\colon S^{(H)}\to S^{(H)}$ equivariantly homotopic to $g^{(H)}$ such that $f_s^{(H)}=\hat{g}_s^{(H)}$. We will show that $f^{(H)}$ and $\hat{g}$ are equivariantly homotopic rel $S_s^{(H)}$. To do this, it suffices to extend the constant homotopy of $f_s^H=g_s^H$ to a $W(H)$-equivariant homotopy between f^H and g^H. According to [8, 2.1] and 2.5, we may assume that $f^{(H)}$ and $\hat{g}$ are both taut over $S_s^{(H)}$ in an invariant neighbourhood U and that $f^{(H)}{}_{|U}=\hat{g}_{|U}$.

Let C be a connected component of $S_{(H)}/G$ and let M denote a compact $W(H)$-invariant submanifold of codimension 0 in $S_{H,C}$ such that $M/W(H)$ is connected and $S_{H,C}\subset U^H\cup M$. Thus, it remains to show how to extend the constant homotopy of $f^H{}_{|\partial M}=\hat{g}^H{}_{|\partial M}$ to a $W(H)$-equivariant homotopy between $f^H{}_{|M}$ and $\hat{g}^H{}_{|M}$. Since the action of $W(H)$ is free on M, the argument of [11, 5.1] applies. In particular, if $\dim W(H)=0$ then we claim that, for a chosen small n-disk D ($n=\dim M$) in $\operatorname{Int} M/W(H)$, there is an equivariant map $\bar{g}\colon S^{(H)}\to S^{(H)}$ equivariantly homotopic rel $S_s^{(H)}$ to $\hat{g}$ and such that

$$\bar{g}^H{}_{|M-p^{-1}(D)}=f^H{}_{|M-p^{-1}(D)},$$

where $p\colon M\to M/W(H)$ is the projection. Now, it is clear by 2.1 that the obstruction for $f^H{}_{|M}$ and $\bar{g}^H{}_{|M}$ to be $W(H)$-equivariantly homotopic rel $M-p^{-1}(D)$ may be identified with $L'(f)_C-L'(g)_C$. Moreover, any integer may be realized as this obstruction (except when $\dim C=0$).

Therefore, by using induction on isotropy types on S it can be proved that f and g are equivariantly homotopic. This proves the injectivity of L'.

For the surjectivity, the same method may be used to construct an equivariant map $f\colon S\to S$ that is taut over each $S_s^{(H)}$, and is such that $L'(f)$ takes any given value in $A'(X)$. ■

COROLLARY 2.8. *If S is the unit sphere in an orthogonal representation of G, then the only fixed point free map up to an equivariant homotopy is the antipodal map.* ■

The set $A'(X)$, where X is the unit sphere SV in an orthogonal G-representation V, previously appeared in [11] and [12]. It was denoted by $A(V, 0)$ in [11] and by $A(V, V)$ in [12]. Both Rubinsztein and Hauschild showed that 2.7 holds in that case, i.e., they proved the existence of a bijection between $[SV, SV]_G$ and $A'(SV)$ (but they did not prove that L' is such a bijection). It is to be noted that the definition of L' is much simpler than the definitions of the bijections they used.

3. Converse to the Lefschetz fixed point theorem. This section contains the proof of Theorem A stated in the introduction.

Let G be a finite group and let X be a compact equivariantly triangulable G-space. Suppose $A \subset X$ is an equivariant closed neighbourhood retract such that the action of G on $X-A$ is free.

LEMMA 3.1. *Let $f\colon (X, A) \to (X, A)$ be an equivariant map. There is an equivariant map f' equivariantly homotopic* rel *A to f which is taut over A and has a finite number of fixed points in $X-A$, each lying in the interior of a maximal simplex of X (for some equivariant triangulation of X).*

Proof. According to 2.5, we may assume that f is taut over A in an invariant neighbourhood U. Choose a fine equivariant triangulation of X and fix an invariant subcomplex X_0 that is a neighbourhood of A in X and is contained in U with its invariant regular neighbourhood N.

Denote the subcomplexes $X_0 \cap (X-\operatorname{Int} X_0)$ and $N \cap (X-\operatorname{Int} N)$ by X_0' and N', respectively. There is an equivariant retraction $r\colon N \to X_0$ for which N is a mapping cylinder of $r_{|N'}$. Therefore, there is an equivariant map $h\colon N' \times I \to N-\operatorname{Int} X_0$ under which $N' \times (0, 1]$ is homeomorphic to $N-X_0$, and which is such that $h(N' \times 0) = X_0'$, $h(N' \times 1) = N'$.

Let $F\colon N' \times I \to X$ be an equivariant homotopy of $f \circ r_{|N'}$ which ends in a simplicial map ($N' \times 1$ being subdivided). Define $f_1\colon N \to X$ by

$$f_{1|X_0} = f_{|X_0} \quad \text{and} \quad f_{1|N-X_0} = F \circ h^{-1}{}_{|N-X_0}.$$

Let $f_2\colon X \to X$ be an equivariant extension of f_1 equivariantly homotopic rel X_0 to f. Without loss of generality, by the equivariant simplicial approximation theorem stated in the introduction, we may assume f_2 to be simplicial on $X-\operatorname{Int} N$ (after subdividing X modulo N). If the chosen equivariant triangulation of X is sufficiently fine, then f_2 is sufficiently close to f. In particular, f_2 has no fixed points in $N-X_0$. At the same time, we can assume that for any simplex s of $X-N$ the stars of s and gs, $g \in G$, are disjoint.

Suppose now that s is a simplex of $X-N$, non-maximal in X and such that $\langle s \rangle \subset f_2(\langle s \rangle)$, $\langle s \rangle$ being the interior of s. In this situation all fixed points of f_2 in $\langle s \rangle$ may be removed to simplexes of greater dimension by means of the Hopf construction [3].

Since the Hopf construction does not change f_2 outside the star of s, it follows that the modification of f_2 may be made equivariantly, i.e., simultaneously for all simplexes of the same equivariant simplex.

Therefore, using an argument similar to that in [3, 8.A.2], we claim that by working up through the simplexes of $X-N$ dimension-by-dimension, after a finite number of applications of the Hopf construction, we obtain an equivariant map f' equivariantly homotopic rel N to f_2 which has a finite number of fixed points in $X-N$, each lying in the interior of a maximal simplex of X. ■

The next lemma is essential in the process of removing isolated fixed points of an equivariant map.

Let M be a compact smooth G-manifold (we still assume that G is finite) and let $f\colon M \to M$ be an equivariant map. Suppose that $Gx_1, \ldots, Gx_n$ are isolated fixed point orbits of f, all lying in $M_e - \partial M$. Suppose further that U is an invariant neighbourhood of $Gx_1 \cup \ldots \cup Gx_n$ in M_e which contains no other fixed points and is such that U/G is connected. Denote by I_{f, x_i} the fixed point index of f around x_i.

LEMMA 3.2. *Assume that all connected components of M which meet U are simply connected and of dimension at least three. If $\sum_{i=1}^{n} I_{f, x_i} = 0$ then there is an equivariant map f' equivariantly homotopic* rel *$M-U$ to f which has no fixed points in U.*

Proof. Choose an invariant neighbourhood of $Gx_1 \cup \ldots \cup Gx_n$ in U of the form $G \times D$, where D is a closed disk in U, and then apply the fixed point theory argument for $f_{|D}$, as is described for example in [3, 8.D.1]. ■

If $\sum_{i=1}^{n} I_{f, x_i} \neq 0$, the fixed point set of $f'_{|U}$ must be nonempty. However, in this case, f' may be required to have only one fixed point orbit in U.

Let G be a compact Lie group of positive dimension and let M be a compact smooth G-manifold. Suppose that $A \subset M$ is an equivariant closed neighbourhood retract such that the action of G on $M-A$ is free.

LEMMA 3.3. *Any equivariant map $f\colon (M, A) \to (M, A)$ is equivariantly homotopic* rel *A to an equivariant map f' which has no fixed points in $M-A$.*

Proof. Assume f to be taut over A in an invariant neighbourhood U. Let $V = \partial M \times I$ be an equivariant collar of ∂M in M with $\partial M = \partial M \times 0$, and let $h\colon V \to I$ be an invariant function for which $h_{|A \cap V} = 0$ and $h_{|V-U} = 1$.

Define $r\colon M \to M$ by

$$r(x, t) = (x, \max(t, h(x, t))) \quad \text{for } (x, t) \in V,$$
$$r(x) = x \quad \text{for } x \in M-V$$

and $f_1\colon M \to M$ as $r \circ f$.

Clearly, f_1 is equivariantly homotopic rel A to f and there is an invariant neighbourhood W of $A \cup \partial M$ in $U \cup V$ such that f_1 has no fixed points in $W-A$.

Denote by C a connected component of $(M-A)/G$. Choose a compact invariant submanifold N of codimension 0 in $M_{e,C}$ such that N/G is connected and $M_{e,C} \subset W \cup N$, $\partial N \cap \partial M = \emptyset$.

Consider the bundle $q\colon N \times_G M \to N/G$. There is a one-to-one correspondence between the equivariant maps $N \to M$ and the cross-sections of q. The maps equivariantly homotopic rel ∂N correspond to the sections which are homotopic rel $\partial N/G$. Moreover, the fixed point free equivariant maps correspond to the sections lying in $N \times_G M - d(N/G)$, where $d\colon N/G \to N \times_G M$ is a section corresponding to the inclusion $N \hookrightarrow M$.

Denote by s_1 the section corresponding to $f_{1|N}$. Since

$$(N \times_G M, N \times_G M - d(N/G)) \to N/G$$

is a bundle pair with the pair of fibres $(M, M - x)$, $x \in N$ being n-connected, where $n = \dim N/G$, it follows that all obstructions for s_1 to be deformable rel N/G into $N \times_G M - d(N/G)$ vanish. ■

Suppose now that G is any compact Lie group and M is a compact smooth G-manifold.

THEOREM 3.4. *For any equivariant map $f\colon M \to M$ there is an equivariant map f' equivariantly homotopic to f which has a finite number of fixed point orbits, each of a type corresponding to a subgroup with a finite Weyl group. Moreover, if H is an isotropy subgroup with a finite Weyl group, C denotes a connected component of $M_{(H)}/G$, and all connected components of M^H meeting $M_{H,C}$ are simply connected and of dimension at least three, then f' may be required to have at most one fixed point orbit in $M_{(H),C}$ and to have no fixed points in $M_{(H),C}$ if, in addition, $L(f)_C = 0$.*

Proof. The construction of f' proceeds by induction on isotropy types on M beginning at a maximal type.

Suppose $g\colon M \to M$ is an equivariant map equivariantly homotopic to f with a finite number of fixed point orbits in $M_s^{(H)}$ none of which lie in the boundary ∂M for some isotropy type (H).

If $W(H)$ is not a finite group, then apply 3.3 for the $W(H)$-equivariant map $g^H\colon (M^H, M_s^H) \to (M^H, M_s^H)$ to remove all fixed points from M_H and then from $M_{(H)}$ by equivariancy. In the case of a finite group $W(H)$ apply first 3.1 to reduce the fixed point set of g^H in M_H to a finite set disjoint with the boundary; since the resulting map is taut over M_s^H, it follows according to 2.1, that all fixed points in $M_{H,C}$ may be removed by the application of 3.2 whenever the assumptions on $L(f)_C$ and on the components of M^H are satisfied. This completes the inductive step of the construction of f'. ■

Theorem A is a simple corollary of 2.6 and 3.4.

Theorem B may be proved in a similar manner, the only difference being the use of another result from the fixed point theory in proving a lemma analogous to 3.2. A somewhat different approach to this theorem is presented in the next section, where, in fact, a more general result about equivariant vector fields is shown.

4. Equivariant vector fields. Let M be a compact smooth G-manifold and let $T(M) \to M$ be its tangent bundle. Denote by $\tilde{M} = M_+ \cup M_-$ the union of two copies of M, sewn together by the identity on the boundary. After the choice of an invariant Riemannian metric on $\tilde{M}$, the exponential map defines an equivariant diffeomorphism $T(\tilde{M}) = T$ over $\tilde{M}$, where T is the equivariant tubular neighbourhood of the diagonal $d(\tilde{M})$ in $\tilde{M} \times \tilde{M}$. In particular, there is a one-to-one correspondence between equivariant cross-sections of the bundles $T(M) \to M$ and $T_{|M} \to M = M_+$, the latter induced by the projection on the first factor. Any cross-section $M \to T_{|M}$ is of the form $x \mapsto (x, f(x))$, where $f\colon M \to \tilde{M}$ is a map sufficiently close (and hence homotopic) to the inclusion. Thus each equivariant vector field $s\colon M \to T(\tilde{M})$ defines an equivariant map $s^*\colon M \to \tilde{M}$ approximating the inclusion, and conversely each such map defines a vector field. Clearly, s is non-singular if and only if s^* is fixed point free.

An equivariant vector field $s\colon N \to T(M)$ defined on a compact smooth G-submanifold $N \subseteq M_+$ is said to be *taut over an invariant closed subspace* $A \subseteq N$ (in an invariant neighbourhood) if $s^*\colon N \to \tilde{M}$ is taut over A, and s is said to be *G-taut* if $s^{(H)}$ is taut over $N_s^{(H)} \cup \partial N_{(H)}$ for each isotropy type (H) on N. Furthermore, s is interior (resp. exterior) on A if there is an equivariant homotopy $F\colon N \times I \to \tilde{M}$ from s^* to the inclusion $N \hookrightarrow \tilde{M}$ satisfying $F(A \times I) \subset M_+$ (resp. $F(A \times I) \subset M_-$).

LEMMA 4.1. *Let $s\colon \partial M \to T(M)$ be a non-singular G-taut equivariant vector field. There is an extension of s to a G-taut equivariant field $t\colon M \to T(M)$ which has at most one singular orbit in $M_{(H),C}$ for each isotropy type (H) and each connected component C of $M_{(H)}/G$. If s is interior on ∂M then t may be chosen interior on M.*

Proof. The construction of t proceeds by induction on isotropy types on M.

Choose an ordering $(H_0), (H_1), \ldots, (H_r)$ for the set of isotropy types on M such that if H_i is subconjugate to H_j then $j \leqslant i$. Define $M_i = \bigcup_{j \leqslant i} M_{(H_j)}$ for $i = 0, 1, \ldots, r$ and set $M_{-1} = \emptyset$.

Suppose we have an already constructed equivariant vector field $t_1\colon M \to T(M)$ which extends s and has the required properties on M_{i-1}. According to [8, 2.1] and 2.5, we may assume that $t_{1|M_i}$ is taut over $M_{i-1} \cup (M_i \cap \partial M)$ in an invariant neighbourhood U. In particular, $t_{1|M_i}$ is taut over M_{i-1} in U_1 for some $U_1 \subset U$ (since $s_{|M_i \cap \partial M}$ is taut over $M_{i-1} \cap \partial M$) and hence is non-singular on $U_1 - M_{i-1}$. Since s is non-singular, it follows that there is an invariant neighbourhood $V_1 \subset U$ of $M_i \cap \partial M$ in which $t_{1|M_i}$ is also non-singular. Set $V = U_1 \cup V_1$.

Suppose $(H_i) = (H)$. For C any connected component of $M_{(H)}/G$, choose a compact smooth $W(H)$-submanifold N of codimension 0 in $M_{H,C}$ such that $N/W(H)$ is connected and $M_{H,C} \subset N \cup V^H$, $\partial N \cap \partial M = \emptyset$. Then $t_N = t_{1|\partial N}^H$ is a non-singular $W(H)$-equivariant vector field on ∂N. Since $W(H)$ acts freely on N, it follows that t_N is induced by a non-singular cross-section $t_N/W(H)$ on $\partial N/W(H)$ of the bundle $T(N)/W(H) \to N/W(H)$.

If $W(H)$ is not a finite group then $t_N/W(H)$ is extendible to a non-singular cross-section on $N/W(H)$ for dimensional reasons. For $W(H)$ a finite group $t_N/W(H)$ may be identified with a vector field on $\partial N/W(H)$ and hence is extendible to a vector field on $N/W(H)$ which has at most one zero. Thus, in both cases, t_N is extendible to a $W(H)$-equivariant vector field t_C on N with at most one singular orbit.

Denote by $t_2\colon M\to T(M)$ a G-equivariant vector field extending $t_{1|V\cup\partial M}$ and all t_C. Then t_2 extends s and has the required properties on M_i. This completes the inductive step of the construction of t. ■

In particular, 4.1 provides examples of equivariant vector fields which have a finite number of singular orbits. For any such vector field s there is a well defined element $\operatorname{Ind}(s)\in A(M)$ by

$$\operatorname{Ind}(s)_C = (-1)^{\dim C}\sum_{x\in M_{H,C}}\operatorname{Ind}_x(s^H),$$

where $\operatorname{Ind}_x(s^H)$ denotes the usual index of s^H at an isolated zero x (if $\dim C = 0$ then we put $\operatorname{Ind}_x(s^H) = 1$).

Assume that $s\colon M\to T(M)$ is an equivariant vector field with a finite number of singular orbits none of which lie on the boundary and is such that both s and $s_{|\partial M}$ are G-taut.

LEMMA 4.2. *If* $\operatorname{Ind}(s) = 0$ *then* $s_{|\partial M}$ *is extendible to a nonsingular G-taut equivariant vector field on* M.

Proof. As in the proof of 4.1, let

$$\emptyset = M_{-1}\subset M_0\subset \ldots \subset M_r = M$$

be a filtration of M by closed invariant subspaces M_i with $M_i - M_{i-1} = M_{(H_i)}$.

Suppose we have an already constructed G-taut equivariant vector field $l\colon M\to T(M)$ which is non-singular on M_{k-1}, has a finite number of singular orbits in $M-M_{k-1}$ for some $k = 0,\ldots,r$, and is such that l and s agree on ∂M and $\operatorname{Ind}(l) = 0$. We will construct a vector field $t\colon M\to T(M)$ which has the same properties, and in addition is non-singular on $M_k - M_{k-1} = M_{(H)}$.

For each $i = k,\ldots,r$ there are invariant neighbourhoods W_i of $M_{i-1}\cup(M_i\cap\partial M)$ in M_i and W_i' of $M_{i-1}\cap\partial M$ in $M_i\cap\partial M$ with equivariant retractions $r_i\colon W_i\to M_{i-1}\cup(M_i\cap\partial M)$ and $r_i'\colon W_i'\to M_{i-1}\cap\partial M$ such that $l^*_{i|W_i} = l_i^*\circ r_i$, $l^*_{i|W_i'} = l^*_{i-1}\circ r_i'$, where $l_i = l_{|M_i}$. Without loss of generality we may assume that the inclusion $(M_k - W_k)/G\to(M_k-M_{k-1})/G$ induces a bijection on sets of path components. Denote $V_i = r_i^{-1}(M_{i-1}\cup W_i')$ and define $r_i''\colon V_i\to M_{i-1}$ as $(\mathrm{id}\cup r_i')\circ r_{i|V_i}$. Then $l^*_{i|V_i} = l^*_{i-1}\circ r_i''$, i.e., l_i is taut over M_{i-1} in V_i, and hence is non-singular on $V_i - M_{i-1}$.

For C a connected component of $M_{(H)}/G$ denote by $Gx_1,\ldots,Gx_n$ the singular orbits of l lying in $M_{(H),C}$. Let $D\subset C$ with $M_{H,D}\cap W_k = \emptyset$ be a closed disk which contains in its interior $p(x_1),\ldots,p(x_n)$, $p\colon M_{H,C}\to C$ being the projection. Then $l_D = l^H_{k|M_{H,D}}$ is induced by the cross-section $l_D/W(H)\colon D\to T(M_H)/W(H)$.

If $W(H)$ is not a finite group, then the zeros of $l_D/W(H)$ may be removed

by a modification of $l_D/W(H)$ in the interior of D for dimensional reasons. If $W(H)$ is a finite group, $l_D/W(H)$ may be identified with a vector field on D, and since its index is

$$(-1)^{\dim C}\,|W(H)|^{-1}\operatorname{Ind}(l)_C = 0,$$

it follows that there is a non-singular vector field on D which agrees with $l_D/W(H)$ on ∂D. Hence, in both cases, we obtain a non-singular $W(H)$-equivariant vector field l_C on $M_{H,C}$ which agrees with l_k^H on $M_{H,C-D}$. Denote by t_k the G-equivariant extension on M_k of l_{k-1} and all l_C. In particular, t_C is non-singular on M_H and agrees with l_k in a neighbourhood of $M_{k-1}\cup(M_k\cap\partial M)$.

Suppose now that we have constructed for some $i = 0,\ldots,r-k-1$, an equivariant vector field $t_{k+i}\colon M_{k+i}\to T(M)$ extending t_k and meeting the following conditions:

i) t_{k+1} has a finite number of singular orbits,

ii) $\operatorname{Ind}(t_{k+i})_C = 0$ for each connected component C of $M_{(H_j)}/G$ and $j\leqslant k+i$,

iii) $t_{k+j} = t_{k+i|M_{k+j}}$ is taut over $M_{k+j-1}\cup(M_{k+j}\cap\partial M)$, $j\leqslant i$,

iv) t_{k+i} agrees with l_{k+i} in a neighbourhood of $M_{k+i}\cap\partial M$ in M_{k+i}.

To define $t_{k+i+1}\colon M_{k+i+1}\to T(M)$ choose a compact smooth G-submanifold N of codimension 0 in $M_{k+i+1}-M_{k+i}$ such that $N\cap\partial M$ is a smooth G-submanifold of codimension 0 in ∂N and $M_{k+i+1}\subset V_{k+i+1}\cup N$. Let $\partial N\times[0,1)$ be an equivariant collar of ∂N in N for which $R\times[0,1)\subset V_{k+i+1}$, where $R = \overline{\partial N-\partial M}$. Choose a smooth invariant function $h\colon\partial N\to I$ with $h^{-1}(0) = \partial N\cap\partial M$ and denote

$$P = \{(x,u)\in\partial N\times[0,1)\colon 0\leqslant u<h(x)\}.$$

Let $F\colon M_{k+i}\times I\to\tilde M$ be an equivariant homotopy from t^*_{k+i} to l^*_{k+i} which is constant in a neighbourhood of $M_{k+i}\cap\partial M$.

Now define t_{k+i+1} by

$$\begin{aligned}
t^*_{k+i+1}(x) &= t^*_{k+i}(r''_{k+i+1}(x)) \quad \text{for } x\in V_{k+i+1}-N,\\
t^*_{k+i+1}(x,u) &= F(r''_{k+i+1}(x,u),\,u/h(x)) \quad \text{for } (x,u)\in P,\\
t^*_{k+i+1}(x) &= l^*_{k+i+1}(x) \quad \text{for } x\in N-P.
\end{aligned}$$

It is an easy exercise to check that t_{k+i+1} admits the properties i)–iv). Using the above procedure $r-k$ times, we obtain a vector field $t_r\colon M_r = M\to T(M)$ which extends t_k and which is the promised G-taut equivariant vector field t. This completes the inductive step of the construction of a non-singular G-taut equivariant vector field on M extending $s_{|\partial M}$. ■

For any compact smooth G-submanifold $N\subset M$ define an element $\chi_M(N)\in A(M)$ by

$$\chi_M(N)_C = \sum_{C'\subset C}\chi(N)_{C'}$$

(recall that $\chi(N) = L(\mathrm{id}_N)\in A(N)$).

Again assume that $s\colon M \to T(M)$ is an equivariant vector field with a finite number of singular orbits. Suppose further that s and $s_{|\partial M}$ are G-taut and $s_{|\partial M}$ is non-singular.

PROPOSITION 4.3. $\operatorname{Ind}(s) = \chi(M)$ *if s is interior on M, and* $\operatorname{Ind}(s) = \chi(M) - \chi_M(\partial M)$ *if s is exterior on ∂M.*

Proof. If s is interior on M then according to 2.1

$$\operatorname{Ind}(s) = L(s^*) = \chi(M)$$

by the G-tautness of s and $s_{|\partial M}$ (note that $\operatorname{Ind}_x(s^H) = (-1)^{\dim C} I_{s^{H*},x}$ for $x \in M_{H,C}$).

If s is exterior on ∂M we proceed as follows. Consider s as a partial cross-section of $T(\tilde{M}) \to \tilde{M}$, defined on $M = M_+$. Then, using 4.1, extend it to a G-taut equivariant field $\tilde{s}\colon \tilde{M} \to T(\tilde{M})$ having a finite number of singular orbits and which is such that $s_- = \tilde{s}_{|M_-}$ is interior on M_-.

Let H be an isotropy subgroup on M where $W(H)$ is a finite group and let C be a connected component of $M_{(H)}/G \subset \tilde{M}_{(H)}/G$. Denote $\tilde{C} = C_+ \cup C_-$, where $C_+ = C$.

If $C_+ \cap C_- = \emptyset$ then, using the first part for $\tilde{s}$, we obtain

$$\begin{aligned}\operatorname{Ind}(s)_C &= \operatorname{Ind}(\tilde{s})_C = \chi(\tilde{M})_C = \chi(M)_C\\ &= \chi(M)_C - \chi_M(\partial M)_C.\end{aligned}$$

For $C_+ \cap C_- \neq \emptyset$ apply the first part for $\tilde{s}$ and s_-:

$$\begin{aligned}\operatorname{Ind}(s)_C &= \operatorname{Ind}(\tilde{s})_{\tilde{C}} - \operatorname{Ind}(s_-)_{C_-}\\ &= \chi(\tilde{M})_{\tilde{C}} - \chi(M_-)_{C_-}\\ &= \chi(M)_C - \chi_M(\partial M)_C. \ \blacksquare\end{aligned}$$

The referee has pointed out to me that a result similar to 4.3 was proved by Hauschild in [13].

Now we are ready to give the proof of the main result of this section.

Suppose W is a compact smooth G-manifold such that the boundary ∂W is a disjoint union of two G-submanifolds M and N.

THEOREM 4.4. *The following conditions are equivalent:*

i) *there is a non-singular equivariant vector field on W which is interior on M and exterior on N,*

ii) $\chi_W(M) = \chi_W(N) = \chi(W)$,

iii) $\chi_W(N) = \chi(W)$.

Proof. Let $W', V \subset W$ be compact smooth G-submanifolds such that $W = W' \cup V$, $V = N \times I$ and $W' \cap V = N \times 1$.

i) $\Rightarrow$ ii) The equivariant cobordisms (W, M, N) and $(W', M, N\times 1)$ are diffeomorphic; hence there is a non-singular equivariant vector field s on W' which is interior on M and exterior on $N \times 1$. Moreover, since any two non-singular equivariant vector fields defined on an invariant submanifold of the boundary which are both interior or exterior are homotopic through non-singular equivariant vector fields, it follows that, without loss of generality, we may assume $s_{|\partial W'}$ to be G-taut. Using 2.4 and 2.5 if necessary, we may assume that s is also G-taut.

Extend $s_{|N\times 1}$ to a G-taut equivariant vector field $s'\colon V \to T(V)$ which is interior on V, has a finite number of singular orbits, and is such that $s'_{|\partial V}$ is G-taut and non-singular. Then s and s' define a G-taut equivariant vector field t on W.

According to 4.3,

$$\chi(W) = \operatorname{Ind}(t) = \operatorname{Ind}(s') = \chi_W(V) = \chi_W(N).$$

Similarly from the existence of the vector field $-s$ it can be deduced that $\chi_W(M) = \chi(W)$.

iii) $\Rightarrow$ i) Suppose $s\colon W' \to T(W)$ is an equivariant vector field that is interior on W' and has a finite number of singular orbits none of which lie in $\partial W'$, and that both s and $s_{|\partial W'}$ are G-taut. Extend $s_{|N\times 1}$ to a G-taut equivariant vector field $s'\colon V \to T(V)$ with a finite number of singular orbits such that $s'_{|\partial V}$ is non-singular, G-taut and exterior on ∂V. Again s and s' define a vector field t on W, and since

$$\begin{aligned}\operatorname{Ind}(t) &= \operatorname{Ind}(s) + \operatorname{Ind}(s')\\ &= \chi_W(W') + \chi_W(V) - \chi_W(\partial V)\\ &= \chi(W) + \chi_W(N) - 2\chi_W(N) = 0,\end{aligned}$$

t follows by 4.2 that $t_{|\partial W}$ is extendible to a non-singular equivariant vector field on W. $\blacksquare$

COROLLARY 4.5. *A compact smooth G-manifold M admits a non-singular equivariant vector field which is interior (exterior) on M if and only if $\chi(M) = 0$, or equivalently $\chi(M) = \chi_M(\partial M)$.* $\blacksquare$

In particular, Theorem B follows.

It is to be noted that our Theorem 4.4 contains as special cases [10, 4.3] and [9, 1.1] (any vector field $\partial M \to T(\partial M) \subset T(M)$ is both interior and exterior on ∂M).

Remark. If $\chi(W) = \chi_W(N)$ then N and W determine the same element in the Burnside ring $A(G)$ [4]. The converse is not true, in general. However, for a compact smooth G-manifold W that is G-connected, i.e., $W_{(H)}/G$ is connected for each H with $W(H)$ a finite group, the group $A(W)$ is canonically isomorphic to a subgroup of $A(G)$ generated by the orbits G/G_x, $x \in W$, and hence 4.4, 4.3 may be interpreted as relations in $A(G)$.

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Remarks on characterization of dimension of separable metrizable spaces *

by

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Abstract. We establish some characterizations of dimension of separable metrizable spaces. For instance, it is shown that a separable metrizable space X is of dimension $\leqslant n$ if and only if X is homeomorphic to a subset S of the $(2n+1)$-dimensional cube I^{2n+1} such that

$$\lim_{\varepsilon\to 0} k(\varepsilon, \|\cdot\|)\varepsilon^p = 0 \quad \text{for} \quad p>n$$

where

$$k(\varepsilon, \|\cdot\|) = \inf\{n\colon \text{ there exists an } \varepsilon\text{-net}\{x_1, \ldots, x_n\} \text{ for } S\}.$$

Dimension is a topological concept, but in many cases it can be characterized by metrics or pseudometrics, [8], [10], [5]. In [12] Szpilrajn established some connections between the concept of dimension and the classical concept of Hausdorff measure. Borsuk [3] has constructed, for each $n \in N$, an n-dimensional pseudomeasure V_n^B of compacta lying in the Hilbert space l_2. This concept is a topological invariant, i.e. if $V_n^B(X)>0$ then $V_n^B(Y)>0$ for every compactum Y homeomorphic to X, [4]. Several connections between dimension and Borsuk pseudomeasure are given in [3], [4]. Since the Borsuk pseudomeasure is defined only on compacta isometrically embedable into l_2, we construct in § 1 of this note a pseudomeasure for the class of all compacta similar to the Borsuk pseudomeasure. This pseudomeasure is shown to have many of the properties possessed by the Borsuk pseudomeasure. In § 2 we establish certain characterizations of dimension of separable metrizable spaces which are related to old results of Szpilrajn [12] and Pontrjagin and Schinirelman [11].

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§ 1. Pseudomeasure and dimension of separable metric spaces. Given a separable metrizable space X. Let $M_{\text{tb}}(X)$ (resp. $P_{\text{tb}}(X)$) denote the set of all totally

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