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FIXED POINT ITERATION PROCESSES FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. Let X be a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm, C a bounded closed convex subset of X, and $T: C \to C$ an asymptotically nonexpansive mapping. It is then shown that the modified Mann and Ishikawa iteration processes defined by $x_{n+1} = t_n T^n x_n + (1-t_n)x_n$ and $x_{n+1} = t_n T^n(s_n T^n x_n + (1-s_n)x_n) + (1-t_n)x_n$, respectively, converge weakly to a fixed point of T.

1. INTRODUCTION

Let C be a nonempty subset of a Banach space X. A mapping $T: C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all x, y in C and $n = 1, 2, \ldots$. This class of mappings, as a natural extension to that of nonexpansive mappings, was introduced by Goebel and Kirk [4] in 1972. They proved that if C is a bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpansive self-mapping T of C has a fixed point. This existence result was recently generalized in [14] to a nearly uniformly convex (NUC) Banach space setting (see [5] for definition).

The study of iterative construction for fixed points of asymptotically nonexpansive mappings began in 1978. Bose [1] first proved that if C is a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition [7] and if $T: C \to C$ is an asymptotically nonexpansive mapping, then $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at x, i.e., $\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0$. This conclusion is still valid [8, 14] if Opial's condition of X is replaced by the condition that X has a Fréchet differentiable norm. Furthermore, in both cases, asymptotic regularity of T at x can be weakened to weak asymptotic regularity of T at x, i.e., w-$\lim_{n\to\infty}(T^n x - T^{n+1} x) = 0$ (see [12, 13]).

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Recently, Schu [10] considered the following modified Mann iteration process:

$$\text{(M)} \qquad x_{n+1} = t_n T^n x_n + (1 - t_n) x_n, \qquad n \geq 1,$$

where $\{t_n\}$ is a sequence of real numbers in $(0, 1)$ which is bounded away from both 0 and 1, i.e., $a \leq t_n \leq b$ for all n and some $0 < a \leq b < 1$. He verified that if C is a bounded closed convex subset of a Banach space X satisfying Opial's condition and if $T: C \to C$ is an asymptotically nonexpansive mapping such that $\sum_{n=1}^{\infty}(k_n - 1)$ converges, then the modified Mann iteration process (M) converges weakly to a fixed point of T. Unfortunately, Schu's theorem does not apply to the L^p spaces if $p \neq 2$ since none of these spaces satisfy Opial's condition (cf. [7]).

In this paper we first show that Schu's theorem remains true if the assumption that X satisfies Opial's condition is replaced by the one that Y has a Fréchet differentiable norm. This result (Theorem 3.1) applies to the L^p spaces for $1 < p < \infty$ since each of these spaces is uniformly convex and uniformly smooth. We then prove the weak convergence of the modified Ishikawa iteration process (cf. Ishikawa [6]):

$$\text{(I)} \qquad x_{n+1} = t_n T^n (s_n T^n x_n + (1 - s_n) x_n) + (1 - t_n) x_n, \qquad n \geq 1,$$

in a uniformly convex Banach space which either satisfies Opial's condition or has a Fréchet differentiable norm.

2. Preliminaries and lemmas

Let X be a Banach space. Recall that X is said to satisfy Opial's condition [7] if for each sequence $\{x_n\}$ in X the condition $x_n \to x$ weakly implies $\overline{\lim}_{n\to\infty} \|x_n - x\| < \overline{\lim}_{n\to\infty} \|x_n - y\|$ for all $y \in X$ different from x. It is known [7] that each l^p $(l \leq p < \infty)$ enjoys this property, while L^p does not unless $p = 2$. It is also known [3] that any separable Banach space can be equivalently renormed so that it satisfies Opial's condition. Recall also that X is said to have a Fréchet differentiable norm if, for each x in $S(X)$, the unit sphere of X, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in S(X)$. In this case, we have

$$\text{(2.1)} \qquad \tfrac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \leq \tfrac{1}{2}\|x + h\|^2 \leq \tfrac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + b(\|h\|)$$

for all x, $h \in X$, where J is the normalized duality map from X to X^* defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},$$

$\langle \cdot, \cdot \rangle$ is the duality pairing between X and X^*, and b is a function defined on $[0, \infty)$ such that $\lim_{t\downarrow 0} b(t)/t = 0$.

Suppose now that C is a bounded closed convex subset of a Banach space X and $\{T_n\}$ is a sequence of Lipschitzian self-mappings of C such that the set F of common fixed points of $\{T_n\}$ is nonempty. Denote by L_n the Lipschitz constant of T_n. In the sequel, we always assume $L_n \geq 1$ for all $n \geq 1$ and use the notations $\overline{\lim} = \limsup$, $\underline{\lim} = \liminf$, $\rightharpoonup$ for weak convergence, $\to$ for strong convergence, and $F(T)$ for the set of fixed points of T.

For a given $x_1 \in C$, we recurrently define the sequence $\{x_n\}$ by

$$x_{n+1} = T_n x_n, \qquad n \geq 1.$$

Lemma 2.1. *Suppose that* $\sum_n (L_n - 1)$ *converges. Then for each* $f \in F$, $\lim_n \|x_n - f\|$ *exists.*

Proof. For all n, $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m+1} - f\| &= \|T_{n+m}x_{n+m} - f\| \leq L_{n+m}\|x_{n+m} - f\| \\ &\leq \left(\prod_{j=n}^{n+m} L_j\right) \|x_n - f\|. \end{aligned}$$

Since $\sum_n (L_n - 1)$ converges, it follows that

$$\overline{\lim_{m\to\infty}} \|x_{n+m+1} - f\| \leq \left(\prod_{j=n}^{\infty} L_j\right) \|x_n - f\|.$$

Consequently,

$$\overline{\lim_n} \|x_n - f\| \leq \underline{\lim_n} \|x_n - f\|.$$

This proves the lemma. □

Lemma 2.2. *Suppose that* X *is uniformly convex and* $\sum_n (L_n - 1)$ *converges. Then* $\lim_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$ *exists for every* f_1, $f_2 \in F$ *and* $0 \leq t \leq 1$.

Proof. We follow an idea of Reich [9]. Set

$$a_n = a_n(t) = \|tx_n + (1-t)f_1 - f_2\|, \qquad S_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_n,$$

and

$$b_{n,m} = \|S_{n,m}(tx_n + (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\|.$$

Then, observing $S_{n,m}x_n = x_{n+m}$, we get

$$\begin{aligned} a_{n+m} &= \|tx_{n+m} + (1-t)f_1 - f_2\| \\ &\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)f_1) - f_2\| \\ &\leq b_{n,m} + \left(\prod_{j=n}^{n+m-1} L_j\right) a_n \leq b_{n,m} + H_n a_n, \end{aligned}$$

where $H_n = \prod_{j=n}^{\infty} L_j$. By a result of Bruck [2], we have

$$\begin{aligned} b_{n,m} &\leq H_n g^{-1}(\|x_n - f_1\| - H_n^{-1}\|S_{n,m}x_n - f_1\|) \\ &\leq H_n g^{-1}(\|x_n - f_1\| - \|x_{n+m} - f_1\| + (1 - H_n^{-1})d), \end{aligned}$$

where $g\colon [0, \infty) \to [0, \infty)$, $g(0) = 0$, is a strictly increasing continuous function depending only on d, the diameter of C. Since $\lim_{n\to\infty} H_n = 1$, it follows from Lemma 2.1 that $\lim_{n,m\to\infty} b_{n,m} = 0$. Therefore,

$$\overline{\lim_{m\to\infty}} a_m \leq \lim_{n,m\leq\infty} b_{n,m} + \underline{\lim_{n\to\infty}} H_n a_n = \underline{\lim_{n\to\infty}} a_n.$$

This completes the proof. □

Lemma 2.3. *Suppose that X is a uniformly convex Banach space with a Fréchet differentiable norm and that $\sum_n(L_n - 1)$ converges. Then for every $f_1, f_2 \in F$, $\lim_{n\to\infty}\langle x_n, J(f_1 - f_2)\rangle$ exists; in particular,*

$$\langle p - q, J(f_1 - f_2)\rangle = 0$$

for all $p, q \in \omega_w(x_n)$. Here, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, i.e., $\omega_w(x_n) = \{y \in X: y = \text{w-}\lim_{k\to\infty} x_{n_k} \text{ for some } n_k \uparrow \infty\}$.

Proof. Taking $x = f_1 - f_2$ and $h = t(x_n - f_1)$ in (2.1), we get

$$\begin{aligned}\tfrac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle &\le \tfrac{1}{2}\|tx_n + (1-t)f_1 - f_2\|^2 \\ &\le \tfrac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle + b(t\|x_n - f_1\|).\end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned}&\tfrac{1}{2}\|f_1 - f_2\|^2 + t\cdot \overline{\lim_{n\to\infty}}\langle x_n - f_1, J(f_1 - f_2)\rangle \\ &\quad\le \lim_{n\to\infty}\tfrac{1}{2}\|tx_n + (1-t)f_1 - f_2\|^2 \\ &\quad\le \tfrac{1}{2}\|f_1 - f_2\|^2 + t\cdot \underline{\lim}_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle + o(t).\end{aligned}$$

This yields

$$\overline{\lim_{n\to\infty}}\langle x_n - f_1, J(f_1 - f_2)\rangle \le \underline{\lim}_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle + o(1).$$

Letting $t \to 0^+$, we see that $\lim_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle$ exists. □

We also need the following known lemmas.

Lemma 2.4 (cf. Schu [10]). *Let X be a uniformly convex Banach space, $\{t_n\}$ a sequence of real numbers in $(0, 1)$ bounded away from 0 and 1, and $\{x_n\}$ and $\{y_n\}$ sequences of X such that $\overline{\lim}_{n\to\infty}\|x_n\| \le a$, $\overline{\lim}_{n\to\infty}\|y_n\| \le a$, and $\lim_{n\to\infty}\|t_nx_n + (1-t_n)y_n\| = a$ for some $a \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.5 [11]. *Let X be a normed space, C a convex subset of X, and $T: C \to C$ a uniformly L-Lipschitzian mapping, i.e., $\|T^nx - T^ny\| \le L\|x - y\|$ for all x, y in C and $n = 1, 2, \ldots$. For any given x_1 in C and sequences $\{t_n\}$ and $\{s_n\}$ in $[0, 1]$, define $\{x_n\}$ by*

$$x_{n+1} = t_nT^n(s_nT^nx_n + (1-s_n)x_n) + (1-t_n)x_n, \qquad n \ge 1.$$

Then we have

$$\|x_n - Tx_n\| \le c_n + c_{n-1}L(1 + 3L + 2L)^2$$

for all $n \ge 2$, where $c_n = \|x_n - T^nx_n\|$.

Lemma 2.6 [14]. *Suppose that C is a bounded closed convex subset of a uniformly convex Banach space and $T: C \to C$ is an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at the origin, i.e., for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x_0$ and $x_n - Tx_n \to 0$ imply $x_0 - Tx_0 = 0$.*

3. Weak convergence

In this section we prove the weak convergence of the modified Mann and the modified Ishikawa iteration processes in a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm.

Theorem 3.1. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, C a bounded closed convex subset of X, and $T: C \to C$ an asymptotically nonexpansive mapping such that $\sum_n (k_n - 1)$ converges. Then for each $x_1 \in C$, the sequence $\{x_n\}$ defined by the modified Mann iteration process* (M) *with $\{t_n\}$ a sequence of real numbers bounded away from* 0 *and* 1 *converges weakly to a fixed point of T.*

Proof. Set $T_n = t_n T^n + (1 - t_n) I$. (Here I is the identity operator of X.) Then it is easily seen that $x_{n+1} = T_n x_n$, $F(T_n) \supseteq F(T)$, and T_n is Lipschitzian with constant $L_n = t_n k_n + (1 - t_n) \geq 1$. Since $L_n - 1 = t_n(k_n - 1) \leq k_n - 1$ and $\sum_n (k_n - 1)$ converges, $\sum_n (L_n - 1)$ also converges. It thus follows from Lemma 2.3 that

$$\langle p - q, J(f_1 - f_2) \rangle = 0 \tag{3.1}$$

for all $p, q \in \omega_w(x_n)$ and $f_1, f_2 \in F(T)$. Moreover, for $f \in F(T)$, we have

$$\overline{\lim_{n\to\infty}} \|T^n x_n - f\| \leq \overline{\lim_{n\to\infty}} k_n \|x_n - f\| = \lim_{n\to\infty} \|x_n - f\|$$

and

$$\lim_{n\to\infty} \|t_n(T^n x_n - f) + (1 - t_n)(x_n - f)\| = \lim_{n\to\infty} \|x_{n+1} - f\|.$$

It follows from Lemma 2.4 that $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$, which implies by Lemma 2.5 that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, which in turn implies by Lemma 2.6 that $\omega_w(x_n)$ is contained in $F(T)$. So to show that $\{x_n\}$ converges weakly to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of just one point. To this end, let p, q be in $\omega_w(x_n)$. Then since p, q belong to $F(T)$, it follows from (3.1) that

$$\|p - q\|^2 = \langle p - q, J(p - q) \rangle = 0.$$

Therefore, $p = q$ and the proof is complete. □

Remark. We do not know whether Theorem 3.1 remains valid if k_n is allowed to approach 1 slowly enough so that $\sum_n (k_n - 1)$ diverges.

Next, we consider the modified Ishikawa iteration process (I) described in §1.

Theorem 3.2. *Let X be a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm, C a bounded closed convex subset of X, and $T: C \to C$ an asymptotically nonexpansive mapping such that $\sum_n (k_n - 1)$ converges. Suppose that x_1 is a given point in C and $\{t_n\}$ and $\{s_n\}$ are real sequences such that $\{t_n\}$ is bounded away from* 0 *and* 1 *and $\{s_n\}$ is bounded away from* 1. *Then the sequence $\{x_n\}$ defined by the modified Ishikawa iteration process* (I) *converges weakly to a fixed point of T.*

Proof. Define a mapping $T_n: C \to C$ by

$$T_n x = t_n T^n (s_n T^n x + (1 - s_n) x) + (1 - t_n) x, \qquad x \in C.$$

Then it is easily seen that $x_{n+1} = T_n x_n$, $F(T_n) \supseteq F(T)$, and T_n is Lipschitzian with constant $L_n = 1 + t_n k_n (1 + s_n k_n - s_n) - t_n \geq 1$ for $k_n \geq 1$. Since $L_n - 1 = t_n(1 + s_n k_n)(k_n - 1) \leq (1 + L)(k_n - 1)$, where $L = \sup_{n\geq 1} k_n$, we see that $\sum_n (L_n - 1)$ converges. Now repeating the arguments in the proof of Theorem 3.1, we arrive at the following conclusions:

(i) $\lim \|x_n - f\|$ exists for every $f \in F(T)$.

(ii) $\langle p - q, J(f_1 - f_2) \rangle = 0$ for every $p, q \in \omega_w(x_n)$ and $f_1, f_2 \in F(T)$.

(iii) $\lim_{n\to\infty} \|x_n - T^n y_n\| = 0$ with $y_n = s_n T^n x_n + (1 - s_n) x_n$.

Since

$$\begin{aligned}\|T^n x_n - x_n\| &\le \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ &\le k_n \|x_n - y_n\| + \|T^n y_n - x_n\| \\ &= k_n s_n \|T^n x_n - x_n\| + \|T^n y_n - x_n\|,\end{aligned}$$

we have

$$\|T^n x_n - x_n\| \le \frac{1}{1 - k_n s_n} \|T^n y_n - x_n\|,$$

from which, together with the facts that $\{s_n\}$ is bounded away from 1 and $\{k_n\}$ converges to 1, we conclude that $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. By Lemma 2.5, we have the following result:

(iv) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

It follows from (iv) and Lemma 2.6 that $\omega_w(x_n) \subset F(T)$. So to show the theorem, it suffices to show that $\omega_w(x_n)$ is a singleton. To this end, we suppose first that X satisfies Opial's condition. Let p, q be in $\omega_w(x_n)$ and $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ chosen so that $x_{n_i} \rightharpoonup p$ and $x_{m_j} \rightharpoonup q$. If $p \ne q$, then Opial's condition of X implies that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| = \lim_{i\to\infty} \|x_{n_i} - p\| < \lim_{i\to\infty} \|x_{n_i} - q\| = \lim_{j\to\infty} \|x_{m_j} - q\| \\ < \lim_{j\to\infty} \|x_{m_j} - p\| = \lim_{n\to\infty} \|x_n - p\|.\end{aligned}$$

This contradiction proves the theorem in case X satisfies Opial's condition. Next, we assume that X has a Fréchet differentiable norm. Then since $\omega_w(x_n) \subset F(T)$, as in the proof of Theorem 3.1, we derive from (ii) that for every p, q in $\omega_w(x_n)$

$$\|p - q\|^2 = \langle p - q, J(p - q)\rangle = 0.$$

This completes the proof. □

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