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FIXED POINT RATIOS IN ACTIONS OF FINITE EXCEPTIONAL GROUPS OF LIE TYPE

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Let G be a finite exceptional group of Lie type acting transitively on a set Ω. For $x \in G$, the fixed point ratio of x is the proportion of elements of Ω which are fixed by x. We obtain new bounds for such fixed point ratios. When a point-stabilizer is parabolic we use character theory; and in other cases, we use results on an analogous problem for algebraic groups in Lawther, Liebeck & Seitz, 2002. These give dimension bounds on fixed point spaces of elements of exceptional algebraic groups, which we apply by passing to finite groups via a Frobenius morphism.

Introduction.

If G is a finite group acting transitively on a set Ω, and $x \in G$, we define the *fixed point ratio* of x to be the proportion of points fixed by x; that is, denoting this quantity by $\mathrm{fpr}(x,\Omega)$,

$$\mathrm{fpr}(x,\Omega) = \frac{\mathrm{fix}_\Omega(x)}{|\Omega|},$$

where $\mathrm{fix}_\Omega(x)$ is the number of fixed points of x on Ω. This can also be expressed in terms of conjugacy classes, as follows: If $\omega \in \Omega$ and $H = G_\omega$, then

$$\mathrm{fpr}(x,\Omega) = \frac{|x^G \cap H|}{|x^G|},$$

where x^G denotes the conjugacy class in G which contains x. (To see the equality of the above two expressions for $\mathrm{fpr}(x,\Omega)$, just count the pairs $\{(\omega, y) : \omega \in \Omega,\ y \in x^G,\ \omega y = \omega\}$ in two different ways.)

Fixed point ratios have been much studied in recent years, and applied to a number of different problems, particularly in the case where G is almost simple. We refer the reader in particular to [**23**, **27**, **30**, **52**], where upper bounds on fixed point ratios are obtained and applied to various problems when G is a classical group; to [**42**], where a general upper bound of $4/3q$ is obtained for groups of Lie type over $\mathbf{F}_q$ (with a few exceptions); and to [**24**, **52**], where these bounds are used to prove the Guralnick-Thompson monodromy group conjecture.

In view of these and possible future applications, it seems important to obtain as strong as possible upper bounds for fixed point ratios. While the bounds in the above references are fairly satisfactory for classical groups, the $4/3q$ bound of [**42**] is still the strongest upper bound for exceptional groups of Lie type which can be found in the literature, apart from groups of rank at most 2, where better bounds are obtained in [**25**]. In this paper we obtain stronger bounds for fixed point ratios of all exceptional groups.

Our main result is Theorem 2 below. This is divided into several cases, giving upper bounds for $\mathrm{fpr}(x,\Omega)$ according to whether x is a semisimple or unipotent element, and also according to whether a point stabilizer is a parabolic or reductive subgroup. In many cases, the bounds given are close to best possible; in particular, this is the case for maximal parabolics. The statement of Theorem 2 is necessarily somewhat involved, with reference to a number of tables, so for convenience we first state the following greatly simplified version, giving an overall bound for all elements and all point stabilizers.

Theorem 1. *Let L be a finite simple exceptional group of Lie type over $\mathbf{F}_q$, and let X be an almost simple group with socle L (that is, $L \triangleleft X \le \mathrm{Aut}\, L$). Suppose X acts faithfully and transitively on a set Ω, and $1 \ne x \in X$. Then*

$$\mathrm{fpr}(x,\Omega) \le \frac{1}{e_L(q)},$$

where $e_L(q)$ is as in Table 1.

$L =$	$E_8(q)$	$E_7(q)$, ${}^2E_6(q)$	$E_6(q), F_4(q), {}^3D_4(q)$	${}^2F_4(q)'$
$e_L(q) =$	$q^8(q^4-1)$	q^6-q^3+1	q^4-q^2+1	q^4
$L =$	$G_2(q) (q \ne 2,4)$	$G_2(4)$	${}^2G_2(q) (q>3)$	${}^2B_2(q)$
$e_L(q) =$	q^2-q+1	$52/7$	q^2-q+1	$\frac{(q^2+1)}{(q^{2/a}+1)}$

Table 1.

For $L = {}^2B_2(q)$, the number a in $e_L(q)$ is the smallest prime divisor of $\log_2 q$.

For all cases except $L = E_8(q)$, $E_7(q)$ or ${}^2F_4(q)'$, the bounds in Theorem 1 are sharp, in the sense that there is an element $x \in X$ and an X-space Ω for which $\mathrm{fpr}(x,\Omega) = 1/e_L(q)$; and for $L = E_8(q)$, $E_7(q)$ or ${}^2F_4(q)'$, $e_L(q)$ is of the correct order of magnitude, as can be seen using Proposition 2.1 below.

Observe that in order to prove Theorem 1, it suffices to prove it in the case where $X = \langle L, x\rangle$ and X acts primitively on Ω. To see this, note that if X acts imprimitively, then the fixed point ratio of x on blocks of imprimitivity is certainly no less than its fixed point ratio on points.

As explained above, despite the sharpness of Theorem 1, it is possible to obtain much stronger bounds for fixed point ratios, and this we do in Theorem 2 below. In order to state Theorem 2, we need to set up some notation. Let G be a simple adjoint algebraic group over an algebraically closed field K of characteristic $p > 0$, and let σ be a Frobenius morphism of G such that the fixed point group $G_\sigma = C_G(\sigma)$ is a finite exceptional group over a field $\mathbf{F}_q$, where q is a power of p. Write L for the simple group $(G_\sigma)'$, and let X be an almost simple group with socle L.

Let Ω be a set on which X acts primitively, and let H be a point stabilizer. Then Ω can be identified with the coset space of H in X, which we denote by X/H. In order to obtain lower bounds on fixed point ratios $\mathrm{fpr}(x, \Omega)$ for $1 \neq x \in X$, it suffices to obtain such bounds just when x is an element of prime order.

Elements of prime order are of the following types: Unipotent elements of order p in G_σ; semisimple elements (of p'-order) in G_σ; and outer automorphisms of L of prime order, not lying in G_σ. The latter are described in Proposition 1.1 in Section 1 below, taken from [**28**, Section 7] (see also [**29**, p. 60]); they are classified as field automorphisms, graph-field automorphisms (which exist only if $G = E_6$, $F_4(p = 2)$, $G_2(p = 3)$ or $B_2(p = 2)$), and graph automorphisms (which exist only if $G = E_6$). The field and graph-field automorphisms are those for which the centralizer has the same type as G, possibly twisted.

Theorem 2 has different bounds for each of these types of elements. Moreover, separate bounds are given for long root elements u_α of G_σ (i.e., nonidentity elements lying in the center of a long root subgroup U_α), for short root elements u_β when these exist, and for unipotent elements which are not long or short root elements.

Theorem 2 is also subdivided into various parts according to the following possibilities for H:

(I) H is a parabolic subgroup of X (i.e., $H \cap L$ is a parabolic subgroup of L);

(II) $H = N_X(M_\sigma)$, where M is a σ-stable reductive subgroup of maximal rank in G (i.e., M contains a maximal torus of G); such maximal subgroups H are classified in [**43**], where they are called *subgroups of maximal rank*;

(III) H is not as in (I) or (II).

Theorem 2. *Let $L = (G_\sigma)'$ be a finite simple exceptional group of Lie type over $\mathbf{F}_q$ as above, let X be an almost simple group with socle L, acting faithfully and primitively on a set Ω, and let $H = X_\alpha$ ($\alpha \in \Omega$), a point stabilizer. Let u be a nonidentity unipotent element of G_σ, let u_α be a long root element and u_β a short root element* (*if these exist*); *let s be a nonidentity semisimple element of G_σ; let ϕ be a field or graph-field automorphism of L*

of prime order, and τ a graph automorphism of prime order (*if these exist*). *According as:*

(I) *H is a parabolic subgroup P,*
(II) *$H = N_X(M_\sigma)$ is a subgroup of maximal rank, or*
(III) *H is not as in* (I) *or* (II),

and $x \in H$ is such that:

(a) *$x = u$,*
(b) *$x = s$,*
(c) *$x = \phi$, or*
(d) *$x = \tau$,*

upper bounds for the fixed point ratio $\mathrm{fpr}(x, \Omega)$ *are given in Table* 2 *below.*

Notation in Table 2. The bounds in Table 2 for parabolic actions are expressed in terms of various polynomials $f_{P,\alpha}(q)$, $f_{P,\beta}(q)$, $g_P(q)$ and $h_P(q)$, which are defined in Tables 7.1A-D at the end of the paper. The symbol $u \sim u_\alpha$ means that u is L-conjugate to u_α. The values of e_G and h_G are given in Table 3 (and if G is not exceptional — which occurs when $L = {}^2B_2(q)$ or ${}^3D_4(q)$ — we set $e_G = h_G = 0$), and $e_L(q)$ is defined in Table 1. We write

$$\mathcal{L}_1 = \{G_2(q), {}^2G_2(q), {}^2B_2(q), {}^3D_4(q)\},$$

and if $H = N_X(M_\sigma)$ and $x = s$ we set $\epsilon_7 = 1$ if $(G, M^0) = (E_7, E_6T_1)$ and 0 otherwise. In addition, there are certain exceptions to the entries in Table 2, marked by single and double daggers: The single daggers indicate two exceptions to Cases (III)(a) and (b), for which upper bounds for $\mathrm{fpr}(x, \Omega)$ are provided in Table 4; the double dagger denotes that in Case (II)(b) separate bounds are given in Table 5 for $q \leq 3$.

	(I) $H = P$	(II) $H = N_X(M_\sigma)$	(III) H other
(a) $x = u$	$\frac{1}{f_{P,\alpha}(q)}$ if $u \sim u_\alpha$ $\frac{1}{f_{P,\beta}(q)}$ if $u \sim u_\beta$ $\frac{1}{g_P(q)}$ otherwise	$\min\left(\frac{2}{q^{e_G}}, \frac{1}{e_L(q)}\right)$	$\min\left(\frac{1}{q^{e_G}}, \frac{1}{e_L(q)}\right)$ (†)
(b) $x = s$	$\frac{1}{h_P(q)}$	$\min\left(\frac{2+\epsilon_7}{q^{h_G}}, \frac{1}{e_L(q)}\right)$ (‡)	$\min\left(\frac{1}{q^{h_G}}, \frac{1}{e_L(q)}\right)$ (†)
(c) $x = \phi$	$\frac{1}{e_L(q)}$ if $L \in \mathcal{L}_1$ $\frac{1}{h_P(q)}$ otherwise	$\frac{1}{e_L(q)}$ if $L \in \mathcal{L}_1$ $\frac{1}{q^{h_G}}$ otherwise	$\frac{1}{e_L(q)}$ if $L \in \mathcal{L}_1$ $\frac{1}{q^{h_G}}$ otherwise
(d) $x = \tau$	$\frac{1}{e_L(q)}$	$\frac{1}{e_L(q)}$	$\frac{1}{e_L(q)}$

Table 2. Upper bounds for $\mathrm{fpr}(x, \Omega)$.

G	e_G	h_G
E_8	24	48
E_7	12	22
E_6	6	12
F_4	4	6
G_2	2	2

Table 3. Values e_G and h_G.

	$(L, H\cap L) = ({}^2E_6(q), F_4(q))$	$(L, H\cap L) = (G_2(4), J_2)$
$x = u$	$\frac{1}{(q^6-q^3+1)}$	$\frac{1}{13}$
$x = s$	$\frac{1}{q^6(q^6-q^3+1)}$	$\frac{7}{52}$

Table 4. Exceptional bounds for (III)(a) and (b).

	L							
q	$E_8(q)$	$E_7(q)$	$E_6(q)$	${}^2E_6(q)$	$F_4(q)$	${}^2F_4(q)'$	$G_2(q)$	${}^3D_4(q)$
2	$\frac{1}{2^{37}}$	$\frac{1}{2^{12}}$	$\frac{1}{2^5}$	$\frac{1}{2^6}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$-$	$\frac{1}{2^5}$
3	$\frac{2}{3^{48}}$	$\frac{1}{3^{19}}$	$\frac{2}{3^{12}}$	$\frac{2}{3^{12}}$	$\frac{2}{3^5}$	$-$	$\frac{2}{3^3}$	$\frac{1}{3^4}$

Table 5. Upper bounds for $\mathrm{fpr}(s, \Omega)$ for $q \leq 3$ in (II)(b).

Remark. The polynomials $f_{P,\alpha}(q)$ have the same degree as the rational functions $1/\mathrm{fpr}(u_\alpha, G_\sigma/P) = |G_\sigma/P|/\mathrm{fix}_{G_\sigma/P}(u_\alpha)$ (and likewise for $f_{P,\beta}(q)$). Precise values of the polynomials $\mathrm{fix}_{G_\sigma/P}(u_\alpha)$ can be read off from Proposition 2.1 in Section 2.

Note that Theorem 1 follows from Theorem 2.

Our methods for handling the three cases (I)-(III) in Theorem 2 are rather different. For technical reasons, we postpone the cases (I)(c,d) and (II)(c,d), where x is an outer automorphism, until the final Section 6 of the paper. For Case (I)(a,b), where $H = P$ is a parabolic subgroup, we use character theory: The permutation character of G_σ on Ω is the induced character $1_P^{G_\sigma}$, and so

$$\mathrm{fpr}(x, \Omega) = \frac{1_P^{G_\sigma}(x)}{1_P^{G_\sigma}(1)}.$$

(We may take $X = G_\sigma$ in Part (I), as we are considering only elements $u, s \in G_\sigma$, and maximal parabolics of L extend to maximal parabolics of G_σ.) In Sections 2 and 3 we investigate the values of $1_P^{G_\sigma}$ on unipotent and semisimple elements, using some sophisticated tools from the character theory of finite groups of Lie type — the Deligne-Lusztig theory, Green functions, Foulkes functions, and so on. As a result we obtain some rather precise upper bounds for fixed point ratios, which are recorded in column (I) of Table 2.

The results in Sections 2 and 3 may have some independent interest, since nowhere else have we found a detailed analysis of the values of the induced characters $1_P^{G_\sigma}$.

In Case (II), where H is a subgroup of maximal rank, we may as above take $X = G_\sigma$, and so $H = M_\sigma$ with M reductive of maximal rank. For this case we use results from the paper [**40**], in which we considered an analogous question for the exceptional algebraic groups G. For these groups the quantity analogous to the fixed point ratio is

$$-f(x, G/M) = \dim \mathrm{fix}_{G/M}(x) - \dim G/M.$$

In [**40**], upper bounds are obtained for $-f(x, G/M)$. By passing from the algebraic groups G to the finite groups G_σ, we are able in Section 4 to use these dimension bounds to obtain the bounds for fixed point ratios recorded in column (II) of Table 2. While basically a straightforward application of Lang's theorem, the process of passing from algebraic to finite groups requires a great deal of careful calculation.

In Case (III), the results [**45**, Theorem 2] and [**48**, Corollary 8] imply that one of the following holds:

(i) $H = N_X(M_\sigma)$ for some maximal closed subgroup M of positive dimension in G (not of maximal rank),
(ii) H is one of a few known local subgroups,
(iii) H is almost simple and of bounded order.

There are not many possibilities under (i) or (ii), and they are dealt with fairly easily using the methods of [**40**]. The subgroups in (iii) are handled in Section 5 using some rather lengthy ad hoc arguments. The upshot for fixed point ratios is recorded in column (III) of Table 2.

For technical reasons, we postpone a few cases in Theorem 2 until the final Section 6. These are the cases in (c) and (d), where x is an outer automorphism, and some cases where L is one of the groups of small rank in $\mathcal{L}_1$.

1. Preliminary results.

In this section we present a variety of results concerning groups of Lie type which will be used in later sections. Some of these may be of independent interest. In particular, Proposition 1.3 provides general upper bounds for the numbers of elements of order 2 or 3 in groups of Lie type; Lemma 1.7 is a general result on the order of a finite unipotent group which is presumably well-known, but for which we have been unable to find a reference; and 1.6 and 1.8 give general upper and lower bounds for the orders of finite reductive groups and their conjugacy classes.

We begin with a well-known result which classifies all outer automorphisms of prime order of finite groups of Lie type. In the terminology of [**28**, Section 7], all such are field, graph-field or graph automorphisms.

Proposition 1.1. *Let $L = L(q)$ be a simple group of Lie type over $\mathbf{F}_q$, and let α be an automorphism of L of prime order. If L is classical with natural module V, suppose that α does not lie in $PGL(V)$; and if L is exceptional, suppose that $\alpha \notin Inndiag(L)$. Then one of the following holds:*

(i) *α is a field or graph-field automorphism, and $C_L(\alpha)$ is of type $L(q^{1/|\alpha|})$ or ${}^2L(q^{1/2})$ (or ${}^3D_4(q^{1/3})$ when $L = D_4(q)$);*

(ii) *α is a graph automorphism and the possibilities are as follows:*

L	$\|\alpha\|$	possible types for $C_L(\alpha)$
$L_n^\epsilon(q)$	2	$PSO_n(q)$ (n odd)
		$PSO_n^{\pm}(q), PSp_n(q)$ (n even, q odd)
		$Sp_n(q)$, $C_{Sp_n(q)}(t)$ (n even, q even)
$D_4(q)$, ${}^3D_4(q)$	3	$G_2(q), A_2^\epsilon(q)$ if $(3,q)=1$
		$G_2(q)$, $C_{G_2(q)}(t)$ if 3 divides q
$E_6^\epsilon(q)$	2	$F_4(q)$, $C_4(q)$ (q odd)
		$F_4(q)$, $C_{F_4(q)}(t)$ (q even)

(*in the last column, t denotes a long root element*).

Proof. By [**28**, Section 7], α is a field, graph-field or graph automorphism, and in the first two cases $C_L(\alpha)$ is as in (i). If α is a graph automorphism of order 3, then $L = D_4(q)$ or ${}^3D_4(q)$ and $C_L(\alpha)$ is as in the table, by [**28**, 9.1]. Finally, the conjugacy classes of graph automorphisms of order 2 are given by [**2**, Section 19] when q is even, and by [**29**, 4.5.1] when q is odd. □

In the proof of the next proposition we shall require the following elementary lemma.

Lemma 1.2.

(i) *If $\{a_1, \ldots, a_l\}$ and $\{b_1, \ldots, b_m\}$ are two sets of distinct integers, all at least 2, then*

$$\frac{\prod_1^l (q^{a_i}-1)}{\prod_1^m (q^{b_i}-1)} < 2q^{\Sigma a_i - \Sigma b_i}.$$

(ii) *If $a_1, \ldots, a_l$ are distinct integers, all at least 2, and $r \geq 3$, then*

$$(q^r+1)\prod_1^l (q^{a_i}+1) < 2q^{r+\Sigma a_i}.$$

(iii) *If* $b \le a$ *then* $\frac{q^a+1}{q^b+1} < q^{a-b}$.

Proof. To prove (i), it is enough to establish that $\prod_2^n \frac{q^i}{q^i-1} < 2$ for any n. Taking natural logarithms, we require $\sum \ln(1+\frac{1}{q^i-1}) < \ln 2$. The left hand side is less than $\sum_{i\ge 2} \frac{1}{2^i-1}$, which is less than $\frac{4}{3}\sum_2^\infty \frac{1}{2^i} = \frac{2}{3}$, and this is less than $\ln 2$, as required. Part (ii) can be proved in similar fashion, and (iii) is trivial. □

The next proposition is similar to but somewhat stronger than a result in [**51**] (see [**51**, 4.1, 4.3]). It is a useful general result which bounds the number of involutions and elements of order 3 in a finite group of Lie type.

For the statement we require a definition: For a finite group D and a positive integer r, denote by $i_r(D)$ the number of elements of order r in D.

Proposition 1.3. *Let* Y *be a simple algebraic group over* K*, and let* N *be the number of positive roots in the root system of* Y*. Suppose that* δ *is a Frobenius morphism of* Y *such that* $S = (Y_\delta)'$ *is a finite simple group of Lie type over* $\mathbf{F}_q$*. If* S *is not of type* ${}^2F_4$, ${}^2G_2$ *or* ${}^2B_2$*, define*

$$N_2 = \dim Y - N, \quad N_3 = \dim Y - \frac{2}{3}N,$$

and if S *is of type* ${}^2F_4$, ${}^2G_2$ *or* ${}^2B_2$ *define*

$$N_2 = \frac{1}{2}(\dim Y - N), \quad N_3 = \frac{1}{2}\left(\dim Y - \frac{2}{3}N\right).$$

(i) *We have*

$$i_2(\text{Aut } S) < 2(q^{N_2} + q^{N_2-1}).$$

(ii) *We have*

$$i_3(\text{Aut } S) < 2(q^{N_3} + q^{N_3-1}).$$

(iii) *The number of unipotent elements in* Y_δ *is equal to* q^{2N}*, unless* S *is of type* ${}^2F_4$, ${}^2G_2$ *or* ${}^2B_2$*, in which case it is* q^N.

Proof. (i) This is essentially careful book-keeping, using well-known information about the conjugacy classes and centralizers of involutions which can be found in [**2**, **29**].

When S is an exceptional group of Lie type, there are few classes of involutions in Aut S. Representatives and centralizers of involutions in Y_δ are given in [**29**, 4.5.1] for q odd and in [**2**] for q even; and the classes of outer involutions are given by 1.1. Using this information it is straightforward to calculate the precise value of $i_2(\text{Aut } S)$, and to verify the conclusion of (i).

Now consider the case where S is a classical group.

Consider first $S = L_n(q)$ with q odd. In $Y_\delta = PGL_n(q)$, the conjugacy classes of involutions are represented by the images (modulo scalars) of matrices

$$\begin{pmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{pmatrix}$$

for $1 \leq m \leq n/2$, and, for n even,

$$\begin{pmatrix} 0 & I_{n/2} \\ \alpha I_{n/2} & 0 \end{pmatrix},$$

where α is a fixed non-square in $\mathbf{F}_q$. The centralizers of these involutions in $PGL_n(q)$ are the images of the subgroups $GL_m(q) \times GL_{n-m}(q)$ for $m < n/2$ and, for n even, the images of $GL_{n/2}(q)^2.2$ and $GL_{n/2}(q^2).2$. Hence for each class of involutions t^{Y_δ}, we have, cancelling terms in

$$|GL_n(q)|/(|GL_m(q)|.|GL_{n-m}(q)|)$$

and using Lemma 1.2(i),

$$|t^{Y_\delta}| < 2q^{\dim t^Y}.\frac{q}{q-1}.$$

(The $q/q-1$ term arises because 1.2(i) applies only when all exponents a_i, b_i are at least 2.) For n even, the dimensions of the involution classes t^Y are $\frac{1}{2}n^2$ (two classes), $\frac{1}{2}n^2-2$, $\frac{1}{2}n^2-8, \ldots, 2n-2$; and for n odd, the dimensions are $\frac{1}{2}(n^2-1), \frac{1}{2}(n^2-9), \ldots, 2n-2$. Hence for n even we have

$$i_2(Y_\delta) < \frac{2q}{q-1}.\left(q^{\frac{1}{2}n^2} + q^{\frac{1}{2}n^2-2} + \cdots + q^{2n-2}\right) \tag{1}$$

while for n odd,

$$i_2(Y_\delta) < \frac{2q}{q-1}.\left(q^{\frac{1}{2}(n^2-1)} + q^{\frac{1}{2}(n^2-9)} + \cdots + q^{2n-2}\right). \tag{2}$$

Now consider involutions in $(\mathrm{Aut}\, S)\backslash Y_\delta$. These are given by 1.1. The involutions t with centralizer of type $O_n^\epsilon(q)$ have $\dim t^Y = \dim Y - \dim(SO_n) = \dim Y - N = N_2$, so that by 1.2(i) the contribution of these involutions to $i_2(\mathrm{Aut}\, S)$ is less than $2q^{N_2}$. The other outer involutions in 1.1 contribute less than $2q^{N_2-n} + 4q^{\frac{1}{2}(n^2-1)}$ (where the first term accounts for the remaining class of graph automorphisms, and is present only if n is even, and the second term accounts for the field and graph-field automorphisms, and is present only if q is square).

Putting all this together, we see that

$$i_2(\mathrm{Aut}\, S) < 2q^{N_2} + (2aq^{N_2-n} + 4bq^{\frac{1}{2}(n^2-1)}) + i_2(Y_\delta), \tag{3}$$

where $a = 1$ if n is even and $a = 0$ otherwise, and $b = 1$ if q is square and $b = 0$ otherwise. Using (1) and (2) it is readily checked that this implies the required inequality $i_2(\mathrm{Aut}\, S) < 2(q^{N_2} + q^{N_2-1})$, provided $n \geq 5$. And for

$n \le 4$ we improve the estimate of Lemma 1.2(i) by calculating the precise value of $i_2(\mathrm{Aut}\, S)$ using the information given above, and the result follows.

The proof for $S = L_n(q)$ with q even follows along the same lines. Here the classes of involutions in Y_δ are represented by matrices j_m $(1 \le m \le n/2)$ having m Jordan blocks of size 2 and the rest of size 1, and by [**2**], the centralizer in $GL_n(q)$ of j_m has order $q^{m(2n-3m)}.|GL_m(q)|.|GL_{n-2m}(q)|$, and j_m^Y has dimension $2m(n-m)$. Hence we obtain inequalities analogous to (1) and (2), and (3) also holds by 1.1. This gives the result for $n \ge 5$, apart from the cases where $S = L_n(2)$ with $n = 5$ or 6. For these groups and for $n \le 4$, we again calculate the precise values of $i_2(\mathrm{Aut}\, S)$ to obtain the result. This completes the proof of (i) for $S = L_n(q)$.

The proof for $S = U_n(q)$ is very similar. For q odd, the classes of involutions in $Y_\delta = PGU_n(q)$ have centralizers the image modulo scalars of $GU_m(q) \times GU_{n-m}(q)$ (if $m = n/2$, $GU_m(q)^2.2$) or $GL_{n/2}(q^2).2$ (one class for each centralizer). For q even involutions in Y_δ are represented by matrices j_m as above, with centralizer of order $q^{m(2n-3m)}.|GU_m(q)|.|GU_{n-2m}(q)|$. And all further outer involutions are given by 1.1. Similar calculations to those above, this time using Parts (ii) and (iii) of Lemma 1.2 as well as Part (i), yield the conclusion.

Now consider $S = PSp_{2m}(q)$ or $P\Omega_n^\epsilon(q)$. For q odd, involutions in $PGSp_{2m}(q)$ have centralizer the image modulo scalars of $Sp_{2l}(q) \times Sp_{2m-2l}(q)$, $Sp_m(q)^2.2$, $Sp_m(q^2).2$, $GL_m(q).2$ or $GU_m(q).2$; and in $PGO_n^\epsilon(q)$, involutions have centralizer the image of $GO_k(q) \times GO_{n-k}(q)$, $GO_{n/2}(q)^2.2$, $GO_{n/2}(q^2).2$, $GL_{n/2}(q).2$ or $GU_{n/2}(q).2$. All further outer involutions are field or graph-field automorphisms, given by 1.1. The result follows, calculating as in the $L_n(q)$ case.

Finally, for q even, by [**2**, Sections 7,8], involutions in $Sp_{2m}(q)$ or $O_{2m}^\epsilon(q)$ are represented by certain elements $a_{m-k}, b_{m-l}, c_{m-k}$ for $0 \le k \le m$, $m-k$ even, and $0 \le l \le m$, $m-l$ odd. For $Y = Sp_{2m}$, we have

$$\dim a_{m-k}^Y = m^2 - k^2,\ \dim b_{m-k}^Y = \dim c_{m-k}^Y = m^2 + m - k^2 - k,$$

and for $Y = SO_{2m}$,

$$\dim a_{m-k}^Y = m^2 - m - k^2 + k,\ \dim b_{m-k}^Y = \dim c_{m-k}^Y = m^2 - k^2.$$

Again, further involutions are field and graph-field automorphisms. Also, the number of simple factors in an involution centralizer is at most 2. The result follows in the usual way.

(ii) This is fairly similar to the proof of (i), and we just give a sketch. First consider S an exceptional group. For $p \ne 3$ the classes and centralizers of elements of order 3 in Y_δ are given in [**29**, 4.7.3], and by 1.1, further outer elements of order 3 are field automorphisms (and graph automorphisms of ${}^3D_4(q)$). Thus we can calculate $i_3(\mathrm{Aut}\, S)$, and the result follows. For $p = 3$, the classes of (unipotent) elements of order 3 are given by the classification of unipotent classes in Y_δ to be found in [**19**, **59**, **60**, **65**, **67**]. A convenient summary can be found in [**38**], from which we read off the labellings of the elements of order 3 in Y; centralizers in Y_δ are then read off from the appropriate references. (The largest classes have corresponding centralizer orders $q^{70}|B_2(q)|$, $q^{37}|A_1(q)|^2$, $q^{21}|A_1(q)|$, $q^{13}|A_1(q)|$, q^4, according as $Y = E_8, E_7, E_6, F_4, G_2$; and the largest classes in ${}^2G_2(q)$, ${}^3D_4(q)$ have centralizer orders q^2, $q^7|A_1(q)|$ respectively.) Thus $i_3(\mathrm{Aut}\, S)$ can be calculated, giving the result.

Now suppose S is a classical group. First consider $S = L_n^\epsilon(q)$. If $q \equiv \epsilon \bmod 3$, then the classes of elements of order 3 in $Y_\delta = PGL_n^\epsilon(q)$ are represented by the images of the matrices $d_{rs} = \mathrm{diag}(\omega I_r, \omega^{-1} I_s, I_{n-r-s})$ ($r \le s \le n-r-s$, ω a cube root of 1), and, when $3|n$, the matrix

$$e = \begin{pmatrix} 0 & I_{n/3} & 0 \\ 0 & 0 & I_{n/3} \\ \alpha I_{n/3} & 0 & 0 \end{pmatrix}$$

(α a fixed non-cube). The elements d_{rs} have centralizers the images of $GL_r^\epsilon(q) \times GL_s^\epsilon(q) \times GL_{n-r-s}^\epsilon(q)$ (if $r = s = n/3$ the centralizer is the image of $GL_{n/3}^\epsilon(q)^3.3$), and e has centralizer the image of $GL_{n/3}^\epsilon(q^3).3$. By 1.1, further automorphisms of order 3 are field automorphisms. The result now follows from calculations as in Part (i).

If $q \equiv -\epsilon \bmod 3$ and $\epsilon = 1$, the classes of elements of order 3 in Y_δ are represented by the images of e and of matrices $f_r = \mathrm{diag}(A, \ldots, A, I_{n-2r})$, where $A \in SL_2(q)$ has order 3 and there are r diagonal blocks A; the centralizer of f_r is the image of $GL_r(q^2) \times GL_{n-2r}(q)$. And when $q \equiv -\epsilon \bmod 3$ and $\epsilon = -1$, representatives are e and d_{rr}, and the centralizer of d_{rr} is the image of $GL_r(q^2) \times GU_{n-2r}(q)$. The result follows in the usual way.

To complete the case where $S = L_n^\epsilon(q)$, suppose $3|q$. The classes of elements of order 3 in Y_δ are represented by matrices $t_{rs} = \mathrm{diag}(J_3, \ldots, J_3, J_2, \ldots, J_2, I_{n-3r-2s})$ where J_i is a unipotent $i \times i$ Jordan block and there are r blocks J_3 and s blocks J_2. By [**72**, p. 34], writing $t = n - 3r - 2s$ we have

$$|C_{GL_n^\epsilon(q)}(t_{rs})| = q^{2ts+2tr+4sr+s^2+2r^2} |GL_r^\epsilon(q)||GL_s^\epsilon(q)||GL_t^\epsilon(q)|,$$

and $\dim t_{rs}^Y = 4rn + 2sn - 6r^2 - 2s^2 - 6rs$. One checks that the maximum possible value of this is $[\frac{2}{3}n^2] \le N_3$. Now the result follows in the usual way.

Now suppose $S = PSp_{2m}(q)$ or $P\Omega_n^\epsilon(q)$. For $(3, q) = 1$, the classes of elements of order 3 are represented by the elements d_{rr} if $q \equiv 1 \bmod 3$ (centralizer $GL_r(q) \times Sp_{2m-2r}(q)$ or $GL_r(q) \times O_{2m-2r}^\epsilon(q)$), and by the elements f_r if $q \equiv -1 \bmod 3$ (centralizer $GU_r(q) \times Sp_{2m-2r}(q)$ or $GU_r(q) \times O_{2m-2r}^\epsilon(q)$). The result follows in this case.

Finally, if $3|q$, by [**72**, p. 34] the classes of elements of order 3 are represented by the elements t_{rs} (with r even for S symplectic, s even for S

orthogonal). The centralizer orders are given in [72]. For $S = PSp_{2m}(q)$ we have

$$\dim t^Y_{rs} = m(4r+s) - 3r^2 - s^2 - 3rs + r + s,$$

the maximum value of which is $[\frac{2}{3}(2m^2+m)] \le N_3$, and the result follows in the usual way. And if $S = P\Omega^\epsilon_n(q)$ then

$$\dim t^Y_{rs} = n(2r+s) - 3r^2 - s^2 - 3rs - r - s,$$

the maximum value of which is again $[\frac{2}{3}\dim Y] \le N_3$, and the result again follows.

(iii) This is a well-known result of Steinberg (see [7, 6.6.1]). □

Next we prove a small generalisation of a result in [36, 5.2.11].

Proposition 1.4. *If Y, Z are simple algebraic groups over K, and Y has a Frobenius morphism δ such that Y'_δ is isomorphic to a subgroup of Z, then* $\mathrm{rank}(Y) \le \mathrm{rank}(Z)$.

Proof. Write $S = Y'_\delta$. The result is trivial if S is soluble or of type $\Omega^+_4(q)$ (see [36, 2.9.2] for a list of the possibilities). So we may assume that S is quasisimple. Say $S = S(q)$, a group of Lie type over $\mathbf{F}_q$.

Suppose first that $Z = Cl(V)$, a classical group with natural module V. Then $R_p(S) \le \dim V$, where $R_p(S)$ denotes the smallest dimension of a nontrivial faithful projective representation of S. The values of $R_p(S)$ are given in [36, 5.4.13], from which it follows that the only possibility with $\mathrm{rank}(Y) > \mathrm{rank}(Z)$ is $Y = A_l$, $Z = B_m, C_m$ or D_m, with $l > m$. However $V \downarrow Z$ is self-dual in this case, so either $\dim V \ge 2(l+1)$ or $\dim V$ is at least the dimension of a self-dual irreducible projective Y-module, which by [36, 5.4.11] is greater that $2(l+1)$. Hence $\dim V \ge 2(l+1)$ in any case, which forces $l < m$, completing the proof for Z classical.

Now suppose Z is of exceptional type. If $q > 2$ then the conclusion is immediate from [44, Theorem 2], so we may assume that $q = 2$. Moreover, [36, 5.2.11] implies that the BN-rank of S is at most $\mathrm{rank}(Z)$. Therefore to complete the proof it remains to exclude the following possible inclusions:

(1) $U_{10}(2), \Omega^-_{18}(2) < E_8$
(2) $U_9(2), SU_9(2), \Omega^-_{16}(2) < E_7$
(3) $U_8(2), \Omega^-_{14}(2) < E_6$
(4) $U_6(2), SU_6(2), \Omega^-_{10}(2)$, ${}^2E_6(2) < F_4$
(5) $U_4(2) < G_2$.

The abelian 3-rank of S is at most that of Z, which by [12] is equal to $\mathrm{rank}(Z)$. This rules out Cases (1), (3) and (5), and also $SU_9(2) < E_7$ and $SU_6(2), \Omega^-_{10}(2) < F_4$. For the rest of Case (2), assuming $\Omega^-_{16}(2) < E_7$, observe that $\Omega^-_{16}(2)$ has a Levi factor $\Omega^-_{14}(2)$ which must lie in a Levi factor of E_7, hence in E_6, which is impossible (we have already excluded Case (3)).

And assuming $U_9(2) < E_7$, note that the group $U_9(2)$ contains a subgroup $3^7.A_9$ (corresponding to $GU_1(2) \wr A_9$), of which the normal 3^7 must lie in a maximal torus of E_7 by [**12**]; this means that A_9 must be a section of the Weyl group $W(E_7)$, which is not the case. To complete the proof we must rule out $U_6(2)$, ${}^2E_6(2) < F_4$ in Case (4). Now $U_6(2)$ contains $3^4.S_6$, so if this were in F_4 the above argument would show that S_6 is a section of $W(F_4)$, which is not the case. Finally, ${}^2E_6(2)$ contains $U_6(2)$, so is not in F_4 either. $\square$

The next two results provide some general estimates for the orders of fixed point groups of Frobenius morphisms on reductive groups. The first is an elementary general fact, which is rather useful.

Proposition 1.5. *Let X be a connected reductive algebraic group. Let σ be a Frobenius endomorphism of X, and let K be a finite, σ-stable subgroup of $Z(X)$. Take σ to act on $Y = X/K$ by $xK \to x^\sigma K$. Then*

$$|X_\sigma| = |Y_\sigma|.$$

Proof. We count the elements of the set $S = \{x \in X : x^\sigma x^{-1} \in K\}$ in two different ways. On the one hand, for each $k \in K$, by Lang's theorem there exists $x \in X$ such that $x^\sigma x^{-1} = k$. Hence there are precisely $|X_\sigma|$ such elements x, so it follows that $|S| = |K||X_\sigma|$. On the other hand, $x^\sigma x^{-1} \in K$ implies that $(xK)^\sigma = xK$, so the number of elements x such that $x^\sigma x^{-1} \in K$ is equal to $|Y_\sigma||K|$. This is also equal to $|S|$, and the conclusion follows. $\square$

Remark. Another way of expressing Proposition 1.5 is simply to say that the order of the fixed point group of a Frobenius morphism on a connected reductive group is independent of the isogeny type of the group.

Proposition 1.6. *Let G be a simple algebraic group in characteristic $p > 0$, and let σ be a Frobenius morphism of G with fixed point group $G_\sigma = G(q)$, of Lie type over $\mathbf{F}_q$; suppose further that $G_\sigma \neq {}^2F_4(q), {}^2G_2(q), {}^2B_2(q)$. Let M be a connected reductive subgroup of G, and set*

$$l = \mathrm{rank}\,(M), \quad z = \mathrm{rank}\,(Z(M)^0).$$

Then

$$(q-1)^l q^{\dim M - l} \leq |M_\sigma| \leq (q+1)^z q^{\dim M - z}.$$

Proof. Write $M = ZE$, where $Z = Z(M)^0, E = M'$. By 1.5, $|M_\sigma| = |Z_\sigma||E_\sigma|$. By [**61**, 2.4(iii)] and its proof, we have

$$(q-1)^z \leq |Z_\sigma| \leq (q+1)^z.$$

Moreover, $|E_\sigma|$ is a monic polynomial in q, and inspection of the orders of quasisimple groups shows that if $d = \mathrm{rank}(E) = l - z$, then

$$(q-1)^d q^{\dim E - d} \leq |E_\sigma| < q^{\dim E}.$$

The conclusion follows. □

Lemma 1.7. *Let G, σ be as in the statement of 1.6, and let U be a connected unipotent σ-stable subgroup of G. Then $|U_\sigma| = q^{\dim U}$.*

Proof. By [4, 15.4], U has a σ-stable filtration by normal subgroups $U = U_0 > U_1 > \dots U_k = 1$, where $\dim U_i/U_{i-1} = 1$ for all i. And by the proof of [48, 1.13], $|(U_i)_\sigma/(U_{i-1})_\sigma| = q$. The result follows. □

Corollary 1.8. *Let G, σ be as in 1.6, and let $x \in G_\sigma$. Write $E = C_G(x)$, and let $a = \dim Z(E^0/R_u(E^0))$. Then*

$$|x^{G_\sigma}| \geq \frac{1}{2}\frac{q^a}{(q+1)^a|E:E^0|}q^{\dim x^G}.$$

Proof. Let $U = R_u(E^0)$ and $F = E^0/U$. Then $|U_\sigma| = q^{\dim U}$ by 1.7, and by 1.6,

$$|F_\sigma| \leq \frac{(q+1)^a}{q^a}q^{\dim F}.$$

From the order formulae for simple groups, and using Lemma 1.2, we have

$$|G_\sigma| > \frac{1}{2}q^{\dim G}.$$

Hence

$$|x^{G_\sigma}| \geq \frac{1}{2}\frac{q^a}{(q+1)^a|E:E^0|}q^{\dim G - \dim U - \dim F}.$$

The result follows. □

2. Proof of Theorem 2(I)(a): Unipotent elements in parabolics.

Let G be a simple algebraic group of exceptional type over the algebraically closed field K of characteristic $p > 0$, and let σ be a Frobenius morphism of G. In this section we prove Theorem 2(I)(a) — the case of unipotent elements in parabolic subgroups. For this case we may assume that $X = G_\sigma$, and moreover that G is simply connected (since $Z(G_\sigma)$ has order coprime to p).

We postpone until the end of the section the cases where G_σ is of type ${}^2F_4$ or ${}^2G_2$. (We also cover ${}^3D_4(q)$ and ${}^2B_2(q)$ at the end of the section.) Excluding these cases, we have $\sigma = q\sigma_0$ where σ_0 is either 1 or a graph automorphism of finite order, and $q = p^a$. Let P be a σ-stable parabolic subgroup of G, so that P_σ is a parabolic subgroup of G_σ. In this section we shall consider fixed point ratios $\mathrm{fpr}(u, G_\sigma/P_\sigma)$ for $u \in G_\sigma$ unipotent. Since the value of the permutation character ${1_{P_\sigma}}^{G_\sigma}$ at an element of G_σ is simply the number of fixed points of the element in the action on G_σ/P_σ, we have

$$\mathrm{fpr}(u, G_\sigma/P_\sigma) = \frac{{1_{P_\sigma}}^{G_\sigma}(u)}{{1_{P_\sigma}}^{G_\sigma}(1)}.$$

Thus we may use character theory to calculate fixed point ratios; we begin by considering the value of ${1_{P_\sigma}}^{G_\sigma}$ at long root elements.

Let T_0 be a fixed maximally split maximal torus of G, and let B be a σ-stable Borel subgroup of G containing T_0; we assume that P contains B. Let $W = N_G(T_0)/T_0$ be the Weyl group of G. Let Φ be the set of roots with respect to T_0, and Φ^+ be the set of positive roots determined by B; let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the corresponding simple system, so that σ_0 permutes the roots in Π. Let α_0 be the highest root of Φ with respect to Π; write $|\alpha|$ for the length of the root α. We shall find it convenient to define the *long height* $lht(\alpha)$ of a long root $\alpha \in \Phi$ by

$$lht\left(\sum n_i\alpha_i\right) = \sum n_i \frac{|\alpha_i|^2}{|\alpha_0|^2}$$

(note that $\sum n_i\alpha_i$ is long if and only if $n_i\frac{|\alpha_i|^2}{|\alpha_0|^2} \in \mathbb{Z}$ for each i); thus if all roots of Φ are long, then the long height coincides with the usual height of a root. We also define the *long root polynomial* L_{Φ,σ_0} of the pair (Φ, σ_0) by

$$L_{\Phi,\sigma_0}(t) = \sum_{\alpha\in\Phi^+ \text{ long},\ \sigma_0(\alpha)=\alpha} t^{lht(\alpha)}.$$

If we abbreviate the polynomial $a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t$ as $n : a_n, a_{n-1}, \ldots, a_1$, then the polynomials L_{Φ,σ_0} for simple systems Φ are given in the following table, in which σ_0 is specified (up to conjugacy) by its order.

Φ	$o(\sigma_0)$	$L_{\Phi,\sigma_0}(t)$
A_n	1	$n : 1, 2, \ldots, n$
A_n	2	$n : 1, 0, 1, 0, \ldots, n - 2\lfloor\frac{n}{2}\rfloor$
B_n	1	$2n - 2 : 1, 1, 2, 2, \ldots, n - 1, n - 1$
C_n	1	$n : 1, 1, \ldots, 1$
D_n	1	$2n - 3 : 1, 1, 2, 2, \ldots, \lfloor\frac{n}{2}\rfloor - 1, \lfloor\frac{n}{2}\rfloor - 1, 3\lfloor\frac{n}{2}\rfloor - n + 1, \lfloor\frac{n}{2}\rfloor + 1, \lfloor\frac{n}{2}\rfloor + 2, \lfloor\frac{n}{2}\rfloor + 2, \ldots, n - 1, n - 1, n$
D_n	2	$2n - 3 : 1, 1, 2, 2, \ldots, \lfloor\frac{n}{2}\rfloor - 1, \lfloor\frac{n}{2}\rfloor - 1, \lfloor\frac{n-1}{2}\rfloor, \lfloor\frac{n}{2}\rfloor - 1, \lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor, \ldots, n - 3, n - 3, n - 2$
D_4	3	$5 : 1, 1, 0, 0, 1$
E_6	1	$11 : 1, 1, 1, 2, 3, 3, 4, 5, 5, 5, 6$
E_6	2	$11 : 1, 1, 1, 0, 1, 1, 2, 1, 1, 1, 2$
E_7	1	$17 : 1, 1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, 6, 7$
E_8	1	$29 : 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8$
F_4	1	$8 : 1, 1, 1, 1, 2, 2, 2, 2$
G_2	1	$3 : 1, 1, 1$

Let P_0 be the normalizer in G of the long root subgroup U_{α_0}; let W_{P_0} be the Weyl group of P_0, so that W_{P_0} is a parabolic subgroup of W. Write

$P = LU_P$, where L is a Levi subgroup of P and U_P its unipotent radical; let $\Phi(L)$ be the root system of L. The following result gives the value of ${1_{P_\sigma}}^{G_\sigma}$ at long root elements if G_σ is not a Suzuki or Ree group.

Proposition 2.1. *With the notation established, if σ_0 preserves root lengths then*

$$ {1_{P_\sigma}}^{G_\sigma}(u_\alpha) = \frac{|(P_0)_\sigma|}{|P_\sigma|}\left(q^{lht(\alpha_0)}L_{\Phi,\sigma_0}(q^{-1}) + q^{lht(\alpha_0)-1}L_{\Phi(L),\sigma_0}(q)\right). $$

Proof. Given $w \in W$, write ${U_w}^-$ for the product of the root subgroups U_α as α runs over the positive roots made negative by w. For each $w \in W$, choose $\dot{w} \in N_G(T_0)$ with $\dot{w}T_0 = w$. Let D_{J_0} be the set of distinguished coset representatives of W_{P_0} in W; thus any $w \in W$ may be written as $w = w_1w_2$ with $w_1 \in W_{P_0}$, $w_2 \in D_{J_0}$ and $\ell(w) = \ell(w_1) + \ell(w_2)$. By the Bruhat decomposition, it follows that any element of G may be written in the form $uh\dot{w}_1v_1\dot{w}_2v_2$, where $u \in U$, $h \in T_0$, $w_1 \in W_{P_0}$, $v_1 \in {U_{w_1}}^-$, $w_2 \in D_{J_0}$ and $v_2 \in {U_{w_2}}^-$. Now $uh\dot{w}_1v_1 \in P_0 = N_G(U_{\alpha_0})$; thus

$$ x_{\alpha_0}(1)^{uh\dot{w}_1v_1\dot{w}_2v_2} = x_{{w_2}^{-1}(\alpha_0)}(t)^{v_2} \qquad \text{for some } t \in K. $$

It follows that if $g = uh\dot{w}_1v_1\dot{w}_2v_2$ then

$$ x_{\alpha_0}(1)^g \in P \iff x_{{w_2}^{-1}(\alpha_0)}(t) \in P^{{v_2}^{-1}} = P \iff {w_2}^{-1}(\alpha_0) \in \Phi(P), $$

where we set $\Phi(P) = \Phi(L) \cup \Phi^+$. It follows that if we set

$$ D_P = \{w \in D_{J_0} : w^{-1}(\alpha_0) \in \Phi(P)\}, $$

then the number of elements $g \in G_\sigma$ with $x_{\alpha_0}(1)^g \in P$ is

$$ |(P_0)_\sigma| \sum_{w \in D_P} |({U_w}^-)_\sigma|. $$

Since the value of ${1_{P_\sigma}}^{G_\sigma}$ at an element x of G_σ is $|P_\sigma|^{-1}$ times the number of elements $g \in G_\sigma$ with $x^g \in P_\sigma$, we see that

$$ {1_{P_\sigma}}^{G_\sigma}(u_\alpha) = \frac{|(P_0)_\sigma|}{|P_\sigma|} \sum_{w \in D_P,\ \sigma_0(w^{-1}(\alpha_0)) = w^{-1}(\alpha_0)} |({U_w}^-)_\sigma|. $$

Now as w ranges over D_{J_0}, $w^{-1}(\alpha_0)$ runs through the long roots of Φ; thus for each long root $\alpha \in \Phi$ there is a unique $w^{(\alpha)} \in D_{J_0}$ such that $(w^{(\alpha)})^{-1}(\alpha_0) = \alpha$; so we may write

$$ {1_{P_\sigma}}^{G_\sigma}(u_\alpha) = \frac{|(P_0)_\sigma|}{|P_\sigma|} \sum_{\alpha \in \Phi(P) \text{ long},\ \sigma_0(\alpha) = \alpha} |({U_{w^{(\alpha)}}}^-)_\sigma|. $$

If α is σ_0-stable, we may write a reduced expression for $w^{(\alpha)}$ in the form $w_{J_1}w_{J_2}\dots w_{J_t}$, where each w_{J_i} is a product of simple reflections corresponding to the roots in a single σ_0-orbit; by [7, 14.1.2(ii)] it follows that

$|(U_{w^{(\alpha)}}{}^{-})_\sigma| = q^{\ell(w^{(\alpha)})}$. Thus

$$1_{P_\sigma}{}^{G_\sigma}(u_\alpha) = \frac{|(P_0)_\sigma|}{|P_\sigma|} \sum_{\alpha \in \Phi(P) \text{ long},\ \sigma_0(\alpha)=\alpha} q^{\ell(w^{(\alpha)})}.$$

Finally, we consider separately the contributions to this sum from positive and negative roots. If α is a positive long root, then there exists a chain of roots from α to α_0 in which each root is higher than its predecessor, and is obtained from it by a simple reflection; the corresponding product of simple reflections is $w^{(\alpha)}$, and so $\ell(w^{(\alpha)}) = lht(\alpha_0) - lht(\alpha)$. Thus the contribution to the above sum from positive long roots α is $q^{lht(\alpha_0)} L_{\Phi,\sigma_0}(q^{-1})$. On the other hand, if α is a negative long root, there exists a chain of roots as above from α to $-\beta$ for some simple long root β; if w is the corresponding product of simple reflections, then $w^{(\alpha)} = w^{(\beta)} w_\beta w$, and so $\ell(w^{(\alpha)}) = (lht(\alpha_0) - 1) + 1 + (lht(-\alpha) - 1) = lht(\alpha_0) - 1 + lht(-\alpha)$. Thus the contribution to the above sum from negative long roots α is $q^{lht(\alpha_0)-1} L_{\Phi(L),\sigma_0}(q)$. The result follows. $\square$

Corollary 2.2. *With the notation established, if either σ is untwisted or all roots of Φ have the same length then*

$$\mathrm{fpr}(u_\alpha, G_\sigma/P_\sigma) = \frac{|(P_0)_\sigma|}{|G_\sigma|} \left(q^{lht(\alpha_0)} L_{\Phi,\sigma_0}(q^{-1}) + q^{lht(\alpha_0)-1} L_{\Phi(L),\sigma_0}(q) \right).$$

The remaining cases of Suzuki and Ree groups are easily dealt with, since tables giving unipotent characters are available; we mention these at the end of this section.

We now turn to considering other values of the permutation character $1_{P_\sigma}{}^{G_\sigma}$; we begin by briefly reviewing the theory by which the values of $1_{P_\sigma}{}^{G_\sigma}$ may be obtained.

Since P_σ is a parabolic subgroup of G_σ, all the constituents of $1_{P_\sigma}{}^{G_\sigma}$ are unipotent characters lying in the principal series. To obtain such irreducible characters of G_σ, we begin with generalized Deligne-Lusztig characters $R_{T,\theta}$ (where T is a σ-stable maximal torus of G and θ is a linear character of T_σ) in which θ is the principal character 1 of T_σ. If T_0 is a fixed maximally split torus of G and $T = {}^g T_0$, then by [7, 3.3.1] $g^{-1}\sigma(g) \in N_G(T_0)$, and so $g^{-1}\sigma(g)$ corresponds to an element w of $W = N_G(T_0)/T_0$; by [7, 3.3.2] the element w is defined up to σ-conjugacy, where $w, w' \in W$ are σ-conjugate if $w' = x^{-1} w \sigma(x)$ for some $x \in W$. In this case we say that T is obtained from T_0 by twisting with w, and may write $T_w = T$; for convenience we write $R_w = R_{T_w,1}$. A two-stage process is then applied to obtain the irreducible unipotent characters (see [26, Section 10]). Firstly, class functions are formed by taking linear combinations of the R_w with coefficients given by values of irreducible characters of the Weyl group; the class functions formed are of the type called almost characters. Secondly, the irreducible unipotent characters are formed by taking linear combinations of almost characters, with coefficients given by entries of nonabelian Fourier transform matrices. However, not all of the almost characters required for the second stage of this process need be obtained from the first stage; this is because the span of the generalized Deligne-Lusztig characters need not contain all class functions of G_σ. Class functions which do lie in this span are called uniform; our next result in this section shows that the permutation character ${1_{P_\sigma}}^{G_\sigma}$ is in fact a uniform function. Let W_P be the Weyl group of P, so that W_P is a standard parabolic subgroup of W.

Lemma 2.3. *With the notation established,*

$$ {1_{P_\sigma}}^{G_\sigma} = \frac{1}{|W_P|} \sum_{w \in W_P} R_w. $$

Proof. Let L be the standard Levi subgroup of P (so that L is also σ-stable). If T is any σ-stable maximal torus of L, and θ is a linear character of T_σ, then we write $R^L_{T,\theta}$ and $R^G_{T,\theta}$ for the generalized Deligne-Lusztig characters of L_σ and G_σ respectively associated with the pair (T, θ). If we denote by $(R^L_{T,\theta})_{P_\sigma}$ the generalized character of P_σ which agrees with $R^L_{T,\theta}$ on L_σ and contains the unipotent radical of P_σ in its kernel, and by ${(R^L_{T,\theta})_{P_\sigma}}^{G_\sigma}$ the result of inducing $(R^L_{T,\theta})_{P_\sigma}$ up to G_σ, then by [7, 7.4.4] we have

$$ {(R^L_{T,\theta})_{P_\sigma}}^{G_\sigma} = R^G_{T,\theta}. $$

Now the Weyl group of L is W_P, and we have

$$ 1_{L_\sigma} = \frac{1}{|W_P|} \sum_{w \in W_P} R^L_{T_w,1}; $$

thus

$$ 1_{P_\sigma} = \frac{1}{|W_P|} \sum_{w \in W_P} (R^L_{T_w,1})_{P_\sigma}. $$

Inducing up to G_σ, we obtain

$$ {1_{P_\sigma}}^{G_\sigma} = \frac{1}{|W_P|} \sum_{w \in W_P} {(R^L_{T_w,1})_{P_\sigma}}^{G_\sigma} = \frac{1}{|W_P|} \sum_{w \in W_P} R^G_{T_w,1} = \frac{1}{|W_P|} \sum_{w \in W_P} R_w $$

as required. □

Now assume that σ is untwisted, so that $\sigma_0 = 1$ and G_σ is an untwisted group $G(q)$. In this case we proceed as follows: Let $\hat{W}$ be the set of irreducible characters of W, and for $\phi \in \hat{W}$ set

$$ R_\phi = \frac{1}{|W|} \sum_{w \in W} \phi(w) R_w. $$

We may invert these equations as in [7, p. 383] to express each R_w as a linear combination of almost characters R_ϕ:

$$R_w = \sum_{\phi \in \hat{W}} \phi(w) R_\phi.$$

Thus by Lemma 2.3 we have

$$1_{P_\sigma}{}^{G_\sigma} = \sum_{\phi \in \hat{W}} \left(\frac{1}{|W_P|} \sum_{w \in W_P} \phi(w) \right) R_\phi.$$

Now if we write

$$1_{W_P}{}^{W} = \sum_{\phi \in \hat{W}} n_\phi \phi,$$

then by [15] we know that

$$1_{P_\sigma}{}^{G_\sigma} = \sum_{\phi \in \hat{W}} n_\phi \chi_\phi,$$

where χ_ϕ is the unipotent character of G_σ corresponding to ϕ.

Lemma 2.4. *With the notation established, if σ is untwisted then*

$$1_{P_\sigma}{}^{G_\sigma} = \sum_{\phi \in \hat{W}} n_\phi R_\phi.$$

Proof. By Frobenius reciprocity we have

$$n_\phi = (1_{W_P}{}^{W}, \phi) = (1_{W_P}, \phi|_{W_P})_{W_P} = \frac{1}{|W_P|} \sum_{w \in W_P} \phi(w);$$

thus

$$1_{P_\sigma}{}^{G_\sigma} = \sum_{\phi \in \hat{W}} n_\phi R_\phi$$

as required. □

This result has a corollary concerning Lusztig's nonabelian Fourier transform matrices, which generalizes a result in [37]; to state this we require a little notation. We recall that the unipotent characters of G_σ occur in families, say $\mathcal{F}_1, \ldots, \mathcal{F}_r$, and that each such family $\mathcal{F}_j$ has associated with it a square matrix M_j. The matrix M_j gives, up to a sign, the inner products between the irreducible unipotent characters in $\mathcal{F}_j$ and the almost characters in $\mathcal{F}_j$; the sign is identically 1 for all families except three, one in E_7 and two in E_8. Let $d = \sum_{j=1}^{r} |\mathcal{F}_j|$ be the number of unipotent characters of G_σ; let M be the $d \times d$ matrix with blocks $M_1, \ldots, M_r$ down the diagonal, and assume that the unipotent characters are numbered $\chi_1, \ldots, \chi_d$ in accordance with the order of the columns of M. For $j = 1, \ldots, d$ let R_j be the class function which is the linear combination of $\chi_1, \ldots, \chi_d$ given by the jth row of M. If $\chi_j = \chi_\phi$ we then have $R_j = R_{i(\phi)}$, where i is the involution on $\hat{W}$ defined in [7, p. 373] as interchanging the pairs of characters involved in the three exceptional families mentioned above, and fixing all other characters in $\hat{W}$; the involution i appears because the effect of the signs on the relevant matrices is to interchange pairs of rows.

Corollary 2.5. *With the notation established, if σ is untwisted and v is the row vector of length d with $v_j = ({1_{P_\sigma}}^{G_\sigma}, \chi_j)$, then $vM = v$.*

Proof. We have

$$v_j = \begin{cases} n_\phi & \text{if } \chi_j = \chi_\phi, \\ 0 & \text{if } \chi_j \text{ is not in the principal series;} \end{cases}$$

by 2.4 we have

$$v_j = ({1_{P_\sigma}}^{G_\sigma}, \chi_j) = \sum_{\phi \in \hat{W}} n_\phi (R_\phi, \chi_j).$$

By [3] we know that the characters of $W(E_7)$ and $W(E_8)$ which are interchanged by the involution i appear in ${1_{W_P}}^W$ with equal multiplicity; thus $n_\phi = n_{i(\phi)}$, and so

$$v_j = \sum_{\phi \in \hat{W}} n_{i(\phi)} (R_\phi, \chi_j) = \sum_{\phi \in \hat{W}} n_\phi (R_{i(\phi)}, \chi_j) = (vM)_j$$

as required. □

Now in this section we are concerned with values at unipotent elements; the restrictions to unipotent elements of almost characters R_ϕ are called Foulkes functions, while the corresponding restrictions of generalized Deligne-Lusztig characters $R_{T,\theta}$ are Green functions. The equations above expressing the R_ϕ as linear combinations of the R_w and vice versa show that the problems of computing all Foulkes functions of G_σ and all Green functions of G_σ are equivalent.

In [53] Lusztig described an algorithm for computing certain functions associated to character sheaves of the algebraic group G; it was later shown by Lusztig [54] and Shoji [66] that the functions computed by this algorithm were in fact the desired Green functions. However, the functions computed in this way are given as linear combinations of other functions, called characteristic functions of irreducible local systems on geometric unipotent classes, and the values of these are in general known only up to a complex scalar of absolute value 1. On unipotent classes containing elements with connected centralizers, or elements with centralizers C such that $|C/C^0| = 2$, it is possible to determine the scalars concerned; for other classes the situation is more complicated.

Using Lusztig's algorithm, Frank Lübeck has computed tables for all finite exceptional groups of Lie type. Each table is a two-dimensional array, with rows indexed by unipotent classes and columns by irreducible characters of the Weyl group, and with all entries being polynomials in q; separate tables are provided for good characteristic and for each bad characteristic. For unipotent classes where the geometric class contains at most two rational classes, the values are known to be those of the Foulkes functions; for the (relatively few) other classes the problem with scalars mentioned above means that in some cases it is not certain that the values given are actually those of the Foulkes functions. The authors are grateful to Lübeck for making these tables available, in CHEVIE-readable format, and for providing the explanation above of the status of Green function computation.

From Lübeck's tables we make two observations:

(i) $R_\phi(u_\alpha)$ is a polynomial in q with nonnegative coefficients;
(ii) if all roots have the same length, then $R_\phi(u_\alpha) \geq |R_\phi(u)|$ for any $u \neq 1$.

(The problem of the uncertainty over certain values in the tables does not create difficulties in (ii): In each of the small number of geometric unipotent classes u^G containing more than two rational classes, the values taken by any R_ϕ are dominated by $R_\phi(u_\alpha)$ to such an extent that it is easy to see that we must have $R_\phi(u_\alpha) \geq |R_\phi(u)|$ for any choice of scalars of absolute value 1. For example, if $G = E_6$ and $\phi = \phi_{6,1}$, then $R_\phi(u_\alpha) = q^8+q^7+q^5+q^4+q$; the only unipotent class of G containing more than two G_σ-classes in G_σ is the class $D_4(a_1)$, which contains three G_σ-classes, the values of R_ϕ on which are given as $2q^3+q$, q and $-q^3+q$. It follows from the way in which the Green functions are obtained from the characteristic functions of irreducible local systems on geometric unipotent classes that the correct values must be of the form $\zeta q + 2\zeta' q^3$, ζq and $\zeta q - \zeta' q^3$ for some ζ, ζ' of absolute value 1, and clearly each of these has absolute value less than $q^8+q^7+q^5+q^4+q$.) Since Lemma 2.4 shows that the permutation character ${1_{P_\sigma}}^{G_\sigma}$ is a nonnegative linear combination of the almost characters R_ϕ, it follows immediately that for $G = E_6$, E_7 or E_8 and for any $u \neq 1$ we have ${1_{P_\sigma}}^{G_\sigma}(u_\alpha) \geq {1_{P_\sigma}}^{G_\sigma}(u)$, and so $\mathrm{fpr}(u, G_\sigma/P_\sigma) \leq \mathrm{fpr}(u_\alpha, G_\sigma/P_\sigma)$ as required.

However, since we wish actually to obtain bounds for the fixed point ratios of root elements and other unipotent elements, we need to calculate values of ${1_{P_\sigma}}^{G_\sigma}$. To do this, Lemma 2.4 shows that in each case we simply need to form the linear combination of Foulkes functions with coefficients obtained from the decomposition of the corresponding permutation character ${1_{W_P}}^W$ in the Weyl group; we may of course treat F_4 and G_2 in this way as well as E_6, E_7 and E_8. The decompositions of the permutation characters ${1_{W_P}}^W$ are straightforward to obtain; for convenience we record them here, using the notation given in [7] for irreducible characters of W.

$${1_{W_{P_1}}}^{W(G_2)} = \phi_{1,0} + \phi_{2,1} + \phi_{2,2} + \phi_{1,3}''$$

$1_{W_{P_2}}{}^{W(G_2)} = \phi_{1,0} + \phi_{2,1} + \phi_{2,2} + \phi_{1,3}{}'$

$1_{W_{P_1}}{}^{W(F_4)} = \phi_{1,0} + \phi_{9,2} + \phi_{8,3}{}' + \phi_{4,1} + \phi_{2,4}{}'$

$1_{W_{P_2}}{}^{W(F_4)} = \phi_{1,0} + 2\phi_{9,2} + 2\phi_{8,3}{}' + \phi_{8,3}{}'' + \phi_{4,1} + \phi_{2,4}{}' + \phi_{12,4} + \phi_{9,6}{}' + \phi_{4,7}{}' + \phi_{6,6}{}' + \phi_{16,5}$

$1_{W_{P_3}}{}^{W(F_4)} = \phi_{1,0} + 2\phi_{9,2} + \phi_{8,3}{}' + 2\phi_{8,3}{}'' + \phi_{4,1} + \phi_{2,4}{}'' + \phi_{12,4} + \phi_{9,6}{}'' + \phi_{4,7}{}'' + \phi_{6,6}{}' + \phi_{16,5}$

$1_{W_{P_4}}{}^{W(F_4)} = \phi_{1,0} + \phi_{9,2} + \phi_{8,3}{}'' + \phi_{4,1} + \phi_{2,4}{}''$

$1_{W_{P_1}}{}^{W(E_6)} = 1_{W_{P_6}}{}^{W(E_6)} = \phi_{1,0} + \phi_{6,1} + \phi_{20,2}$

$1_{W_{P_2}}{}^{W(E_6)} = \phi_{1,0} + \phi_{6,1} + \phi_{20,2} + \phi_{30,3} + \phi_{15,4}$

$1_{W_{P_3}}{}^{W(E_6)} = 1_{W_{P_5}}{}^{W(E_6)} = \phi_{1,0} + \phi_{6,1} + 2\phi_{20,2} + \phi_{64,4} + \phi_{60,5} + \phi_{30,3} + \phi_{15,4}$

$1_{W_{P_4}}{}^{W(E_6)} = \phi_{1,0} + \phi_{6,1} + 3\phi_{20,2} + 2\phi_{64,4} + 3\phi_{60,5} + \phi_{81,6} + \phi_{24,6} + 2\phi_{30,3} + 2\phi_{15,4} + \phi_{80,7} + \phi_{60,8} + \phi_{10,9}$

$1_{W_{P_1}}{}^{W(E_7)} = \phi_{1,0} + \phi_{7,1} + \phi_{27,2} + \phi_{56,3} + \phi_{35,4}$

$1_{W_{P_2}}{}^{W(E_7)} = \phi_{1,0} + \phi_{7,1} + \phi_{27,2} + \phi_{21,3} + \phi_{189,5} + \phi_{105,6} + \phi_{56,3} + \phi_{35,4} + \phi_{120,4} + \phi_{15,7}$

$1_{W_{P_3}}{}^{W(E_7)} = \phi_{1,0} + \phi_{7,1} + 2\phi_{27,2} + \phi_{21,3} + 2\phi_{189,5} + \phi_{210,6} + \phi_{105,6} + \phi_{168,6} + 2\phi_{56,3} + 2\phi_{35,4} + \phi_{120,4} + \phi_{105,5} + \phi_{315,7} + \phi_{280,8} + \phi_{70,9}$

$1_{W_{P_4}}{}^{W(E_7)} = \phi_{1,0} + \phi_{7,1} + 3\phi_{27,2} + 2\phi_{21,3} + 5\phi_{189,5} + 2\phi_{210,6} + 3\phi_{105,6} + 4\phi_{168,6} + 2\phi_{189,7} + 2\phi_{378,9} + 2\phi_{210,10} + \phi_{210,13} + 3\phi_{56,3} + 3\phi_{35,4} + 3\phi_{120,4} + \phi_{15,7} + 2\phi_{105,5} + 2\phi_{405,8} + 2\phi_{216,9} + \phi_{420,10} + \phi_{84,12} + \phi_{512,11} + \phi_{512,12} + 3\phi_{315,7} + 3\phi_{280,8} + \phi_{280,9} + 2\phi_{70,9}$

$1_{W_{P_5}}{}^{W(E_7)} = \phi_{1,0} + \phi_{7,1} + 2\phi_{27,2} + 2\phi_{21,3} + 3\phi_{189,5} + \phi_{210,6} + 2\phi_{105,6} + 2\phi_{168,6} + \phi_{189,7} + \phi_{378,9} + \phi_{210,10} + 2\phi_{56,3} + 2\phi_{35,4} + 2\phi_{120,4} + \phi_{15,7} + \phi_{105,5} + \phi_{405,8} + \phi_{216,9} + \phi_{315,7} + \phi_{280,8} + \phi_{70,9}$

$1_{W_{P_6}}{}^{W(E_7)} = \phi_{1,0} + \phi_{7,1} + 2\phi_{27,2} + \phi_{21,3} + \phi_{189,5} + \phi_{168,6} + \phi_{56,3} + \phi_{35,4} + \phi_{120,4} + \phi_{105,5}$

$1_{W_{P_7}}{}^{W(E_7)} = \phi_{1,0} + \phi_{7,1} + \phi_{27,2} + \phi_{21,3}$

$1_{W_{P_1}}{}^{W(E_8)} = \phi_{1,0} + \phi_{8,1} + \phi_{35,2} + \phi_{560,5} + \phi_{112,3} + \phi_{84,4} + \phi_{210,4} + \phi_{50,8} + \phi_{700,6} + \phi_{400,7}$

$1_{W_{P_2}}{}^{W(E_8)} = \phi_{1,0} + \phi_{8,1} + \phi_{35,2} + 2\phi_{560,5} + \phi_{567,6} + \phi_{3240,9} + 2\phi_{112,3} + 2\phi_{84,4} + \phi_{210,4} + \phi_{50,8} + 2\phi_{700,6} + 2\phi_{400,7} + \phi_{2240,10} + \phi_{1400,11} + \phi_{1400,7} + \phi_{1344,8} + \phi_{448,9} + \phi_{1400,8} + \phi_{1050,10} + \phi_{175,12}$

$1_{W_{P_3}}{}^{W(E_8)} = \phi_{1,0} + \phi_{8,1} + 2\phi_{35,2} + 4\phi_{560,5} + 2\phi_{567,6} + 3\phi_{3240,9} + \phi_{4536,13} + \phi_{2835,14} + 3\phi_{112,3} + 3\phi_{84,4} + 2\phi_{210,4} + \phi_{50,8} + \phi_{160,7} + 4\phi_{700,6} + 3\phi_{400,7} + \phi_{300,8} + \phi_{2268,10} + \phi_{972,12} + 2\phi_{2240,10} + 2\phi_{1400,11} + \phi_{4096,11} + \phi_{4096,12} + \phi_{4200,12} + \phi_{3360,13} + 3\phi_{1400,7} + 3\phi_{1344,8} + \phi_{1008,9} + 2\phi_{448,9} + 2\phi_{1400,8} + 2\phi_{1050,10} + \phi_{1575,10} + \phi_{175,12}$

$1_{W_{P_4}}{}^{W(E_8)} = \phi_{1,0} + \phi_{8,1} + 3\phi_{35,2} + 9\phi_{560,5} + 5\phi_{567,6} + 13\phi_{3240,9} + \phi_{525,12} + 9\phi_{4536,13} + 5\phi_{2835,14} + 4\phi_{6075,14} + 3\phi_{4200,15} + \phi_{4200,21} + \phi_{2835,22} + 5\phi_{112,3} +$

$5\phi_{84,4}+4\phi_{210,4}+2\phi_{50,8}+2\phi_{160,7}+10\phi_{700,6}+6\phi_{400,7}+4\phi_{300,8}+6\phi_{2268,10}+5\phi_{972,12}+\phi_{1296,13}+9\phi_{2240,10}+7\phi_{1400,11}+2\phi_{840,13}+7\phi_{4096,11}+7\phi_{4096,12}+8\phi_{4200,12}+3\phi_{840,14}+5\phi_{3360,13}+2\phi_{2800,13}+\phi_{700,16}+\phi_{2100,16}+3\phi_{5600,15}+3\phi_{3200,16}+10\phi_{1400,7}+10\phi_{1344,8}+4\phi_{1008,9}+6\phi_{448,9}+6\phi_{1400,8}+6\phi_{1050,10}+4\phi_{1575,10}+2\phi_{175,12}+2\phi_{4480,16}+2\phi_{3150,18}+2\phi_{4200,18}+\phi_{4536,18}+\phi_{5670,18}+\phi_{420,20}+\phi_{2688,20}+3\phi_{7168,17}+\phi_{1344,19}+2\phi_{2016,19}+\phi_{5600,19}$

${1_{W_{P_5}}}^{W(E_8)}=\phi_{1,0}+\phi_{8,1}+2\phi_{35,2}+6\phi_{560,5}+3\phi_{567,6}+8\phi_{3240,9}+\phi_{525,12}+5\phi_{4536,13}+3\phi_{2835,14}+2\phi_{6075,14}+2\phi_{4200,15}+4\phi_{112,3}+4\phi_{84,4}+3\phi_{210,4}+2\phi_{50,8}+\phi_{160,7}+7\phi_{700,6}+5\phi_{400,7}+2\phi_{300,8}+3\phi_{2268,10}+3\phi_{972,12}+6\phi_{2240,10}+5\phi_{1400,11}+\phi_{840,13}+3\phi_{4096,11}+3\phi_{4096,12}+4\phi_{4200,12}+2\phi_{840,14}+2\phi_{3360,13}+\phi_{2800,13}+\phi_{700,16}+\phi_{5600,15}+\phi_{3200,16}+6\phi_{1400,7}+6\phi_{1344,8}+2\phi_{1008,9}+4\phi_{448,9}+4\phi_{1400,8}+4\phi_{1050,10}+2\phi_{1575,10}+2\phi_{175,12}+\phi_{4480,16}+\phi_{3150,18}+\phi_{4200,18}+\phi_{420,20}+\phi_{7168,17}+\phi_{1344,19}+\phi_{2016,19}$

${1_{W_{P_6}}}^{W(E_8)}=\phi_{1,0}+\phi_{8,1}+2\phi_{35,2}+4\phi_{560,5}+2\phi_{567,6}+3\phi_{3240,9}+\phi_{4536,13}+3\phi_{112,3}+3\phi_{84,4}+2\phi_{210,4}+\phi_{50,8}+\phi_{160,7}+4\phi_{700,6}+2\phi_{400,7}+2\phi_{300,8}+\phi_{2268,10}+\phi_{972,12}+2\phi_{2240,10}+\phi_{1400,11}+\phi_{840,13}+\phi_{4096,11}+\phi_{4096,12}+\phi_{4200,12}+\phi_{840,14}+3\phi_{1400,7}+3\phi_{1344,8}+\phi_{1008,9}+2\phi_{448,9}+\phi_{1400,8}+\phi_{1050,10}+\phi_{1575,10}$

${1_{W_{P_7}}}^{W(E_8)}=\phi_{1,0}+\phi_{8,1}+2\phi_{35,2}+2\phi_{560,5}+\phi_{567,6}+2\phi_{112,3}+2\phi_{84,4}+\phi_{210,4}+\phi_{160,7}+\phi_{700,6}+\phi_{300,8}+\phi_{1400,7}+\phi_{1344,8}+\phi_{448,9}$

${1_{W_{P_8}}}^{W(E_8)}=\phi_{1,0}+\phi_{8,1}+\phi_{35,2}+\phi_{112,3}+\phi_{84,4}$

On taking the corresponding linear combinations of Foulkes functions, we obtain the values of ${1_{P_\sigma}}^{G_\sigma}$ on unipotent elements. In E_6, E_7 and E_8 we observe that, as noted above, the maximum value of ${1_{P_\sigma}}^{G_\sigma}$ on nonidentity elements occurs on the class of root elements. Table 7.1A gives lower bounds for the reciprocal of the fixed point ratio for root elements and for other nonidentity unipotent elements; in each case the lower bound given for the element x is a polynomial $f(q)$ in q such that the polynomial ${1_{P_\sigma}}^{G_\sigma}(1)-f(q){1_{P_\sigma}}^{G_\sigma}(x)$ always takes positive values but is of smaller degree than ${1_{P_\sigma}}^{G_\sigma}(1)$. In F_4 and G_2, the presence of short root elements makes for complications in characteristic 2 and 3 respectively; here we observe that the maximum value of ${1_{P_\sigma}}^{G_\sigma}$ on nonidentity elements always occurs at a root element, and in fact if ${1_{P_\sigma}}^{G_\sigma}(u)>{1_{P_\sigma}}^{G_\sigma}(u_\alpha)$ for some nonidentity unipotent element u and some parabolic subgroup P, then u must be a short root element. (The problem of the uncertainty over certain values in the tables provided by Lübeck does not in fact affect these statements, as it is possible to see as above that values on root elements dominate all others, irrespective of the choice of scalars of absolute value 1; of course, in many cases this question does not even arise, because the Green functions have been independently obtained by another method – for example, the full character table of $G_2(q)$ is given in all characteristics in [**9**, **19**, **20**].) Accordingly, for these groups Tables 7.1B,C give bounds for long root elements, short root elements and other nonidentity unipotent elements. This completes the treatment of untwisted groups.

Now consider the case $G_\sigma = {}^2E_6(q)$. Here we have $\sigma_0 = w_0$, the longest word in the Weyl group $W = W(E_6)$, and the almost characters are defined by

$$R_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(w_0w)R_w$$

for $\phi \in \hat{W}$. Inverting these equations gives

$$R_w = \sum_{\phi\in\hat{W}}\phi(w_0w)R_\phi;$$

thus by Lemma 2.3 we have

$$1_{P_\sigma}{}^{G_\sigma} = \sum_{\phi\in\hat{W}}\left(\frac{1}{|W_P|}\sum_{w\in W_P}\phi(w_0w)\right)R_\phi.$$

For each choice of P we may calculate the coefficients $\frac{1}{|W_P|}\sum_{w\in W_P}\phi(w_0w)$ which appear, and hence form the appropriate linear combination of Foulkes functions. Here we observe that the maximum value of $1_{P_\sigma}{}^{G_\sigma}$ on nonidentity elements always occurs at a long root element; nevertheless, in Table 7.1B we give bounds for long root elements, short root elements and other nonidentity unipotent elements. (In fact, the irreducible unipotent characters of G_σ lying in the principal series are labelled by the irreducible characters of $W(F_4)$, and the parabolic permutation characters $1_{P_\sigma}{}^{G_\sigma}$ for $P = P_{1,6}$, P_2, $P_{3,5}$ and P_4 are given by the expressions above for $1_{W_{P_i}}{}^{W(F_4)}$ with $i = 4$, 1, 3 and 2 respectively.)

The remaining twisted groups may be handled more simply, because irreducible unipotent characters have already been obtained. If $G_\sigma = {}^3D_4(q)$, the characters in the principal series are labelled by the irreducible characters of $W(G_2)$; there are two maximal parabolic subgroups, $(P_{1,3,4})_\sigma$ and $(P_2)_\sigma$, with permutation characters $\phi_{1,0}+\phi_{2,1}+\phi_{2,2}+\phi_{1,3}{}''$ and $\phi_{1,0}+\phi_{2,1}+\phi_{2,2}+\phi_{1,3}{}'$ respectively. In this case the unipotent characters are given in [**67**]; using them we obtain the bounds for long root elements, short root elements and other nonidentity unipotent elements given in Table 7.1C. If instead $G_\sigma = {}^2F_4(q)$, there are again two maximal parabolic subgroups, $(P_{1,4})_\sigma$ and $(P_{2,3})_\sigma$, with permutation characters $1+\epsilon'+\rho_2{}'+\rho_2{}''+\rho_2$ and $1+\epsilon''+\rho_2{}'+\rho_2{}''+\rho_2$ respectively, in the notation of [**7**]. Here the unipotent characters are given in [**55**]; again, we obtain the bounds for long root elements, short root elements and other nonidentity unipotent elements given in Table 7.1C. Finally, if $G_\sigma = {}^2B_2(q)$ or ${}^2G_2(q)$, the Borel subgroup B_σ is a maximal subgroup of G_σ, and the permutation character $1_{B_\sigma}{}^{G_\sigma}$ is the sum of the principal and the Steinberg characters of G_σ; since the second of these is zero on nonidentity unipotent elements, we have $\mathrm{fpr}(u, G_\sigma/B_\sigma) = \frac{1}{q^2+1}$ or $\frac{1}{q^3+1}$ respectively for any nonidentity unipotent element $u \in G_\sigma$. This completes the proof of Theorem 2(I)(a). □

It is convenient at this point to handle a case in Part (I)(d) of Theorem 2, using the character-theoretic methods of this section. This is the case of the fixed point ratio of a graph automorphism τ when $G = E_6$ and $p = 2$, for parabolic actions.

Proposition 2.6. *Let $G = E_6$ with $p = 2$, so that $L = G'_\sigma = E_6(q)$ or ${}^2E_6(q)$, and let τ be a graph automorphism of L of order 2. If P is a maximal $\langle\sigma, \tau\rangle$-stable parabolic subgroup of G, then*

$$\mathrm{fpr}(\tau, G_\sigma/P_\sigma) \le \frac{1}{k_P(q)},$$

where $k_P(q)$ is as in the table below.

L	P	$k_P(q)$
${}^2E_6(q)$	$P_{1,6}$	$q^8(q-1)$
	P_2	$q^6 - q^3 + 1$
	$P_{3,5}$	$q^{10}(q-1)$
	P_4	$q^6(q^2-1)(q-1)$
$E_6(q)$	$P_{1,6}$	q^9
	P_2	q^9
	$P_{3,5}$	$\frac{1}{3}q^{11}$
	P_4	q^9

Proof. Let δ be the standard graph automorphism of G centralizing F_4. By Proposition 1.1, we may take $\tau = \delta$ or $r\delta$, where r is a long root element in F_4. Write $G_\sigma.2 = G_\sigma\langle\delta\rangle$. Let P be a standard $\langle\delta, \sigma\rangle$-stable parabolic subgroup of G, so that P_σ is a standard parabolic subgroup of G_σ; write $P_\sigma.2 = P_\sigma\langle\delta\rangle$. We consider fixed point ratios $f(u\delta, G_\sigma.2/P_\sigma.2)$, where $u \in G_\sigma$ is such that $u\delta$ is unipotent (i.e., has order a power of 2); in particular, this includes the case where $u = 1$ or r, i.e., where $u\delta = \tau$.

As before, we have

$$\mathrm{fpr}(u\delta, G_\sigma.2/P_\sigma.2) = \frac{{1_{P_\sigma.2}}^{G_\sigma.2}(u\delta)}{{1_{P_\sigma.2}}^{G_\sigma.2}(1)},$$

so that we may use character theory to calculate fixed point ratios.

In [**18**], Digne and Michel developed a Deligne-Lusztig theory for the complex characters of a non-connected reductive group over a finite field. They defined characters which they called generalized Deligne-Lusztig characters; these are extensions of (ordinary) Deligne-Lusztig characters for the relevant connected group. They showed that classes of δ, σ-stable maximal tori of $G\langle\delta\rangle$ may be taken to be parametrized by conjugacy classes of W_δ, the group of fixed points under δ of the Weyl group W of G; writing T_w for a maximal torus of G_σ corresponding to $w \in W_\delta$, the character $R_{T_w,1}$ depends only on the class of $w \in W_\delta$, and there are results on scalar products of such characters similar to those in the connected case.

Malle built upon the work of Digne and Michel in [**57**], and in particular considered the decomposition of the $R_{T_w,1}$. By analogy with the connected case, linear combinations of $R_{T_w,1}$ are formed with coefficients given by the character table of W_δ, to create class functions of the type called almost characters; Fourier transform matrices then relate almost characters to irreducible unipotent characters. Malle gives details of the Fourier transform matrices required for several small rank cases, including $E_6(q).2$ and ${}^2E_6(q).2$. He then goes on to calculate Green functions for certain cases in which the order of δ is equal to the characteristic of the underlying field, so that there are unipotent elements lying in the outer coset(s); the cases of $E_6(q).2$ and ${}^2E_6(q).2$ in characteristic 2 are not treated in [**57**], but are covered by a separate paper [**58**], which first gives details of the outer unipotent classes which occur.

Using the above, and the known decompositions of the permutation characters into irreducible constituents, it is possible to determine the values ${1_{P_\sigma.2}}^{G_\sigma.2}(u\delta)$ in the twisted case for all unipotent elements $u\delta$; doing so and comparing with the values ${1_{P_\sigma.2}}^{G_\sigma.2}(1)$ gives $\mathrm{fpr}(u\delta, G_\sigma.2/P_\sigma.2) \leq \frac{1}{k_P(q)}$ with $k_P(q)$ as in the table above.

However, there is a complication in the untwisted case, caused by the fact that not all extensions of unipotent characters need occur in a family. In particular the subcuspidal characters induced from the Levi factor ${}^2A_5$ of the twisted group have scalar product zero with all almost characters formed as described above, and are thus orthogonal to the space of uniform functions. This does not create difficulties in the twisted case, since such characters do not occur as constituents of the permutation characters ${1_{P_\sigma.2}}^{G_\sigma.2}$. In the untwisted case, though, the Fourier transform matrices are described by means of a natural correspondence between unipotent characters of the untwisted and twisted groups, under which the unipotent character $\chi_{\phi_{64,4}}$ of $E_6(q)$ is paired with such a subcuspidal character of ${}^2E_6(q)$; and the extension to $E_6(q).2$ of $\chi_{\phi_{64,4}}$ does occur as a constituent of some of the permutation characters ${1_{P_\sigma.2}}^{G_\sigma.2}$. The solution to this problem is to apply the criterion given in [**57**, Proposition 9] to show that most of the outer unipotent classes, including in particular the two containing elements of prime order, are in fact uniform (as defined just before Lemma 2.3), so that the extension to $E_6(q).2$ of $\chi_{\phi_{64,4}}$ takes the value zero on such classes. It is now possible to proceed as in the twisted case, and obtain the bounds $\mathrm{fpr}(u\delta, G_\sigma.2/P_\sigma.2) \leq \frac{1}{k_P(q)}$ with $k_P(q)$ as in the table. $\square$

3. Proof of Theorem 2(I)(b): Semisimple elements in parabolics.

We continue with the notation of the previous section, so that G, σ, q, σ_0, G_σ, P, P_σ, T_0, B, Φ, Π and α_0 are as before; however, we do not assume that G is simply connected. Our focus in this section is on the fixed point ratios $\mathrm{fpr}(s, G_\sigma/P_\sigma)$ for $s \in G_\sigma$ semisimple; as before, we have

$$\mathrm{fpr}(s, G_\sigma/P_\sigma) = \frac{1_{P_\sigma}{}^{G_\sigma}(s)}{1_{P_\sigma}{}^{G_\sigma}(1)}.$$

Since Lemma 2.3 gives $1_{P_\sigma}{}^{G_\sigma} = |W_P|^{-1}\sum_{w\in W_P} R_w$, we must consider the values $R_w(s)$.

We first require further notation. Since $W = N_G(T_0)/T_0$ and T_0 is σ-stable, we have an action of σ on W. We recall that elements $w, w' \in W$ are said to be σ-conjugate if there exists $x \in W$ with $w' = x^{-1}w\sigma(x)$; the G_σ-classes of σ-stable maximal tori of G are in natural correspondence with the σ-conjugacy classes in W, and we have $R_w = R_{w'}$ if and only if w and w' are σ-conjugate. Let $w_1, w_2, \dots, w_c$ be representatives of the σ-conjugacy classes in W. For each $w \in W$ choose $\dot{w} \in N_G(T_0)$ with $\dot{w}T_0 = w$; take $g_w \in G$ with ${g_w}^{-1}\sigma(g_w) = \dot{w}$, and set $T_w = {}^{g_w}T_0$. The torus T_w is then σ-stable, and is said to be obtained from T_0 by twisting with w; for $1 \le i \le c$ write $T_i = T_{w_i}$, so that $(T_1)_\sigma, (T_2)_\sigma, \dots, (T_c)_\sigma$ are representatives of the G_σ-classes of maximal tori of G_σ.

Now assume that the action of σ on W is such that there exists $w^* \in W$ with

$$\sigma(w) = {}^{w^*}w \qquad \text{for all } w \in W.$$

(This hypothesis is certainly satisfied if either σ is untwisted or $G_\sigma = {}^2E_6(q)$, as we may take $w^* = 1$ or w_0 respectively, where w_0 is the long word in W; the remaining cases are easily dealt with, and will be mentioned briefly at the end of this section.) In this case, w and w' are σ-conjugate if and only if ww^* and $w'w^*$ are conjugate; thus $w_1w^*, w_2w^*, \dots, w_cw^*$ are conjugacy class representatives in W. Let $C_i = (w_iw^*)^W$ be the ith conjugacy class of W, so that $C_i{w^*}^{-1}$ is the σ-conjugacy class containing w_i. By Lemma 2.3 we have

$$1_{P_\sigma}{}^{G_\sigma} = \frac{1}{|W_P|}\sum_{i=1}^{c} |W_P \cap C_i{w^*}^{-1}| R_{w_i}.$$

We next consider the semisimple element s. We recall that each semisimple class in G_σ may be associated with a pair $(J, [w])$, where J is a proper subset of $\Pi \cup \{\alpha_0\}$ (determined up to conjugacy in W), W_J is the subgroup of W generated by reflections in the roots in J, and $[w] = W_Jw$ is a conjugacy class representative of $N_W(W_J)/W_J$, as explained in [16, 21, 22]. This association has the following properties: If $s \in G_\sigma$ has class associated with $(J, [w])$, then s lies in $T_{ww^{*-1}}$, and if we set $t = s^{g_{ww^{*-1}}} \in T_0$, then $W_J = (C_G(t)^0 \cap N)/T_0$.

For H a σ-stable subgroup of G, let $\epsilon_H = (-1)^{r_H}$, where r_H is the relative rank of H.

Proposition 3.1. *With the notation established,*

$$R_{w_i}(s) = \frac{|W|}{|C_i|}.\frac{|W_J w \cap C_i|}{|W_J|}.\frac{\epsilon_{C_G(s)^0}\epsilon_{T_i}|(C_G(s)^0)_\sigma|_{p'}}{|(T_i)_\sigma|}.$$

Proof. By [7, 7.2.8] we have

$$R_{w_i}(s) = \frac{1}{|(C_G(s)^0)_\sigma|}\sum_{x\in G_\sigma,\ s^x\in T_i} Q^{C_G(s)^0}_{{}^xT_i}(1),$$

while by [7, 7.5.1] we have

$$Q^{C_G(s)^0}_{{}^xT_i}(1) = \frac{\epsilon_{C_G(s)^0}\epsilon_{T_i}|(C_G(s)^0)_\sigma|_{p'}}{|(T_i)_\sigma|}.$$

We must therefore determine $|(C_G(s)^0)_\sigma|^{-1}|\{x \in G_\sigma : s^x \in T_i\}|$.

Let $r = |C_G(s)/C_G(s)^0|$, and $m = |C_{G_\sigma}(s)|/|(C_G(s)^0)_\sigma|$; thus m is the number of σ-stable cosets of $C_G(s)^0$ in $C_G(s)$. For convenience write $z = g_{ww^{*-1}}$ and $y = g_{w_i}$; set $t = s^z \in T_0$, so that $|C_W(t)| = r|W_J|$. Define $\sigma' : G \to G$ by $\sigma'(g) = {}^{\dot w\dot w^{*-1}}\sigma(g) = {}^{z^{-1}}\sigma({}^z g)$. Given $g \in C_G(t)$ we have ${}^z g \in C_G(s)$; since s is σ-stable we have $\sigma({}^z g) \in C_G(s)$, so that $\sigma'(g) \in C_G(t)$. Thus σ' preserves $C_G(t)$; and $C_G(t)_{\sigma'} = C_{G_\sigma}(s)^z$.

Now assume that $s^x = s' \in T_i$; set $t' = (s')^y \in T_0$, so that $t' = t^{z^{-1}xy}$. Since $t, t' \in T_0$ are G-conjugate, there exists $w' \in W$ with $t^{w'} = t'$; thus $(z^{-1}xy).(\dot w')^{-1} \in C_G(t)$, so that $x = zc\dot w'y^{-1}$ for some $c \in C_G(t)$. It follows that

$$|\{x \in G_\sigma : s^x \in T_i\}| = \frac{1}{|C_W(t)|}|\{(c, w') \in C_G(t) \times W : zc\dot w'y^{-1} \in G_\sigma\}|.$$

Since

$$\begin{aligned} zc\dot w'y^{-1} \in G_\sigma &\iff zc\dot w'y^{-1} = \sigma(z)\sigma(c)\sigma(\dot w')\sigma(y)^{-1} \\ &\iff \dot w'\dot w_i\sigma(\dot w')^{-1} = c^{-1}\dot w\dot w^{*-1}\sigma(c) \\ &\iff \dot w'\dot w_i\sigma(\dot w')^{-1}\dot w^*\dot w^{-1} = c^{-1}\sigma'(c), \end{aligned}$$

we have

$$\begin{aligned} &\frac{|\{x \in G_\sigma : s^x \in T_i\}|}{|(C_G(s)^0)_\sigma|} \\ &= \frac{m}{|C_{G_\sigma}(s)|.|C_W(t)|}|\{(c, w') \in C_G(t)\times W : \dot w'\dot w_i\sigma(\dot w')^{-1}\dot w^*\dot w^{-1} = c^{-1}\sigma'(c)\}| \\ &= \frac{m}{r|W_J|}|\{w' \in W : \dot w'\dot w_i\sigma(\dot w')^{-1}\dot w^*\dot w^{-1} = c^{-1}\sigma'(c) \text{ for some } c \in C_G(t)\}|. \end{aligned}$$

Given $g \in C_G(t)$, by [7, 3.5.3] we may choose $\dot{w}_g \in N \cap C_G(t)^0 g$; then $w_g \in C_W(t)$. Set $v_g = {w_g}^{-1}.{}^{w}w_g \in W$. Since the map $x \mapsto x^{-1}\sigma'(x)$ from $C_G(t)^0$ to itself is surjective by Lang's theorem, we have

$$\begin{aligned}
\{c^{-1}\sigma'(c) : c \in C_G(t)^0 g\} &= \{c^{-1}\sigma'(c) : c = c_0\dot{w}_g, c_0 \in C_G(t)^0\} \\
&= \{\dot{w}_g^{-1}.{c_0}^{-1}\sigma'(c_0).\sigma'(\dot{w}_g) : c_0 \in C_G(t)^0\} \\
&= \dot{w}_g^{-1}.C_G(t)^0.\sigma'(\dot{w}_g) \\
&= C_G(t)^0\dot{w}_g^{-1}\sigma'(\dot{w}_g) \\
&= C_G(t)^0\dot{v}_g,
\end{aligned}$$

since ${w_g}^{-1}\sigma'(w_g) = {w_g}^{-1}.{}^{ww^{*-1}}\sigma(w_g) = {w_g}^{-1}.{}^{w}w_g = v_g$. Thus if we set

$$W(g) = \{w' \in W : \dot{w}'\dot{w}_i\sigma(\dot{w}')^{-1}\dot{w}^*\dot{w}^{-1} = c^{-1}\sigma'(c) \text{ for some } c \in C_G(t)^0 g\},$$

then

$$\begin{aligned}
W(g) &= \{w' \in W : \dot{w}'\dot{w}_i\sigma(\dot{w}')^{-1}\dot{w}^*\dot{w}^{-1} \in C_G(t)^0\dot{v}_g \cap N\} \\
&= \{w' \in W : w'w_i\sigma(w')^{-1}w^*w^{-1} \in W_J v_g\} \\
&= \{w' \in W : w'w_iw^*{w'}^{-1} \in W_J v_g w\} \\
&= \{w' \in W : {}^{w'}(w_iw^*) \in W_J.w^{w_g}\}.
\end{aligned}$$

Now it is clear that premultiplication by w_g gives a bijection from the set $\{w' \in W : {}^{w'}(w_iw^*) \in W_J.w^{w_g}\}$ to the set $\{w' \in W : {}^{w'}(w_iw^*) \in W_J w\}$. Thus $|W(g)|$ is independent of $g \in C_G(t)$. Since the number of cosets of $C_G(t)^0$ in $C_G(t)$ which contain elements of the form $c^{-1}\sigma'(c)$ with $c \in C_G(t)$ is $\frac{r}{m}$, we have

$$\begin{aligned}
&|\{w' \in W : \dot{w}'\dot{w}_i\sigma(\dot{w}')^{-1}\dot{w}^*\dot{w}^{-1} = c^{-1}\sigma'(c) \text{ for some } c \in C_G(t)\}| \\
&= \frac{r}{m}|\{w' \in W : {}^{w'}(w_iw^*) \in W_J w\}|.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{|\{x \in G_\sigma : s^x \in T_i\}|}{|(C_G(s)^0)_\sigma|} &= \frac{m.\frac{r}{m}|\{w' \in W : {}^{w'}(w_iw^*) \in W_J w\}|}{r|W_J|} \\
&= \frac{|C_W(w_iw^*)|.|W_J w \cap C_i|}{|W_J|} \\
&= \frac{|W|}{|C_i|}.\frac{|W_J w \cap C_i|}{|W_J|}.
\end{aligned}$$

The result follows. □

Corollary 3.2. *With the notation established, we have*

$$1_{P_\sigma}{}^{G_\sigma}(s) = \sum_{i=1}^{c} \frac{|W|}{|C_i|}.\frac{|W_P \cap C_i w^{*-1}|}{|W_P|}.\frac{|W_J w \cap C_i|}{|W_J|}.\frac{\epsilon_{C_G(s)^0}\epsilon_{T_i}|(C_G(s)^0)_\sigma|_{p'}}{|(T_i)_\sigma|}.$$

We observe that all quantities in the above expression may be calculated for given w^*, P and $(J,[w])$. In [21, 22] Fleischmann and Janiszczak list all possibilities for $(J,[w])$, and give the centralizers $C_{G_\sigma}(s)$, if G is of type E_n; similar information for $G = F_4$ or G_2 may be obtained from [8, 19, 63, 65]. For a given G_σ, it is thus straightforward (if somewhat lengthy in the cases of E_7 and E_8) to work through all types of semisimple class s^{G_σ} and calculate the values ${1_{P_\sigma}}^{G_\sigma}(s)$ for all maximal σ-stable parabolic subgroups P. This has been done using the computer package Maple, which facilitates the manipulation of the polynomials in q which occur. It is observed that in all cases the value ${1_{P_\sigma}}^{G_\sigma}(s)$ is a polynomial in q in which all coefficients are nonnegative integers. Moreover, for each choice of G_σ and P, there is a single pair $(J',[w']) \neq (\Pi,[1])$ such that if s' is any associated semisimple element, and s is any semisimple element associated with any other pair $(J,[w]) \neq (\Pi,[1])$, then ${1_{P_\sigma}}^{G_\sigma}(s') - {1_{P_\sigma}}^{G_\sigma}(s)$ is a polynomial in q which takes positive values for any $q > 1$. The elements s' are as follows:

G	E_8	E_7	E_6	$F_4\,(G_\sigma = F_4(q))$	$G_2\,(G_\sigma = G_2(q))$
$C_G(s')$	E_7A_1	E_6T_1	D_5T_1	B_4	A_2

We may therefore use the values ${1_{P_\sigma}}^{G_\sigma}(s')$ to obtain the bounds for $\mathrm{fpr}(s, G_\sigma/P_\sigma)$ required for Theorem 2(I)(b).

Though our calculations are now complete (except for the small twisted groups, which are handled at the end of the section below), we now offer some comments which explain in some cases why ${1_{P_\sigma}}^{G_\sigma}(s)$ is a polynomial in q in which all coefficients are nonnegative integers. In the case where $w^* = 1$ (and so σ is untwisted), it is possible to simplify the expression in Corollary 3.2 above somewhat. Note that

$$\begin{aligned} {1_{W_P}}^W(w_i w^*) &= \frac{1}{|W_P|} \sum_{w \in W,\ {}^w(w_i w^*) \in W_P} 1 \\ &= \frac{1}{|W_P|} |C_W(w_i w^*)|.|W_P \cap C_i| \\ &= \frac{|W|}{|W_P|}.\frac{|W_P \cap C_i|}{|C_i|}; \end{aligned}$$

thus if $w^* = 1$ we have

$$\begin{aligned} {1_{P_\sigma}}^{G_\sigma}(s) &= \sum_{i=1}^{c} {1_{W_P}}^W(w_i).\frac{|W_J w \cap C_i|}{|W_J|}.\frac{\epsilon_{C_G(s)}\epsilon_{T_i}|C_{G_\sigma}(s)|_{p'}}{|(T_i)_\sigma|} \\ &= \frac{1}{|W_J|} \sum_{w' \in W_J} {1_{W_P}}^W(w'w)\frac{\epsilon_{C_G(s)}\epsilon_{T_{w'w}}|C_{G_\sigma}(s)|_{p'}}{|(T_{w'w})_\sigma|} \\ &= \frac{1}{|W_J|} \sum_{w' \in W_J} {1_{W_P}}^W(w'w) Q_{T_{w'w}}^{C_G(s)}(1), \end{aligned}$$

where we assume (as we may) that the elements $g_{w'w}$ for $w' \in W_J$ are chosen so that each maximal torus $T_{w'w}$ lies in $C_G(s)$.

If $w = 1$ as well, so that we may take $s \in T_0$, we may simplify further. Recall that for $\phi \in \hat{W}_J$, by [7, 11.1.1] we have

$$\frac{1}{|W_J|} \sum_{w'' \in W_J} \phi(w'') Q_{T_{w''}}^{C_G(s)}(1) = P_\phi^{C_G(s)}(q),$$

where $P_\phi^{C_G(s)}(t)$ is the fake degree corresponding to ϕ. Inverting these equations gives

$$Q_{T_{w''}}^{C_G(s)}(1) = \sum_{\phi \in \hat{W}_J} \phi(w''^{-1}) P_\phi^{C_G(s)}(q).$$

Thus

$$\begin{aligned}
{1_{P_\sigma}}^{G_\sigma}(s) &= \frac{1}{|W_J|} \sum_{w' \in W_J} {1_{W_P}}^W(w') \sum_{\phi \in \hat{W}_J} \phi(w'^{-1}) P_\phi^{C_G(s)}(q) \\
&= \sum_{\phi \in \hat{W}_J} \left(\frac{1}{|W_J|} \sum_{w' \in W_J} {1_{W_P}}^W(w') \phi(w'^{-1}) \right) P_\phi^{C_G(s)}(q) \\
&= \sum_{\phi \in \hat{W}_J} ({1_{W_P}}^W|_{W_J}, \phi)_{W_J} P_\phi^{C_G(s)}(q) \\
&= \sum_{\phi \in \hat{W}_J} ({1_{W_P}}^W, \phi^W)_W P_\phi^{C_G(s)}(q),
\end{aligned}$$

using Frobenius reciprocity. Since we may write $P_\phi^{C_G(s)}(t) = \sum_i n_i(\phi) t^i$, where $n_i(\phi)$ is the multiplicity of ϕ in the ith graded component of the regular W_J-module, we have

$${1_{P_\sigma}}^{G_\sigma}(s) = \sum_i \left(\sum_{\phi \in \hat{W}_J} ({1_{W_P}}^W, \phi^W)_W n_i(\phi) \right) q^i,$$

which is indeed a polynomial in q with all coefficients nonnegative integers.

Finally in this section we consider the cases which are not covered by the above. If $G_\sigma = {}^3D_4(q)$ or ${}^2F_4(q)$, as already stated the unipotent characters are given in [67] or [55] respectively; it is thus straightforward to calculate the values ${1_{P_\sigma}}^{G_\sigma}(s)$ for all semisimple $s \in G_\sigma$. If $G_\sigma = {}^2B_2(q)$ or ${}^2G_2(q)$, again the only permutation character involved is ${1_{B_\sigma}}^{G_\sigma}$ which is the sum of the principal and Steinberg characters of G_σ. In the former case the centralizer of any nonidentity semisimple element lying in B_σ is merely a torus, so the value of the Steinberg character is 1 and thus $\mathrm{fpr}(s, G_\sigma/B_\sigma) = \frac{2}{q^2+1}$. In the latter case, the only nonidentity semisimple elements in B_σ whose centralizer is not merely a torus are involutions with centralizer $A_1(q)$, so the value of the Steinberg character is q and thus $\mathrm{fpr}(s, G_\sigma/B_\sigma) = \frac{q+1}{q^3+1} = \frac{1}{q^2-q+1}$ (as may be checked from the character table in [**73**]).

4. Proof of Theorem 2(II)(a,b): Maximal rank subgroups.

We now embark on the proof of Theorem 2(II)(a,b). Let G be an adjoint exceptional algebraic group of rank l over the algebraically closed field K of characteristic $p > 0$, and let σ be a Frobenius morphism of G such that $G_\sigma = G(q)$, a finite exceptional group of Lie type over $\mathbf{F}_q$, where $q = p^a$. Assume that G_σ is not ${}^2F_4(q)$ or ${}^2G_2(q)$ (these cases will be dealt with later in Section 6).

For Theorem 2(II)(a,b) we need to obtain bounds for unipotent and semisimple elements of G_σ in actions where the point stabilizer is a subgroup H of maximal rank. Since for such a subgroup, $HL = G_\sigma$, we may assume that $X = G_\sigma$. Then $H = M_\sigma$, where M is reductive of maximal rank in G. The possibilities for M are given by [**43**] (see Tables A and B, p. 302, and 2.2 and 2.3), and for convenience we record them here.

Proposition 4.1. *The possibilities for the maximal rank subgroup M are as follows:*

G	M^0	M/M^0
E_8	$A_1E_7, D_8, A_8, A_2E_6, D_4D_4,$	$1, 1, Z_2, Z_2, S_3 \times Z_2,$
	$A_4A_4, A_2^4, A_1^8, T_8$	$Z_4, GL_2(3), AGL_3(2), 2.O_8^+(2)$
E_7	$T_1E_6, A_1D_6, A_7, A_2A_5,$	$Z_2, 1, Z_2, Z_2,$
	$A_1^3D_4, A_1^7, T_7$	$S_3, L_3(2), 2 \times Sp_6(2)$
E_6	$T_1D_5, T_2D_4, A_1A_5, A_2^3, T_6$	$1, S_3, 1, S_3, O_6^-(2)$
F_4	$A_1C_3, B_4, C_4 (p = 2), D_4, A_2A_2$	$1, 1, 1, S_3, Z_2$
G_2	A_1A_1, A_2	$1, Z_2$

We wish to bound the fixed point ratios $\mathrm{fpr}(u, G_\sigma/H)$ and $\mathrm{fpr}(s, G_\sigma/H)$, where u is a unipotent and s a semisimple element of G_σ. To this end we may assume that u and s both have prime order and lie in H.

In the proof we shall make heavy use of the main result in [**40**], namely [**40**, Theorem 2]; for $x = s, u$ this result provides strong lower bounds for the quantities

$$\dim G/M - \dim \mathrm{fix}_{G/M}(x) = \dim x^G - \dim(x^G \cap M)$$

(the equality is a consequence of [**40**, 1.14]). We shall also freely use several of the preliminary results given in [**40**, Section 1], particularly those on conjugacy classes. Here is one particular consequence:

Proposition 4.2. *Let u_α be a long root element in G_σ (or a short root element when $(G, p) = (F_4, 2)$ or $(G_2, 3)$), and let u be a nonidentity unipotent element which is not* Aut (G_σ)*-conjugate to u_α. Let s be a nonidentity* *semisimple element of G_σ, with $D = C_G(s)$. Then lower bounds for the sizes of the classes $|s^{G_\sigma}|$, $|u_\alpha^{G_\sigma}|$ and $|u^{G_\sigma}|$ are given in Table 6 below.*

G	$\lvert s^{G_\sigma}\rvert \geq$	$\lvert u^{G_\sigma}\rvert \geq$	$\lvert u_\alpha^{G_\sigma}\rvert \geq$
E_8	q^{112}, if $D \rhd E_7$ q^{128}, if $D \not\rhd E_7$	q^{92}	q^{58}
E_7	q^{53}, if $D^0 = T_1E_6$ q^{64}, if $D^0 \neq T_1E_6$	q^{52}	q^{34}
E_6	q^{31}, if $D = T_1D_5$ q^{40}, if $D \neq T_1D_5$	q^{31}	q^{22}
F_4	q^{16}, if $D = B_4$ q^{28}, if $D \neq B_4$	q^{21}	q^{16}
G_2	$q^3(q^3-1)$, if $D = A_2$ q^8, if $D \neq A_2$	$(q^2-1)(q^6-1)$	q^6-1

Table 6.

Proof. First consider semisimple elements. Inspection of subsystems shows that the possibilities for $D = C_G(s)$ of the largest few dimensions are:

$$\begin{array}{ll} G = E_8: & D \rhd E_7, D = D_8, D = T_1D_7 \\ G = E_7: & D^0 = T_1E_6, D \rhd D_6, D^0 = A_7 \\ G = E_6: & D = T_1D_5, D \rhd A_5, D = T_2D_4 \\ G = F_4: & D = B_4, D \rhd C_3, D = T_1B_3, D = A_2A_2 \end{array}$$

For the two possibilities of largest dimension, we calculate $|s^{G_\sigma}|$ directly, and see that the bound in the conclusion holds. And for possibilities of smaller dimension, 1.8 gives the result.

For unipotent elements the argument is similar: By [**40**, 1.7], for $G = E_6, E_7, E_8$, the two smallest classes are those with labels A_1 (long root elements) and $2A_1$, and for these we calculate $|u^{G_\alpha}|$ directly; while for the rest we use [**40**, 1.7] and 1.8. □

Lemma 4.3. *The conclusion of Theorem 2(II)(a,b) holds if $H = M_\sigma$, where the maximal rank reductive subgroup M is one of the following:*

$$\begin{array}{ll} G = E_8: & M^0 = T_8, A_1^8, A_2^4 \\ G = E_7: & M^0 = T_7, A_1^7 \\ G = E_6: & M^0 = T_6. \end{array}$$

Proof. Suppose first $G = E_8, M^0 = T_8 = T$. Then

$$|M_\sigma| \leq |T_\sigma||W(E_8)| \leq (q+1)^8|2.O_8^+(2)| < (q+1)^8.2^{30}.$$

For $1 \neq s \in G_\sigma$ semisimple, we have $|s^{G_\sigma}| \geq q^{112}$ by 4.2, and hence, writing $\Omega = G_\sigma/M_\sigma$, we have

$$\mathrm{fpr}(s,\Omega) = \frac{|s^{G_\sigma} \cap M_\sigma|}{|s^{G_\sigma}|} < \frac{|M_\sigma|}{|s^{G_\sigma}|} < \frac{1}{q^{48}},$$

as required. For $1 \neq u \in G_\sigma$ unipotent and not a root element, we have $|u^{G_\sigma}| > q^{92}$ by 4.2, yielding

$$\mathrm{fpr}(u,\Omega) < \frac{|M_\sigma|}{q^{92}} < \frac{1}{q^{24}},$$

as required. Finally, for u a root element, $|u^{G_\sigma}| > q^{58}$ by 4.2, which as above gives $\mathrm{fpr}(u,\Omega) < 1/q^{24}$, except when $q = 2$. For $q = 2$, observe that a root element $w \in N_G(T)$ corresponds to a reflection in $W(E_8)$ (see [**40**, 1.13(iii)]). It follows that if $r(W(E_8))$ denotes the number of reflections in $W(E_8)$, then

$$|u^{G_\sigma} \cap M_\sigma| \leq |T_\sigma|.r(W(E_8)) \leq (q+1)^8.120,$$

which is enough to give the desired bound $\mathrm{fpr}(u,\Omega) < 1/q^{24}$.

All other cases in the hypothesis of the lemma follow using the same arguments: The bound $\mathrm{fpr}(x,\Omega) < |M_\sigma|/|x^{G_\sigma}|$ gives the conclusion except for some small values of q when x is a root element and M^0 is a maximal torus, in which case the bound is strengthened by replacing $|M_\sigma|$ by the number of root elements in M_σ. □

We assume for the rest of this section that the maximal subgroup H in Theorem 2 is not one of the subgroups in Lemma 4.3.

Lemma 4.4. *The conclusion of Theorem* 2(II)(a) *holds if u is a long root element* (*or a short root element when* $(G,p) = (F_4,2)$ *or* $(G_2,3)$).

Proof. Suppose u is as in the statement. We may assume that $u \in H = M_\sigma$.

Exclude for the moment the case where $(G,M^0,p) = (F_4,D_4,2)$. We then claim that $u \in M^0$. For otherwise, by 4.1, M is one of the following:

$$\begin{array}{ll}
G = E_8: & M^0 = A_8, A_2E_6, D_4D_4, A_4A_4 \\
G = E_7: & M^0 = T_1E_6, A_7, A_2A_5, A_1^3D_4 \\
G = E_6: & M^0 = T_2D_4, A_2^3 \\
G = F_4: & M^0 = D_4 (p \neq 2), A_2A_2 \\
G = G_2: & M^0 = A_2.
\end{array}$$

However, [**40**, 1.13(iii)] shows that none of these has a root element in $M \backslash M^0$.

Thus $u \in M^0$. Now by [**40**, 1.13(ii)], u lies in a simple factor M_0 of M^0, and is a long root element therein (or a short root element for $(G,M,p) = (F_4,B_4,2)$). Now the result follows from the list of possibilities for M given in 4.1, together with the sizes of the long root classes given by [**40**, 1.12]; for example, when $G = E_8$, these results imply that

$$\mathrm{fpr}(u,\Omega) \leq \frac{|u_\alpha^{E_7(q)}| + |u_\alpha^{A_1(q)}|}{|u_\alpha^{E_8(q)}|} < \frac{2}{q^{24}}$$

as required.

Finally, we deal with the excluded case $(G, M, p) = (F_4, D_4, 2)$. In this case [**40**, 1.13(iii)] shows that there is one class of root involutions in $M \backslash M^0$, centralizing B_3 in D_4. Hence as above we obtain

$$\mathrm{fpr}(u,\Omega) \leq \frac{|u_\alpha^{D_4(q)}| + |D_4(q) : B_3(q)|}{|u_\alpha^{F_4(q)}|} < \frac{1}{q^4 - q^2 + 1},$$

as required. □

The next result is our main tool for passing from the dimension bounds of [**40**, Theorem 2] to the bounds for the finite groups G_σ that we require. In the statement, we denote by f the order of the fundamental group of G (so $f = 2, 3$ for $G = E_7, E_6$ respectively, and $f = 1$ otherwise). Reference is also made to a double coset space $W(D)\backslash W(G)/W(M)$: Here $D = C_G(s)$ for some semisimple element $s \in G$, and we observe that some conjugate of s lies in a maximal torus T of M^0. Replacing s by this conjugate, we have $T \leq D \cap M$, so we have a well-defined double coset space $N_D(T)\backslash N_G(T)/N_M(T)$, which we identify with $W(D)\backslash W(G)/W(M)$.

Lemma 4.5. *Let $u, s \in G_\sigma$ be unipotent and semisimple elements of prime order, and let $C = C_G(u), D = C_G(s)$. Write $z = \dim Z(D^0)$ and $y = \dim Z(C^0/R_u(C^0))$. Let l' be the semisimple rank of M, and $\Omega = G_\sigma/M_\sigma$. Then*

$$\mathrm{fpr}(s,\Omega) < \frac{\left(|W(D)\backslash W(G)/W(M)| + \frac{|M/M^0|}{o(s)}\right).|M/M^0|.2(q+1)^z f}{q^{\dim s^G - \dim(s^G \cap M) + z - l'}(q-1)^{l'}}$$

and

$$\mathrm{fpr}(u,\Omega) < \frac{u_p(M).|M/M^0|.2(q+1)^y |C : C^0|}{q^{\dim u^G - \dim(u^G \cap M) + y - l'}(q-1)^{l'}},$$

where $u_p(M)$ denotes the number of classes of elements of order p in M.

Proof. We have $|D : D^0| \leq f$ by [**68**, II, 4.4], and by 1.8,

$$|s^{G_\sigma}| \geq \frac{1}{2}\frac{q^z}{(q+1)^z f} q^{\dim s^G}. \tag{1}$$

By [**40**, 1.3], $s^G \cap M^0 = \bigcup_{i \in I} s_i^M$, where $|I| \leq |W(D)\backslash W(G)/W(M)|$. Also by Lang's theorem [**68**, I, 2.2], $(s_i^{M^0})_\sigma$ breaks up into M_σ^0-classes corresponding to the σ-classes in E/E^0, where $E = C_{M^0}(s_i)$; if the representatives are

s_{ij} corresponding to $w_{ij} \in E/E^0$, then $|s_{ij}^{M_\sigma^0}| = f_{ij}(q)/c_{ij}$, where $f_{ij}(q)$ is a monic polynomial of degree $\dim s_i^M$, and c_{ij} is the order of the σ-centralizer of w_{ij} in E/E^0. Note that $\sum_j \frac{1}{c_{ij}} = 1$.

Write $M^0 = TN$, where $T = Z(M^0), N = (M^0)'$. By 1.5 and 1.6, $|M_\sigma^0| \leq |T_\sigma| q^{\dim N}$. And if $C_{ij} = C_{M^0}(s_{ij})$, then $|(C_{ij}^0)_\sigma| = |T_\sigma||C_N(s_{ij})_\sigma^0|$, whence

$$|(s_i^{M^0})_\sigma| \leq \frac{|M_\sigma^0|}{\min_j |(C_{ij}^0)_\sigma|} \leq \frac{q^{\dim N}}{|C_N(s_{ij})_\sigma^0|}.$$

By 1.6,

$$|C_N(s_{ij})_\sigma^0| \geq \frac{(q-1)^{l'}}{q^{l'}} q^{\dim C_N(s_{ij})}$$

and hence

$$|(s_i^{M^0})_\sigma| \leq \frac{q^{\dim N + l'}}{(q-1)^{l'} q^{\dim C_N(s_i)}}.$$

Therefore

$$|(s_i^{M^0})_\sigma| \leq \frac{q^{\dim M - \dim C_M(s_i) + l'}}{(q-1)^{l'}}. \tag{2}$$

It follows that

$$|(s_i^{M})_\sigma| \leq \frac{q^{\dim M - \dim C_M(s_i) + l'} |M/M^0|}{(q-1)^{l'}} \tag{3}$$

and hence

$$|(s^G \cap M^0)_\sigma| \leq \frac{|W(D)\backslash W(G)/W(M)|.|M/M^0|.q^{\dim(s^G \cap M^0) + l'}}{(q-1)^{l'}}. \tag{4}$$

Now consider the case where $s \in M - M^0$. For this case, we first verify that inequality (2) still holds (with s replacing s_i). If M^0 is semisimple then by [**40**, 1.4], $C_{M^0}(s)$ is semisimple of rank at most $l' - 1$, so by 1.6,

$$|C_{M_\sigma}(s)| \geq \frac{(q-1)^{l'-1}}{q^{l'-1}} q^{\dim C_M(s)},$$

whence (2) holds. And if M^0 is not semisimple, then either $M^0 = E_6 T_1$ (with $G = E_7$) or $M^0 = T_2 D_4$ (with $G = E_6$). In the first case, s acts as a graph automorphism of order 2 on the E_6 factor, and inverts T_1, so by [**40**, 1.4], $C_{M^0}(s)^0$ is simple of rank $4 = l' - 2$, so

$$|(s^{M^0})_\sigma| \leq \left(\frac{q+1}{q} q^{\dim M} \right) \Big/ \left(\frac{(q-1)^{l'-2}}{q^{l'-2}} q^{\dim C_M(s)} \right),$$

which again yields the bound in (2) (since $(q-1)^2/q^2 < q/(q+1)$). Finally, if $M^0 = T_2 D_4$ then $M/M^0 \cong S_3$, so s has order 2 or 3. For $o(s) = 3$,

$|M_\sigma| \le (q^2+q+1)|{}^3D_4(q)|$, and by 1.1, $C_M(s)$ has semisimple rank at most 2, giving

$$|(s^{M^0})_\sigma| \le \frac{q^2+q+1}{(q-1)^2} q^{\dim s^M} < \frac{q^4}{(q-1)^4} q^{\dim s^M},$$

which implies (2). And for $o(s) = 2$, $|M_\sigma| \le (q^2-1)|{}^2D_4(q)|$, and by [**40**, 1.4], $C_M(s)$ has semisimple rank at most 3, giving (2) in similar fashion.

Thus (2) holds for $s \in M - M^0$. Using [**40**, 1.4 and 1.10], we see that the number of classes of elements of prime order in $M - M^0$ is at most $|M/M^0|$; therefore the above considerations give

$$|(s^G \cap (M - M^0))_\sigma| < |M/M^0| \cdot \frac{q^{\dim(s^G\cap(M-M^0))+l'}}{(q-1)^{l'}} \cdot \frac{|M/M^0|}{|C_{M/M^0}(s)|}.$$

Combining this with (4) we obtain
(5)

$$|(s^G \cap M)_\sigma| \le \frac{\left(|W(D)\backslash W(G)/W(M)| + \frac{|M/M^0|}{o(s)}\right).|M/M^0|.q^{\dim(s^G\cap M)+l'}}{(q-1)^{l'}}.$$

Finally, $\mathrm{fpr}(s,\Omega) \le |(s^G \cap M)_\sigma|/|s^{G_\sigma}|$, from which the conclusion follows using (1) and (5).

The proof for unipotent elements is entirely similar to the above, except that for unipotent elements we use $u_p(M)$ to bound the number of M-classes in $u^G \cap M$. □

Remark. If $M = M^0$, then in the statement of 4.5, the extra term $\frac{|M/M^0|}{o(s)}$ does not have to be included, as the proof shows.

Lemma 4.6. *The conclusion of Theorem 2(II)(a) holds for unipotent elements u.*

Proof. We have $H = M_\sigma$; write $\Omega = (G_\sigma : H)$. By 4.4, we may assume u is not a long root element.

Consider first $G = E_8$. Here $\dim u^G \ge 92$ by [**40**, 1.7], and $\dim u^G - \dim(u^G \cap M) \ge 40$ by [**40**, Theorem 2(II)(a)]. Arguing as in (1) in the proof of the previous lemma, we have

$$\mathrm{fpr}(u,\Omega) \le \frac{|H|}{|u^{G_\sigma}|} \le 2|H|\frac{(q+1)^8}{q^8} q^{-\dim u^G}.$$

This yields the desired bound $\mathrm{fpr}(u,\Omega) < 2/q^{24}$, unless $|H| > q^{76}/(q+1)^8$. We may suppose the latter bound to hold, in which case by 4.1, we have $M^0 = A_1E_7, D_8, A_2E_6$ or A_8. Using [**40**, 1.8 and 1.10] for these groups, we have $u_p(M).|M : M^0| \le 130$. Also, writing $C = C_G(u)$, we have $|C : C^0| \le$

60 by [**60**]. Hence 4.5 gives

$$\frac{|(u^G \cap M)_\sigma|}{|u^{G_\sigma}|} < \frac{130.2(q+1)^8.60}{q^{40}(q-1)^8}.$$

If $q \geq 3$ this yields the required bound $\mathrm{fpr}(u,\Omega) < 2/q^{24}$.

To complete the E_8 case, assume $q = 2$. Here u is an involution. The involution classes in M_σ and their sizes are given by [**2**]. For $M^0 = A_8, D_8$ or A_2E_6, M_σ has at most 9 involution classes, the largest of which has size $|D_8(q) : C_{D_8(q)}(c_8)|$ (notation of [**40**, 1.10]), which is less than q^{64}. Since the smallest non-root involution class in G_σ has size $|E_8(q) : q^{78}B_6(q)|$, it follows that

$$f(u,\Omega) < \frac{9q^{64}}{|E_8(q) : q^{78}B_6(q)|} < \frac{2}{q^{24}}.$$

Finally, for $M^0 = A_1E_7$, the involution classes in M and the corresponding classes in G are given in the proof of [**40**, 4.6], and the result follows easily.

Next consider $G = E_7$. By 4.1 and 4.3, we have $M^0 = T_1E_6, A_1D_6, A_7$, A_2A_5 or $A_1^3D_4$. Assume first that $p \geq 5$. Using [**40**, 1.4, 1.8 and 1.10], we deduce that $u_p(M).|M : M^0| \leq 39.6$. Also $\dim u^G - \dim u^M \geq 20$ by [**40**, Theorem 2], and $|C : C^0| \leq 6$ by [**60**]. Hence 4.5 gives

$$\mathrm{fpr}(u,\Omega) \leq \frac{39.6.2(q+1)^7.6}{q^{20}(q-1)^7}$$

which, for $q \geq 5$, is less than $2/q^{12}$, as required.

For $p = 2$, we argue in similar fashion to the E_8 case above, using [**2**]. If $M = (T_1E_6).2$, then M_σ has 4 non-root involution classes, with E_6-centralizers $F_4, C_{F_4}(u_\alpha), U_{24}B_3T_1, U_{27}A_2A_1$, and the corresponding classes in G are $3A_1'', 4A_1, 2A_1, 3A_1'$ respectively (see the proofs of [**40**, 4.1 and 4.6]). It follows that $\mathrm{fpr}(u,\Omega) = |u^{M_\sigma}|/|u^{G_\sigma}| < 2/q^{12}$. If $M = A_1D_6$ then by [**40**, 1.10], M_σ has 13 classes of involutions, the largest two of which are represented by c_6 and u_0c_6, where $c_6 \in D_6$ is as in [**40**, 1.10], and $1 \neq u_0 \in A_1$. The smallest non-root involution class in G_σ has centralizer $q^{42}.B_4(q)A_1(q)$. Hence

$$\mathrm{fpr}(u,\Omega) < \frac{|A_1(q)D_6(q) : C(u_0c_6)| + 12|A_1(q)D_6(q) : C(c_6)|}{|E_7(q) : q^{42}.B_4(q)A_1(q)|} < \frac{2}{q^{12}}.$$

If $M = A_7.2$ then M_σ has 6 involution classes, the largest of which has M^0-centralizer $C_{C_4}(u_\alpha)$ (u_α a long root element of C_4). The conclusion follows in the usual way. Finally, for $M^0 = A_2A_5$ or $A_1^3D_4$, the conclusion follows by the same method, estimating $|(u^G \cap M)_\sigma|$ by multiplying the largest unipotent class size of M_σ by $u_2(M_\sigma) \leq 10$ or 23.

To complete the E_7 case, it remains to handle $p = 3$. This is similar to the $p = 2$ case, and we give just a sketch. For $M^0 = T_1E_6$, the classes of (non-root) elements of order 3 in M_σ are in E_6, and are represented by $2A_1, 3A_1, A_2, A_2 + A_1, 2A_2, A_2 + 2A_1$; since E_6 is a Levi subgroup of G, the labels of the corresponding classes in G are the same (with label $3A_1'$ for the second class by [**38**]). The centralizers in M_σ and in G_σ of all these elements can be read off from [**40**, 1.7], and the result follows. For $M^0 = A_7$ or A_1D_6 we argue as follows. If u is not in class $2A_1$ or $3A_1''$ of G, then $|u^{G_\sigma}| > q^{64}$ (see [**40**, 1.7] and the proof of 4.2), and we see that $\mathrm{fpr}(u, \Omega) < i_3(M_\sigma)/q^{64} < 2/q^{12}$, where $i_3(M_\sigma)$ is the number of elements of order 3 in M_σ, estimated as above. And if u is in class $2A_1$ or $3A_1''$ of G, then analysis of Jordan blocks on V_{56} using [**38**] shows that for $M^0 = A_7$, u is in class $2A_1$ of M^0, and for $M = A_1D_6$, $u = u_0u_1$ with u_0 in the A_1 factor and u_1 in class kA_1 ($k \leq 3$) of D_6, and the result again follows. Finally, for the cases $M^0 = A_2A_5, A_1^3D_4$ we argue just as in the $p = 2$ analysis above.

The cases $G = E_6, G_2$ are handled with very similar arguments, and are left to the reader. As for F_4, the same is true except for the case where $M = B_4$. When $p = 2$, the involution classes in B_4 and the corresponding classes in F_4 are given by [**63**] (see the proof of [**40**, 4.6] for a list); and for $p \neq 2$, classes of elements of order p in B_4 are labelled either by Levi subgroups which are also Levi in F_4, or by the Levi subgroups A_1A_1, A_3, A_1B_2, B_4 of B_4; consideration of actions on $V_{F_4}(\lambda_4)$ (see [**47**, Section 2]) shows that the corresponding classes in F_4 are $\widetilde{A}_1, B_2, C_3(a_1), F_4(a_1)$ respectively. The conclusion follows quickly. □

Lemma 4.7. *The conclusion of Theorem* 2(II)(b) *holds for semisimple elements s.*

Proof. Let $H = M_\sigma$, $\Omega = G_\sigma/H$ and $D = C_G(s)$. In most of the cases of interest we have $|M/M^0| \leq 2$, in which case the following bound is a consequence of 4.5:

$$(\dagger) \qquad f(s, \Omega) < \frac{|W(G) : W(M)|.|M/M^0|.2(q+1)^z f}{q^{\dim s^G - \dim(s^G \cap M) + z - l'}(q-1)^{l'}} \quad (\text{if } |M/M^0| \leq 2)$$

(note that we have not included the $\frac{|M/M^0|}{o(s)}$ term in the statement of 4.5: This is because if $o(s) > 2$ then no conjugate of s lies in $M \backslash M^0$, while if $o(s) = 2$ then it is easy to check that $|W(D)\backslash W(G)/W(M)| + 1 < |W(G) : W(M)|$).

Consider $G = E_8$. Arguing as at the beginning of the proof of the previous lemma, we may assume that $M^0 = A_1E_7, D_8, A_2E_6$ or A_8.

If $M \neq A_1E_7, D_8$, or if $C_G(s)$ has no factor E_7 or D_8, then [**40**, Theorem 2(II)(b)] gives $\dim s^G - \dim(s^G \cap M) \geq 65$, so by (†) we have

$$\mathrm{fpr}(s, \Omega) < \frac{|W(E_8) : W(A_2E_6).2|.2.2.(q+1)^8}{q^{65}(q-1)^8} = \frac{2^7.35(q+1)^8}{q^{65}(q-1)^8},$$

and the bounds in Theorem 2(II)(b) follow from this.

Thus we assume now that $M = A_1E_7$ or D_8 and $C_G(s)$ has a factor E_7 or D_8.

If $C_G(s) = D_8$ then $|s| = 2$ and q is odd; M is also the centralizer of an involution, say t, and $\langle s, t\rangle$ is a Klein 4-group in G. From the classification of Klein 4-groups in [10, 3.7], we see that $s^G \cap M$ consists of two M-classes, and either

$$M = D_8,\ M \cap D = (D_4D_4).2 \text{ or } (A_7T_1).2, \quad \text{or}$$

$$M = A_1E_7,\ M \cap D = A_1A_1D_6 \text{ or } (A_7T_1).2.$$

Similarly, if $C_G(s) = X_1E_7$ (where $X_1 = A_1$ or T_1) then either

$$M = D_8,\ (M \cap D)^0 = A_7T_1 \text{ or } X_1A_1D_6, \quad \text{or}$$

$$M = A_1E_7,\ M \cap D = X_1A_1D_6 \text{ or } E_6T_2.$$

The conclusion now follows by direct calculation of $|(s^G \cap M)_\sigma|$ and $|s^{G_\sigma}|$; the only close call is the case where $M = A_1E_7$, $D \rhd E_7$, when

$$\frac{|(s^G \cap M)_\sigma|}{|s^{G_\sigma}|} \leq \frac{|(A_1E_7)(q) : (A_1A_1D_6)(q)| + |(A_1E_7)(q) : (q-1)^2.E_6(q).2| + |(A_1E_7)(q) : (q+1)^2.{}^2E_6(q).2|}{|E_8(q) : (A_1E_7)(q)|},$$

which we check is less than $2/q^{48}$, as required.

Now let $G = E_7$. Consider first $M^0 = T_1E_6$. If $s \in M\backslash M^0$ then s is an involution, $C_{M^0}(s) = F_4$ or C_4, and $C_G(s)^0 = T_1E_6$ or A_7 respectively (see [11, 2.15]). Now consider $s \in M^0$. Here M^0 and D share a common maximal torus, so, as explained at the beginning of [40, Section 5], the possible intersections of the maximal rank subgroups D, M are determined by properties of the root systems, to study which we may assume $p = 0$. Then $M = C_G(t)$ where t is an involution, and $t \in D$.

Suppose now that $D^0 = T_1E_6, A_1D_6$ or T_1D_6. By the above, up to G-conjugacy the possibilities for $(M\cap D)^0$ are $T_1A_1A_5$, T_2A_5, T_2D_5 and F_4 (the first does not occur when $D^0 = T_1E_6$, by an argument in the proof of [40, 5.7]; the second only occurs when $D^0 = T_1D_6$; and the last is for $s \in M\backslash M^0$). Taking fixed point groups, we can calculate $\frac{|(s^G\cap M)_\sigma|}{|s^{G_\sigma}|}$ as above, and see that it is less than $3/q^{22}$, as required: The only close call occurs when $D^0 = T_1E_6$ and $(M \cap D)^0 = T_2D_5$; here $|W(D)\backslash W(G)/W(M)| = 2$, so $s^G \cap M$ contains two M-classes, and we have

$$\text{fpr}(s, \Omega) < \frac{|(T_1E_6)_\sigma.2 : (T_2D_5)_\sigma| + |(T_1E_6)_\sigma.2 : (F_4)_\sigma|}{|G_\sigma : (T_1E_6)_\sigma|},$$

the right hand side of which is actually greater than $2/q^{22}$, but less than $3/q^{22}$.

If instead $D^0 \neq T_1E_6, A_1D_6$ or T_1D_6, then by [40, Theorem 2], $\dim s^G - \dim(s^G \cap M) \geq 27$. In the notation of 4.5, set $z = \dim Z(D^0)$. If $z \geq 3$ then $\dim D \leq \dim T_3D_4 = 31$, so $\dim s^G \geq \dim G - 31 = 102$ while $\dim s^M \leq \dim M - 7 = 72$ (this is clear for $s \in M^0$, and follows from [40, 1.4] for

$s \in M \backslash M^0$), giving $\dim s^G - \dim(s^G \cap M) \geq 30$. Hence the right hand side of (†) is at most

$$\frac{|W(G) : W(E_6).2|.2.2(q+1)^7.2}{q^{30+7-6}(q-1)^6} = \frac{224(q+1)^7}{q^{31}(q-1)^6}.$$

And if $z \leq 2$ the right hand side of (†) is at most

$$\frac{224(q+1)^2}{q^{27+2-6}(q-1)^6},$$

which is larger. Hence by (†), we have

$$\mathrm{fpr}(s, \Omega) \leq \frac{224(q+1)^2}{q^{27+2-6}(q-1)^6}.$$

This yields the bounds required for Theorem 2 in this case.

Next let $M^0 = A_7$. If $D^0 \neq T_1E_6$ then [**40**, Theorem 2] gives $\dim s^G - \dim(s^G \cap M) \geq 31$. If also $z \geq 3$ then $\dim s^G \geq 102$ while $\dim s^M \leq \dim M - 7 = 56$, giving $\dim s^G - \dim s^M \geq 46$. Hence as above, (†) gives

$$\mathrm{fpr}(s, \Omega) \leq \frac{|W(E_7) : W(A_7).2|.2.2(q+1)^2.2}{q^{31-5}(q-1)^7} = \frac{288(q+1)^2}{q^{26}(q-1)^7}.$$

This gives the result. And if $D^0 = T_1E_6$ we can similarly use 4.5 to get

$$\mathrm{fpr}(s, \Omega) \leq \frac{(|W(E_6).2 \backslash W(E_7)/W(A_7).2| + 1).2.2(q+1).2}{q^{27-6}(q-1)^7}.$$

Observe that the relevant number of double cosets is 1, as $Sp_6(2) = O_6^+(2)O_6^-(2)$ (see [**13**, p. 46]). The required bounds follow.

Entirely similar calculations handle the remaining possibilities $M^0 = A_1D_6, A_2A_5$ and $A_1^3D_4$.

Now suppose $G = E_6$. Here $M^0 = T_1D_5, T_2D_4, A_1A_5$ or A_2^3.

Consider first $M^0 = T_1D_5$. Assume D does not have a normal subgroup D_5 or A_5. Then by [**40**, Theorem 2], $\dim s^G - \dim(s^G \cap M) \geq 20$. As above, (†) gives

$$\mathrm{fpr}(s, \Omega) \leq \frac{|W(E_6) : W(D_5)|.2(q+1)^2.3}{q^{17}(q-1)^5},$$

which yields the bounds required for Theorem 2.

Now assume D has a factor A_5. Here [**40**, Theorem 2] gives $\dim s^G - \dim(s^G \cap M) \geq 16$. As $|D : D^0|$ is not divisible by 3, then we lose the number 3 ($= f$) in the numerator above. We may also replace $|W(E_6) : W(D_5)|$ by $|W(D) \backslash W(E_6)/W(D_5)|$, which is equal to the inner product $\left(1_{W(D)}^{W(E_6)}, 1_{W(D_5)}^{W(E_6)}\right)$, and from the induced characters given in Section 2 we see that this is at most 3. Hence

$$\mathrm{fpr}(s, \Omega) \leq \frac{3.2(q+1)^2}{q^{13}(q-1)^5},$$

which is enough to give the bound required in this case.

Finally, assume that $D = D_5T_1$. As above, the possibilities for $D \cap M$ depend only root systems, and not on the characteristic. To calculate these possibilities we take $p = 0$ for now. Then $M = C_G(t)$ for an involution t. Moreover, $t \in D$, and the image of t in the associated orthogonal group SO_{10} is either an involution of the form $\mathrm{diag}(-1^4, 1^6)$ or $\mathrm{diag}(-1^8, 1^2)$, or a matrix $\mathrm{diag}(-i^5, i^5)$, where i is a fourth root of unity. The first involution $\mathrm{diag}(-1^4, 1^6)$ has centralizer A_1A_5 in G, since its action on the irreducible 27-dimensional module $V_G(\lambda_1)$ is $\mathrm{diag}(-1^{12}, 1^{15})$; the second involution has G-centralizer T_1D_5 and D-centralizer $T_2D_4.2$; and the third element has G-centralizer T_1D_5 and D-centralizer $T_2A_4.2$. Resuming our calculations in the finite group G_σ, and taking fixed points, we obtain the desired upper bound $2/q^{12}$ for $\frac{|(s^G \cap M)_\sigma|}{|s^{G_\sigma}|}$.

This completes the proof for $M = T_1D_5$. The remaining possibilities T_2D_4, A_1A_5, A_2^3 for M^0 are treated in similar fashion, and we leave this to the reader.

Now let $G = F_4$. By [**40**, Theorem 2], $\dim s^G - \dim(s^G \cap M) \geq 8$. If $M = B_4$ then as before we may take $z = 2$ in 4.5, giving

$$\mathrm{fpr}(s, \Omega) \leq \frac{|W(F_4) : W(B_4)|.2(q+1)^2}{q^6(q-1)^4} = \frac{6(q+1)^2}{q^6(q-1)^4}.$$

This gives the conclusion, except when $q = 2$. For $q = 2$, [**39**] gives much information about the action of $G_\sigma = F_4(2)$ on $\Omega = F_4(2)/B_4(2)$: The rank is 5, and in the notation of [**13**, p. 167], the permutation character is $1a + 1105a + 1377a + 23205a + 44200a$. The fixed point ratios of all elements of $F_4(2)$ can be read off from this permutation character, and the result follows. The remaining possibilities for M are dealt with using the same methods.

Finally, the case $G = G_2$ is similar and straightforward, and we leave it to the reader. □

5. Proof of Theorem 2, Part (III).

Continue to let G be an exceptional algebraic group of rank l over the algebraically closed field K of characteristic $p > 0$, and let σ be a Frobenius morphism of G such that G_σ is a finite exceptional group of Lie type over $\mathbf{F}_q$, where $q = p^a$. Assume that $G_\sigma \neq {}^2F_4(q)$ or ${}^2G_2(q)$ (these cases will be dealt with in Section 6). Let X be an almost simple group with socle $L = G'_\sigma$.

In this section we handle Case (III) of Theorem 2, in which H is a maximal subgroup of X which is not parabolic or of maximal rank. Write $\Omega = X/H$, and let s, u be nonidentity semisimple and unipotent elements of prime order in H. Also let ϕ be a field or graph-field automorphism of L of prime order, and τ a graph automorphism of prime order (if these exist).

For the case where $G = G_2$, we shall use the fact that the maximal subgroups of $G_\sigma = G_2(q)$ are known by [**14**, **34**].

The first result is taken from [**45**, Theorem 2], and classifies the possibilities for H into various types.

Proposition 5.1. *One of the following holds:*

(1) $H = N_X(M_\sigma)$ *for some maximal* σ*-stable closed subgroup* M *of* G *of positive dimension* (*not parabolic or of maximal rank*);
(2) H *is one of the local subgroups given in* [**11**, Theorem 1(II)];
(3) $G = E_8$ *and* $F^*(H) = \mathrm{Alt}_5 \times \mathrm{Alt}_6$;
(4) H *is of the same type as* G (*possibly twisted*) *over a subfield of* $\mathbf{F}_q$;
(5) H *is almost simple, and not of type* (1) *or* (4).

We shall deal with each of the cases in 5.1 separately. First it is convenient to handle root elements.

Lemma 5.2. *Assume that Case* (*4*) *of* 5.1 *does not hold. Then the conclusion of Theorem* 2(III)(a) *holds for* u *a long root element* (*or a short root element if* $(G, p) = (F_4, 2)$ *or* $(G_2, 3)$).

Proof. Suppose u is a long root element. Observe that $u^X = u^{G_\sigma}$. The conclusion certainly holds if $|H| < |u^{G_\sigma}|/q^{e_G}$, so we may assume that

$$(*) \qquad |H| \geq \frac{|u^{G_\sigma}|}{q^{e_G}}.$$

Lower bounds for $|u^{G_\sigma}|$ are given in 4.2.

Assume first that H is not almost simple. If H is local then [**11**], together with ($*$), implies that either $G = G_2$, $p \neq 2$ and $H = 2^3.L_3(2)$, or $G = E_7$, $p \neq 2$ and $H = M_\sigma$, where $M = (2^2 \times D_4).S_3$. In the latter case, by [**40**, 1.13], we have $u \in M^0 = D_4$, and moreover, u is a root element of M^0. This D_4 lies in a subgroup A_7 of G. If V_{56} denotes the 56-dimensional G-module $V(\lambda_7)$, then by [**47**, Section 2], $V_{56} \downarrow A_7 = V_{A_7}(\lambda_2) \oplus V_{A_7}(\lambda_6)$, and hence

$$V_{56} \downarrow D_4 = V_{D_4}(\lambda_2) \oplus V_{D_4}(\lambda_2).$$

It follows that if $u \in D_4$ then $C_{V_{56}}(u)$ has dimension $2 \dim C_{D_4}(u)$, which by [**40**, 1.12] is 36. However, as u is a root element of G, $\dim C_{V_{56}}(u) = 44$ by [**38**], so in fact $u \notin H$ in this case. For the case where $G = G_2$ and $H = 2^3.L_3(2)$, observe that ($*$) forces $q = 3$; but elements of order 3 in H act on $V_7 = V_{G_2}(\lambda_1)$ as $J_3^2 \oplus J_1$, so by [**38**] are not root elements (in fact they are in the class $G_2(a_1)$).

Thus we can assume H is non-local. Clearly Case (3) of 5.1 is impossible by ($*$). Therefore by [**45**, Theorem 2], $H = M_\sigma$, where M is one of the following:

$$\begin{array}{ll} G=E_8: & M=G_2F_4 \\ G=E_7: & M=G_2C_3 \text{ or } A_1F_4 \\ G=E_6: & M=(A_2G_2).2 \\ G=F_4: & M=A_1G_2. \end{array}$$

By [**40**, 1.13(iii)], $u \in M^0$, and by [**40**, 1.13(ii)], u lies in one of the simple factors of M^0 and is a long root element therein. Thus $|u^{G_\sigma} \cap H|$ is equal to the number of long root elements in the two factors of H, which is given by [**40**, 1.12], and it follows that $|u^{G_\sigma} \cap H|/|u^{G_\sigma}| < 1/q^{e_G}$, as required.

We have dealt with the case where H is not almost simple, so assume now that H is almost simple.

Suppose $p \neq 2$. Denote by Lie(p) the set of all simple groups of Lie type in characteristic p. If $F^*(H) \notin$ Lie(p), the list of possible isomorphism types for $F^*(H)$ is given by [**49**]. Using this together with the bound $(*)$, we see that the only possibilities in which $H \backslash F^*(H)$ can have an element of order p are: $G_\sigma = F_4(3)$, $H = D_4(2).3$ or ${}^3D_4(2).3$, or $G_\sigma = G_2(3)$, $H = L_2(8).3$. Suppose x is a long root element in $H \backslash F^*(H)$. In the case where $F^*(H) = D_4(2)$, x permutes transitively three commuting subgroups S_3, and so for some involution t we have $\langle x, x^t \rangle \cong \mathrm{Alt}_4$; this is not possible as x is a long root element. Similarly, in the other cases x acts on a Sylow 2-subgroup of $L_2(8)$ and we obtain the same contradiction.

Thus we have $u \in F^*(H)$. By Baer's theorem we can find $h \in F^*(H)$ such that $\langle u, u^h \rangle$ is not nilpotent. Then $\langle u, u^h \rangle \cong SL_2(p^e)$ lies in a fundamental SL_2 in G. At this point we apply the main result of [**1**], which, together with $(*)$, gives the conclusion.

Now suppose $F^*(H) \in$ Lie(p) (with $p \neq 2$). Let U_α be a long root group in G containing u, and define

$$\overline{H} = \langle H, U_\alpha \rangle.$$

Then by [**46**, 6.4], H and $\overline{H}$ stabilize the same subspaces of $L(G)$. We are assuming that Case (4) of 5.1 does not hold. Hence [**48**, Theorem 4] implies that $F^*(H)$ acts reducibly on some G-composition factor of $L(G)$. Since u acts on $L(G)$ with exactly one Jordan block of size 3 and the rest of size 1 or 2, it follows that H also acts reducibly. Therefore so does $\overline{H}$, and in particular, $\overline{H}$ is proper in G. Now U_α is σ-stable, as it is the unique root group containing u, and hence $\overline{H}$ is σ-stable. It follows that $H = M_\sigma$ for some σ-stable maximal closed subgroup M of G of positive dimension. Using $(*)$ and [**45**, Theorem 2], we see that one of the following holds:

$$\begin{array}{ll} G=E_6: & M=F_4, C_4\,(p\neq 2) \text{ or } G_2\,(p\neq 7) \\ G=F_4: & M=G_2\,(p=7). \end{array}$$

In all cases, [**40**, 1.13] implies that u is a long root element of M, the number of which is given by [**40**, 1.12]. For $M \neq F_4$, we check that

$$\mathrm{fpr}(u,\Omega) = \frac{|u^{M_\sigma}|}{|u^{G_\sigma}|} < \frac{1}{q^{e_G}},$$

and for $M = F_4$ we similarly check that for $G'_\sigma = E_6^\epsilon(q)\,(\epsilon = \pm)$, we have

$$\mathrm{fpr}(u,\Omega) = \frac{1}{q^6+\epsilon q^3+1},$$

which gives the conclusion.

To complete the proof we must deal with the case where $p = 2$ (and H is almost simple). The main result of [**71**] gives a list of possible isomorphism types for $F^*(H)$, and identifies u as a root involution for each type; we also use ($*$) together with [**49**] to pare down this list when $F^*(H) \notin \mathrm{Lie}(2)$. The upshot is that one of the following holds:

(a) $F^*(H) \in \mathrm{Lie}(2)$ and $u \in F^*(H)$ is a long or short root element;
(b) $F^*(H) = D_n(2^e)$ and $u \notin F^*(H)$ is a reflection;
(c) $\langle F^*(H), u\rangle = S_c$, and u is a transposition, where $c \leq 17, 13, 12, 10, 5$ according as $G = E_8, E_7, E_6, F_4, G_2$;
(d) $F^*(H) = Fi_{22}, \Omega_7(3), U_4(3), L_4(3)$, and u is a root involution (also $G = E_6$ for the first three possibilities).

In Cases (c) and (d), a simple check shows that $|u^H|/|u^{G_\sigma}| < 1/q^{e_G}$.

In Cases (a) and (b), let $F^*(H) = H(q_0)$, a group of Lie type over $\mathbf{F}_{q_0}$, where q_0 is a power of 2. Now H contains two root elements u, u^h with product $a = uu^h$ of order $q_0 + 1$, and a lies in a torus T_1 of a fundamental SL_2 in G. The weights of T_1 on $L(G)$ are $\pm 2, \pm 1, 0$. Hence if $q_0 > 2$ then a and T_1 stabilize the same subspaces of $L(G)$. Thus H and $\overline{H} = \langle H, T_1\rangle$ stabilize the same subspaces of $L(G)$. The weights ± 2 of T_1 occur exactly once on $L(G)$, so we see as before that $\overline{H}$ is reducible on $L(G)$, and deduce that $H = M_\sigma$ with M maximal σ-stable of positive dimension. Now the conclusion follows as above.

This leaves the case where $q_0 = 2$. By 1.4, the rank of $F^*(H) = H(2)$ is at most that of G. For each possible subgroup $H(2)$ of $G(q)$ which satisfies Lagrange's theorem and ($*$), we use [**40**, 1.12] to calculate the number $|u^H|$ of root elements (or reflections in Case (b)) in H, and check again that $|u^H|/|u^{G_\sigma}| < 1/q^{e_G}$, as required. □

In view of 5.2, we assume from now on that the unipotent element u is not a long root element (or a short root element when $(G, p) = (F_4, 2)$ or $(G_2, 3)$).

Now suppose that $x = s, u, \phi$ or τ is an element which violates the conclusion of Theorem 2(III); in other words, there is a maximal subgroup H as in Theorem 2(III), such that $\mathrm{fpr}(x, X/H)$ is greater than the upper bound stated in Theorem 2. We may take $x \in H$ and replace X by the group $\langle L, x\rangle$ (so $x^X = x^L$). In particular, if $x = u$ then $X = L$, and if $x = s$ then $X = L$ or G_σ, and $s^L = s^{G_\sigma}$.

Then, excluding the exceptional cases in Table 4 of Theorem 2, when $x = s$ or ϕ we have $|x^L \cap H| \geq |x^L|/q^{h_G}$; when $x = u$ we have $|x^L \cap H| \geq |x^L|/q^{e_G}$; and when $x = \tau$ we have $|x^L \cap H| \geq |x^L|/e_L(q)$. The conjugacy classes of field and graph-field automorphisms ϕ, and of graph automorphisms τ are given in 1.1, and the next lemma follows from this, together with 4.2.

Lemma 5.3. *As above, assume that x violates the conclusion of Theorem 2(III). Exclude the case where $(G'_\sigma, H) = ({}^2E_6(q), F_4(q))$.*

(i) *If $x = s$, then, writing $D = C_G(s)$, we have the following lower bounds for $|s^L \cap H|$:*

$$\begin{array}{ll} G = E_8: & |s^L \cap H| > q^{64}, \text{ and } |s^L \cap H| > q^{80} \text{ if } D \not\geq E_7 \\ G = E_7: & |s^L \cap H| > q^{31}, \text{ and } |s^L \cap H| > q^{42} \text{ if } D \not\geq E_6 \\ G = E_6: & |s^L \cap H| > q^{19}, \text{ and } |s^L \cap H| > q^{28} \text{ if } D \not\geq D_5 \\ G = F_4: & |s^L \cap H| > q^{10}, \text{ and } |s^L \cap H| > q^{22} \text{ if } D \not\geq B_4. \end{array}$$

(ii) *If $x = u$, we have the following lower bounds for $|u^L \cap H|$:*

$$\begin{array}{ll} G = E_8: & |u^L \cap H| > q^{68} \\ G = E_7: & |u^L \cap H| > q^{40} \\ G = E_6: & |u^L \cap H| > q^{25} \\ G = F_4: & |u^L \cap H| > q^{17}. \end{array}$$

(iii) *If $x = \phi$, of order r say, then $C_L(x) = L(q^{1/r})$ is a group of the same type as G (possibly twisted) over $\mathbf{F}_{q^{1/r}}$, and*

$$|x^L \cap H| > |L : L(q^{1/r})|/q^{h_G}.$$

(iv) *If $x = \tau$ (so $G = E_6$), then $C_{G_\sigma}(\tau) = F_4(q)$, $C_4(q)(p \neq 2)$ or $C_{F_4(q)}(t)$ ($p = 2, t$ a long root element of $F_4(q)$), and*

$$|x^L \cap H| > |L : C_L(\tau)|/e_L(q).$$

Remark. The following observation will be useful, in the special case where $q = 2$, $x = s$, $G = E_6$ and $D = T_1D_5$. In this case, the fact that $1 \neq s \in Z(D_\sigma)$ implies that $G_\sigma = {}^2E_6(2).3$ and $D_\sigma = {}^2D_5(2) \times 3$, with s a central element of order 3. Then $s \in G_\sigma \backslash G'_\sigma$, so s is an outer automorphism of $F^*(H)$. In particular, this case does not occur if $F^*(H)$ is a simple group which has no outer automorphism of order 3.

Lemma 5.4. *The conclusion of Theorem 2(III) holds in the case where $G = E_6$ and $F^*(H) = M_\sigma$, where $M = F_4$ or C_4 ($p \neq 2$).*

Proof. Note that $H = M_\sigma$, unless $x = \phi$ or τ, in which case $H = M_\sigma\langle x\rangle$. First consider the unipotent case $x = u$. For $M = F_4$, [**38**, Table A, p. 4130] lists the unipotent classes of M together with the corresponding classes in G, from which it follows that with one exception, $u^G \cap M = u^M$; the exception occurs for $p = 2$ with the class $2A_1$ in G, which is represented by two classes in M, namely $\widetilde{A}_1$ and $\widetilde{A}_1^{(2)}$. The required bounds for $\mathrm{fpr}(u, \Omega)$ follow easily.

For $M = C_4$ (still with $x = u$) we use the fact that the total number of unipotent elements in M_σ is q^{32}, by 1.3(iii). Therefore we may assume that $|u^{G_\sigma}| \leq q^{32}q^{e_G} = q^{38}$. It follows from the unipotent class classification (see [**40**, 1.7]) that u lies in class $2A_1$ of G. As shown in the last paragraph of the proof of [**40**, 6.2], u must act as $J_2^2 \oplus J_1^4$ on the usual module for C_4. Thus $\dim u^G = 32, \dim u^M = 14$, and the result follows easily on taking fixed point groups under σ.

Now we consider the case where $x = s$, a semisimple element. Observe that $M = C_G(\tau)$, where τ is an involutory graph automorphism of G. We refer the reader to the proof of the corresponding result for algebraic groups in [**40**, 6.2]. We shall handle in turn the various possibilities for $D = C_G(s)$: These are

$$D = T_1D_5,\ D \triangleright A_5,\ D = T_2D_4,\ D \triangleright A_4,\ D \triangleright A_3,\ \text{and } D \leq A_2^3.$$

Suppose first that $D = T_1D_5$. As τ inverts T_1, and $s \in M = C_G(\tau)$, it follows that s is an involution and $p \neq 2$. We must have $C_{F_4}(s) = B_4$ and $C_{C_4}(s) = C_2C_2$, with $s^G \cap M = s^M$, and the required bound $\mathrm{fpr}(s, \Omega) < 1/q^{12}$ follows.

If $D = A_1A_5$ then again s is an involution, and $C_{F_4}(s) = A_1C_3$, $C_{C_4}(s) = A_1C_3$. The bound follows in the C_4 case, while for $M = F_4$ we have

$$\mathrm{fpr}(s, \Omega) = \frac{|F_4(q) : (A_1C_3)(q)|}{|E_6^\epsilon(q) : (A_1A_5^\epsilon)(q)|} = \frac{1}{q^6(q^6 + \epsilon q^3 + 1)},$$

as in the conclusion of Theorem 2 (see Table 3 when $\epsilon = -1$). The argument for $D = T_1A_5$ is the same, replacing A_1 by T_1.

Next let $D = T_2D_4$. Here $C_M(s) = C_D(\tau) = B_3T_1, B_2B_1T_1$ for $M = F_4, C_4$ respectively, and $s^G \cap M$ falls into at most two M-classes (as T_1 can only be inverted in G). The required bounds follow.

At this point the proof is complete for $M = C_4$, since for the remaining possibilities for D we have $|s^{G_\sigma}| > q^{48} > |C_4(q)|q^{12}$, giving $\mathrm{fpr}(s, \Omega) < 1/q^{12}$. So assume from now on that $M = F_4$.

Now suppose $D \triangleright A_4$. Then $D = A_4T_2$ (not $A_4A_1T_1$, for τ inverts the T_1, which would force s to have order 2, hence have centralizer A_5A_1). Now $C_G(A_4)' = A_1$, so τ normalizes an A_5 centralizing this A_1 and containing the A_4. We established above that $C_{A_5}(\tau) = C_3$. However, $C_{A_4}(\tau)$ must be a subgroup $B_2 = SO_5$ acting irreducibly in this A_4, whereas C_3 does not contain such an SO_5, a contradiction.

Next consider $D \triangleright A_3$. We have $N_G(A_3)^0 = A_1A_1A_3T_1 < A_1A_5$. Now τ inverts the T_1 factor, so s lies diagonally in the A_1A_1, and hence in fact

$D = A_3T_3$. Then $C_D(\tau) = D_2T_2$ or C_2T_2, and the former is impossible as A_1A_1 (both long root SL_2s) is not a Levi subgroup of F_4. Moreover, $N_G(A_3)$ and $N_{F_4}(C_2)$ both induce groups of order 8 on T_3 and T_2 respectively, so $s^G \cap F_4 = s^{F_4}$, and so

$$\mathrm{fpr}(s,\Omega) \le \frac{|(A_3T_3)_\sigma : (C_2T_2)_\sigma|}{|G_\sigma : M_\sigma|},$$

giving the required bound.

Lastly, suppose $D \le A_2^3$. If $D = A_2^3$ then τ interchanges two of the factors and centralizes a diagonal subgroup $\widetilde{A}_2$ of their product, which is a short root A_2 in F_4. Since s has order 3 in this case, it follows that $C_{F_4}(s) = A_2\widetilde{A}_2$, $s^G \cap M = s^M$, and the result follows.

Now suppose $D < A_2^3$. We now make a general observation. The number of F_4-classes in $s^G \cap M$ is at most $|W(G) : W(F_4)| = 45$, and hence, taking s with $|C_{M_\sigma}(s)|$ minimal, we have

$$(**) \qquad \mathrm{fpr}(s,\Omega) \le 45\frac{|M_\sigma : C_{M_\sigma}(s)|}{|G_\sigma : D_\sigma|} = 45\frac{|\tau^{D_\sigma}|}{|G_\sigma : M_\sigma|}.$$

If D contains two factors A_2, these are interchanged by τ, and so $C_{F_4}(s) = \widetilde{A}_2T_2$ or $\widetilde{A}_2A_1T_1$. Hence by $(**)$

$$\mathrm{fpr}(s,\Omega) \le 45\frac{|(A_2^2A_1T_1)_\sigma : (\widetilde{A}_2T_2)_\sigma|}{|G_\sigma : M_\sigma|},$$

which gives the result. Therefore D has at most one factor A_2, so $D \le A_2A_1A_1T_2$, and now the required bound follows directly from $(**)$.

It remains to consider the cases where $x = \phi$, a field or graph-field automorphism, or τ, a graph automorphism. Now ϕ is a Frobenius morphism of both G and M, and hence by a standard argument using Lang's theorem (see [**28**, 7.2]), the coset $M_\sigma\phi$ contains only one M_σ-conjugacy class of elements of (prime) order $|\phi| = r$. Therefore

$$\mathrm{fpr}(\phi,\Omega) \le \frac{|\phi^{M_\sigma}|}{|\phi^L|} \le \frac{|F_4(q) : F_4(q^{1/r})|}{|L : E_6^\delta(q^{1/r})|} \quad \text{or} \quad \frac{|C_4(q) : C_4(q^{1/r})|}{|L : E_6^\delta(q^{1/r})|}$$

and the result follows for $x = \phi$.

Finally, suppose $x = \tau$. If $C_G(\tau) \ne F_4$, then by 1.1 we have $|C_{G_\sigma}(\tau)| < q^{36}$, and the bound $\mathrm{fpr}(\tau,\Omega) \le i_2(M_\sigma)/|L : C_L(\tau)|$ gives the result, using 1.3(i) to bound $i_2(M_\sigma)$.

So assume that $C_G(\tau) = F_4$. This case requires a fairly delicate analysis. First consider the case where $M = C_4$ (so $p \ne 2$). Then there exists $c \in M$ such that τc centralizes M, and so $\tau^G \cap M\tau$ consists of all elements τct, where t is an involution in F_4 with centralizer A_1C_3 (not B_4, as B_4 does not lie in C_4); hence $\mathrm{fpr}(\tau,\Omega) = |M_\sigma : (A_1C_3)_\sigma|/|L : (F_4)_\sigma|$, giving the required bound.

Now suppose $M = F_4 = C_G(\tau)$. For p odd, $\tau^G \cap M\tau$ consists of all elements τt with t an involution in M with centralizer B_4, and the required bounds follow. Now let $p = 2$. The involution classes in $M\tau$ (apart from $\{\tau\}$ itself) are of the form τC, where C is one of the involution classes A_1,$\widetilde{A}_1$, $\widetilde{A}_1^{(2)}$, $A_1\widetilde{A}_1$ in F_4. Of these, we claim that only the class τC with $C = \widetilde{A}_1$ lies in τ^G. For by 1.1, we know that $\tau u_{\alpha_0}(1) \notin \tau^G$ (where α_0 is the highest root of E_6). Conjugating this by $u_{\alpha_1}(1)$, we see that $\tau u_{\alpha_0}(1)u_{\alpha_1}(1)u_{\alpha_6}(1) \notin \tau^G$, and this is of the form τc with c in class $A_1\widetilde{A}_1$ of F_4. Finally, for $C = \widetilde{A}_1^{(2)}$, we can take the representative as $\tau U_{2342}(1)U_{1232}(1)$ (where $a_1a_2a_3a_4$ denotes the F_4-root $\sum a_i\alpha_i$). As an element of E_6 this is $\tau U_{122321}(1)U_{111221}(1)U_{112211}(1)$, which is $(\tau U_{122321}(1))^{U_{111221}(1)}$, hence is not in τ^G.

Thus $\tau^G \cap M = \tau C$, where C is the class $\widetilde{A}_1$ of F_4, and hence

$$\mathrm{fpr}(\tau,\Omega) = \frac{|F_4(q) : q^{15}C_3(q)|}{|E_6^\epsilon(q) : F_4(q)|}$$

giving the result. □

Lemma 5.5. *The conclusion of Theorem 2(III) holds if H is as in Case (1) of 5.1.*

Proof. In this case, $H = N_X(M_\sigma)$ for some maximal σ-stable closed subgroup M of G of positive dimension and not parabolic or of maximal rank. We exclude the possibilities $G = E_6$, $M = F_4$ or C_4 dealt with in the previous lemma. By [45, Theorem 1] and the bounds in 5.3, the possibilities for M are as follows:

G	M	x
E_8	G_2F_4	$x = s$, $D = C_G(s) \triangleright E_7$
E_7	A_1F_4	
	G_2C_3	$x = s$, $D \triangleright E_6$
E_6	A_2G_2	$x = s$, $D \triangleright D_5$, or $x = \tau$
	$B_3\,(p = 2)$	$x = s$, $D \triangleright D_5$
F_4	$A_1G_2\,(p \neq 2)$	$x = s$, $D = B_4$
	$G_2\,(p = 7)$	$x = s$, $D = B_4$
G_2	$A_1\,(p \geq 7)$	

Consider first $G = E_8$. Here $M = G_2F_4$ and $C_G(s) = A_1E_7$ or T_1E_7. By [40, 1.3], $s^G \cap M$ splits into at most $|W(G) : W(E_7)| = 240$ M-classes, with representatives s_i $(1 \leq i \leq k)$, say. Now $|W(M)| = 2^93^3$, so it follows that for each i, $W(C_M(s_i))$ has order at least $2^93^3/240$, hence at least 58. Also $C_M(s_i)$ is connected, and it follows easily that $|C_{M_\sigma}(s_i)| > q^{10}$. Consequently, estimating $|s^{G_\sigma} \cap M_\sigma|$ as in the proof of 4.5, we deduce that

$$|s^{G_\sigma} \cap M_\sigma| < \frac{240.|G_2(q)|.|F_4(q)|}{q^{10}},$$

which contradicts 5.3.

Likewise, for $M = G_2C_3 < E_7$, we have $|W(G) : W(E_6)| = 56$, $|W(M)| = 2^6 3^2$, so $|W(C_M(s_i))| > 10$. In this case $|C_M(s_i) : C_M(s_i)^0| \le 2$, so it follows that $C_M(s_i)$ contains $A_1^3T_2$, C_2T_3 or A_2T_3. The result follows as before, except when $q = 2$. In this case s has order 3, and we have $|s^{G_\sigma} \cap M_\sigma| < i_3(G_2(q) \times C_3(q))$. By 1.3 this is less than q^{31}, contrary to 5.3.

A similar argument handles the case where $G = E_6, M = A_2G_2$ and $x = s$. Here $|W(C_M(s_i))| \ge 3$, so either $C_M(s_i)$ is connected and contains $A_1^2T_2$ or A_2T_2, or $|C_M(s_i)/C_M(s_i)^0| = 3$ and $|s| = 3$. In the first case we obtain the result as before, and in the second we use $|s^{G_\sigma} \cap M_\sigma| \le i_3(M_\sigma)$ together with 1.3 to contradict 5.3. And for $M = B_3$, we have

$$|s^{G_\sigma} \cap M_\sigma| < \frac{|W(G) : W(D_5)|.|B_3(q)|}{(T_3)_\sigma} \le \frac{27|B_3(q)|}{(q-1)^3}.$$

This contradicts 5.3 provided $q > 5$. And for $q \le 5$ we have $r = |s| \le 5$, and using $|s^{G_\sigma} \cap M_\sigma| < i_r(B_3(q))$ together with 1.3 gives the result.

When $M = A_2G_2$ with $x = \tau$, we use the bound $\mathrm{fpr}(\tau, \Omega) \le i_2(M_\sigma)/|\tau^L|$ to obtain the result.

Next consider $G = F_4, M = A_1G_2$. By [**47**, Section 2] we have

$$L(G) \downarrow A_1G_2 = L(A_1G_2) \oplus (V(4) \otimes V(\lambda_1)).$$

Using this we check that all involutions in M not in the A_1 factor act on $L(G)$ as $(1^{24}, (-1)^{28})$, hence have centralizer A_1C_3. As $C_G(s) = B_4$, it follows that $s^G \cap M \subseteq A_1$ giving a contradiction to 5.3. And if $M = G_2\,(p = 7)$ then $|s^{G_\sigma} \cap H| \le i_2(H) < q^{10}$, contrary to 5.3.

The case with $G = G_2$ is trivial (just use estimates for class sizes in G_σ, compared to $|M|$), so it remains to deal with the case $G = E_7, M = A_1F_4$. For the unipotent case $x = u$, we may assume that $|u^L| < q^{e_G}|u^L \cap H| < q^{67}$, and hence u is in one of the unipotent classes $2A_1, 3A_1'', 3A_1', A_2$ of G (see [**40**, 1.7]). Write $u = u_0u_1$ with $u_0 \in A_1, u_1 \in F_4$. There are at most 20 unipotent classes in F_4 (see [**63**, **65**]), so we may assume that

$$|u^H| > \frac{|u^{G_\sigma}|}{20q^{e_G}} > \frac{q^{40}}{20}.$$

It follows that u_1 lies in one of the classes $C_3(a_1), \ldots, F_4$ of F_4, listed in order as in [**38**, Table 4]. However, by [**38**], elements in each of these classes have more than one Jordan block of size 5 or more on $L(F_4)$, whereas u has at most one such block on $L(G)$, a contradiction.

Now suppose $x = s$, a semisimple element. If $|s| = 2$ then by 1.3, $|s^{G_\sigma} \cap H| \le i_2(H) < q^{31}$, contrary to 5.3. Hence s has odd (prime) order. Moreover, we have $|s^{G_\sigma}| < |H|q^{h_G} < q^{77}$, whence (by [**40**, 1.1] for example) $C_G(s)$ must be T_1E_6 or T_1D_6. Write F for the factor F_4 of M. The proof of [**40**, 6.3] shows that there is a rank 1 torus $T_1 < F$ such that $s \in T_1$, and moreover that $p=2$ and $C_G(s)=T_1E_6$. There is an element $t\in T_1$ of order 3 such that $C_G(s)=C_G(t)$. As in [**40**, 6.3], we see that $s^{G_\sigma}\cap F_\sigma$ splits into at most three F_σ-classes, with F_4-centralizers T_1B_3, T_1C_3 or A_2A_2; moreover, the centralizer A_2A_2 does not occur when $D=T_1E_6$. It follows that

$$\mathrm{fpr}(s,G_\sigma/M_\sigma)<2\frac{|M_\sigma:(T_1B_3)_\sigma|}{|s^{G_\sigma}|}<\frac{1}{q^{h_G}},$$

as required.

Finally, if $x=\phi$, a field automorphism, then ϕ is a Frobenius morphism of M, so the coset $M_\sigma\phi$ has only one class of elements of (prime) order $|\phi|=r$, and $C_{M_\sigma}(\phi)=A_1(q^{1/r})F_4(q^{1/r})$. The result follows. □

Lemma 5.6. *The conclusion of Theorem 2(III) holds if H is as in Case (2) or (3) of 5.1.*

Proof. In this case the only possibility for H which satisfies the bounds in 5.3 is $G=G_2$, $H=2^3.L_3(2)$ (with $p\neq 2$). For $x=u$, we have the result unless $|u^{G_\sigma}|\le q^{e_G}|u^{G_\sigma}\cap H|$, which forces $q=3$. But elements of order 3 in H act as $J_3^2\oplus J_1$ on $V_7=V_G(\lambda_1)$, so by [**38**] are in class $G_2(a_1)$, which gives $|u^{G_\sigma}|>q^2(q^2-1)(q^6-1)/6$, contradicting the above inequality.

Now consider the semisimple case $x=s$. If s has order 7 then $C_G(s)=A_1T_1$ or T_2, so $|s^{G_\sigma}|>q^9$; and if s has order 3 then it acts on V_7 as $(\alpha,\alpha,\alpha^{-1},\alpha^{-1},1^3)$, where α is a cube root of 1, so has centralizer A_1T_1 in G (rather than A_2). The required bound follows easily for these cases. Finally, if s is an involution then $|s^{G_\sigma}|>q^8$ and we use $\mathrm{fpr}(s,\Omega)\le i_2(H)/|s^{G_\sigma}|$ together with 1.3. □

Lemma 5.7. *The conclusion of Theorem 2(III) holds if H is as in Case (4) of 5.1.*

Proof. In this case, by [**46**, 5.1], there are three kinds of subgroups H of the same type as G:

(A) $G_\sigma=G(q)$, $H=G_\delta=G(q_0)$, where $\delta^r=\sigma$ and $q_0^r=q$;
(B) $G_\sigma=\mathrm{Inndiag}(E_6(q))$, $H=G_{\tau\sigma}=\mathrm{Inndiag}({}^2E_6(q^{1/2}))$, where τ is a graph automorphism of G;
(C) $G_\sigma=F_4(q)$ or $G_2(q)$ and $H={}^2F_4(q)$ or ${}^2G_2(q)$ respectively, where $q=2^{2a+1}$ or 3^{2a+1}.

First observe that if $x=\phi$ or τ, then x acts as a field, graph-field or graph automorphism of H, and the result follows using 1.1. So assume from now on that $x=s$ or u.

We handle Cases (A) and (B) together. If $x=u$ is unipotent, then its class in G_σ and in H is determined by its the labelling of its class in G. Hence $|u^L|$ and $|u^L\cap H|$ can be worked out using the lists of classes and centralizers of unipotent elements to be found in [**59**, **60**, **63**, **65**]. In particular, $|u^L| / |u^L \cap H| = f(q_0)$ is a rational function of q_0 of degree $(r-1)\dim u^G$, and $f(q_0)$ is easily seen to be greater than q^{e_G}, as required.

When $x = s$ is semisimple, let $C = C_G(s)$, and $C^0 = DT$ with D semisimple, T a central torus. Note that $|C : C^0| \leq f$, the order of the fundamental group of G. Then $C_{G_\sigma}(s)$ contains $(DT)_\sigma$ (or $(DT)_{\tau\sigma}$) with index at most f, and likewise for $C_H(s)$. Moreover, $s^{G_\sigma} \cap H$ consists of at most f H-classes (see [**68**, I,3.4 and II,4.4]). It follows that $|s^{G_\sigma}| / |s^{G_\sigma} \cap H| = g(q_0)$ is a rational function of q_0 of degree $(r-1)\dim s^G$, which is easily seen to be greater than q^{h_G}, as required.

Finally, consider Case (C). The conjugacy classes in ${}^2F_4(q)$ and the corresponding classes in $F_4(q)$ are given explicitly in [**64**], and the result follows in this case. In ${}^2G_2(q)$, by [**73**], unipotent elements of prime order have centralizers of size q^3 or $2q^2$, and correspond respectively to elements of $G_2(q)$ with centralizer orders q^6 (class $A_1^{(3)}$ in [**38**]) or $2q^4$ (class $G_2(a_1)$). And semisimple elements in ${}^2G_2(q)$ are either regular, or involutions with centralizer $2 \times L_2(q)$, and correspond to regular elements or involutions with centralizer A_1A_1 in G_2. The result follows. □

Lemma 5.8. *The conclusion of Theorem 2(III) holds if H is as in Case (5) of 5.1, with $F^*(H) \notin \mathrm{Lie}(p)$.*

Proof. In this case the possibilities for $S = F^*(H)$ are given by [**49**].

If $S \cong \mathrm{Alt}_c$, the bounds of 5.3 imply that one of the following holds:

$$\begin{array}{ll} G = E_7: & c = 12 \text{ or } 13, q = 2, x = s, D = T_1E_6 \\ G = E_6: & 9 \leq c \leq 12, q = 2, \text{ and if } x = s \text{ then } D = T_1D_5 \\ G = F_4: & 7 \leq c \leq 10, q = 2, x = u. \end{array}$$

In each case, x has order $r = 2$ or 3, and using $|x^L \cap H| \leq i_r(S_c)$, we obtain the result using 1.3 and also the remark after 5.3.

Next suppose S is a sporadic group. Here [**49**] and the bounds of 5.3 (together with the remark after 5.3) imply that one of the following holds:

$$\begin{array}{ll} G = E_6: & S = Fi_{22} (q = 2 \text{ or } 4) \\ G = F_4: & S = J_2 (q = 2) \\ G = G_2: & S = J_1 (q = 11),\ J_2 (q = 4). \end{array}$$

First consider $G = E_6, S = Fi_{22}$. If $q = 2$ then $L = G'_\sigma = {}^2E_6(2)$ by Lagrange's theorem. The embeddings of Fi_{22} in ${}^2E_6(2)$ are identified and studied in [**31**]; there are precisely three conjugacy classes of subgroups Fi_{22} in ${}^2E_6(2)$, permuted by $\mathrm{Out}(L) \cong S_3$. Referring to [**13**, pp. 192, 156], we check that the irreducible character χ_{1938} of L restricts to each subgroup Fi_{22} as a sum $\chi_1 + \chi_{78} + \chi_{429} + \chi_{1430}$ of irreducible characters. From this it is easy to see which classes in a subgroup $Fi_{22}.2$ lie in which classes of $L.2$, hence to calculate $|x^L \cap H|/|x^L|$ for each $x \in L.2$ of prime order, and to show that it satisfies the required bound. If $q = 4$ then $G'_\sigma = E_6(4)$ by Lagrange, and $G_\sigma = \mathrm{Inndiag}(E_6(4)) = E_6(4).3$. The bound in 5.3 shows that $x = s$ and $D = T_1D_5$. Then $D_\sigma = 3 \times D_5(4)$, with central 3-elements lying outside the simple group $E_6(4)$. But this means that $x \notin E_6(4)$, whereas $x \in S < E_6(4)$, a contradiction.

For $G = F_4$, $S = J_2$ with $q = 2$, the bound in 5.3 implies that $x = u$, and we obtain the result using $|x^L \cap H| \le i_2(H)$.

When $G = G_2$, $S = J_1$ with $q = 11$, the only case which does not yield to trivial bounds has $x = s$ of order 3, with $D = A_2$ and $D_\sigma = SU_3(11)$. But from [**32**] we see that an element of order 3 in J_1 has trace 1 on the 7-dimensional G_2-module $V(\lambda_1)$, whereas x acts on $V(\lambda_1)$ as $(\alpha, \alpha, \alpha, \alpha^{-1}, \alpha^{-1}, \alpha^{-1}, 1)$ (where α is a cube root of 1), hence has trace -2.

For $G_\sigma = G_2(4)$ with $H = J_2$, we use [**13**, p. 97] to see that $\mathrm{fpr}(s, \Omega) \le 7/52$ (with equality for $3A$-elements of J_2, which have centralizer $SL_3(4)$ in $G_2(4)$), and $\mathrm{fpr}(u, \Omega) \le 1/13$ (with equality for $2A$-elements of J_2), as in the conclusion of Theorem 2.

Now suppose that $S \in \mathrm{Lie}(p')$. Then [**49**] and 5.3 show that one of the following holds:

$$\begin{array}{ll} G = E_6: & S = L_4(3), U_4(3), \Omega_7(3), G_2(3), \text{ all with } q = 2 \\ G = F_4: & S = L_4(3) (q = 2),\ {}^3D_4(2) (q = 3) \\ G = G_2: & S = L_2(13), U_3(3). \end{array}$$

For $G = E_6$, the remark after 5.3 shows that $S = \Omega_7(3)$. If x has order $r = 2$ or 3 we use $|x^L \cap H| \le i_r(H)$ and 1.3 to obtain the result. And if x has order greater than 3 then $x = s$ and $|s^{G_\sigma}| \ge |G_\sigma : (T_2D_4)_\sigma| > 2^{46}$, while $|H| < 2^{34}$, giving the conclusion in this case.

Now consider $G = F_4, S = L_4(3)$. For $x = s$ of order greater than 3, we have $|s^{G_\sigma}| > 2^6|H|$, giving the conclusion. And for x of order $r = 2$ or 3 we use $|x^L \cap H| \le i_r(H)$. For $S = {}^3D_4(2), q = 3$, 5.3 forces $x = s$ and $D = B_4$, so x is an involution. From [**32**] we see that the largest class of involutions in S have trace 1 on the 25-dimensional module $V = V_G(\lambda_4)$, whereas involutions with centralizer B_4 have trace -7. Hence $x^{G_\sigma} \cap H$ lies in the smallest class of involutions of S, and the result follows in this case.

Lastly, for $G = G_2$ there are unique classes of subgroups $L_2(13), U_3(3)$. The classes of elements of these subgroups can be identified in G_2 using [**13**] and the action on the 7-dimensional module $V_G(\lambda_1)$. Hence the fixed point ratios can be calculated, and the result follows easily. □

Lemma 5.9. *The conclusion of Theorem 2(III) holds if H is as in Case (5) of 5.1 with $F^*(H) = H(q_0) \in \mathrm{Lie}(p)$, a group of Lie type over $\mathbf{F}_{q_0}$ where $q_0 > 2$.*

Proof. Suppose $S = F^*(H) = H(q_0) \in \mathrm{Lie}(p)$, where q_0 is a power of p and $q_0 > 2$. By the conditions of 4.1(5), $H(q_0)$ is not of the same type as G, and also $H \neq M_\sigma$ for any positive-dimensional closed subgroup M of G. For $G = G_2$ the maximal subgroups of G_σ are known, and none falls into this category; thus $G \neq G_2$.

By [**48**, Theorem 1], one of the following holds:

(a) $q_0 \leq 9$,

(b) $S = A_1(q_0)$, ${}^2B_2(q_0)$, ${}^2G_2(q_0)$ or $A_2^\epsilon(16)$ ($\epsilon = \pm$).

First consider Case (b). For $S = A_1(q_0)$ with $q_0 = p^a$, choose a prime r dividing $p^{2a}-1$ but not dividing p^i-1 for $1 \leq i < 2a$ (by [**74**], such a prime exists, except when $a = 1$, or $p^{2a} = 2^6$; we can assume neither of these hold, by orders). Then r divides $|G_\sigma|$, and it follows that if $q = p^b$ then $2a$ is at most kb, where k is the largest integer such that a factor $q^k - 1$ occurs in the order formula for $|G_\sigma|$. Therefore $q_0 \leq q^{15}, q^9, q^6$ or q^6, according as $G = E_8, E_7, E_6$ or F_4, respectively. This contradicts the bounds in 5.3, except when $G = F_4$. In this case, q_0 cannot be q^6, as this would mean that S had an element of order $(q^6 + 1)/(2, q - 1)$, whereas $F_4(q)$ has no such element. Hence the above argument using the prime r shows that $q_0 \leq q^4$, which gives a contradiction using 5.3 again.

The argument for $S = {}^2B_2(q_0)$ or ${}^2G_2(q_0)$ is similar. Reasoning as above using [**74**], we obtain, for $G = E_8, E_7, E_6, F_4$ respectively, $q_0 \leq q^6, q^3, q^3, q^3$ ($S = {}^2B_2(q_0)$), and $q_0 \leq q^5, q^3, q^2, q^2$ ($S = {}^2G_2(q_0)$). Except for $G = F_4$, this contradicts 5.3. And for $G = F_4$, 5.3 implies that $|x| = 2$ and we use $|x^L \cap H| < i_2(H)$ to obtain a contradiction.

Finally for (b), if $S = A_2^\epsilon(16)$, then 5.3 forces $q = 2$ and $G = E_6$ or F_4. But $E_6^\pm(2)$ does not contain $A_2^\epsilon(16)$: For $\epsilon = -$ this is implied by Lagrange; and $L_3(16)$ contains $L_2(16) \times 5$ lying in a parabolic, whereas no parabolic of $E_6^\pm(2)$ contains such a subgroup.

Now consider Case (a), $q_0 \leq 9$. For $q_0 > 2$, [**44**, Theorem 3] determines all maximal subgroups H such that $F^*(H) = H(q_0)$ as above, and $\mathrm{rk}(S) > \frac{1}{2}\mathrm{rk}(G)$ (where $\mathrm{rk}(S)$ denotes the untwisted rank of S, i.e., the rank of the corresponding simple algebraic group). The conclusion is that for such subgroups, either $H = N_X(M_\sigma)$ with M of positive dimension, or $G = E_8$ and $S = {}^2A_5(5)$ or ${}^2D_5(3)$. In the latter cases the bound of 5.3 is violated. Hence

$$\mathrm{rk}(S) \leq \frac{1}{2}\mathrm{rk}(G).$$

It is shown in [**50**, 1.2] that for such subgroups, $|H| < q^{60}$ or $q^{30} \cdot 4 \log_p q$ if $G = E_8$ or E_7, so these cases are out by 5.3.

Thus $G = E_6$ or F_4. Further, the bounds in 5.3 force either $q_0 > q$, or (G, S) to be one of $(E_6, B_3(q)$ or $C_3(q)), (F_4, B_2(q)$ or $G_2(q))$. Moreover, in these cases, $x = s$, $D = T_1D_5$ if $G = E_6$, and $D = B_4$ if $G = F_4$. In the F_4 cases, x is an involution, and using $|x^L \cap H| < i_2(H)$ we see that 5.3 is violated. And for $G = E_6$, x has order r dividing $q \pm 1$, and we check that $i_r(S) < q^{19}$ (use 1.3 for $r = 2, 3$ and check directly for $r = 5, 7$), again contradicting 5.3.

Therefore $q_0 > q$. Since $q_0 \le 9$, the only possibilities for (q, q_0) are $(3, 9), (2, 4)$ and $(2, 8)$.

Consider the first two cases. Here $q_0 = q^2$, so S is a group over $\mathbf{F}_{q^2}$ of untwisted rank at most $\frac{1}{2}\mathrm{rank}(G)$, of order dividing $|G_\sigma| = |G(q)|$, and satisfying the bounds of 5.3. Inspection shows that the only such possibilities for S are among the following:

$$\begin{array}{ll} G = E_6 : & S = A_3(q^2), B_2(q^2), G_2(q^2) \\ G = F_4 : & S = A_2^\epsilon(q^2), B_2(q^2), G_2(q^2). \end{array}$$

Consider $G = E_6$. As $A_3(q^2)$ has an element of order $(q^8 - 1)/((q^2 - 1)(4, q^2 - 1))$, and G_σ has no such element (see [**5**]), we have $S = B_2(q^2)$ or $G_2(q^2)$. For $x = u$ now use 1.3(iii); for $x = s$, we have $q \ne 2$ by the remark after 5.3, so $q = 3$ and $|s| = 2$ and the result follows using $i_2(H)$ in 1.3; and for $x = \phi$ or τ we also have $|x| = 2$, which again gives the result using $i_2(H)$ in 1.3.

Now let $G = F_4$. Here $S = G_2(q^2)$ is ruled out, as $G_2(q^2)$ contains an element of order $q^4 + q^2 + 1$, whereas $F_4(q)$ has no such element (see [**5**]). If $S = A_2^\epsilon(q^2)$, then 5.3 implies that $x = s$ and $D = B_4$, whence $q = 3$. Now $i_2(H) < 3^{10}$, giving a contradiction by 5.3. Lastly, let $S = B_2(q^2)$. If $x = u$ then $i_p(H) < q^{16}$ by 1.3(iii), contrary to 5.3. Therefore $x = s$, and 5.3 implies that $q = 3$ and $D = B_4$. We argue that in fact, $B_2(9) \not\le F_4(3)$. For suppose $X = B_2(9) < F_4(3)$, and consider the possible nontrivial composition factors for X on the irreducible 25-dimensional $F_4(3)$-module V_{25} over $\mathbf{F}_3$. The irreducible X-modules over $\mathbf{F}_3$ of dimension 25 or less are $V(10) \oplus V(10)^{(3)}$ of dimension 10, $V(02) \oplus V(02)^{(3)}$ of dimension 20, $V(01) \otimes V(01)^{(3)}$ of dimension 16, and $V(10) \otimes V(10)^{(3)}$ of dimension 25. The involutions of $F_4(3)$ act as either $(-1^{16}, 1^9)$ or $(-1^{12}, 1^{13})$ on V_{25}. However, one checks that on any set of composition factors for X of dimension adding to 25, one of the involutions in X does not act as either of these possibilities. Therefore $B_2(9) \not\le F_4(3)$.

This finishes the case where $(q, q_0) = (3, 9)$ or $(2, 4)$. Finally, consider the other case, in which $q = 2, q_0 = 8$. Here Lagrange restricts us to the following possibilities:

$$\begin{array}{ll} G = E_6 : & S = A_2^\epsilon(8), A_3^\epsilon(8), B_2(8) \\ G = F_4 : & S = B_2(8). \end{array}$$

Now $A_3^\epsilon(8)$ has an element of order $(2^{12} - 1)/(2^3 - \epsilon)$, whereas $E_6^\epsilon(2)$ has no such element ([**5**]); and $B_2(8)$ has an element of order 65, which likewise cannot lie in $E_6^\epsilon(2)$. This leaves us with $S = A_2^\epsilon(8)$. Here 5.3 implies that $x = s$ and $D = T_1 D_5$. As we have seen before, this means that x has order 3 and is an outer automorphism of $G'_\sigma = {}^2E_6(2)$, hence acts as an outer automorphism of S. All such are conjugate to a field automorphism of S, so $|x^{G_\sigma} \cap H| \le |S : A_2^\epsilon(2)| < 2^{19}$, contrary to 5.3. □

Lemma 5.10. *The conclusion of Theorem 2(III) holds if H is as in Case (5) of 5.1 with $p = 2$ and $F^*(H) = H(2)$, a group of Lie type over $\mathbf{F}_2$.*

Proof. First consider $G = E_8$. When $q = 2$, $x = s$ and $D = T_1E_7$ we have $|s| = 3$. Hence 5.3 implies the following bounds: If $q = 2$ and $x = s$, then either $i_3(H) > q^{64}$ or $|s^{G_\sigma} \cap H| > q^{80}$; if $q \ge 4$ and $x = s$ then $|s^{G_\sigma} \cap H| > q^{64}$; and if $x = u$ then $i_2(H) > q^{68}$. Noting that $\mathrm{rk}(S) \le 8$ by 1.4, we deduce that $H(2)$ is one of the following:

$$\begin{array}{l} A_8^\epsilon(2), D_7^\epsilon(2), D_8(2), B_7(2) \text{ all with } q = 2, \\ E_7(2) \text{ with } q = 2 \text{ or } 4. \end{array}$$

Suppose $H(2) \neq A_8^\epsilon(2)$. Then $H(2)$ contains $D_6(2)$. The latter contains $(S_3 \times S_3 \times D_4(2)).2 = (3^2 \times D_4(2)).Dih_8$ (where Dih_8 indicates a dihedral group of order 8 induced on the 3^2). Using [**40**, 1.2] we see that either $C_G(3^2)^0 = T_2E_6$, or $C_G(3^2)'$ is a product of classical groups. In the first case, $N_G(T_2E_6) = (T_2E_6).(S_3 \times 2)$ does not induce Dih_8, so this does not occur. In the second, the subgroup $D = D_4(2)$ lies in a factor $D_n\,(4 \le n \le 6)$ of $C_G(3^2)^0$. Since $H^1(D_4(2), V(\lambda_1)) = 0$ (see [**33**]), D lies in a natural connected subgroup D_4 of D_n. Now

$$L(G) \downarrow D_4 = V(\lambda_1)^8 \oplus V(\lambda_3)^8 \oplus V(\lambda_4)^8 \oplus V$$

(see [**62**, 1.8]), where V has composition factors $V(\lambda_2)$ (of dimension 26) and 0^{30}. Pick an element $t \in D$ of order 15. Then t lies in a subgroup A_3 of D_4, and in a torus $T_1 = \{\mathrm{diag}(\alpha, \alpha^2, \alpha^4, \alpha^8) : \alpha \in K\}$ of this A_3. One checks that $C_V(t) = C_V(T_1)$, from which it follows that D and $D_4 = \langle D, T_1\rangle$ stabilize the same subspaces of $L(G)$. Therefore H and $\overline{H} = \langle H, D_4\rangle$ stabilize the same subspaces of $L(G)$.

We claim that H is reducible on $L(G)$. For suppose otherwise. Since by [**48**, Theorem 4], $H(2) = F^*(H)$ is not irreducible on $L(G)$, we must have $L(G) \downarrow H(2) = V_1 \oplus \cdots \oplus V_t$, a direct sum of conjugate modules V_i of dimension $248/t$, permuted transitively by an outer automorphism of $H(2)$ of order t. It follows that $t = 2$. Pick an involution $y \in H - H(2)$. Since y interchanges V_1 and V_2, it has 124 Jordan blocks of size 2 in its action on $L(G)$. But there is no such involution in E_8, by [**38**].

Therefore H is reducible on $L(G)$, from which we deduce that $\overline{H} \neq G$. Moreover, $\overline{H}$ is σ-stable, since D_4 is uniquely determined by D, hence is σ-stable. Therefore $H = M_\sigma$ for some σ-stable maximal closed subgroup M of positive dimension in G, contrary to our earlier assumption.

Finally, suppose $H(2) = A_8^\epsilon(2)$. Here 5.3 forces $x = s$ and $D = T_1E_7$. Then s has order 3, so $|x^{G_\sigma} \cap H| \le i_3(H)$, giving a contradiction by 1.3 and 5.3.

Next consider $G = E_7$. Here the bounds in 5.3 imply that $H(2)$ is of one of the following types:

$$A_c^\epsilon(2)\,(c = 6, 7),\ B_d(2)\,(d = 5, 6, 7),\ D_e^\epsilon(2)\,(e = 5, 6, 7),\ F_4(2),\ E_6^\epsilon(2)$$

with $q = 2$ or 4.

If $H(2)$ contains $A_4(2)$, then we can choose an element $t \in H(2)$ of order 31. Because $\langle t\rangle$ is a Sylow subgroup of G_σ, we know that t lies in a subsystem subgroup A_4 of G, and indeed lies in a rank 1 torus

$$T_1 = \{t_\alpha = \mathrm{diag}(\alpha, \alpha^2, \alpha^4, \alpha^8, \alpha^{-15}) : \alpha \in K\} < A_4.$$

From [47, Section 2] we see that if V_{56} denotes the 56-dimensional G-module $V(\lambda_7)$, then $V_{56} \downarrow A_4$ has nontrivial composition factors $V_{A_4}(\lambda_i)$ for $i = 1, 2, 3, 4$. One checks that on the sum of these four modules, t_α has just the eigenvalues $\alpha^{\pm k}$ with $0 \le k \le 15$. It follows that the 31-element t and the torus T_1 fix precisely the same subspaces of V_{56}. Therefore, if we define $\overline{H} = \langle H, T_1\rangle$, then $\overline{H}$ and H fix the same subspaces of V_{56}. By calculating the dimensions of irreducible H-modules of dimension up to 56, we see easily that H is not irreducible on V_{56} except possibly if $H(2) = A_7(2)$, in which case $V_{56} \downarrow H(2)$ could be the sum of two 28-dimensional modules permuted by an outer automorphism of order 2. Excluding the latter possibility, we have $\overline{H} < G$, and now we obtain $H = M_\sigma$ with M of positive dimension as above. And if $H(2) = A_7(2)$, we redefine $\overline{H} = \langle H(2), T_1\rangle$. Then $\overline{H} < G$ and contains $A_7^\epsilon(2)$, from which it follows that $\overline{H} = A_7$, a subsystem subgroup. The embedding of $H(2)$ in this A_7 is determined by [46, 5.1], and we conclude that $H = N_{G_\sigma}(H(2))$ lies in $N_G(A_7)$. Hence $H = M_\sigma$ again.

This leaves the cases $H(2) = {}^2A_6(2)$, ${}^2A_7(2)$, ${}^2D_5(2)$, $F_4(2)$ and ${}^2E_6(2)$ to deal with. A subgroup ${}^2E_6(2)$ must act on V_{56} with nontrivial composition factors just V_{27} and V_{27}^* (where $V_{27} = V_{E_6}(\lambda_1)$). But the acting group on V_{27} is a triple cover $3.{}^2E_6(2)$, which is not simple, so $H(2) \ne {}^2E_6(2)$. And if $H(2) = F_4(2)$ then the only nontrivial composition factors of $V_{56} \downarrow H(2)$ are $V_{26}^i = V_{F_4}(\lambda_i)$ ($i = 1$ or 4). Since $H^1(V_{26}^i, V(0)) = 0$ by [33], it follows that H fixes a nonzero vector in V_{56}, hence lies in a positive-dimensional proper subgroup of E_7, giving a contradiction in the usual way.

Next consider $H(2) = {}^2A_7(2) = U_8(2)$. This contains a subgroup $3 \times SU_6(2)$ with centre 3^2, and from the centralizers of 3-elements in $G_\sigma = E_7(2)$ we see that the $SU_6(2)$ lies in a subsystem group A_5 of G. If $V_{56} = V_G(\lambda_7)$, then $V_{56} \downarrow A_5$ is a direct sum of submodules $V(\lambda_i)$ ($i \in \{1, 2, 3, 4, 5\}$) and trivial modules, so A_5 and $SU_6(2)$ fix the same subspaces of V_{56}. The group $\langle H(2), A_5\rangle$ is reducible on V_{56} and normalized by H (as A_5 is uniquely determined by $SU_6(2)$), so we see in the usual way that $H = M_\sigma$ for some σ-stable maximal subgroup M of positive dimension, contrary to assumption.

Now let $H(2) = U_7(2)$. Here 5.3 implies that $q = 2, x = s$ and $D = T_1E_6$, so s has order 3. Let M be a subgroup $SU_6(2)$ of H, with centre $\langle t\rangle$. Then $C_G(t) = A_2A_5$ or T_1E_6. In the former case, M and A_5 fix the same subspaces of V_{56} and we argue as in the previous paragraph. So suppose $C_G(t) = T_1E_6$ (so s is conjugate to t). As

$$V_{56} \downarrow E_6 = V(\lambda_1) \oplus V(\lambda_6) \oplus 0^2,$$

(see [**47**, Section 2]), t acts on V_{56} as $(\alpha^{27}, (\alpha^{-1})^{27}, 1^2)$, where $\alpha \in K$ is a cube root of 1. It follows that

$$V_{56} \downarrow U_7(2) = V_7 \oplus V_7^* \oplus \wedge^2 V_7 \oplus \wedge^2 V_7^*,$$

where V_7 is the usual module for $U_7(2)$. Moreover, $s \in U_7(2)$ must act as $(\alpha^6, 1)$ or $((\alpha^{-1})^6, 1)$ on V_7. It follows that $|s^{G_\sigma} \cap H| \le 2|U_7(2) : SU_6(2)|$, contrary to 5.3.

It remains to consider $H(2) = {}^2D_5(2)$. Again 5.3 forces $q = 2, x = s$ and $D = T_1E_6$. The action of s on V_{56} is given above. The nontrivial irreducible modules for $H(2)$ of dimension 56 or less are $V(\lambda_i)$ for $i = 1, 2, 4, 5$, of dimension 10, 44, 16, 16 respectively. It is easy to check that there are no combinations possible, of dimension adding to 56, on which an element $s \in H(2)$ of order 3 can act with only 2-dimensional fixed space.

Now let $G = E_6$. Here the bound of 5.3 implies that one of the following holds:

$$\begin{array}{ll} q = 2: & H(2) = {}^2A_5(2),\ B_4(2),\ D_5^\epsilon(2) \text{ or } F_4(2) \\ q = 4: & H(2) = B_d(2), D_d^\epsilon(2) (d = 5, 6) \text{ or } F_4(2). \end{array}$$

As before, if $H(2) = F_4(2)$ then H fixes a 1-space of V_{27}, leading to $H = M_\sigma$, a contradiction.

Consider $q = 2$. The case $H(2) = B_4(2)$ is ruled out for $x = s$ using the remark after 5.3 and the bound for $i_3(B_4(2))$ in 1.3, and for $x = u$ or τ using the bound for $i_2(H)$ in 1.3. If $H(2) = D_5^\epsilon(2)$ then H contains a subgroup $3 \times D_4^{-\epsilon}(2)$, with centre $\langle t\rangle$, say. Then $C_G(t)^0 = T_2D_4$, so $D_4^{-\epsilon}(2) < D_4$ and we see in the usual way that these two subgroups of G fix the same subspaces of V_{27}, leading to $H = M_\sigma$ with M of positive dimension, a contradiction. And if $H(2) = U_6(2)$, the remark after 5.3 forces s to be an outer automorphism of $G'_\sigma = {}^2E_6(2)$, so H contains $U_6(2).3$, which contains $3 \times U_5(2)$. Say $\langle t\rangle$ is the centre of the group. The only possibility for $C_{G'_\sigma}(t)$ is ${}^2D_5(2)$. Then $U_5(2) < A_4 < D_5 < G$, and the subgroups $U_5(2)$ and A_4 fix the same subspaces of V_{27}, giving a contradiction in the usual way.

For $q = 4$, Lagrange forces $G'_\sigma = E_6(4)$. If $x = s$ and $D = T_1D_5$ then $D_\sigma = 3 \times D_5(4)$, with central 3-element lying in $G_\sigma \backslash G'_\sigma$, so this is not possible. Therefore 5.3 implies that either $x = s$ and $|s^{G_\sigma} \cap H| > 4^{28}$, or $x \in \{u, \phi, \tau\}$ and $i_2(H) \ge |u^L \cap H| > 4^{21}$. It follows that $x = s$ and $H(2) = D_6^\epsilon(2)$ or $B_6(2)$. However, both of these contain a subgroup $3 \times D_5^\epsilon(2)$, whereas G'_σ contains no such subgroup.

Finally, when $G = F_4$, 5.3 implies that $H(2)$ is $B_4(2)$ or $D_4^{\epsilon}(2)$, with $q = 2$. However, the classes of such subgroups in $F_4(2)$ are determined in [**41**], and all are of the form M_σ. This completes the proof. □

This completes the proof of Theorem 2(III), apart from the case where $G_\sigma = {}^2F_4(q)$, ${}^2G_2(q)$, ${}^3D_4(q)$ or ${}^2B_2(q)$, which we shall deal with in the next section.

6. Completion of proof of Theorem 2.

By the work in the previous sections, what remains for us to do in order to complete the proof of Theorem 2 is the following:

(i) To prove Theorem 2(c,d) (the case of outer automorphisms);
(ii) to prove Theorem 2 for $L = {}^2F_4(q)'$, ${}^2G_2(q)$, ${}^3D_4(q)$ and ${}^2B_2(q)$.

We carry this out in this section.

We adopt the notation of previous sections, except that in this section our simple exceptional group $L = G'_\sigma = G(q)$ is also allowed to be ${}^2F_4(q)'$, ${}^2G_2(q)$, ${}^3D_4(q)$ or ${}^2B_2(q)$. Let $1 \neq x \in \mathrm{Aut}\, L$ be of prime order. As usual we assume that $X = \langle L, x\rangle$, H is a maximal subgroup of X containing x, and $\Omega = X/H$.

We shall prove the bounds for $\mathrm{fpr}(x, \Omega)$ required for the conclusion of Theorem 2 in Cases (i), (ii) above. First we deal with some of the outer automorphisms.

Lemma 6.1. *The conclusion of Theorem 2(c) holds when $x = \phi$, a field or graph-field automorphism of L of prime order.*

Proof. Suppose $x = \phi$, of prime order r. Observe that ϕ extends to a Frobenius morphism of G such that $\sigma = \phi^r$ or $(\tau\phi)^r$. In Section 5 we handled the case where H is as in (III) of Theorem 2, i.e., not parabolic or reductive of maximal rank. (We did not do this when $L ={}^2 F_4(q)'$ or ${}^2G_2(q)$, but we will cover this in Lemma 6.2 below.) Thus we assume that H is parabolic or of maximal rank.

Suppose first that H is parabolic, so $N_{G_\sigma}(H) = P_\sigma$ where P is a σ-stable parabolic subgroup of G. We claim that ϕ normalizes P. For suppose that P_σ lies in another parabolic subgroup P' of G. Now a Borel subgroup of P_σ contains a regular unipotent element, which lies in a unique Borel subgroup of G. Therefore P and P' share a common Borel subgroup, and since both contain P_σ it follows that $P = P'$. Hence ϕ normalizes P, as claimed.

Now ϕ is a Frobenius morphism of the connected group P. Hence by a standard argument using Lang's theorem, the coset $P_\sigma\phi$ has just one P_σ-class of elements of order r (cf. [**28**, 7.2]), and hence $\phi^{G_\sigma} \cap P_\sigma\phi = \phi^{P_\sigma}$. Moreover, $C_{P_\sigma}(\phi)$ is the corresponding parabolic of the group $C_{G_\sigma}(\phi) = G^{\epsilon}(q^{1/r})$.

Thus, writing $P_\sigma = P(q)$ we have

$$\mathrm{fpr}(\phi, G_\sigma/P_\sigma) = \frac{|G^\epsilon(q^{1/r}) : P(q^{1/r})|}{|G(q) : P(q)|}.$$

Routine computation shows that this is less than $\frac{1}{h_P(q)}$, as required for Theorem 2, except when $G = G_2$ and $r = 2$, in which case it is less that $1/e_L(q)$, as required.

Now suppose that $H = N_X(M_\sigma) = N_X(M)$, where M is a σ-stable reductive subgroup of maximal rank in G. By [**43**], either the subgroup M is as in 4.1; or $(G, p) = (F_4, 2)$ and $M^0 = B_2B_2$ or T_4, with ϕ a graph-field automorphism (i.e., with fixed point group of type ${}^2F_4$); or $(G, p) = (G_2, 3)$ and $M^0 = T_2$ with ϕ a graph-field automorphism. The cases where M^0 is a maximal torus, or is as in 4.3 are quickly ruled out as in that proof.

In the action of M_σ on $L(G)$, there is a unique summand on which M_σ has the same composition factors as it does on $L(M)$. It follows that $M^\phi = M$. As above using Lang's theorem we see that $\phi^{G_\sigma} \cap H\phi$ falls into at most $|M/M^0|$ H-classes. By 1.6, if $z = \dim Z(M^0)$ and $l = \mathrm{rank}(G)$, we have $|M_\sigma^0| \le (q+1)^z q^{\dim M - z}$, while $|M_\phi^0| \ge (q^{1/r} - 1)^l q^{(\dim M - l)/r}$, from which we obtain the bound

$$\mathrm{fpr}(\phi, \Omega) \le \frac{|M/M^0|.(q+1)^z.q^{\dim M - z}}{(q^{1/r} - 1)^l.q^{(\dim M - l)/r}.|\phi^L|}.$$

This gives the bound required for Theorem 2(c) in all cases in 4.1. □

Lemma 6.2. *The conclusion of Theorem 2 holds when $L = {}^2F_4(q)'$ or ${}^2G_2(q)$.*

Proof. First consider $L = {}^2F_4(q)'$. The maximal subgroups of L are determined in [**56**]. For $q > 2$ they are just parabolics, subgroups ${}^2F_4(q_0)$ with $q = q_0^r$, together with the maximal rank subgroups $({}^2B_2(q) \times {}^2B_2(q)).2$, $B_2(q).2, {}^2A_2(q)$ and some maximal torus normalizers. And for $q = 2$ they are these, and also $L_2(25)$ and $L_3(3).2$. The parabolics have been dealt with in Section 2 (together with 6.1). Suppose now that H is one of the maximal rank subgroups or ${}^2F_4(q_0)$.

By 6.1 we may assume that $x = s$ or u, a semisimple or unipotent element in G_σ. The conjugacy classes of G_σ are determined in [**64**], from which we deduce the following information:

(1) ${}^2F_4(q)$ has two classes of involutions, with centralizers $q^9 SL_2(q)$ and $q^{10}\,{}^2B_2(q)$;
(2) either $|s| = 3$ and $C_{G_\sigma}(s) = SU_3(q)$, or $|s^{G_\sigma}| > q^{20}/3$.

Now for $x = u$ we use $\mathrm{fpr}(u, G_\sigma/H) \le i_2(H)/|u^{G_\sigma}|$, which together with 1.3 gives the conclusion. And for $x = s$ the crude bound $\mathrm{fpr}(s, G_\sigma/H) < |H|/|s^{G_\sigma}|$ is sufficient.

Finally, for $q = 2$ the subgroups $L_2(25)$ and $L_3(3).2$ are easily dealt with using [**13**, p. 74].

Now consider $L = {}^2G_2(q)$, $q > 3$. The conjugacy classes of L are found in [**73**], and the maximal subgroups are determined in [**34**]: These are Borel subgroups, subfield subgroups ${}^2G_2(q_0)$, involution centralizers $2 \times L_2(q)$, and some maximal torus normalizers (of maximal order $6(q + \sqrt{3q} + 1)$).

Suppose $H = 2 \times L_2(q)$. If $x = u$ then $|C_L(u)| = 2q^2$, so $\mathrm{fpr}(u, \Omega) = 2(q^2 - 1)/(q(q^3 + 1)(q - 1)) < 1/q^2$; if $x = s$ is an involution then $\mathrm{fpr}(x, \Omega) = (1 + q(q-1))/q^2(q^2 - q + 1)$; and if $x = s$ has order greater than 2 then $|x^{G_\sigma}| \geq |G_\sigma : T_\sigma|$ for some maximal torus T, and the result follows easily. Other subgroups H are handled simply using the bound $\mathrm{fpr}(x, \Omega) \leq i_r(H)/|x^{G_\sigma}|$ and we leave this to the reader. $\square$

Lemma 6.3. *The conclusion of Theorem 2 holds when $L = {}^3D_4(q)$ or ${}^2B_2(q)$.*

Proof. For ${}^2B_2(q)$ the conjugacy classes and maximal subgroups are given in [**70**]: The maximal subgroups are Borel subgroups, subfield subgroups and torus normalizers (of maximal order $4(q + \sqrt{2q} + 1)$). For $H = B$, a Borel subgroup, elements of L fix at most 2 points, while a field automorphism ϕ of order a (where a divides $\log_2 q$) fixes $q^{2/a} + 1$ points, giving $\mathrm{fpr}(\phi, \Omega) = (q^{2/a} + 1)/(q^2 + 1)$. For other maximal subgroups, just use the fact that the smallest semisimple and unipotent classes of elements of prime order in L have sizes $|L|/(q + \sqrt{2q} + 1)$ and $(q^2 + 1)(q - 1)$, respectively, and the result follows easily.

Now consider $L = {}^3D_4(q)$. The conjugacy classes of elements of L can be found in [**17**, **67**]; and the classes of outer automorphisms of prime order are given in 1.1. Long root elements of L have centralizer $q^9SL_2(q^3)$; other unipotent classes have size at least q^{16}. And apart from involutions with centralizer $(SL_2(q) \circ SL_2(q^3)).(2, q - 1)$ and elements with centralizer $(SL_3^\epsilon(q) \circ (q^2 + \epsilon q + 1)).(3, q - \epsilon)$, semisimple classes have size at least $q^{17}(q - 1)$.

The maximal subgroups of L are classified in [**35**]: These are parabolics, maximal rank subgroups $(SL_2(q) \circ SL_2(q^3)).(2, q - 1)$ and $(SL_3^\epsilon(q) \circ (q^2 + \epsilon q + 1)).(3, q - \epsilon)$, subgroups $G_2(q)$, ${}^3D_4(q_0)$, $PGL_3^\delta(q)$ ($q \equiv \delta \bmod 3$), and the torus normalizers $(q^2 \pm q + 1)^2.SL_2(3)$, $(q^4 - q^2 + 1).4$. The possibility $H = {}^3D_4(q_0)$ is handled as in 5.7, while the torus normalizers are small enough to be dealt with easily by counting.

Suppose H is parabolic, say $H \cap L = P_\sigma$, where P is a parabolic subgroup of the ambient algebraic group $G = D_4$ (and $L = G_\sigma$). The case where $x \in L$ was handled in Sections 2 and 3, and the case where x is a field automorphism is dealt with as in the proof of 6.1. So let x be a graph automorphism of order 3. For $p \neq 3$, x lifts to a semisimple automorphism of G, so by [**69**, 7.5] stabilizes a maximal torus T of P. Hence we see as in the proof of [**40**, 3.1] that $C_H(x)$ is a parabolic subgroup of $C_L(x) = G_2(q)$

or $A_2^\epsilon(q)$. Now there are 3 classes of elements of order 3 in Tx, represented by x, xy and xy^{-1}, where y is an element of order 3 in T. From the action on $V_{D_4}(\lambda_2)$ we see that xy and xy^{-1} are not G-conjugate to x, and hence $x^G \cap P = x^P$. It follows that $\mathrm{fpr}(x, \Omega) = |H : C_H(x)|/|L : C_L(x)|$, and the result follows easily from this.

Now suppose $p = 3$ (and x is a graph automorphism of order 3). Note that $P = P_2$ or P_{134}.

First assume $P = P_2$ and $P = P^x$. Here $P = N_G(U)$, where U is a long root subgroup of G and $U = Z(R_u(P))$, so x normalizes U, inducing an automorphism. But $\mathrm{Aut}(U)$ is the multiplicative group of the base field, so contains no elements of order 3. Hence $U < C_G(x)$.

If $C_G(x) = G_2$, then $C_G(x)$ contains just one class of root groups, hence is transitive on the conjugates of P stabilized by x. It follows that $x^G \cap Px = x^P$ and we proceed as for $p \neq 3$. Otherwise, $C_G(x) = C_{G_2}(u)$ for u a long root element of G_2, so this is the derived group of a parabolic of G_2. Here we check that there are two classes of long root groups, the center and noncentral root groups in the unipotent radical. Then $x^G \cap Px = x_1^P \cup x_2^P$, where $C_P(x_1) = C_G(x) = U_5A_1T_1$ and $C_P(x_2) = U_4U_1T_1$. In both cases the stabilizers are connected and an easy check gives the desired inequality.

Now assume $P = P_{134}$ and $P = P^x$. We first determine the conjugacy classes of outer automorphisms. Let τ be a graph automorphism for which $Px = P\tau$ and set $Q = R_u(P)$. Modulo Q the elements of order 3 in $P\tau$ are represented by τ and τu, where u is a long root element of the Levi group (which is centralized by τ). Now $Q/[Q, Q]$ is the direct sum of 3 copies of U_2 which are permuted transitively by τ and fixed by u. Similarly, $[Q, Q]/Z(Q)$ is the sum of 3 copies of U_1 with similar action.

It follows that each element of order 3 in $P\tau$ is P-conjugate to an element of either $Z(Q)\tau$ or $Z(Q)\tau u$. The A_1 Levi factor centralizes τ and acts on $Z(Q)$ as on the natural module. So elements of $Z(Q)\tau$ are P-conjugate to either τ or $\tau U_{1211}(1)$.

Now consider $Z(Q)\tau u$. We may take $u = U_{0100}(1)$, which induces a transvection on $Z(Q)$. Conjugating by elements in a torus centralizing τu and elements of $Z(Q)$ we see that all elements in the coset are conjugate to $\tau u = \tau U_{0100}(1)$ or to $\tau U_{0100}(1)U_{1111}(1)$.

We have therefore shown that elements of order 3 in Px are conjugate to τ, $\tau U_{0100}(1)$, $\tau U_{1211}(1)$, or $\tau U_{0100}(1)U_{1111}(1)$. There are two classes of elements of order 3 in $G\tau$, with representatives τ and τv for v a long root element in $C_G(\tau)$. The last 3 representatives are all of this latter type. This is clear for the first two. For the last representative, note that $U_{0100}(1)U_{1111}(1)$ is a regular unipotent element in an A_2 centralized by τ. Using this and a consideration of the action on $L(G)$ we see from the dimension of the fixed point space that the assertion holds.

We can now complete the argument. If x is conjugate to τ, then from the above we have $x^G \cap Px = x^P$ and we proceed as before. On the other hand if x is conjugate to τu, then $x^G \cap Px$ is the union of 3 conjugacy classes. Each class has a connected centralizer, as is easily checked, and the centralizer has dimension at least 5. At this point we easily get the necessary bounds.

Now suppose $H = G_2(q)$. If x is a long root element of L then it is a long root element of H also (see [**40**, 1.13]), so

$$\mathrm{fpr}(x,\Omega) = \frac{|G_2(q) : q^5 SL_2(q)|}{|{}^3D_4(q) : q^9 SL_2(q^3)|} = \frac{1}{q^4 - q^2 + 1},$$

as required for Theorem 1. For other unipotent classes, we simply use 1.3(iii) to get $i_p(H) = q^{12}$, and hence $\mathrm{fpr}(x,\Omega) \le 1/q^4$. For x semisimple the result is clear using the above information on semisimple classes, except when x is an involution, in which case we have

$$\mathrm{fpr}(x,\Omega) = \frac{|G_2(q)|}{|SL_2(q)|^2} . \frac{|SL_2(q)|\,|SL_2(q^3)|}{|{}^3D_4(q)|},$$

giving the conclusion. When x is a field automorphism, the result follows as in 6.1. Now let x be a graph automorphism of order 3. If $C_L(x) \ne G_2(q)$ then by 1.1, $|x^L| > q^{20}/2$ and the result follows easily, so assume $C_L(x) = G_2(q) = H$. For $p \ne 3$ there are two classes of elements of order 3 in G_2 with centralizers A_2 and A_1T_1. If y belongs to the latter class then consideration of actions on $L(D_4)$ shows that xy is not conjugate to x. Hence $x^L \cap Hx$ consists of x, together with elements xy with y in the 3-element class of H having centralizer A_2, and the required bound follows easily.

Now assume $p = 3$ in this case. Let $xy \in x^L \cap Hx$, where y is an element of order 3 in H. In the notation of [**38**], y lies in one of the classes $A_1, \widetilde{A}_1, \widetilde{A}_1^{(3)}, G_2(a_1)$ of the algebraic group G_2. We know that x is not conjugate to xy with y a long root element of D_4 (see 1.1), so y is not in class A_1. If y is in class $G_2(a_1)$ then y is a regular element in a maximal unipotent subgroup of a maximal rank $A_2 < G_2$. If we multiply y by a root element u in the centre of this maximal unipotent subgroup, we again obtain a regular unipotent element of A_2, so xy is conjugate to xyu with u a long root element centralizing xy; therefore by the previous observation, xy is not conjugate to x. If y lies in class $\widetilde{A}_1^{(3)}$, then y lies in $Z(U)$ for U a maximal unipotent subgroup of G_2. The group $Z(U)$ has the form $U_\alpha U_\beta$, where α is a long root and β a short root in the G_2 system. All elements of $Z(U) \backslash (U_\alpha \cup U_\beta)$ are conjugate by a maximal torus. Hence xy is conjugate to $xyU_\alpha(c)$ for suitable choice of c, and consequently xy is not conjugate to x. It follows that x is only conjugate to itself and to xy with $y \in \widetilde{A}_1$ a short root element. Hence $|x^L \cap H| = 1 + |y^H| \le q^6$, which gives the result.

Finally, when $H = (SL_2(q) \circ SL_2(q^3)).(2, q-1)$, $(SL_3^{\epsilon}(q) \circ (q^2 + \epsilon q + 1)).(3, q-\epsilon)$, or $PGL_3^{\delta}(q)$, the argument is similar and much easier and we leave it to the reader. $\square$

Lemma 6.4. *The conclusion of Theorem 2(d) holds when $G = E_6$ and $x = \tau$, an involutory graph automorphism.*

Proof. Suppose $x = \tau$. By 1.1, $C_G(\tau) = F_4$ or $C_4\,(p \neq 2)$, $C_{F_4}(t)\,(p = 2)$, where t is a long root element of F_4.

Suppose H is parabolic, say $H = N_X(P_\sigma)$, where P is a τ-stable parabolic in G. The case where $p = 2$ was handled in 2.6, so assume $p \neq 2$. Then by [**69**, 7.5], τ normalizes a maximal torus and Borel subgroup of P, and we see as in the proof of [**40**, 5.1] that $C_P(\tau)$ is a parabolic subgroup of $C_G(\tau)$. The result follows easily from this: For example, suppose $P = P_2$. This has Levi factor A_5, on which τ centralizes C_3 or D_3, and it follows that if $C_G(\tau) = F_4$ then $C_P(\tau)$ is a C_3-parabolic of F_4, while if $C_G(\tau) = C_4$ then $C_P(\tau)$ is an A_3-parabolic of C_4. Therefore

$$\mathrm{fpr}(\tau, \Omega) = \frac{|F_4(q) : C_3(q)\text{-parabolic}|}{|E_6^{\epsilon}(q) : A_5^{\epsilon}(q)\text{-parabolic}|} \text{ or } \frac{|C_4(q) : A_3(q)\text{-parabolic}|}{|E_6^{\epsilon}(q) : A_5^{\epsilon}(q)\text{-parabolic}|}$$

giving the result.

The case where H is as in (III) of Theorem 2 was handled in Section 5, so it remains to consider the case where $H = N_X(M_\sigma)$, where M is reductive of maximal rank (and τ-stable). We certainly have $|\tau^{G_\sigma} \cap H| \leq i_2(M_\sigma\langle\tau\rangle)$. If $C_G(\tau) \neq F_4$ then $|\tau^{G_\sigma}| > q^{40}$, and the conclusion is clear using 1.3. So assume $C_G(\tau) = F_4$. Here the conclusion follows in the same way using 1.3, unless $M = T_1D_5$ or A_1A_5.

Consider $M = T_1D_5$. If $p \neq 2$ then as τ inverts T_1, it centralizes an involution $t \in T_1$, and so $C_M(\tau) = C_{F_4}(t)$, which must be B_4 (not A_1C_3, as this does not lie in D_5). Therefore $\mathrm{fpr}(\tau, \Omega) = |M_\sigma : B_4(q)|/|G_\sigma : F_4(q)|$, giving the result. And when $p = 2$, the outer involution classes of $D_5\langle\tau\rangle$ are, in the notation of [**2**] (see [**40**, 1.10]), b_1, b_3 and b_5. Here b_1 is a conjugate of τ and $C_{D_5}(b_1) = B_4$; and $b_3 = b_1u_\alpha$ with u_α a root element of D_5, so b_3 is not G-conjugate to τ (see 1.1). Finally, b_5 acts as J_2^5 on the usual D_5-module V_{10}, and $L(E_6) \downarrow D_5T_1 = L(D_5T_1) \oplus V(\lambda_4) \oplus V(\lambda_5)$, with b_5 interchanging the 16-dimensional modules $V(\lambda_4)$ and $V(\lambda_5)$; if b_5 were conjugate to τ it would centralize a 36-dimensional subspace of $L(D_5)$, but this is clearly not the case as $L(D_5)$ involves $V(\lambda_2) = \wedge^2 V_{10}/N$, where N is 1-dimensional. We conclude that only b_1 is conjugate to τ, and the result follows as before.

Lastly, consider $M = A_1A_5$. If $p \neq 2$ then τ centralizes the involution in $Z(M)$, so $C_M(\tau)$ is an involution centralizer in F_4, hence is A_1C_3, giving $\mathrm{fpr}(\tau, \Omega) = |M_\sigma : (A_1C_3)(q)|/|G_\sigma : F_4(q)|$. If $p = 2$ there are four M-classes of involutions in $M\tau$, with representatives $\tau, \tau u, \tau u', \tau u u'$, where u, u' are long root elements in A_5, A_1 respectively. We know by 1.1 that τu and $\tau u'$

are not G-conjugate to τ. We claim that neither is $\tau uu'$. To see this, embed $uu' \in A_1 \times A_1 < A_3$, with τ inducing a graph automorphism (fixing C_2) on A_3. From the action on $L(A_3)$ we see that $\tau uu'$ cannot centralize a group of type C_2, whence it is conjugate to $\tau u''$ with u'' a root element centralizing τ. The claim follows. $\square$

This completes the proof of Theorem 2.

7. The tables of polynomials for Theorem 2.

This section consists of Tables 7.1A-D containing the polynomials $f_{P,\alpha}(q)$, $f_{P,\beta}(q)$, $g_P(q)$ and $h_P(q)$ which define the bounds in the conclusion of Theorem 2. Recall that $L = G'_\sigma$, a simple group of exceptional Lie type over $\mathbf{F}_q$.

Our convention about labelling maximal parabolics in twisted groups is standard: We label according to the corresponding twisted root system, as described in [**6**, 13.3.8]. For example, the maximal parabolics of ${}^2E_6(q)$ are labelled according to the root system F_4: Thus P_1, P_2, P_3, P_4 correspond respectively to the E_6-parabolics P_2, P_4, P_{35}, P_{16}.

P	poly	$L = E_8(q)$	$E_7(q)$	$E_6(q)$
P_1	$f_{P,\alpha}(q) =$	$q^{11}(q^5-1)(q^2-1)$	$q^6(q^2-1)$	q^4-q^2+1
	$g_P(q) =$	$q^{23}(q^3-1)(q^2-1)$	$q^5(q^5-1)(q^2-1)$	$q^4(q^2-1)$
	$h_P(q) =$	$q^{33}(q-1)^2$	$q^4(q^5-2)(q^3-1)$	$q(q^3-1)(q^2-1)$
P_2	$f_{P,\alpha}(q) =$	$q^{21}(q-1)$	$q^{10}(q-1)$	$q^5(q-1)$
	$g_P(q) =$	$q^{31}(q^2-1)(q-1)$	$q^{15}(q-1)$	$q^3(q^3-1)(q^2-1)$
	$h_P(q) =$	$q^{37}(q^3-3)(q-1)$	$q^{11}(q^4-1)(q^2-1)$	$q^3(q^3-2)(q^2-1)$
P_3	$f_{P,\alpha}(q) =$	$q^{17}(q^5-1)(q-1)$	$q^{11}(q-1)$	$q^6(q-1)$
	$g_P(q) =$	$q^{34}(q^2-q-1)$	$q^{16}(q^2-q-1)$	$q^7(q^2-1)(q-1)$
	$h_P(q) =$	$q^{38}(q^6-2q^5+2q^3-4)$	$q^9(q^4-3)(q^3-3)(q^2-1)$	$q^7(q^2-2)(q-1)$
P_4	$f_{P,\alpha}(q) =$	$q^{23}(q^2-q-1)$	$q^{12}(q-1)^2$	$\frac{1}{2}q^7(q-1)(q-\frac{1}{2})$
	$g_P(q) =$	$\frac{1}{2}q^{38}(q-1)^2$	$q^{17}(q-1)^4$	$q^9(q-1)^3$
	$h_P(q) =$	$q^{45}(q^3-4q^2+4q+1)$	$\frac{1}{3}q^{19}(q^2-\frac{4}{3})(q-1)$	$q^{10}(q^2-3q+3)$
P_5	$f_{P,\alpha}(q) =$	$q^{23}(q-1)^2$	$q^{10}(q^2-1)(q-1)$	$q^6(q-1)$
	$g_P(q) =$	$q^{36}(q-1)^3$	$\frac{1}{2}q^{18}(q-1)(q-\frac{1}{2})$	$q^7(q^2-1)(q-1)$
	$h_P(q) =$	$q^{44}(q^3-3q^2+2q+1)$	$\frac{1}{4}q^{18}(q^2-\frac{3}{4})(q-\frac{1}{2})$	$q^7(q^2-2)(q-1)$
P_6	$f_{P,\alpha}(q) =$	$q^{17}(q^3-1)(q^2-1)$	$q^5(q^3-1)(q^2-1)$	q^4-q^2+1
	$g_P(q) =$	$q^{34}(q-1)^2$	$q^{13}(q^2-1)(q-1)$	$q^4(q^2-1)$
	$h_P(q) =$	$\frac{1}{2}q^{42}(q^2-q-\frac{3}{2})$	$\frac{1}{2}q^{11}(q^3-\frac{3}{4})(q^2-\frac{1}{4})(q-\frac{1}{2})$	$q(q^3-1)(q^2-1)$
P_7	$f_{P,\alpha}(q) =$	$q^{11}(q^4-1)(q^3-1)$	q^6-q^3+1	
	$g_P(q) =$	$q^{25}(q^4-1)(q-1)$	$q^9(q-1)$	
	$h_P(q) =$	$q^{24}(q^5-2)(q^4-4)(q^3-2)$	$\frac{1}{2}q^7(q^4-1)$	
P_8	$f_{P,\alpha}(q) =$	$q^8(q^4-1)$		
	$g_P(q) =$	$q^{18}(q^2-1)$		
	$h_P(q) =$	$q^9(q^6-1)(q^5-1)(q^4-1)$		

Table 7.1A. $L = E_8(q), E_7(q), E_6(q)$.

P	poly	$L = {}^2E_6(q)$	$F_4(q)$
P_1	$f_{P,\alpha}(q) =$	$q^6 - q^3 + 1$	$q^2(q^3 - 1)$
	$f_{P,\beta}(q) =$	$q^9(q - 1)$	$q^5(q - 1)\ (p \neq 2)$, $q^4 - q^2 + 1\ (p = 2)$
	$g_P(q) =$	$q^7(q^3 - 1)(q^2 - 1)$	$q^7(q - 1)\ (p \neq 2)$, $q^4(q^2 - 1)\ (p = 2)$
	$h_P(q) =$	$(q^7 - 1)(q^3 - 1)$	$q^4 - q^2 + 1$
P_2	$f_{P,\alpha}(q) =$	q^6	$q^6(q - 1)$
	$f_{P,\beta}(q) =$	$q^3(q^3 - 1)(q^2 - 1)$	$\frac{1}{2}q^7(q - 1)(q - \frac{1}{2})\ (p \neq 2)$, $q^5(q - 1)\ (p = 2)$
	$g_P(q) =$	$q^{10}(q - 1)$	$q^{10}(q - 3)\ (p \neq 2)$, $q^7(q - 1)^2\ (p = 2)$
	$h_P(q) =$	$q^6(q^2 - 1)$	$q^3(q^2 - 1)(q - 1)$
P_3	$f_{P,\alpha}(q) =$	$q^8(q - 1)$	$q^5(q - 1)$
	$f_{P,\beta}(q) =$	$\frac{1}{2}q^{13}(q - 1)$	$q^6(q - 1)^3\ (p \neq 2)$, $q^6(q - 1)\ (p = 2)$
	$g_P(q) =$	$q^{12}(q^3 - 1)(q - 1)$	$q^{10}(q - 3)\ (p \neq 2)$, $q^7(q - 1)^2\ (p = 2)$
	$h_P(q) =$	$q^{11}(q^2 - 1)(q - 1)$	$q^5(q^2 - 2q + 2)$
P_4	$f_{P,\alpha}(q) =$	$q^9(q - 1)$	$q^4 - q^2 + 1$
	$f_{P,\beta}(q) =$	$q^8(q^3 - 1)(q - 1)$	$q^2(q^2 - 1)^2\ (p \neq 2)$, $q^2(q^3 - 1)\ (p = 2)$
	$g_P(q) =$	$q^{14}(q - 1)$	$q^7(q - 1)\ (p \neq 2)$, $q^4(q^2 - 1)\ (p = 2)$
	$h_P(q) =$	$q^{11}(q - 1)$	$q^2(q^3 - 2)$

Table 7.1B. $L = {}^2E_6(q), F_4(q)$.

P	poly	$L = G_2(q)$	${}^3D_4(q)$	${}^2F_4(q)'$
P_1	$f_{P,\alpha}(q) =$	q^3+1	q^5	q^4
	$f_{P,\beta}(q) =$	$q^2(q-1)\ (p \neq 3)$, $q^2\ (p=3)$	$q^2(q^3-1)$	q^6+1
	$g_P(q) =$	$\frac{1}{3}q^4\ (p \neq 3)$, $q^3+1\ (p=3)$	q^8	$q^4(q^2-1)$
	$h_P(q) =$	$\frac{1}{2}(q^3+1)$	$q^5(q^2-2q+2)$	$\frac{1}{3}q^8$
P_2	$f_{P,\alpha}(q) =$	q^2	q^4	q^6+1
	$f_{P,\beta}(q) =$	$q^3\ (p \neq 3)$, $q^3+1\ (p=3)$	$q^2(q^3-1)$	$q^5(q-1)$
	$g_P(q) =$	$q^4\ (p \neq 3)$, $q^3+1\ (p=3)$	$q^6(q^2-1)$	$q^5(q^2-1)$
	$h_P(q) =$	q^2-q+1	$q^2(q^3-2)$	$(q^6+1)(q^2+1)$

Table 7.1C. $L = G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)'$.

P	poly	$L = {}^2G_2(q)$	${}^2B_2(q)$
B (Borel sgp)	$f_{P,\alpha}(q) =$	q^3+1	q^2+1
	$g_P(q) =$	q^3+1	q^2+1
	$h_P(q) =$	q^2-q+1	q^2+1

Table 7.1D. $L = {}^2G_2(q)$, ${}^2B_2(q)$.

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