DOI: 10.2478/awutm-2019-0017

Analele Universităţii de Vest,
Timişoara
Seria Matematică – Informatică
LVII, 2, (2019), 86– 102

Fixed Point Results for Generalized Contraction Mappings and Cyclical Mappings in b-Metric Spaces Endowed with a Digraph

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Abstract. In this paper, we establish a fixed point theorem for generalized contraction mappings in b-metric spaces endowed with a digraph. As an application of this result, we obtain fixed points of cyclical mappings in the setting of b-metric spaces. Our results extend and generalize several existing results in the literature.

AMS Subject Classification (2000). 54H25; 47H10.
Keywords. b-metric; digraph; cyclical mappings; fixed point.

1 Introduction

In 1922, S. Banach [4] proved the well known Banach contraction theorem in a complete metric space, which became very famous due to its wide applications. In particular, it is an important tool for solving existence and uniqueness problems in many branches of mathematics and applied sciences. Several authors generalized this result in many directions. In 1989, I. A. Bakhtin [3] introduced the concept of b-metric spaces as a generalization of metric spaces and studied some fixed point results in the setting of b-metric spaces.

In recent investigations, the study of fixed point theory combining a graph is a new development in the domain of contractive type single valued and multi valued theory. In 2005, Echenique [12] studied fixed point theory by using graphs. Later on, Espinola and Kirk [13] applied fixed point results in graph theory. Afterwards, combining fixed point theory and graph theory, a series of articles (see [1, 2, 6–8, 16, 27] and references therein) have been dedicated to the improvement of fixed point theory. Many important results of [10, 18, 21–24, 26] have become the source of motivation for many researchers that do research in fixed point theory. The main purpose of this article is to obtain a fixed point theorem for generalized contraction mappings in the framework of b-metric spaces with a graph. Furthermore, we apply our result to derive fixed points of cyclical mappings in metric spaces and b-metric spaces. Finally, some examples are provided to justify the validity of our main result.

2 Some Basic Concepts

We begin with some basic notations, definitions and results in b-metric spaces.

Definition 2.1. [3] *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It is to be noted that the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above fact.

Example 2.1. [20] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.2. [25] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Definition 2.2. [9] *Let (X,d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n,x)=0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x(n\to\infty)$.

(ii) (x_n) is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n,x_m)=0$.

(iii) (X,d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 2.1. [9] In a b-metric space (X,d), the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

Definition 2.3. [15] *Let (X,d) be a b-metric space. A subset $A \subseteq X$ is said to be open if and only if for any $a \in A$, there exists $\epsilon > 0$ such that the open ball $B(a,\epsilon) \subseteq A$. The family of all open subsets of X will be denoted by τ.*

Theorem 2.1. [17] *τ defines a topology on (X,d).*

Theorem 2.2. [17] *Let (X,d) be a b-metric space and τ be the topology defined above. Then for any nonempty subset $A \subseteq X$ we have*

(i) A is closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$;

(ii) if we define $\overline{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \overline{A}$ and for any $\epsilon > 0$, we have $B(x,\epsilon)\cap A \neq \emptyset$.

Theorem 2.3. [17] *Let (X,d) be a b-metric space and τ be the topology defined above. Let $\emptyset \neq A \subseteq X$. The following properties are equivalent:*

(i) A is compact;

(ii) For any sequence (x_n) in A, there exists a subsequence (x_{n_k}) of (x_n) which converges and $\lim_{n_k\to\infty} x_{n_k} \in A$.

Definition 2.4. [15] *A subset A is called sequentially compact if and only if for any sequence (x_n) in A, there exists a subsequence (x_{n_k}) of (x_n) which converges and $\lim\limits_{n_k\to\infty} x_{n_k} \in A$. Also A is called totally bounded if for any $\epsilon > 0$, there exist $x_1, x_2, \cdots, x_n \in A$ such that $A \subseteq B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots B(x_n, \epsilon)$.*

Theorem 2.4. [17] *Let (X, d) be a b-metric space and τ be the topology defined above. Let $\emptyset \neq A \subseteq X$. Then*

(i) A is compact if and only if A is sequentially compact.

(ii) If A is compact, then A is totally bounded.

Corollary 2.5. *Every closed subset of a complete b-metric space is complete.*

Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and ρ be a binary relation over X. Denote $S = \rho \cup \rho^{-1}$. Then

$$x, y \in X,\ xSy \Leftrightarrow x\rho y \text{ or } y\rho x.$$

Definition 2.5. *We say that (X, d, S) is regular if the following condition holds:*

If the sequence (x_n) in X and the point $x \in X$ are such that $x_n S x_{n+1}$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} d(x_n, x) = 0$, then there exists a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} S x$ for all $i \geq 1$.

Definition 2.6. *Let (X, d) be a b-metric space and ρ be a binary relation over X. Then the mapping $T : X \to X$ is called comparative if T maps comparable elements into comparable elements, that is,*

$$x, y \in X,\ xSy \Rightarrow Tx\, S\, Ty.$$

Let Ψ be a class of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(ψ_1) ψ is a nondecreasing function;
(ψ_2) $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Remark 2.2. [19] For each $\psi \in \Psi$, we see that the following assertions hold:

(i) $\lim\limits_{n\to\infty} \psi^n(t) = 0$, for all $t > 0$;

(ii) $\psi(t) < t$ for each $t > 0$;

(iii) $\psi(0) = 0$.

We next review some basic notions in graph theory.

Let (X, d) be a b-metric space. We assume that G is a digraph with the set of vertices $V(G) = X$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x, x) : x \in X\}$. We also assume that G has no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [5, 11, 14]. If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x,\ x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

Definition 2.7. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Then the mapping $T : X \to X$ is called edge preserving if*

$$x, y \in X,\ (x, y) \in E(\tilde{G}) \Rightarrow (Tx, Ty) \in E(\tilde{G}).$$

3 Main Results

Theorem 3.1. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Assume that $T : X \to X$ is edge preserving and there exists $\psi \in \Psi$ with $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$ such that*

$$sd(Tx, Ty) \leq \psi(M_s(x, y)) \tag{3.1}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where

$$M_s(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\}.$$

Suppose also that the following property holds:

$(*)$ *If* (x_n) *is a sequence in* X *such that* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(\tilde{G})$ *for all* $n \geq 1$*, then there exists a subsequence* (x_{n_i}) *of* (x_n) *such that* $(x_{n_i}, x) \in E(\tilde{G})$ *for all* $i \geq 1$.

If there exists $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(\tilde{G})$*, then* T *has a fixed point in* X*. Moreover,* T *has a unique fixed point in* X *if the graph* G *has the following property:*

$(**)$ *If* x, y *are fixed points of* T *in* X*, then* $(x, y) \in E(\tilde{G})$.

Proof. Suppose there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(\tilde{G})$. Define the sequence (x_n) in X such that $x_n = Tx_{n-1}$, $n = 1, 2, \cdots$. Since T is edge preserving, it follows that $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n = 0, 1, 2, \cdots$. We assume that $x_n \neq x_{n-1}$ for every $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then $x_{n-1} = x_n = Tx_{n-1}$ and hence x_{n-1} is a fixed point of T.

Note that

$$\begin{aligned} M_s(x_{n-1}, x_n) &= max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\} \\ &\leq max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\} \\ &= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{3.2}$$

By (ψ_1), it follows that

$$\psi(M_s(x_{n-1}, x_n)) \leq \psi(max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \tag{3.3}$$

For any natural number n, we have by applying conditions (3.1) and (3.3) that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq sd(Tx_{n-1}, Tx_n) \leq \psi(M_s(x_{n-1}, x_n)) \\ &\leq \psi(max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned} \tag{3.4}$$

We shall show that (x_n) is a Cauchy sequence in (X, d).

If $max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then from condition (3.4) and using $\psi(t) < t$ for each $t > 0$, we obtain

$$d(x_n, x_{n+1}) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

which is a contradiction. Therefore,

$$max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n).$$

Thus, it follows from condition (3.4) that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)), \ for \ all \ n \in \mathbb{N}.$$

By repeated use of (ψ_1), we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_1, x_0)), \ for \ all \ n \geq 0.$$

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + s^{m-n-1} d(x_{m-2}, x_{m-1}) + s^{m-n-1} d(x_{m-1}, x_m) \\ &\leq s\psi^n(d(x_1, x_0)) + s^2\psi^{n+1}(d(x_1, x_0)) + \cdots \\ &\quad + s^{m-n-1}\psi^{m-2}(d(x_1, x_0)) + s^{m-n}\psi^{m-1}(d(x_1, x_0)) \\ &= \frac{1}{s^{n-1}} \sum_{i=n}^{m-1} s^i \psi^i(d(x_1, x_0)) \\ &\leq \sum_{i=n}^{m-1} s^i \psi^i(d(x_1, x_0)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$, it follows that

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0.$$

This proves that (x_n) is a Cauchy sequence in (X, d). As (X, d) is complete, there exists $u \in X$ such that $x_n \to u$ i.e., $\lim_{n \to \infty} d(x_n, u) = 0$.

By property $(*)$, there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, u) \in E(\tilde{G})$ for all $i \geq 1$.

Again, using condition (3.1), we have

$$sd(x_{n_i+1}, Tu) = sd(Tx_{n_i}, Tu) \leq \psi(M_s(x_{n_i}, u)), \tag{3.5}$$

where $M_s(x_{n_i}, u) = max\{d(x_{n_i}, u), d(x_{n_i}, x_{n_i+1}), d(u, Tu), \frac{d(x_{n_i}, Tu) + d(u, x_{n_i+1})}{2s}\}$.

Suppose that $d(u, Tu) \neq 0$. Let $\epsilon = \frac{d(u,Tu)}{2s} > 0$. Since $x_{n_i} \to u$, there exists $k_1 \in \mathbb{N}$ such that

$$d(x_{n_i}, u) < \frac{d(u, Tu)}{2s}, \ for \ each \ i \geq k_1. \tag{3.6}$$

Then, for each $i \geq k_1$

$$\begin{aligned} d(x_{n_i}, Tu) &\leq s[d(x_{n_i}, u) + d(u, Tu)] \\ &< s[\frac{d(u, Tu)}{2s} + d(u, Tu)] \\ &\leq (\frac{s}{2} + s)d(u, Tu) \\ &= \frac{3s}{2} d(u, Tu). \end{aligned}$$

As $x_n \to u$, there exists $k_2 \in \mathbb{N}$ such that

$$d(x_{n_i+1}, u) < \frac{d(u, Tu)}{2s}, \; for\ each\ i \geq k_2.$$

Put $k = max\{k_1, k_2\}$. Then, for $i \geq k$, we have

$$\begin{aligned} \frac{d(x_{n_i}, Tu) + d(u, x_{n_i+1})}{2s} &< \frac{1}{2s}[\frac{3s}{2} + \frac{1}{2s}]d(u, Tu) \\ &\leq \frac{1}{2s}[\frac{3s}{2} + \frac{s^2}{2s}]d(u, Tu) \\ &= d(u, Tu). \end{aligned} \tag{3.7}$$

Again, for $i \geq k$, we have

$$\begin{aligned} d(x_{n_i}, x_{n_i+1}) &\leq s[d(x_{n_i}, u) + d(u, x_{n_i+1})] \\ &< s[\frac{1}{2s} + \frac{1}{2s}]d(u, Tu) \\ &= d(u, Tu). \end{aligned} \tag{3.8}$$

Thus, for $i \geq k$, it follows from conditions (3.6), (3.7) and (3.8) that

$$max\{d(x_{n_i}, u), d(x_{n_i}, x_{n_i+1}), d(u, Tu), \frac{d(x_{n_i}, Tu) + d(u, x_{n_i+1})}{2s}\} = d(u, Tu).$$

Therefore, for $i \geq k$, we obtain from (3.5) that

$$sd(x_{n_i+1}, Tu) \leq \psi(d(u, Tu)). \tag{3.9}$$

By using condition (3.9), for $i \geq k$, we have

$$\begin{aligned} d(u, Tu) &\leq sd(u, x_{n_i+1}) + sd(x_{n_i+1}, Tu) \\ &\leq sd(u, x_{n_i+1}) + \psi(d(u, Tu)). \end{aligned}$$

Taking limit as $i \to \infty$, we get

$$d(u, Tu) \leq \psi(d(u, Tu)),$$

which is a contradiction, since $d(u, Tu) > 0$. Therefore, $d(u, Tu) = 0$ and so, $Tu = u$, i.e., u is a fixed point of T.

For uniqueness, assume that $v(\neq u) \in X$ is another fixed point of T. Then, by property $(**)$, we have $(u, v) \in E(\tilde{G})$. Then,

$$d(u, v) = d(Tu, Tv) \leq sd(Tu, Tv)) \leq \psi(M_s(u, v)),$$

where

$$M_s(u, v) = max\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv) + d(v, Tu)}{2s}\} = d(u, v).$$

Thus,

$$0 < d(u, v) \leq \psi(d(u, v)),$$

which is a contradiction, since $\psi(t) < t$ for each $t > 0$.

So, it must be the case that, $d(u, v) = 0$ and hence, $u = v$.

Therefore, T has a unique fixed point in X. □

Corollary 3.2. *Let (X, d) be a complete metric space and ρ be a binary relation over X. Assume that $T : X \to X$ is a comparative map and*

$$x, y \in X,\ xSy \Rightarrow d(Tx, Ty) \leq \psi(M_1(x, y)),$$

where $\psi \in \Psi$ and $S = \rho \cup \rho^{-1}$. Suppose also that the following conditions hold:

(i) (X, d, S) is regular;

(iii) there exists $x_0 \in X$ such that $x_0 S\, Tx_0$.

Then T has a fixed point in X. Moreover, T has a unique fixed point in X if the following property holds:
If x, y are fixed points of T in X, then xSy.

Proof. The proof follows from Theorem 3.1 by taking $G = (V(G), E(G))$ where $V(G) = X,\ E(G) = \{(x, y) \in X \times X : xSy\} \cup \Delta$ and $s = 1$. □

Corollary 3.3. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping. Assume that there exists $\psi \in \Psi$ with $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$ such that*

$$d(Tx, Ty) \leq \frac{1}{s}\psi(M_s(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. $\square$

Corollary 3.4. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be such that*

$$sd(Tx, Ty) \leq h\,max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\}$$

for all $x, y \in X$, where $0 < h < \frac{1}{s}$ is a constant. Then T has a unique fixed point in X.

Proof. The proof can be obtained from Theorem 3.1 by taking $G = G_0$ and $\psi(t) = ht$ for each $t \geq 0$, where $h \in (0, \frac{1}{s})$ is a fixed number. $\square$

Corollary 3.5. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta[d(x, Ty) + d(y, Tx)] \quad (3.10)$$

for all $x, y \in X$, where α, β, γ, $\delta > 0$ and $\alpha + \beta + \gamma + 2s\delta < \frac{1}{s^2}$. Then T has a unique fixed point in X.

Proof. Condition (3.10) gives that

$$sd(Tx, Ty) \leq (\alpha s + \beta s + \gamma s + 2s^2\delta)M_s(x, y)$$

for all $x, y \in X$. Taking $h = \alpha s + \beta s + \gamma s + 2s^2\delta$, it follows that $h \in (0, \frac{1}{s})$. Now applying Corollary 3.4, we obtain the desired result. $\square$

Finally, we provide some examples to justify the validity of our main result.

Example 3.1. Let $X = [0, \infty)$ with $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : x,\, y \in [0, 1]\}$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{3}, & \text{if } 0 \leq x \leq 1, \\ 3x, & \text{if } x > 1. \end{cases} \quad (3.11)$$

Then, $(x, y) \in E(\tilde{G})$ implies $(Tx, Ty) \in E(\tilde{G})$ i.e., T is edge preserving. Take $\psi(t) = \frac{t}{4}$ for each $t \geq 0$.

If x, $y \in [0,1]$, then $(x,y) \in E(\tilde{G})$ and

$$\begin{aligned} sd(Tx,Ty) &= s\left|\frac{x}{3} - \frac{y}{3}\right|^2 = \frac{2}{9}|x-y|^2 \\ &= \frac{8}{9}\cdot\frac{1}{4}d(x,y) \le \psi(d(x,y)) \le \psi(M_s(x,y)). \end{aligned}$$

Thus,

$$sd(Tx,Ty) \le \psi(M_s(x,y)),$$

for all x, $y \in X$ with $(x,y) \in E(\tilde{G})$.

Let (x_n) be a sequence in X and $x \in X$ be such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \ge 1$. Then, either (x_n) is a constant sequence or $x_n \in C$ for all $n \ge 1$, where $C = [0,1]$. In the former case, $(x_n, x) \in \Delta$ for all $n \ge 1$. But in the latter case, $x \in C$, C being closed and hence $(x_n, x) \in E(\tilde{G})$. This proves that property $(*)$ holds. Moreover, $(x_0, Tx_0) \in E(\tilde{G})$ for $x_0 = 1$. Furthermore, the graph G has the property $(**)$. Thus, we have all the conditions of Theorem 3.1 and 0 is the unique fixed point of T in X.

Remark 3.1. It is valuable to note that in Example 3.1, the condition

$$sd(Tx,Ty) \le \psi(M_s(x,y))$$

does not hold for all x, $y \in X$.
In fact, for $x = 0$, $y = 4$, we have $sd(Tx,Ty) = 2d(0,12) = 288$ and

$$\begin{aligned} M_s(x,y) &= max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\} \\ &= max\left\{16,\ 0,\ 64,\ \frac{144+16}{4}\right\} \\ &= 64, \end{aligned}$$

which implies that,

$$sd(Tx,Ty) > \psi(M_s(x,y)).$$

We now examine the strength of the hypothesis made in Theorem 3.1. The following example shows that the second part of Theorem 3.1 shall fall through by dropping the property $(**)$ of the graph G.

Example 3.2. Let $X = \{2,\ 4,\ 6,\ 8\}$ with $d(x,y) = |x-y|^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(2,6),\ (4,8)\}$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 6, & \text{if } x \in \{2,6\}, \\ 4, & \text{if } x \in \{4,8\}. \end{cases}$$

Obviously, T is edge preserving.

Take $\psi(t) = \frac{t}{4}$ for each $t \geq 0$. Then it is easy to verify that

$$sd(Tx, Ty) \leq \psi(M_s(x, y)),$$

for all x, $y \in X$ with $(x, y) \in E(\tilde{G})$.

Moreover, property $(*)$ holds because if (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} = x$ for all $i \geq 1$ and consequently, it follows that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$. Furthermore, $(x_0, Tx_0) \in E(\tilde{G})$ for $x_0 = 4$. We find that 4 and 6 are fixed points of T in X but $(4, 6) \notin E(\tilde{G})$. It is worth mentioning that there does not exist unique fixed point of T due to lack of property $(**)$ of the graph G.

The following example shows that Theorem 3.1 is invalid without property $(*)$.

Example 3.3. Let $X = [0, 1]$ with $d(x, y) = |x - y|^3$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 4$. Let G be a digraph such that $V(G) = X$ and $E(G) = \{(0, 0)\} \cup \{(x, y) : (x, y) \in (0, 1] \times (0, 1],\ x \geq y\}$.

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{3}, & \text{if } x \in (0, 1] \\ 1, & \text{if } x = 0. \end{cases}$$

Take $\psi(t) = \frac{t}{6}$ for each $t \geq 0$. Then T is edge preserving and

$$\begin{aligned} sd(Tx, Ty) &= \frac{4}{27} d(x, y) = \frac{8}{9} \cdot \frac{1}{6} d(x, y) \\ &\leq \psi(d(x, y)) \leq \psi(M_s(x, y)), \end{aligned}$$

for all x, $y \in X$ with $(x, y) \in E(\tilde{G})$.

We now verify that property $(*)$ does not hold.

Taking $x_n = \frac{1}{n}$, we observe that (x_n) is a sequence in X with $x_n \to 0$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$ but there exists no subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, 0) \in E(\tilde{G})$. Moreover, $(x_0, Tx_0) \in E(\tilde{G})$ for $x_0 = 1$. Thus, we have all the conditions of Theorem 3.1 except property $(*)$ and T has no fixed point in X due to lack of property $(*)$.

4 Fixed points for cyclical mappings

In this section, we obtain some fixed point results for generalized cyclic contraction mappings in b-metric spaces.

Theorem 4.1. *Let (M,d) be a complete b-metric space with the coefficient $s \geq 1$ and let A, B be nonempty closed subsets of M. Suppose that $T : M \to M$ satisfies the following conditions:*

(i) $T(A) \subseteq B,\ T(B) \subseteq A$;

(ii) there exists $\psi \in \Psi$ with $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$ such that

$$sd(Tx, Ty) \leq \psi(M_s(x, y)),$$

for all $x \in A$, $y \in B$.

Then T has a unique fixed point in $A \cap B$.

Proof. Let us put $X = A \cup B$. Then, in view of hypothesis (i), T is a selfmap of X. As A, B are closed, X is a closed subset of M and hence (X, d) is a complete b-metric space. We define a graph $G = (X, E(G))$ where $E(G) = (A \times B) \cup \Delta$ and $\Delta = \{(x, x) : x \in X\}$. It is easy to verify that $E(\tilde{G}) = (A \times B) \cup (B \times A) \cup \Delta$ and condition (3.1) of Theorem 3.1 holds for all x, $y \in X$ with $(x, y) \in E(\tilde{G})$ and T is edge preserving.

We now verify property $(*)$ of Theorem 3.1.

Let (x_n) be a sequence in X such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$. Denote $I = \{n \in \mathbb{N} : (x_n, x_{n+1}) \in E(G)\}$ and $J = \{n \in \mathbb{N} : (x_n, x_{n+1}) \in E(G^{-1})\}$. As $I \cup J = \mathbb{N}$, at least one of these subsets is infinite. We assume that I is infinite. So, it may be written as a strictly increasing sequence of ranks: $I = \{n(k) : k \geq 0\}$ where $k \longmapsto n(k)$ is strictly increasing and hence $\lim_{k\to\infty} n(k) = \infty$. Denote $m(k) = n(k) + 1$, for $k \geq 0$, which is also a strictly increasing sequence of ranks which tends to infinity. The sequences $(x_{n(k)} : k \geq 0)$ and $(x_{m(k)} : k \geq 0)$ have the properties $\lim_{k\to\infty} x_{n(k)} = \lim_{k\to\infty} x_{m(k)} = x,\ x_{n(k)} \in A,\ x_{m(k)} \in B$, for all $k \geq 0$. As $x \in X = A \cup B$, either $x \in A$ or $x \in B$. If $x \in A$, then $(x_{m(k)}, x) \in B \times A \subseteq E(\tilde{G}), \forall k \geq 0$. On the otherhand, if $x \in B$, then $(x_{n(k)}, x) \in A \times B \subseteq E(\tilde{G}), \forall k \geq 0$. Thus, in any case, we get a subsequence fulfilling the property $(*)$.

Moreover, taking $x_0 \in A$, A being nonempty, it follows that $Tx_0 \in B$ and so $(x_0, Tx_0) \in A \times B \subseteq E(\tilde{G})$. By applying first part of Theorem 3.1, there exists $u \in X$ such that $Tu = u$.

It is to be noted that $u \in A \cap B$. Because, if $u \in A$, then $Tu \in B$ and so $u(= Tu) \in B$ i.e., $u \in A \cap B$. The case $u \in B$ may be treated similarly.

Finally, let $v(\neq u)$ be another fixed point of T in X. Then $u, v \in A \cap B$ and consequently, it follows that $(u, v) \in A \times B \subseteq E(\tilde{G})$. Thus, property $(**)$ of Theorem 3.1 also holds. By applying second part of Theorem 3.1, there exists a unique fixed point of T in $A \cap B$. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Let (M, d) be a complete metric space and let A, B be nonempty closed subsets of M. Suppose that $T : M \to M$ satisfies the following conditions:*

(i) $T(A) \subseteq B,\ T(B) \subseteq A$;

(ii) there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq \psi(M_1(x, y)),$$

for all $x \in A$, $y \in B$.

Then T has a unique fixed point in $A \cap B$.

Remark 4.1. Corollary 4.2 is a generalization of Theorem 3.9 in [26]. Thus, Theorem 4.1 is an extension of Theorem 3.9 in [26] in metric spaces to b-metric spaces.

The following result (see [18, 22]) is an immediate consequence of Corollary 4.2.

Corollary 4.3. *Let (M, d) be a complete metric space and let A, B be nonempty closed subsets of M. Suppose that $T : M \to M$ satisfies the following conditions:*

(i) $T(A) \subseteq B,\ T(B) \subseteq A$;

(ii) there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x \in A$, $y \in B$.

Then T has a unique fixed point in $A \cap B$.

Corollary 4.4. *Let (M, d) be a complete metric space and let A, B be nonempty closed subsets of M. Suppose that $T : M \to M$ satisfies the following conditions:*

(i) $T(A) \subseteq B,\ T(B) \subseteq A$;

(ii) *there exists a constant* $k \in (0,1)$ *such that*

$$d(Tx,Ty) \leq k\, max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\},$$

for all $x \in A$, $y \in B$.

Then T *has a unique fixed point in* $A \cap B$.

Proof. The result follows from Corollary 4.2 by taking $\psi(t) = kt$ for each $t \geq 0$, where $k \in (0,1)$ is a constant. □

Corollary 4.5. *Let* (M,d) *be a complete metric space and let* A, B *be nonempty closed subsets of* M. *Suppose that* $T : M \to M$ *satisfies the following conditions:*

(i) $T(A) \subseteq B,\ T(B) \subseteq A$;

(ii) *there exist* α, β, γ, $\delta > 0$ *with* $\alpha+\beta+\gamma+2\delta < 1$ *such that*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta[d(x,Ty)+d(y,Tx)],$$

for all $x \in A$, $y \in B$.

Then T *has a unique fixed point in* $A \cap B$.

Proof. Condition (ii) implies that

$$d(Tx,Ty) \leq (\alpha+\beta+\gamma+2\delta) M_1(x,y),$$

for all $x \in A$, $y \in B$. The result follows from Corollary 4.4 by taking $k = \alpha+\beta+\gamma+2\delta$. □

Acknowledgement

The authors are grateful to the anonymous reviewers for their valuable comments and helpful suggestions.

References

[1] **S. Aleomraninejad, S. Rezapour, and N. Shahzad**, Fixed point results on subgraphs of directed graphs, *Mathematical Sciences*, **2013**, (2013), 7:41

[2] **M. R. Alfuraidan and M. A. Khamsi**, Caristi fixed point theorem in metric spaces with a graph, *Abstract and Applied Analysis*, **2014**, (2014), Article ID 303484

[3] **I. A. Bakhtin**, The contraction mapping principle in almost metric spaces, *Funct. Anal., Gos. Ped. Inst. Unianowsk*, **30**, (1989), 26-37

[4] **S. Banach**, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3**, (1922), 133-181

[5] **J. A. Bondy and U. S. R. Murty**, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976

[6] **I. Beg, A. R. Butt, and S. Radojevic**, The contraction principle for set valued mappings on a metric space with a graph, *Comput. Math. Appl.*, **60**, (2010), 1214-1219

[7] **F. Bojor**, Fixed point of φ-contraction in metric spaces endowed with a graph, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **37**, (2010), 85-92

[8] **F. Bojor**, Fixed points of Kannan mappings in metric spaces endowed with a graph, *An. St. Univ. Ovidius Constanta*, **20**, (2012), 31-40

[9] **M. Boriceanu**, Strict fixed point theorems for multivalued operators in b-metric spaces, *Int. J. Mod. Math.*, **4**, (2009), 285-301

[10] **S. Czerwik**, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav*, **1**, (1993), 5-11

[11] **G. Chartrand, L. Lesniak, and P. Zhang**, *Graph and digraph*, CRC Press, New York, NY, USA, 2011

[12] **F. Echenique**, A short and constructive proof of Tarski's fixed point theorem, *Internat. J. Game Theory*, **33**, (2005), 215-218

[13] **R. Espinola and W. A. Kirk**, Fixed point theorems in R-trees with applications to graph theory, *Topology Appl.*, **153**, (2006), 1046-1055

[14] **J. I. Gross and J. Yellen**, *Graph theory and its applications*, CRC Press, New York, NY, USA, 1999

[15] **N. Hussain, R. Saadati, and R. P. Agrawal**, On the topology and wt-distance on metric type spaces, *Fixed Point Theory and Applications*, **2014**, (2014), 88

[16] **J. Jachymski**, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.*, **136**, (2008), 1359-1373

[17] **M. A. Khamsi and N. Hussain**, KKM mappings in metric type spaces, *Nonlinear Anal.*, **73**, (2010), 3123-3129

[18] **W. A. Kirk, P. S. Srinivasan, and P. Veeramani**, Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory*, **4**, (2003), 79-89

[19] **P. Kaushik and S. Kumar**, Fixed point results for (α, ψ, ξ)-contractive compatible multi-valued mappings, *Journal of Nonlinear Analysis and Applications*, **2016**, (2016), doi:10.5899/2016/jnaa-00305

[20] **S. K. Mohanta**, Common fixed points in b-metric spaces endowed with a graph, *Math. Vesnik*, **68**, (2016), 140-154

[21] **J. J. Nieto and R. R. López**, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin.(Engl. Ser.)*, **23**, (2007), 2205-2212

[22] **M. Păcurar and I. A. Rus**, Fixed point theory for cyclic φ-contractions, *Nonlinear Anal.*, **72**, (2010), 1181-1187

[23] **M. A. Petric**, Some results concerning cyclical contractive mappings, *General Mathematics*, **18**, (2010), 213-226

[24] **I. A. Rus**, Cyclic representations and fixed points, *Ann. T. Popoviciu Seminar Funct. Eq. Approx. Convexity*, **3**, (2005), 171-178

[25] **J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, and W. Shatanawi**, Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces, *Fixed Point Theory and Applications*, **2013**, (2013), 159

[26] **B. Samet and M. Turinici**, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, *Communications in Mathematical Analysis*, **13**, (2012), 82-97

[27] **J. Tiammee and S. Suantal**, Coincidence point theorems for graph-preserving multi-valued mappings, *Fixed Point Theory and Applications*, **2014**, (2014), 2014:70

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