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Fixed point results for weak contractions in partially ordered *b*-metric space

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Abstract

Objectives: We explore the existence of a fixed point as well as the uniqueness of a mapping in an ordered *b*-metric space using a generalized $(\check{\psi}, \hat{\eta})$-weak contraction. In addition, some results are posed on a coincidence point and a coupled coincidence point of two mappings under the same contraction condition. These findings generalize and build on a few recent studies in the literature. At the end, we provided some examples to back up our findings.

Result: In partially ordered *b*-metric spaces, it is discussed how to obtain a fixed point and its uniqueness of a mapping, and also investigated the existence of a coincidence point and a coupled coincidence point for two mappings that satisfying generalized weak contraction conditions.

Keywords: $(\check{\psi}, \hat{\eta})$-weak contraction, Fixed point, Coincidence and coupled coincidence points, Ordered *b*-metric space

Mathematics Subject Classification: 54H25, 47H10

Introduction

In a wide range of pure and applied mathematics problems, fixed points of mappings that satisfy contractive conditions in extended metric spaces are extremely useful. First, Ran and Reuings [32] described the existence of fixed points in this direction for certain maps in ordered metric space and exhibited matrix linear equations applications. Following that, Nieto et al. [28, 29] expanded the result of [32] to nondecreasing mappings and used their findings to obtain differential equations solutions. Agarwal et al. [3] and O'Regan et al. [30] examined the influence of generalized contractions in ordered spaces at the same time. Bhaskar and Lakshmikantham [11] first developed coupled fixed point theory for some maps, then used the results to find a unique solution to periodic boundary value problems. Following that, Lakshmikantham and Ćirić [22], which were the extensions of [11] involving monotone property to a function in the space, pioneered the idea of coupled coincidence, common fixed point results. [19, 25, 34–37] provide additional information on coupled fixed point effects in various spaces under various contractive conditions.

A *b*-metric space is one of several generalizations of a standard metric space proposed by Bakhtin in his work [9], and widely used by Czerwik in his work [14, 15]. Following that, a lot of progress was made in acquiring the results of fixed points to single valued as well as multi-valued operators in the space, as evidenced by [1, 2, 4–8, 10, 13, 16–18, 20, 21, 23, 24, 26, 27, 31, 38–41].

We demonstrate some fixed points results for mappings in ordered *b*-metric space that satisfy a generalized weak contraction in this paper. The results from [10, 11, 19, 22, 33] are expanded here as well as some examples noted to support the findings at the end of our work.

Preliminaries

The following definitions are subsequently used in our study.

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Definition 2.1 [15] A b-metric is a mapping $\eth : \mathscr{E} \times \mathscr{E} \to [0, +\infty)$ that satisfies the properties below for all $\varepsilon, \wp, \zeta$ in $\mathscr{E}$ and some $s \geq 1$,

(a) $\eth(\varepsilon, \wp) = 0$ if and if $\varepsilon = \wp$,
(b) $\eth(\varepsilon, \wp) = \eth(\wp, \varepsilon)$,
(c) $\eth(\varepsilon, \wp) \leq s(\eth(\varepsilon, \zeta) + \eth(\zeta, \wp))$.

A b-metric space is specified as $(\mathscr{E}, \eth, s)$.

Example 2.2 The space $L_q[0, 1]$, where $0 < q < 1$ of all real functions $f(t), t \in [0, 1]$ such that $\int_0^1 |f(t)|^q dt < \infty$ is a b-metric space if we take $\eth(\varepsilon, \wp) = \int_0^1 (|f(t) - g(t)|^q dt)^{\frac{1}{q}}$, for all $\varepsilon, \wp \in L_q[0, 1]$.

Note 2.3 Every metric space is a b-metric space with $s = 1$, but in general a b-metric space need not necessarily be a metric space, as in below example 2.4 is b-metric space but not a metric space. Thus, the class of b-metric spaces is larger than the class of metric spaces.

Example 2.4 Let $\mathscr{E} = \mathbb{R}$ and define the mapping $\eth : \mathscr{E} \times \mathscr{E} \to \mathbb{R}^+$ by $\eth(\varepsilon, \wp) = |\varepsilon - \wp|^2$, for all $\varepsilon, \wp \in \mathscr{E}$. Then $(\mathscr{E}, \eth)$ is a b-metric space with coefficient $s = 2$.

The generalization of the above Example 2.4 is as follows:

Example 2.5 Let $(\mathscr{E}, d)$ be a metric space and $q \geq 1$ be a given real number. Then $\eth(\varepsilon, \wp) = [d(\varepsilon, \wp)]^q$ is a b-metric on $\mathscr{E}$ with parameter $s \leq 2^{q-1}$.

Definition 2.6 [10, 15] In a b-metric space,

(1) if $\eth(\varepsilon_n, \varepsilon) \to 0$ as $n \to +\infty$ then $\{\varepsilon_n\}$ is said to be convergent to ε.
(2) if $\eth(\varepsilon_n, \varepsilon_m) \to 0$ as $n, m \to +\infty$ then $\{\varepsilon_n\}$ is a Cauchy sequence.
(3) if $(\mathscr{E}, \eth, s)$ is a complete b-metric space then very Cauchy sequence is convergent.

Definition 2.7 [15, 33] If $\mathscr{E}$ is a partial ordered set with respect to an ordered relation $\preceq$ and $\eth$ is a metric on it, then $(\mathscr{E}, \eth, \preceq)$ is a partially ordered metric space. $(\mathscr{E}, \eth, \preceq)$ is complete partially ordered b-metric space, despite the fact that $\eth$ is complete.

Definition 2.8 [33] Let $\mathscr{h} : \mathscr{E} \to \mathscr{E}$ be a mapping. If $\mathscr{h}(\varepsilon) \preceq \mathscr{h}(\wp)$ for all $\varepsilon, \wp \in \mathscr{E}$ with $\varepsilon \preceq \wp$, then $\mathscr{h}$ is called monotone nondecreasing mapping.

Definition 2.9 [12] Let $\mathscr{h}, \mathscr{I} : \mathscr{A} \to \mathscr{A}$ be two mappings, and $\mathscr{A} \neq \emptyset \subseteq \mathscr{E}$. If $\mathscr{h}\varepsilon = \mathscr{I}\varepsilon = \varepsilon$ ($\mathscr{h}\varepsilon = \mathscr{I}\varepsilon$) for $\varepsilon \in \mathscr{A}$, then ε is called a common fixed point (coincidence point) of $\mathscr{h}$ and $\mathscr{I}$.

Definition 2.10 [12] If $\mathscr{h}\mathscr{I}\varepsilon = \mathscr{I}\mathscr{h}\varepsilon$ for all $\varepsilon \in \mathscr{A}$, then $\mathscr{h}$ and $\mathscr{I}$ are commuting.

Definition 2.11 [12, 33] The two self mappings $\mathscr{h}$ and $\mathscr{I}$ are known to be compatible, if $\lim_{n \to +\infty} d(\mathscr{I}\mathscr{h}\varepsilon_n, \mathscr{h}\mathscr{I}\varepsilon_n) = 0$ for every sequence $\{\varepsilon_n\}$ in $\mathscr{E}$ such that $\lim_{n \to +\infty} \mathscr{h}\varepsilon_n = \lim_{n \to +\infty} \mathscr{I}\wp_n = \mu$, for some $\mu \in \mathscr{A}$.

Definition 2.12 [12, 33] If $\mathscr{h}\varepsilon = \mathscr{I}\varepsilon$ for some $\varepsilon \in \mathscr{A}$, then $\mathscr{h}\mathscr{I}\varepsilon = \mathscr{I}\mathscr{h}\varepsilon$, the mappings $\mathscr{h}$ and $\mathscr{I}$ are called weakly compatible.

Definition 2.13 [33] If $\mathscr{h}\varepsilon \preceq \mathscr{h}\wp$ implies $\mathscr{I}\varepsilon \preceq \mathscr{I}\wp$ for each $\varepsilon, \wp \in \mathscr{E}$, then the mapping $\mathscr{I}$ is called monotone $\mathscr{h}$ -nondecreasing.

Definition 2.14 [11] Let $\mathscr{I} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ and $\mathscr{h} : \mathscr{E} \to \mathscr{E}$ are two mappings,

(a) a point $(\varepsilon, \wp) \in \mathscr{E} \times \mathscr{E}$ is coupled coincidence point of $\mathscr{I}$ and $\mathscr{h}$, if $\mathscr{I}(\varepsilon, \wp) = \mathscr{h}\varepsilon$ and $\mathscr{I}(\wp, \varepsilon) = \mathscr{h}\wp$. In particular, if $\mathscr{h}$ is an identity mapping, then $(\varepsilon, \wp)$ is a coupled fixed point of $\mathscr{I}$.
(b) a point $\varepsilon \in \mathscr{E}$ is a common fixed point of $\mathscr{I}$ and $\mathscr{h}$, if $\mathscr{I}(\varepsilon, \varepsilon) = \mathscr{h}\varepsilon = \varepsilon$.
(c) if $\mathscr{I}(\mathscr{h}\varepsilon, \mathscr{h}\wp) = \mathscr{h}(\mathscr{I}\varepsilon, \mathscr{I}\wp)$ for all $\varepsilon, \wp \in \mathscr{E}$, then $\mathscr{I}$ and $\mathscr{h}$ are commuting each other.
(d) If every two elements of $\mathscr{A} \subseteq \mathscr{E}$ are comparable, then the set $\mathscr{A}$ is called a well ordered set.

Definition 2.15 A self mapping $\check{\psi}$ on $[0, +\infty)$ that meets the conditions below is known as an altering distance function:

(a) $\check{\psi}$ is a non-decreasing and continuous function,
(b) $\check{\psi}(\ell) = 0$ if and only if $\ell = 0$.

As seen above, the symbol $\hat{\Phi}$ represents the set of all altering distance functions.

Similarly, $\hat{\Psi}$: {$\hat{\eta}|\hat{\eta}$ *is a lower semi-continuous self mapping on* $[0,+\infty)$ and, $\hat{\eta}(\ell)=0$ if and only if $\ell=0$}.

The presented lemmas under here are frequently used in our main results.

Lemma 2.16 [27] *Let* $\hbar:\mathscr{E}\to\mathscr{E}$ *be a mapping, and* $\mathscr{E}\neq\emptyset$. *Then* $\mathscr{M}\subseteq\mathscr{E}$ *occurs, resulting in* $\hbar\mathscr{M}=\hbar\mathscr{E}$, *where* $\hbar:\mathscr{M}\to\mathscr{E}$ *is one-to-one.*

Lemma 2.17 [4] *Let* $\{\varepsilon_n\}$ *and* $\{\wp_n\}$ *be two sequences and b-convergent to* ε *and* $\wp$ *in a b-metric space* $(\mathscr{E},\eth,\mathrm{s},\preceq)$, *where* $\mathrm{s}>1$. *Then*

$$\frac{1}{\mathrm{s}^2}\eth(\varepsilon,\wp)\le\lim_{n\to+\infty}\inf\eth(\varepsilon_n,\wp_n)\le\lim_{n\to+\infty}\sup\eth(\varepsilon_n,\wp_n)\le\mathrm{s}^2\eth(\varepsilon,\wp).$$

In particular, if $\varepsilon=\wp$, *then* $\lim_{n\to+\infty}\eth(\varepsilon_n,\wp_n)=0$. *In addition, for every* $\tau\in\mathscr{E}$, *we get*

$$\frac{1}{\mathrm{s}}\eth(\varepsilon,\tau)\le\lim_{n\to+\infty}\inf\eth(\varepsilon_n,\tau)\le\lim_{n\to+\infty}\sup\eth(\varepsilon_n,\tau)\le\mathrm{s}d(\varepsilon,\tau).$$

$$\mathscr{P}(\varepsilon_{n-1},\varepsilon_n)=\max\left\{\frac{\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)\left[1+\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1})\right]}{1+\eth(\varepsilon_{n-1},\varepsilon_n)},\frac{\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_n)+\eth(\varepsilon_n,\mathscr{I}\varepsilon_{n-1})}{2s},\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1}),\eth(\varepsilon_n,\mathscr{I}\varepsilon_n),\eth(\varepsilon_{n-1},\varepsilon_n)\right\} \quad (6)$$
$$\leqslant\max\left\{\eth(\varepsilon_n,\varepsilon_{n+1}),\frac{\eth(\varepsilon_{n-1},\varepsilon_n)+\eth(\varepsilon_n,\varepsilon_{n+1})}{2},\eth(\varepsilon_{n-1},\varepsilon_n)\right\}\leqslant\max\{D_n,D_{n-1}\}.$$

Main results

We start this section with the following fixed point theorem in an ordered *b*-metric space.

Theorem 3.1 *Suppose* $(\mathscr{E},\eth,\mathrm{s},\preceq)$ *is a complete partially ordered b-metric space with* $\mathrm{s}>1$. *A mapping* $\mathscr{I}:\mathscr{E}\to\mathscr{E}$ *is continuous and nondecreasing with respect to* $\preceq$. *If* $\varepsilon_0\in\mathscr{E}$ *is such that* $\varepsilon_0\preceq\mathscr{I}\varepsilon_0$ *and the following contraction condition is fulfilled, then* $\mathscr{I}$ *has a fixed point in* $\mathscr{E}$.

$$\check{\psi}(\mathrm{s}\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp))\le\check{\psi}(\mathscr{P}(\varepsilon,\wp))-\hat{\eta}(\mathscr{P}(\varepsilon,\wp)), \quad (1)$$

for $\check{\psi}\in\hat{\Phi},\hat{\eta}\in\hat{\Psi}$ *and for any* $\varepsilon,\wp\in\mathscr{E}$ *such that* $\varepsilon\preceq\wp$ *and where*

$$\mathscr{P}(\varepsilon,\wp)=\max\left\{\frac{\eth(\wp,\mathscr{I}\wp)\left[1+\eth(\varepsilon,\mathscr{I}\varepsilon)\right]}{1+\eth(\varepsilon,\wp)},\frac{\eth(\varepsilon,\mathscr{I}\wp)+\eth(\wp,\mathscr{I}\varepsilon)}{2s},\eth(\varepsilon,\mathscr{I}\varepsilon),\eth(\wp,\mathscr{I}\wp),\eth(\varepsilon,\wp)\right\}. \quad (2)$$

Proof For some $\varepsilon_0\in\mathscr{E}$ with $\mathscr{I}\varepsilon_0=\varepsilon_0$, then the result is trivial. Assuming that $\varepsilon_0\prec\mathscr{I}\varepsilon_0$, we describe a sequence $\{\varepsilon_n\}\subset\mathscr{E}$ by $\varepsilon_{n+1}=\mathscr{I}\varepsilon_n$ for all $n\ge 0$. However, we can deduce the following as $\mathscr{I}$ is nondecreasing,

$$\varepsilon_0\prec\mathscr{I}\varepsilon_0=\varepsilon_1\preceq\mathscr{I}\varepsilon_1=\varepsilon_2\preceq\ldots\preceq\mathscr{I}\varepsilon_{n-1}=\varepsilon_n\preceq\mathscr{I}\varepsilon_n=\varepsilon_{n+1}\preceq\ldots\ldots \quad (3)$$

If $\varepsilon_{n_0}=\varepsilon_{n_0+1}$ for $n_0\in\mathbb{N}$, then ε_{n_0} is a fixed point of $\mathscr{I}$ from (3). Otherwise, for all $n\ge 1$, $\varepsilon_n\neq\varepsilon_{n+1}$. For $n\ge 1$, let $D_n=\eth(\varepsilon_{n+1},\varepsilon_n)$. We know that for every $n\ge 1$, $\varepsilon_{n-1}\prec\varepsilon_n$ and, then the equation (1) becomes

$$\check{\psi}(D_n)=\check{\psi}(\eth(\varepsilon_n,\varepsilon_{n+1}))=\check{\psi}(\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n))\le\check{\psi}(\mathrm{s}\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n))\le\check{\psi}(\mathscr{P}(\varepsilon_{n-1},\varepsilon_n))-\hat{\eta}(\mathscr{P}(\varepsilon_{n-1},\varepsilon_n)). \quad (4)$$

From (4), we get

$$\eth(\varepsilon_n,\varepsilon_{n+1})=\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n)\le\frac{1}{\mathscr{s}}\mathscr{P}(\varepsilon_{n-1},\varepsilon_n), \quad (5)$$

where

If $\max\{D_n,D_{n-1}\}=D_n$ for certain $n\ge 1$, equation (5) is then accompanied by

$$\eth(\varepsilon_n,\varepsilon_{n+1})\le\frac{1}{\mathscr{s}}\eth(\varepsilon_n,\varepsilon_{n+1}),$$

this is a contradiction. Thus, $\max\{D_n,D_{n-1}\}=D_{n-1}$ for $n\ge 1$. Hence, equation (5) becomes

$$\eth(\varepsilon_n,\varepsilon_{n+1})\le\frac{1}{\mathscr{s}}\eth(\varepsilon_n,\varepsilon_{n-1}).$$

Since $\frac{1}{\mathscr{s}}\in(0,1)$, then $\{\varepsilon_n\}$ is a Cauchy sequence from [1, 6, 8, 18]. Also, the completeness of $\mathscr{E}$ gives that $\varepsilon_n\to\mu\in\mathscr{E}$.

We may also deduce the following from the continuity of $\mathscr{I}$,

$$\mathscr{I}\mu = \mathscr{I}(\lim_{n\to+\infty} \varepsilon_n) = \lim_{n\to+\infty} \mathscr{I}\varepsilon_n = \lim_{n\to+\infty} \varepsilon_{n+1} = \mu. \tag{7}$$

As a result, $\mathscr{I}$ in $\mathscr{E}$ has a fixed point μ. □

The continuity assumption on $\mathscr{I}$ is extracted from Theorem 3.1 and used to derive the following theorem.

Theorem 3.2 *In Theorem 3.1, if $\mathscr{E}$ satisfies below condition, then $\mathscr{I}$ has a fixed point.*

$$\begin{aligned}&\text{If a non-decreasing sequence } \{\varepsilon_n\}\\ &\subseteq \mathscr{E} \quad \text{and } \varepsilon_n \to \sigma \text{ then } \varepsilon_n \le \sigma,\\ &\text{for each } n \in \mathbb{N}, \text{ i.e., } \sigma = \sup \varepsilon_n.\end{aligned} \tag{8}$$

Proof We have an increasing sequence $\{\varepsilon_n\} \subseteq \mathscr{E}$ that eventually converges to some $\sigma \in \mathscr{E}$ as a result of Theorem 3.1. But by the hypotheses for all n, $\varepsilon_n \preceq \sigma$, which means that $\sigma = \sup \varepsilon_n$.

We can now assert that σ is a fixed point of $\mathscr{I}$. Assume that $\mathscr{I}\sigma \neq \sigma$. Let

$$\mathscr{P}(\varepsilon_n, \sigma) = \max\left\{\frac{\eth(\sigma, \mathscr{I}\sigma)\left[1 + \eth(\varepsilon_n, \mathscr{I}\varepsilon_n)\right]}{1 + \eth(\varepsilon_n, \sigma)}, \frac{\eth(\varepsilon_n, \mathscr{I}\sigma) + \eth(\sigma, \mathscr{I}\varepsilon_n)}{2s}, \eth(\varepsilon_n, \mathscr{I}\varepsilon_n), \eth(\sigma, \mathscr{I}\sigma), \eth(\varepsilon_n, \sigma)\right\} \tag{9}$$

then taking limit as $n \to +\infty$ in the equation (9) and making use of $\lim_{n\to+\infty} \varepsilon_n = \sigma$, we get

$$\lim_{n\to+\infty} \mathscr{P}(\varepsilon_n, \sigma) = \max\{\eth(\sigma, \mathscr{I}\sigma), 0\} = \eth(\sigma, \mathscr{I}\sigma). \tag{10}$$

Since, $\varepsilon_n \preceq \sigma$ for each n, then we obtain the following from equations (1) and (9)

$$\begin{aligned}\check{\psi}(\eth(\varepsilon_{n+1}, \mathscr{I}\sigma)) &= \check{\psi}(\eth(\mathscr{I}\varepsilon_n, \mathscr{I}\sigma)) \le \check{\psi}(s\eth(\mathscr{I}\varepsilon_n, \mathscr{I}\sigma))\\ &\le \check{\psi}(\mathscr{P}(\varepsilon_n, \sigma)) - \hat{\eta}(\mathscr{P}(\varepsilon_n, \sigma)).\end{aligned} \tag{11}$$

Take limit as $n \to +\infty$ in (11) and from equation (10) as well as the properties of $\check{\psi}, \hat{\eta}$, we have

$$\check{\psi}(\eth(\sigma, \mathscr{I}\sigma)) \le \check{\psi}(\eth(\sigma, \mathscr{I}\sigma)) - \hat{\eta}(\eth(\sigma, \mathscr{I}\sigma)) < \check{\psi}(\eth(\sigma, \mathscr{I}\sigma)). \tag{12}$$

This is a contradiction to $\mathscr{I}\sigma \neq \sigma$. Hence, $\mathscr{I}\sigma = \sigma$. □

In the above theorems, the fixed point is unique if $\mathscr{E}$ meets the following condition.

$$\begin{aligned}&\text{There exists a } \sigma \text{ in } \mathscr{E} \text{ that is comparable to}\\ &\varepsilon \text{ and } \wp, \text{ for each } \varepsilon, \wp \in \mathscr{E}.\end{aligned} \tag{13}$$

Theorem 3.3 *If $\mathscr{E}$ assumes the condition (13) in Theorem 3.1 & 3.2, then $\mathscr{I}$ has a unique fixed point in $\mathscr{E}$.*

Proof Theorems 3.1 & 3.2 show that the set of fixed points of $\mathscr{I}$ is nonempty. Assume $\varepsilon^* \neq \wp^*$ are fixed points of $\mathscr{I}$ to ensure uniqueness. Following that,

$$\begin{aligned}\check{\psi}(\eth(\mathscr{I}\varepsilon^*, \mathscr{I}\wp^*)) &\le \check{\psi}(s\eth(\mathscr{I}\varepsilon^*, \mathscr{I}\wp^*))\\ &\le \check{\psi}(\mathscr{P}(\varepsilon^*, \wp^*)) - \hat{\eta}(\mathscr{P}(\varepsilon^*, \wp^*)),\end{aligned} \tag{14}$$

where

$$\mathscr{P}(\varepsilon^*, \wp^*) = \max\left\{\frac{\eth(\wp^*, \mathscr{I}\wp^*)\left[1 + \eth(\varepsilon^*, \mathscr{I}\varepsilon^*)\right]}{1 + \eth(\varepsilon^*, \wp^*)}, \frac{\eth(\varepsilon^*, \mathscr{I}\wp^*) + \eth(\wp^*, \mathscr{I}\varepsilon^*)}{2s}, \eth(\varepsilon^*, \mathscr{I}\varepsilon^*), \eth(\wp^*, \mathscr{I}\wp^*), \eth(\varepsilon^*, \wp^*)\right\}. \tag{15}$$

Therefore from equations (14) and (15), we have

$$\begin{aligned}\check{\psi}(\eth(\varepsilon^*, \wp^*)) &= \check{\psi}(\eth(\mathscr{I}\varepsilon^*, \mathscr{I}\wp^*)) \le \check{\psi}(\eth(\varepsilon^*, \wp^*))\\ &\quad - \hat{\eta}(\eth(\varepsilon^*, \wp^*)) < \check{\psi}(\eth(\varepsilon^*, \wp^*)),\end{aligned} \tag{16}$$

this contradicts to $\varepsilon^* \neq \wp^*$. Hence, $\varepsilon^* = \wp^*$. □

Now, we have the below corollary from Theorems 3.1 to 3.3.

Corollary 3.4 *Let $(\mathscr{E}, \eth, \preceq)$ be a partially ordered b-metric space. Suppose the mappings $\mathscr{I}, \hbar : \mathscr{E} \to \mathscr{E}$ are continuous such that*

(C_1).

$$\check{\psi}(s\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp)) \leq \check{\psi}(\mathscr{P}_{\hbar}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon,\wp)), \quad (17)$$

for every $\varepsilon,\wp \in \mathscr{E}$ *with* $\hbar\varepsilon \preceq \hbar\wp$, $s > 1$, $\check{\psi} \in \hat{\Phi}$, $\hat{\eta} \in \hat{\Psi}$ *and, where*

$$\mathscr{P}_h(\varepsilon,\wp) = \max\left\{\frac{\eth(h\wp,\mathscr{I}\wp)[1+\eth(h\varepsilon,\mathscr{I}\varepsilon)]}{1+\eth(h\varepsilon,h\wp)}, \frac{\eth(h\varepsilon,\mathscr{I}\wp)+\eth(h\wp,\mathscr{I}\varepsilon)}{2s}, \eth(h\varepsilon,\mathscr{I}\varepsilon), \eth(h\wp,\mathscr{I}\wp), \eth(h\varepsilon,h\wp)\right\}. \quad (18)$$

(C_2). $\mathscr{I}\mathscr{E} \subset \hbar\mathscr{E}$ and $\hbar\mathscr{E} \subseteq \mathscr{E}$ is complete,
(C_3). $\mathscr{I}$ is monotone $\hbar$-non-decreasing and
(C_4). $\mathscr{I}$ and $\hbar$ are compatible.

If for some $\varepsilon_0 \in \mathscr{E}$ such that $\hbar\varepsilon_0 \preceq \mathscr{I}\varepsilon_0$, then a pair of mappings $(\mathscr{I},\hbar)$ has a coincidence point in $\mathscr{E}$.

Proof By Lemma 2.16, there exists $\mathscr{M} \subset \mathscr{E}$ such that $\hbar\mathscr{M} = \hbar\mathscr{E}$ and $\hbar : \mathscr{M} \to \mathscr{E}$ is one-to-one. Now define a map $\mathscr{k} : \hbar\mathscr{M} \to \hbar\mathscr{M}$ by $\mathscr{k}(\hbar\varepsilon) = \mathscr{I}\varepsilon$, $\varepsilon \in \mathscr{M}$. Since $\hbar$ is one-to-one on $\mathscr{M}$, $\mathscr{k}$ is well defined. Then, $\hbar\mathscr{M} = \hbar\mathscr{E}$ is complete and then (17) becomes

$$\check{\psi}(s\eth(\mathscr{k}(\hbar\varepsilon),\mathscr{k}(\hbar\wp))) \leq \check{\psi}(\mathscr{P}_{\hbar}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon,\wp)), \quad (19)$$

for every $\varepsilon,\wp \in \mathscr{E}$ with $\hbar\varepsilon \preceq \hbar\wp$ and, where

$$\mathscr{P}_h(\varepsilon,\wp) = \max\Big\{\frac{\eth(h\wp,k\eth(h\wp))[1+\eth(h\varepsilon,k\eth(h\varepsilon))]}{1+\eth(h\varepsilon,h\wp)}, \frac{\eth(h\varepsilon,k\eth(h\wp))+\eth(h\wp,k\eth(h\varepsilon))}{2s}, \eth(h\varepsilon,k\eth(h\varepsilon)), \eth(h\wp,k\eth(h\wp)), \eth(h\varepsilon,h\wp)\Big\}. \quad (20)$$

Let $\varepsilon_0 \in \mathscr{M}$ such that $\hbar\varepsilon_0 \preceq \mathscr{I}\varepsilon_0 = \mathscr{k}(\hbar\varepsilon_0)$. Choose $\varepsilon_1 \in \mathscr{M}$ such that $\hbar\varepsilon_1 = \mathscr{I}\varepsilon_0 = \mathscr{k}(\hbar\varepsilon_0)$. By continuing this process, we obtain a sequence $\{\hbar\varepsilon_n\} \subset \hbar\mathscr{M}$ such that $\hbar\varepsilon_{n+1} = \mathscr{I}\varepsilon_n = \mathscr{k}(\hbar\varepsilon_n)$ for $n \geq 0$. By using the similar argument as in the proof of Theorem 3.1, we obtain that $\{\hbar\varepsilon_n\} \subset \hbar\mathscr{M}$ is a b-Cauchy sequence. Since $\hbar\mathscr{M}$ is complete, there exists $\upsilon \in \hbar\mathscr{M}$ such that $\lim_{n\to+\infty} \hbar\varepsilon_n = \upsilon \in \hbar\mathscr{E}$. Then

$$\lim_{n\to+\infty} \hbar\varepsilon_n = \lim_{n\to+\infty} \mathscr{I}\varepsilon_{n-1} = \upsilon.$$

From the condition (C_4), we have

$$\lim_{n\to+\infty} \eth(\hbar(\mathscr{I}\varepsilon_n),\mathscr{I}(\hbar\varepsilon_n)) = 0. \quad (21)$$

Furthermore, the triangular inequality of b-metric, we have

$$\eth(\mathscr{I}\upsilon,\hbar\upsilon) \leq s\eth(\mathscr{I}\upsilon,\mathscr{I}(\hbar\varepsilon_n)) + s^2\eth(\mathscr{I}(\hbar\varepsilon_n),\hbar(\mathscr{I}\varepsilon_n)) + s^2\eth(\hbar(\mathscr{I}\varepsilon_n),\hbar\upsilon). \quad (22)$$

Taking $n \to +\infty$ in (22) and the continuity of $\mathscr{I}$, $\hbar$ and (21), we get $\eth(\mathscr{I}\upsilon,\hbar\upsilon) = 0$. That is $\mathscr{I}\upsilon = \hbar\upsilon$. Therefore, υ is a coincidence point of $\mathscr{I}$, $\hbar$.

The following result can get from Corollary 3.4 by weakening its hypotheses.

Corollary 3.5 *If* $\mathscr{E}$ *satisfies the following condition in Corollary* 3.4,

$$\text{for very nondecreasing sequence } \{\hbar\varepsilon_n\} \subseteq \mathscr{E} \text{ such that } \hbar\varepsilon_n \to \hbar\sigma, \text{ then } \hbar\varepsilon_n \leq \hbar\sigma\ (n \geq 0), \text{ i.e., } \hbar\sigma = \sup \hbar\varepsilon_n. \quad (23)$$

then, if $\hbar\mu \preceq \hbar(\hbar\mu)$ *for some coincidence point* μ, *a coincidence point exists for the weakly compatible mappings* $(\mathscr{I},\hbar)$. *Moreover,* $(\mathscr{I},\hbar)$ *has only one common fixed point if and only if the set of common fixed points is well ordered.* □

Proof A pair of mappings $(\mathscr{I},\hbar)$ has a coincidence point, according to Theorem 3.3 and Corollary 3.4.

Next, assume that a pair of mappings $(\mathscr{I},\hbar)$ is weakly compatible. Let $\upsilon \in \mathscr{E}$ be a point with $\upsilon = \mathscr{I}\mu = \hbar\mu$. Then, $\mathscr{I}\upsilon = \mathscr{I}(\hbar\mu) = \hbar(\mathscr{I}\mu) = \hbar\upsilon$.

Therefore,

$$\mathscr{P}_{\hbar}(\mu,\nu) = \max\left\{\frac{\eth(h\nu,I\nu)[1+\eth(h\mu,I\mu)]}{1+\eth(h\mu,h\nu)}, \frac{\eth(h\mu,I\nu)+\eth(h\nu,I\mu)}{2s}, \eth(h\mu,I\mu),\eth(h\nu,I\nu),\eth(h\mu,h\nu)\right\}$$
$$= \max\left\{0, \frac{\eth(I\mu,I\nu)}{s}, \eth(I\mu,I\nu)\right\} = \eth(I\mu,I\nu). \tag{24}$$

Thus from equation (17), we get

$$\check{\psi}(\eth(\mathscr{I}\mu,\mathscr{I}\upsilon)) \le \check{\psi}(\mathscr{P}_{\hbar}(\mu,\upsilon)) - \hat{\eta}(\mathscr{P}_{\hbar}(\mu,\upsilon)) \le \check{\psi}(\eth(\mathscr{I}\mu,\mathscr{I}\upsilon)) - \hat{\eta}(\eth(\mathscr{I}\mu,\mathscr{I}\upsilon)). \tag{25}$$

By the property of $\hat{\eta}$, we get $\eth(\mathscr{I}\mu,\mathscr{I}\upsilon) = 0$ implies that $\mathscr{I}\upsilon = \hbar\upsilon = \upsilon$.

Finally, we can deduce from Theorem 3.3 that $(\mathscr{I},\hbar)$ has only one common fixed point if and only if the common fixed points of $(\mathscr{I},\hbar)$ is well ordered. □

Remark 3.6 Theorems 3.1 to 3.3 are respectively the extension of Theorems 2.1,.2.2 & 2.3 of [27].

Remark 3.7 Corollaries 3.4 & 3.5 are the generalizations of Corollaries 2.1 & 2.2 of [12] respectively.

Definition 3.8 Consider a partially ordered b-metric space, $(\mathscr{E},\eth,\preceq)$. A mapping $\mathscr{I}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ is known to be a generalized $(\check{\psi},\hat{\eta})$-contractive mapping with regards to $\hbar:\mathscr{E}\to\mathscr{E}$, if

$$\check{\psi}(s^k\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\zeta,\Im))) \le \check{\psi}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\zeta,\Im)) - \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\zeta,\Im)), \tag{26}$$

for all $\varepsilon,\wp,\zeta,\Im\in\mathscr{E}$ with $\hbar\varepsilon\preceq\hbar\zeta$ and $\hbar\wp\succeq\hbar\Im$, $k>2$, $s>1$, $\check{\psi}\in\hat{\Phi}$, $\hat{\eta}\in\hat{\Psi}$ and where

$$\mathscr{P}_h(\varepsilon,\wp,\zeta,\Im) = \max\left\{\frac{\eth(h\zeta,\mathscr{I}\,\eth(\zeta,\Im))\left[1+\eth(h\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp))\right]}{1+\eth(h\varepsilon,h\zeta)}, \frac{\eth(h\varepsilon,\mathscr{I}\,\eth(\zeta,\Im))+\eth(h\zeta,\mathscr{I}\,\eth(\varepsilon,\wp))}{2s}, \eth(h\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp)),\eth(h\zeta,\mathscr{I}\,\eth(\zeta,\Im)),\eth(h\varepsilon,h\zeta)\right\}$$

Theorem 3.9 *Suppose that $(\mathscr{E},\eth,\preceq)$ is a complete partially ordered b-metric space. A mapping $\mathscr{I}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ satisfies the condition (26) and $\mathscr{I}$, $\hbar$ are continuous, $\mathscr{I}$ has mixed $\hbar$-monotone property and also commutes with $\hbar$. Assume that, if for some $(\varepsilon_0,\wp_0)\in\mathscr{E}\times\mathscr{E}$ such that $\hbar\varepsilon_0\preceq\mathscr{I}(\varepsilon_0,\wp_0)$, $\hbar\wp_0\succeq\mathscr{I}(\wp_0,\varepsilon_0)$ and $\mathscr{I}(\mathscr{E}\times\mathscr{E})\subseteq\hbar(\mathscr{E})$, then $\mathscr{I}$ and $\hbar$ have a coupled coincidence point in $\mathscr{E}$.*

Proof From Theorem 2.2 of [7], there exist two sequences $\{\varepsilon_n\}$ and $\{\wp_n\}$ in $\mathscr{E}$ such that

$$\hbar\varepsilon_{n+1} = \mathscr{I}(\varepsilon_n,\wp_n),\quad \hbar\wp_{n+1} = \mathscr{I}(\wp_n,\varepsilon_n), n\ge 0.$$

In particular, the sequences $\{\hbar\varepsilon_n\}$ and $\{\hbar\wp_n\}$ are non-decreasing and non-increasing in $\mathscr{E}$. Put $\varepsilon=\varepsilon_n,\wp=\wp_n,\zeta=\varepsilon_{n+1},\Im=\wp_{n+1}$ in (26), we get

$$\check{\psi}(s^k\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})) = \check{\psi}(s^k\eth(\mathscr{I}(\varepsilon_n,\wp_n),\mathscr{I}(\varepsilon_{n+1},\wp_{n+1}))) \le \check{\psi}(\mathscr{P}_{\hbar}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1})) - \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1})), \tag{27}$$

where

$$\mathscr{P}_{\hbar}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1}) \le \max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})\}. \tag{28}$$

Therefore from (27), we have

$$\check{\psi}(s^k\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})) \le \check{\psi}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})\}) - \hat{\eta}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})\}). \tag{29}$$

Similarly by taking $\varepsilon=\wp_{n+1},\wp=\varepsilon_{n+1},\zeta=\varepsilon_n,\Im=\varepsilon_n$ in (26), we get

$$\check{\psi}(s^k\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})) \le \check{\psi}(\max\{\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}) - \hat{\eta}(\max\{\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}). \tag{30}$$

We know that $\max\{\check{\psi}(l_1),\check{\psi}(l_2)\} = \check{\psi}\{\max\{l_1,l_2\}\}$ for $l_1,l_2\in[0,+\infty)$. Then by adding (29) and (30) together we get,

$$\check{\psi}(s^k\Gamma_n) \le \check{\psi}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2}),\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}) - \hat{\eta}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2}),\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}), \tag{31}$$

where

$$\mathscr{P}(\varepsilon,\wp,\zeta,\Im)=\max\left\{\frac{\eth(\zeta,\mathscr{I}\,\eth(\zeta,\Im))\left[1+\eth(\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp))\right]}{1+\eth(\varepsilon,\zeta)},\frac{\eth(\varepsilon,\mathscr{I}\,\eth(\zeta,\Im))+\eth(\zeta,\mathscr{I}\,\eth(\varepsilon,\wp))}{2s},\right.$$
$$\left.\eth(\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp)),\eth(\zeta,\mathscr{I}\,\eth(\zeta,\Im)),\eth(\varepsilon,\zeta)\right\}.$$

$$\Gamma_n=\max\{\eth(\hbar\,\varepsilon_{n+1},\hbar\,\varepsilon_{n+2}),\eth(\hbar\,\wp_{n+1},\hbar\,\wp_{n+2})\}. \tag{32}$$

Let us denote,

$$\varkappa_n=\max\{\eth(\hbar\,\varepsilon_n,\hbar\,\varepsilon_{n+1}),\eth(\hbar\,\varepsilon_{n+1},\hbar\,\varepsilon_{n+2}),\eth(\hbar\,\wp_n,\hbar\,\wp_{n+1}),\eth(\hbar\,\wp_{n+1},\hbar\,\wp_{n+2})\}. \tag{33}$$

Hence from equations (29)-(32), we obtain

$$s^k\Gamma_n\le\varkappa_n. \tag{34}$$

Now to claim that

$$\Gamma_n\le\lambda\Gamma_{n-1}, \tag{35}$$

for $n\ge 1$ and $\lambda=\frac{1}{s^k}\in[0,1)$.

Suppose that if $\varkappa_n=\Gamma_n$ then from (34), we get $s^k\Gamma_n\le\Gamma_n$ this leads to $\Gamma_n=0$, since $s>1$ and thus (35) holds. Suppose $\varkappa_n=\max\{\eth(\hbar\,\varepsilon_n,\hbar\,\varepsilon_{n+1}),\eth(\hbar\,\wp_n,\hbar\,\wp_{n+1})\}$, i.e., $\varkappa_n=\Gamma_{n-1}$ then (34) follows (35).

Now from (34), we obtain that $\Gamma_n\le\lambda^n\delta_0$ and hence,

$$\eth(\hbar\,\varepsilon_{n+1},\hbar\,\varepsilon_{n+2})\le\lambda^n\Gamma_0\ \text{ and }\ \eth(\hbar\,\wp_{n+1},\hbar\,\wp_{n+2})\le\lambda^n\Gamma_0, \tag{36}$$

which shows that $\{\hbar\,\varepsilon_n\}$ and $\{\hbar\,\wp_n\}$ in $\mathscr{E}$ are Cauchy sequences by Lemma 3.1 of [20]. Therefore, we can conclude from Theorem 2.2 of [5] that, $\mathscr{I}$ and $\hbar$ have a coincidence point in $\mathscr{E}$. □

Corollary 3.10 *Suppose that $(\mathscr{E},\eth,\preceq)$ is a complete partially ordered b-metric space. A continuous mapping $\mathscr{I}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ has a mixed monotone property and is satisfying the below contraction conditions for all $\varepsilon,\wp,\zeta,\Im\in\mathscr{E}$ such that $\varepsilon\preceq\zeta$ and $\wp\succeq\Im$, $k>2$, $s>1$, $\check{\psi}\in\hat{\Phi}$ and $\hat{\eta}\in\hat{\Psi}$:*

(i). $$\check{\psi}(s^k\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\zeta,\Im)))\le\check{\psi}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im))-\hat{\eta}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)),$$

(ii). $$\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\zeta,\Im))\le\frac{1}{s^k}\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)-\frac{1}{s^k}\hat{\eta}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)),$$

where

If there exists $(\varepsilon_0,\wp_0)\in\mathscr{E}\times\mathscr{E}$ such that $\varepsilon_0\preceq\mathscr{I}(\varepsilon_0,\wp_0)$ and $\wp_0\succeq\mathscr{I}(\wp_0,\varepsilon_0)$, then $\mathscr{I}$ has a coupled fixed point in $\mathscr{E}$.

Theorem 3.11 *The unique coupled common fixed point for $\mathscr{I}$ and $\hbar$ exists in Theorem 3.9, if for every $(\varepsilon,\wp),(\kappa,\ell)\in\mathscr{E}\times\mathscr{E}$ there exists some $(\Lambda,\Upsilon)\in\mathscr{E}\times\mathscr{E}$ such that $(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda))$ is comparable to $(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\wp,\varepsilon))$ and to $(\mathscr{I}(\kappa,\mathscr{I}),\mathscr{I}(\ell,\kappa))$.*

Proof The existence of a coupled coincidence point for $\mathscr{I}$ and $\hbar$ is guaranteed by the Theorem 3.9. Let $(\varepsilon,\wp),(\kappa,\ell)\in\mathscr{E}\times\mathscr{E}$ are two coupled coincidence points of $\mathscr{I}$ and $\hbar$. Now, we assert that $\hbar\,\varepsilon=\hbar\,\kappa$ and $\hbar\,\wp=\hbar\,\ell$. By the hypotheses $(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda))$ is comparable to $(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\wp,\varepsilon))$ and to $(\mathscr{I}(\kappa,\mathscr{I}),\mathscr{I}(\ell,\kappa))$ for some $(\Lambda,\Upsilon)\in\mathscr{E}\times\mathscr{E}$.

Now, assume the following

$$(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\wp,\varepsilon))\le(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda))$$
$$\text{and }(\mathscr{I}(\kappa,\ell),\mathscr{I}(\ell,\kappa))\le(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda)).$$

Suppose $\Lambda_0=\Lambda$ and $\Upsilon_0=\Upsilon$ then there is a point $(\Lambda_1,\Upsilon_1)\in\mathscr{E}\times\mathscr{E}$ such that

$$\hbar\,\Lambda_1=\mathscr{I}(\Lambda_0,\Upsilon_0),\ \ \hbar\,\Upsilon_1=\mathscr{I}(\Upsilon_0,\Lambda_0)\ \ (n\ge 1).$$

As by applying the preceding argument repeatedly, we have the sequences $\{\hbar\,\Lambda_n\}$ and $\{\hbar\,\Upsilon_n\}$ in $\mathscr{E}$ such that

$$\hbar\,\Lambda_{n+1}=\mathscr{I}(\Lambda_n,\Upsilon_n),\ \ \hbar\,\Upsilon_{n+1}=\mathscr{I}(\Upsilon_n,\Lambda_n)\ \ (n\ge 0).$$

Define the sequences in the same way $\{\hbar\,\varepsilon_n\}$, $\{\hbar\,\wp_n\}$ and, $\{\hbar\,\kappa_n\}$, $\{\hbar\,\ell_n\}$ in $\mathscr{E}$ by setting $\varepsilon_0=\varepsilon$, $\wp_0=\wp$ and $\kappa_0=\kappa$, $\ell_0=\ell$. Further, we have that

$$\hbar\,\varepsilon_n\to\mathscr{I}(\varepsilon,\wp),\ \hbar\,\wp_n\to\mathscr{I}(\wp,\varepsilon),$$
$$\hbar\,\kappa_n\to\mathscr{I}(\kappa,\ell),\ \hbar\,\ell_n\to\mathscr{I}(\ell,\kappa)\ \ (n\ge 1). \tag{37}$$

Thus by induction, we get

$$(\hbar\,\varepsilon_n,\hbar\,\wp_n)\le(\hbar\,\Lambda_n,\hbar\,\Upsilon_n)\ \text{ for every } n. \tag{38}$$

As a consequence of (26), we have

$$\check{\psi}(\eth(\hslash\,\varepsilon,\hslash\,\Lambda_{n+1})) \leq \check{\psi}(s^k\eth(\hslash\,\varepsilon,\hslash\,\Lambda_{n+1})) = \check{\psi}(s^k\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\Lambda_n,\Upsilon_n))) \leq \check{\psi}(\mathscr{P}_{\hslash}(\varepsilon,\wp,\Lambda_n,\Upsilon_n)) - \hat{\eta}(\mathscr{P}_{\hslash}(\varepsilon,\wp,\Lambda_n,\Upsilon_n)), \tag{39}$$

where

$$\begin{aligned}\mathscr{P}_h(\varepsilon,\wp,\Lambda_n,\Upsilon_n) &= \max\left\{\frac{\eth(h\Lambda_n,\mathscr{I}\,\eth(\Lambda_n,\Upsilon_n))\left[1+\eth(h\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp))\right]}{1+\eth(h\varepsilon,h\Lambda_n)}, \frac{\eth(h\varepsilon,\mathscr{I}\,\eth(\Lambda_n,\Upsilon_n))+\eth(h\Lambda_n,\mathscr{I}\,\eth(\varepsilon,\wp)}{2s},\right.\\ &\qquad \left.\eth(h\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp)),\eth(h\Lambda_n,\mathscr{I}\,\eth(\Lambda_n,\Upsilon_n)),\eth(h\varepsilon,h\Lambda_n)\right\}\\ &= \max\left\{0,\frac{\eth(h\varepsilon,h\Lambda_n)}{s},\eth(h\varepsilon,h\Lambda_n)\right\} = \eth(h\varepsilon,h\Lambda_n).\end{aligned}$$

Therefore from (39), we have

$$\check{\psi}(\eth(\hslash\,\varepsilon,\hslash\,\Lambda_{n+1})) \leq \check{\psi}(\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n)) - \hat{\eta}(\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n)). \tag{40}$$

As by the similar argument, we acquire that

$$\check{\psi}(\eth(\hslash\,\wp,\hslash\,\Upsilon_{n+1})) \leq \check{\psi}(\eth(\hslash\,\wp,\hslash\,\Upsilon_n)) - \hat{\eta}(\eth(\hslash\,\wp,\hslash\,\Upsilon_n)). \tag{41}$$

Hence from (40) and (41), we have

$$\begin{aligned}\check{\psi}(\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_{n+1}),\eth(\hslash\,\wp,\hslash\,\Upsilon_{n+1})\}) &\leq \check{\psi}(\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n),\eth(\hslash\,\wp,\hslash\,\Upsilon_n)\})\\ &\quad - \hat{\eta}(\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n),\eth(\hslash\,\wp,\hslash\,\Upsilon_n)\})\\ &< \check{\psi}(\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n),\eth(\hslash\,\wp,\hslash\,\Upsilon_n)\}).\end{aligned} \tag{42}$$

Thus the property of $\check{\psi}$ implies,

$$\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_{n+1}),\eth(\hslash\,\wp,\hslash\,\Upsilon_{n+1})\} < \max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n),\eth(\hslash\,\wp,\hslash\,\Upsilon_n)\}.$$

Hence, $\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n),\eth(\hslash\,\wp,\hslash\,\Upsilon_n)\}$ is a decreasing sequence of positive reals and bounded below and by a result, we have

$$\lim_{n\to+\infty}\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n),\eth(\hslash\,\wp,\hslash\,\Upsilon_n)\} = \Gamma,\ \Gamma \geq 0.$$

Therefore as $n \to +\infty$ in equation (42), we get

$$\check{\psi}(\Gamma) \leq \check{\psi}(\Gamma) - \hat{\eta}(\Gamma), \tag{43}$$

from which we get $\hat{\eta}(\Gamma) = 0$, this implies that $\Gamma = 0$. Therefore,

$$\lim_{n\to+\infty}\max\{\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n),\eth(\hslash\,\wp,\hslash\,\Upsilon_n)\} = 0.$$

Hence, we have

$$\lim_{n\to+\infty}\eth(\hslash\,\varepsilon,\hslash\,\Lambda_n) = 0 \text{ and } \lim_{n\to+\infty}\eth(\hslash\,\wp,\hslash\,\Upsilon_n) = 0. \tag{44}$$

From the similar argument as above, we obtain that

$$\lim_{n\to+\infty}\eth(\hslash\,\mathscr{k},\hslash\,\Lambda_n) = 0 \text{ and } \lim_{n\to+\infty}\eth(\hslash\,\mathscr{I},\hslash\,\Upsilon_n) = 0. \tag{45}$$

Therefore from (44) and (45), we get $\hslash\,\varepsilon = \hslash\,\mathscr{k}$ and $\hslash\,\wp = \hslash\,\mathscr{I}$. Since $\hslash\,\varepsilon = \mathscr{I}(\varepsilon,\wp)$ and $\hslash\,\wp = \mathscr{I}(\wp,\varepsilon)$ and, the commutative property of $\mathscr{I}$ and $\hslash$ implies that

$$\begin{aligned}\hslash\,(\hslash\,\varepsilon) &= \hslash\,(\mathscr{I}(\varepsilon,\wp)) = \mathscr{I}(\hslash\,\varepsilon,\hslash\,\wp) \text{ and } \hslash\,(\hslash\,\wp)\\ &= \hslash\,(\mathscr{I}(\wp,\varepsilon)) = \mathscr{I}(\hslash\,\wp,\hslash\,\varepsilon).\end{aligned} \tag{46}$$

If $\hslash\,\varepsilon = \Lambda^*$ and $\hslash\,\wp = \Upsilon^*$, then from (46), we get

$$\hslash\,(\Lambda) = \mathscr{I}(\Lambda^*,\Upsilon^*) \text{ and } \hslash\,(\Upsilon^*) = \mathscr{I}(\Upsilon^*,\Lambda^*), \tag{47}$$

which exhibits that (Λ^*,Υ^*) is a coupled coincidence point of $\mathscr{I}$, $\hslash$. Hence, $\hslash\,(\Lambda^*) = \hslash\,\mathscr{k}$ and $\hslash\,(\Upsilon^*) = \hslash\,\mathscr{I}$ which in turn gives that $\hslash\,(\Lambda) = \Lambda^*$ and $\hslash\,(\Upsilon^*) = \Upsilon^*$. Therefore from (47), (Λ^*,Υ^*) is a coupled common fixed point of $\mathscr{I}$, $\hslash$.

Let $(\Lambda_1^*,\Upsilon_1^*)$ be another coupled common fixed point of $\mathscr{I}$, $\hslash$. Then, $\Lambda_1^* = \hslash\,\Lambda_1^* = \mathscr{I}(\Lambda_1^*,\Upsilon_1^*)$ and $\Upsilon_1^* = \hslash\,\Upsilon_1^* = \mathscr{I}(\Upsilon_1^*,\Lambda_1^*)$. But $(\Lambda_1^*,\Upsilon_1^*)$ is a coupled common fixed point of $\mathscr{I}$ and $\hslash$ then, $\hslash\,\Lambda_1^* = \hslash\,\varepsilon = \Lambda$ and $\hslash\,\Upsilon_1^* = \hslash\,\wp = \Upsilon^*$. Therefore, $\Lambda_1^* = \hslash\,\Lambda_1^* = \hslash\,\Lambda = \Lambda$ and $\Upsilon_1^* = \hslash\,\Upsilon_1^* = \hslash\,\Upsilon^* = \Upsilon^*$. Hence the uniqueness. □

Theorem 3.12 *In Theorem 3.11, if* $\hbar\varepsilon_0 \preceq \hbar\wp_0$ *or* $\hbar\varepsilon_0 \succeq \hbar\wp_0$, *then a unique common fixed point of* $\mathscr{I}$ *and* $\hbar$ *can be found.*

Proof Assume that $(\varepsilon, \wp) \in \mathscr{E}$ is a unique coupled common fixed point of $\mathscr{I}$ and $\hbar$. Then to demonstrate that $\varepsilon = \wp$. Suppose that $\hbar\varepsilon_0 \preceq \hbar\wp_0$, then we get by induction that, $\hbar\varepsilon_n \preceq \hbar\wp_n$ for $n \geq 0$. From Lemma 2 of [21], we have

$$\begin{aligned}
\check{\psi}(s^{k-2}\eth(\varepsilon,\wp)) &= \check{\psi}(s^k\frac{1}{s^2}\eth(\varepsilon,\wp)) \leq \lim_{n\to+\infty}\sup\check{\psi}(s^k\eth(\varepsilon_{n+1},\wp_{n+1}))\\
&= \lim_{n\to+\infty}\sup\check{\psi}(s^k\eth(\mathscr{I}(\varepsilon_n,\wp_n),\mathscr{I}(\wp_n,\varepsilon_n)))\\
&\leq \lim_{n\to+\infty}\sup\check{\psi}(\mathscr{P}_\hbar(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n)) - \lim_{n\to+\infty}\inf\hat{\eta}(\mathscr{P}_\hbar(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n))\\
&\leq \check{\psi}(\eth(\varepsilon,\wp)) - \lim_{n\to+\infty}\inf\hat{\eta}(\mathscr{P}_\hbar(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n))\\
&< \check{\psi}(\eth(\varepsilon,\wp)),
\end{aligned}$$

a contradiction. Hence, $\varepsilon = \wp$.

The result can also be similar in the case of $\hbar\varepsilon_0 \succeq \hbar\wp_0$. □

Remark 3.13 While $s = 1$ and the result of [19], the condition

$$\check{\psi}(\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\eth,\Im))) \leq \check{\psi}(\max\{\eth(\hbar\varepsilon,\hbar\eth),\eth(\hbar\wp,\hbar\Im)\}) - \hat{\eta}(\max\{\eth(\hbar\varepsilon,\hbar\eth),\eth(\hbar\wp,\hbar\Im)\})$$

is equivalent to,

$$\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\eth,\Im)) \leq \varphi(\max\{\eth(\hbar\varepsilon,\hbar\eth),\eth(\hbar\wp,\hbar\Im)\}),$$

where $\check{\psi} \in \hat{\Phi}$, $\hat{\eta} \in \hat{\Psi}$ and φ is a continuous self mapping on $[0,+\infty)$ with $\varphi(y) < y$ for every $y > 0$ with $\varphi(y) = 0$ if and only if $y = 0$. Hence the results found here are generalized and extended the results of [11, 18, 22, 25, 27] and several comparable results.

Now depending on the type of a metric, some examples are shown here under.

Example 3.14 Let $\mathscr{E} = \{e_1,e_2,e_3,e_4,e_5,e_6\}$ and $\eth : \mathscr{E}\times\mathscr{E} \to \mathscr{E}$ be a metric defined by

$$\begin{aligned}
&(\varepsilon,\wp) = (\wp,\varepsilon) = 0,\ \text{if } \varepsilon = \wp = \{e_1,e_2,e_3,e_4,e_5,e_6\}\\
&\quad\text{and } \varepsilon = \wp,\ (\varepsilon,\wp) = (\wp,\varepsilon) = 3,\ \textit{if}\ \varepsilon = \wp = \{e_1,e_2,e_3,e_4,e_5\}\\
&\quad\text{and } \varepsilon \neq \wp,\ (\varepsilon,\wp) = (\wp,\varepsilon) = 12,\ \textit{if}\ \varepsilon = \{e_1,e_2,e_3,e_4\}\\
&\quad\text{and } \wp = e_6,\ (\varepsilon,\wp) = (\wp,\varepsilon) = 20,\ \textit{if}\ \varepsilon = e_5 \text{ and } \wp = e_6,\ \text{with usual order} \leq .
\end{aligned}$$

A self-mapping $\mathscr{I}$ on $\mathscr{E}$ defined by $\mathscr{I}e_1 = \mathscr{I}e_2 = \mathscr{I}e_3 = \mathscr{I}e_4 = \mathscr{I}e_5 = 1, \mathscr{I}e_6 = 2$ has a fixed point with $\check{\psi}(y) = \frac{y}{2}$ and $\hat{\eta}(y) = \frac{y}{4}$ where $y \in [0,+\infty)$.

Proof When $s = 2$, $(\mathscr{E},\eth,\leq)$ is a complete partially ordered b-metric space. Let $\varepsilon,\wp \in \mathscr{E}$ such that $\varepsilon < \wp$ then we'll look at the cases below.

Case 1. If $\varepsilon,\wp \in \{e_1,e_2,e_3,e_4,e_5\}$ then $\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp) = \eth(e_1,e_1) = 0$. Hence,

$$\check{\psi}(2\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp)) = 0 \leq \check{\psi}(\mathscr{P}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}(\varepsilon,\wp)).$$

Case 2. If $\varepsilon \in \{e_1,e_2,e_3,e_4,e_5\}$ and $\wp = e_6$, then $\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp) = \eth(e_1,e_2) = 3$, $\mathscr{P}(e_6,e_5) = 20$ and $\mathscr{P}(\varepsilon,e_6) = 12$, for $\varepsilon \in \{e_1,e_2,e_3,e_4\}$. Hence,

$$\check{\psi}(2\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp)) \leq \frac{\mathscr{P}(\varepsilon,\wp)}{4} = \check{\psi}(\mathscr{P}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}(\varepsilon,\wp)).$$

As a result, all of the conditions of Theorem 3.1 are met, and hence $\mathscr{I}$ has a fixed point. □

Example 3.15 Let us define a metric $\eth$ with usual order $\leq$ by

$$\eth(\varepsilon,\wp) = \begin{cases} 0 & ,\ \textit{if}\ \varepsilon = \wp\\ 1 & ,\ \textit{if}\ \varepsilon \neq \wp \in \{0,1\}\\ |\varepsilon - \wp| & ,\ \textit{if}\ \varepsilon,\wp \in \{0,\frac{1}{2n},\frac{1}{2m} : n \neq m \geq 1\}\\ 6 & ,\ \textit{otherwise.}\end{cases}$$

where $\mathscr{E} = \{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n},\ldots\}$. A self-mapping $\mathscr{I}$ on $\mathscr{E}$ by $\mathscr{I}0 = 0, \mathscr{I}\frac{1}{n} = \frac{1}{12n}(n \geq 1)$ has a fixed point with $\check{\psi}(y) = y$ and $\hat{\eta}(y) = \frac{4y}{5}$ for $y \in [0,+\infty)$.

Proof $\eth$ is clearly discontinuous, and $(\mathscr{E}, \eth, \preceq)$ is a complete partially ordered b-metric space for $s = \frac{12}{5}$. Now we'll look at the following cases for $\varepsilon, \wp \in \mathscr{E}$ with $\varepsilon < \wp$.

Case 1. Suppose $\varepsilon = 0$ and $\wp = \frac{1}{n}$ $(n > 0)$, then $\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp) = \eth(0, \frac{1}{12n}) = \frac{1}{12n}$ and $\mathscr{P}(\varepsilon, \wp) = \frac{1}{n}$ and $\mathscr{P}(\varepsilon, \wp) = \{1, 6\}$. Thus,

$$\check{\psi}\left(\frac{12}{5}\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)\right) \leq \frac{\mathscr{P}(\varepsilon, \wp)}{5} = \check{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).$$

Case 2. Let $\varepsilon = \frac{1}{m}$ and $\wp = \frac{1}{n}$ where $m > n \geq 1$, then

$$\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp) = \eth(\frac{1}{12m}, \frac{1}{12n}), \mathscr{P}(\varepsilon, \wp) \geq \frac{1}{n} - \frac{1}{m} \text{ or } \mathscr{P}(\varepsilon, \wp) = 6.$$

Thus,

$$\check{\psi}\left(\frac{12}{5}\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)\right) \leq \frac{\mathscr{P}(\varepsilon, \wp)}{5} = \check{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).$$

Hence, we have the conclusion from Theorem 3.1 as all assumptions are fulfilled. □

Example 3.16 Define a metric $d : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$, where $\mathscr{E} = \{\tilde{\ell}/\tilde{\ell} : [a_1, a_2] \to [a_1, a_2]$ is continuous$\}$ by

$$\eth(\tilde{\ell}_1, \tilde{\ell}_2) = \sup_{y \in [a_1, a_2]} \{|\tilde{\ell}_1(y) - \tilde{\ell}_2(y)|^2\}$$

for any $\tilde{\ell}_1, \tilde{\ell}_2 \in \mathscr{E}$, $0 \leq a_1 < a_2$ with $\tilde{\ell}_1 \preceq \tilde{\ell}_2$ implies $a_1 \leq \tilde{\ell}_1(y) \leq \tilde{\ell}_2(y) \leq a_2, y \in [a_1, a_2]$. A self-mapping $\mathscr{I}$ on $\mathscr{E}$ defined by $\mathscr{I}\tilde{\ell} = \frac{\tilde{\ell}}{5}, \tilde{\ell} \in \mathscr{E}$ has a unique fixed point with $\check{\psi}(y) = y$ and $\hat{\eta}(y) = \frac{y}{3}$ for any $y \in [0, +\infty]$.

Proof As $\min(\tilde{\ell}_1, \tilde{\ell}_2)(y) = \min\{\tilde{\ell}_1(y), \tilde{\ell}_2(y)\}$ is continuous and all other assumptions of Theorem 3.3 are fulfilled for $s = 2$. Hence, $0 \in \mathscr{E}$ is a unique fixed point of $\mathscr{I}$. □

Limitations

We examined a fixed point, a coincidence point and a couple coincidence point for mappings that are satisfying generalized $(\check{\psi}, \hat{\eta})$-weak contractions in a partially ordered b-metric space. The findings in this paper are generalized and extended a few well-known results in the current literature. Some examples are shown at the end to support the results obtained here.

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Authors' contributions

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