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Fixed point results via a Hausdorff controlled type metric

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Abstract

In this paper, we establish that every controlled metric space (X, d_α) induces a Hausdorff controlled metric $(H_\alpha, CLD(X))$ on the class of closed subsets of X which is also complete if (X, d_α) is complete. Furthermore, we define multivalued almost F -contractions on Hausdorff controlled metric spaces and prove some fixed point results.

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1 Introduction and preliminaries

We denote by $P(X)$, $CLB(X)$, $CLD(X)$ and by $K(X)$ the class of all nonempty subsets of X , the class of all nonempty closed and bounded subsets of X , the class of all nonempty closed subsets of X , and the class of all nonempty compact subsets of X . For $\mathcal{A}, \mathcal{B} \in CLB(X)$, let

$$H(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(b, \mathcal{A}) \right\},$$

where $d(a, \mathcal{B}) = \inf\{d(a, b) : b \in \mathcal{B}\}$. Then H is a metric on $CLB(X)$, which is called the Pompeiu–Hausdorff metric induced by d . In 1969, Nadler [1] proved that every multivalued contraction on a complete metric space has a fixed point. Since then, many researchers extended it multi-directionally (see, for example [2–14]). Berinde and Berinde in [15] introduced the idea of multivalued almost contractions (originally called multivalued (δ, L) -weak contractions) and proved the following fixed point theorem.

Theorem 1.1 ([15]) *Let $T : X \rightarrow CLB(X)$ be a multivalued almost contraction mapping on a complete metric space (X, d) , that is, there exist two constants $0 < \delta < 1$ and $L \geq 0$ such that, for all $x, y \in X$, it satisfies*

$$H(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx). \quad (1)$$

Then T has a fixed point.

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Wardowski [16] extended the Banach contraction principle by introducing F -contractions and established fixed point theorems in metric spaces as follows.

Definition 1.1 ([16]) Let us consider a function $F : (0, \infty) \rightarrow \mathbb{R}$ and the following axioms:

- (F1) F is strictly non-decreasing;
- (F2) for each sequence $\{a_n\} \subset (0, \infty)$ of positive real numbers, $\lim_{n \rightarrow \infty} a_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(a_n) = -\infty$;
- (F3) for each sequence $\{a_n\} \subset (0, \infty)$ of positive real numbers, $\lim_{n \rightarrow \infty} a_n = 0$, there exists $l \in (0, 1)$ such that $\lim_{n \rightarrow \infty} (a_n)^l F(a_n) = 0$;
- (F4) $F(\inf \mathcal{A}) = \inf F(\mathcal{A})$ for all $\mathcal{A} \subset (0, \infty)$ with $\inf \mathcal{A} > 0$.

We denote by \mathcal{F} the family of all functions F satisfying (F1)–(F3), and by \mathcal{F}^* the family of all functions F satisfying (F1)–(F4).

Example 1.1 ([16]) Let $F : (0, \infty) \rightarrow \mathbb{R}$ be defined by

- (i) $F(\alpha) = \ln \alpha$;
- (ii) $F(\alpha) = \alpha + \ln \alpha$.

Clearly, F in (i) and (ii) satisfies (F1)–(F4).

Definition 1.2 ([16]) A mapping $T : X \rightarrow X$ on a metric space (X, d) is called F -contraction, if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{2}$$

for all $x, y \in X$ with $d(x, y) > 0$.

If we take $F(\alpha) = \ln \alpha$ in (2), we obtain

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \quad \text{for all } x, y \in X, Tx \neq Ty. \tag{3}$$

Clearly for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Thus, T is an ordinary contraction with contractive constant $c = e^{-\tau}$, but its converse is not true in general.

By combining the ideas of Wardowski and Nadler, Altun et al. [17] introduced the idea of multivalued F -contractions and obtained some fixed point results for this type of mappings on complete metric spaces.

Definition 1.3 ([17]) Let $T : X \rightarrow CLB(X)$ be a multivalued mapping on a metric space (X, d) . Then T is called a multivalued F -contraction, if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y)), \tag{4}$$

for all $x, y \in X$ with $H(x, y) > 0$.

By putting $F(a) = \ln a$, then every multivalued contraction in the sense of Nadler is also a multivalued F -contraction.

Theorem 1.2 ([17]) *Let $T : X \rightarrow K(X)$ be a multivalued F -contraction on a complete metric space (X, d) . Then T has a fixed point in X .*

Theorem 1.3 ([17]) *Let $T : X \rightarrow CLB(X)$ be a multivalued F -contraction on a complete metric space (X, d) . If $F \in \mathcal{F}^*$, then T has a fixed point in X .*

Altun et al. [18] established the concept of multivalued almost F -contractions and proved some fixed point results as follows.

Definition 1.4 ([18]) *A multivalued mapping $T : X \rightarrow CLB(X)$ on a metric space (X, d) is called a multivalued almost F -contraction, if $F \in \mathcal{F}$ and there exist two constants $\tau > 0$ and $\gamma \geq 0$ such that*

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y)) + \gamma d(y, Tx), \tag{5}$$

for all $x, y \in X$ with $H(x, y) > 0$.

By putting $F(a) = \ln a$, then every multivalued almost contraction (1) is a multivalued almost F -contraction.

Theorem 1.4 ([18]) *Let $T : X \rightarrow CLB(X)$ be a multivalued almost F -contraction on a complete metric space (X, d) . If $F \in \mathcal{F}^*$, then T has a fixed point in X .*

Remark 1.1 Theorem 1.4 generalized Theorem 1.1 and Theorem 1.3, because

- (i) If we take $F(a) = \ln a$, $\tau = -\ln \delta$ and $\gamma = \frac{1}{\delta}$, where $\delta \in (0, 1)$ in equation (5). Then we get equation (1).
- (ii) If we take $\gamma = 0$ in equation (5), we get equation (4).

In recent times, Kamran et al. in [19] established the idea of extended b -metric spaces, which generalized b -metric spaces (see [20, 21]) simply by replacing a constant s by a function depending on the left hand side of the triangle inequality.

Definition 1.5 ([19]) *Let X be a nonempty set and $\theta : X \times X \rightarrow [1, \infty)$. Then a mapping $d_\theta : X \times X \rightarrow [0, \infty)$ is called an extended b -metric, if for all $x, y, z \in X$, it satisfies the following axioms:*

- (i) $d_\theta(x, y) = 0$ iff $x = y$,
- (ii) $d_\theta(x, y) = d_\theta(y, x)$,
- (iii) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair (X, d_θ) is called an extended b -metric space.

Since then, many authors proved several fixed point results in the context of extended b -metric spaces; see [22–31]. In [32], Mlaiki et al. introduced the concept of controlled type metric spaces as a generalization of b -metric spaces, which is different from extended b -metrics space and is very useful to prove existence and uniqueness theorems for different types of integral and differential equations.

Definition 1.6 ([32]) *Let X be a nonempty set and $\alpha : X \times X \rightarrow [1, \infty)$. Then a mapping $d_\alpha : X \times X \rightarrow [0, \infty)$ is called a controlled metric, if for all $x, y, z \in X$, it satisfies the following axioms:*

- (i) $d_\alpha(x, y) = 0$ iff $x = y$,
- (ii) $d_\alpha(x, y) = d_\alpha(y, x)$,
- (iii) $d_\alpha(x, z) \leq \alpha(x, y)d_\alpha(x, y) + \alpha(y, z)d_\alpha(y, z)$.

The pair (X, d_α) is called a controlled metric space.

Remark 1.2 Every b -metric space is a controlled metric space, if we take $\alpha(x, y) = s \geq 1$ for all $x, y \in X$. Generally, a controlled metric space is not an extended b -metric space [32], if we take same functions $\alpha = \theta$ as follows.

Example 1.2 ([32]) Let $X = \{1, 2, \dots\}$. Define $d_\alpha : X \times X \rightarrow [0, \infty)$ as:

$$d_\alpha(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \frac{1}{x}, & \text{if } x \text{ is even and } y \text{ is odd;} \\ \frac{1}{y}, & \text{if } x \text{ is odd and } y \text{ is even;} \\ 1, & \text{otherwise.} \end{cases}$$

Hence (X, d_α) is a controlled metric space, where $\alpha : X \times X \rightarrow [1, \infty)$ is defined as:

$$\alpha(x, y) = \begin{cases} x, & \text{if } x \text{ is even and } y \text{ is odd;} \\ y, & \text{if } x \text{ is odd and } y \text{ is even;} \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, d_α is not an extended b -metric for the same function $\alpha = \theta$.

In this paper, we define a generalized Hausdorff metric on the class of nonempty closed subsets of controlled metric spaces. Also we prove that if (X, d_α) is complete, then $(H_\alpha, CLD(X))$ is complete, too. Moreover, we define multivalued almost F -contractions on controlled metric spaces and prove some fixed point results, which generalize many pre-existing results in the literature.

2 Main results

We denote by $\alpha(x, \mathcal{A}) = \inf_{a \in \mathcal{A}} \alpha(x, a)$, and $d_\alpha(x, \mathcal{A}) = \inf_{a \in \mathcal{A}} d_\alpha(x, a)$, for $\mathcal{A} \subset X$.

Lemma 2.1 *Let (X, d_α) be a controlled metric space. Then*

$$d_\alpha(x_1, \mathcal{A}) \leq \alpha(x_1, x_2)d_\alpha(x_1, x_2) + \alpha(x_2, \mathcal{A})d_\alpha(x_2, \mathcal{A}), \tag{6}$$

for all $x_1, x_2 \in X$ and $a \in \mathcal{A} \subset X$, where $\alpha(x_2, \mathcal{A}) = \inf_{a \in \mathcal{A}} \alpha(x_2, a)$.

Proof From axiom of definition, we have

$$d_\alpha(x_1, a) \leq \alpha(x_1, x_2)d_\alpha(x_1, x_2) + \alpha(x_2, a)d_\alpha(x_2, a), \quad \text{for all } x_1, x_2, a \in X.$$

By taking infimum of both sides over \mathcal{A} , we get

$$\inf_{a \in \mathcal{A}} d_\alpha(x_1, a) \leq \alpha(x_1, x_2)d_\alpha(x_1, x_2) + \inf_{a \in \mathcal{A}} \alpha(x_2, a) \inf_{a \in \mathcal{A}} d_\alpha(x_2, a).$$

Since $\alpha(x_2, \mathcal{A}) = \inf_{a \in \mathcal{A}} \alpha(x_2, a)$,

$$d_\alpha(x_1, \mathcal{A}) \leq \alpha(x_1, x_2)d_\alpha(x_1, x_2) + \alpha(x_2, \mathcal{A})d_\alpha(x_2, \mathcal{A}). \quad \square$$

Now we will introduce the Pompeiu–Hausdorff metric.

Definition 2.1 Let (X, d_α) be a controlled metric space. Then the function $H_\alpha : CLD(X) \times CLD(X) \rightarrow [0, \infty)$ is defined by

$$H_\alpha(\mathcal{A}, \mathcal{B}) = \begin{cases} \max\{\sup_{a \in \mathcal{A}} d_\alpha(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d_\alpha(b, \mathcal{A})\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise,} \end{cases}$$

where $\mathcal{A}, \mathcal{B} \in CLD(X)$.

Lemma 2.2 For all $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset CLD(X)$, we have

$$H_\alpha(\mathcal{A}, \mathcal{C}) \leq \max\left\{\sup_{a \in \mathcal{A}} \alpha(a, b), \alpha(b, \mathcal{A})\right\} H_\alpha(\mathcal{A}, \mathcal{B}) + \max\left\{\alpha(b, \mathcal{C}), \sup_{c \in \mathcal{C}} \alpha(c, b)\right\} H_\alpha(\mathcal{B}, \mathcal{C}).$$

Proof Assume that $H_\alpha(\mathcal{A}, \mathcal{B})$ and $H_\alpha(\mathcal{B}, \mathcal{C})$ are finite. From Lemma 2.1 for $a \in \mathcal{A}, b \in \mathcal{B}$, we have

$$d_\alpha(a, \mathcal{C}) \leq \alpha(a, b)d_\alpha(a, b) + \alpha(b, \mathcal{C})d_\alpha(b, \mathcal{C}).$$

As $d_\alpha(b, \mathcal{C}) \leq H_\alpha(\mathcal{B}, \mathcal{C})$, therefore we have

$$\begin{aligned} d_\alpha(a, \mathcal{C}) &\leq \alpha(a, b)d_\alpha(a, b) + \alpha(b, \mathcal{C})H_\alpha(\mathcal{B}, \mathcal{C}), \\ d_\alpha(a, \mathcal{C}) &\leq \alpha(a, b)d_\alpha(a, \mathcal{B}) + \alpha(b, \mathcal{C})H_\alpha(\mathcal{B}, \mathcal{C}). \end{aligned}$$

Hence by taking supremum over $a \in \mathcal{A}$, we get

$$\sup_{a \in \mathcal{A}} d_\alpha(a, \mathcal{C}) \leq \sup_{a \in \mathcal{A}} \alpha(a, b)H_\alpha(\mathcal{A}, \mathcal{B}) + \alpha(b, \mathcal{C})H_\alpha(\mathcal{B}, \mathcal{C}).$$

Analogously,

$$\sup_{c \in \mathcal{C}} d_\alpha(c, \mathcal{A}) \leq \alpha(b, \mathcal{A})H_\alpha(\mathcal{A}, \mathcal{B}) + \sup_{c \in \mathcal{C}} \alpha(c, b)H_\alpha(\mathcal{B}, \mathcal{C}).$$

So

$$\begin{aligned} \max\left\{\sup_{a \in \mathcal{A}} d_\alpha(a, \mathcal{C}), \sup_{c \in \mathcal{C}} d_\alpha(c, \mathcal{A})\right\} &\leq \max\left\{\sup_{a \in \mathcal{A}} \alpha(a, b), \alpha(b, \mathcal{A})\right\} H_\alpha(\mathcal{A}, \mathcal{B}) \\ &\quad + \max\left\{\alpha(b, \mathcal{C}), \sup_{c \in \mathcal{C}} \alpha(c, b)\right\} H_\alpha(\mathcal{B}, \mathcal{C}). \end{aligned}$$

Therefore, by Definition 2.1, we get

$$H_\alpha(\mathcal{A}, \mathcal{C}) \leq \max \left\{ \sup_{a \in \mathcal{A}} \alpha(a, b), \alpha(b, \mathcal{A}) \right\} H_\alpha(\mathcal{A}, \mathcal{B}) + \max \left\{ \alpha(b, \mathcal{C}), \sup_{c \in \mathcal{C}} \alpha(c, b) \right\} H_\alpha(\mathcal{B}, \mathcal{C}).$$

Moreover, if $H_\alpha(\mathcal{A}, \mathcal{B})$ or $H_\alpha(\mathcal{B}, \mathcal{C})$ is infinite, the condition is obvious. □

Theorem 2.1 *Let (X, d_α) be a controlled metric space, then the function $H_\alpha : CLD(X) \times CLD(X) \rightarrow [0, \infty]$ is a generalized controlled metric space in $CLD(X)$.*

Proof Let $H_\alpha(\mathcal{A}, \mathcal{B}) = 0$, for $\mathcal{A}, \mathcal{B} \in CLD(X)$. This implies

$$\max \left\{ \sup_{a \in \mathcal{A}} d_\alpha(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d_\alpha(b, \mathcal{A}) \right\} = 0.$$

Then $d_\alpha(a, \mathcal{B}) = 0$ for all $a \in \mathcal{A}$, hence $a \in \mathcal{B}$, i.e., $\mathcal{A} \subset \mathcal{B}$. In the same way, we see that $\mathcal{B} \subset \mathcal{A}$ and consequently $\mathcal{A} = \mathcal{B}$. Conversely, if $\mathcal{A} = \mathcal{B}$, then $H_\alpha(\mathcal{A}, \mathcal{B}) = 0$. Of course $H_\alpha(\mathcal{A}, \mathcal{B}) = H_\alpha(\mathcal{B}, \mathcal{A})$ for all $\mathcal{A}, \mathcal{B} \in CLD(X)$. Finally, in view of Lemma 2.2, the proof is complete. □

Definition 2.2 $a \in \bar{\mathcal{A}}$, where $\bar{\mathcal{A}}$ is the closure of a set $\mathcal{A} \subset X$, if and only if there exists a sequence $\{a_n\}$ in \mathcal{A} such that $a = \lim_{n \rightarrow \infty} a_n$, for $n = 0, 1, 2, \dots$

Denote for $\varepsilon > 0$ and $\mathcal{A} \subset X$,

$$\mathcal{A}_\varepsilon = \{x \in X : d_\alpha(x, \mathcal{A}) \leq \varepsilon\}.$$

Lemma 2.3 *If $x \in \bar{\mathcal{A}}_\varepsilon$, then $d_\alpha(x, \mathcal{A}) \leq \lim_{n \rightarrow \infty} \alpha(x_n, \mathcal{A})\varepsilon$, where*

$$\alpha(x_n, \mathcal{A}) = \inf_{a \in \mathcal{A}} \alpha(x_n, a).$$

Proof Let $x \in \bar{\mathcal{A}}_\varepsilon$, then there exists a sequence $\{x_n\}$ in \mathcal{A}_ε such that $\lim_{n \rightarrow \infty} x_n = x$, for $n = 0, 1, 2, \dots$. From Lemma 2.1, we have

$$d_\alpha(x, \mathcal{A}) \leq \alpha(x, x_n)d_\alpha(x, x_n) + \alpha(x_n, \mathcal{A})d_\alpha(x_n, \mathcal{A}).$$

By letting $n \rightarrow \infty$ in the above inequality, we get

$$d_\alpha(x, \mathcal{A}) \leq \lim_{n \rightarrow \infty} \alpha(x_n, \mathcal{A})\varepsilon.$$

It proves the lemma. □

Definition 2.3 The upper topological limit of a sequence $\{\mathcal{A}_l\}$, for $l = 1, 2, \dots$ in controlled metric space X is denoted by $\overline{Lt}\mathcal{A}_l$ determined by

$$a \in \overline{Lt}\mathcal{A}_l, \quad \text{if and only if} \quad \liminf_{l \rightarrow \infty} d_\alpha(a, \mathcal{A}_l) = 0.$$

Theorem 2.2 A point $a \in \overline{Lt}A_l$, if and only if there exists a subsequence $\{a_{n_l}\} \subset A$ such that $\lim_{l \rightarrow \infty} a_{n_l} = a$ and $a_{n_l} \in A_{n_l}$, for $l = 1, 2, 3, \dots$

Proof First, let us suppose that $a \in \overline{Lt}A_l$, then there exists a subsequence $\{A_{n_l}\}$ of A_l such that $\lim_{l \rightarrow \infty} d_\alpha(a, A_{n_l}) = 0$. Hence for every l there exists a strictly increasing sequence of positive integers $\{p_l\}$ with

$$d_\alpha(a, A_{n_l}) < \frac{1}{l}, \quad \text{for all } n \geq p_l.$$

Therefore, we can find a sequence $\{a_{n_l}\}$ of points such that $a_{n_l} \in A_{n_l}$ and $d_\alpha(a, a_{n_l}) < \frac{1}{l}$, for $p_l \leq n < p_{l+1}$. Hence $\lim_{l \rightarrow \infty} a_{n_l} = a$.

Conversely, let us assume that $a_{n_l} \rightarrow a$ and $a_{n_l} \in A_{n_l}$, $l = 1, 2, 3, \dots$. Hence

$$d_\alpha(a, A_{n_l}) \leq d_\alpha(a, a_{n_l}) \rightarrow 0$$

and $\lim_{l \rightarrow \infty} \inf d_\alpha(a, A_l) = 0$. This implies that $a \in \overline{Lt}A_l$. □

Theorem 2.3 $L = \overline{Lt}A_l$ is closed.

Proof Suppose that x is a limit point of L . Then there exists a sequence $x_m \in L - \{x\}$ that converges to x . By Theorem 2.2 for $x_m \in L$, there exists a subsequence $\{x_{m_l}\} \subset A$ such that $\lim_{l \rightarrow \infty} x_{m_l} = x$ and $x_{m_l} \in A_{m_l}$, for $l = 1, 2, 3, \dots$. Now by the triangular inequality, we have

$$d_\alpha(x_{m_l}, x) \leq \alpha(x_{m_l}, x_l)d_\alpha(x_{m_l}, x_l) + \alpha(x_l, x)d_\alpha(x_l, x).$$

Clearly $\lim_{l \rightarrow \infty} x_{m_l} = x$. It follows that $\{x_{m_l}\}$ converges to x and $x_{m_l} \in A_{m_l}$, for $l = 1, 2, 3, \dots$. Therefore, by Theorem 2.2, $x \in L$. Hence L is closed. □

Corollary 2.1

$$\overline{Lt}A_l = \bigcap_{l=1}^{\infty} \overline{\bigcup_{n=0}^{\infty} A_{l+n}}.$$

Proof First, let us assume that $x \in \overline{Lt}A_l$, then there exists $\{x_{n_l}\} \subset A$ such that $\lim_{l \rightarrow \infty} x_{n_l} = x$ and $x_{n_l} \in A_{n_l}$, for $l = 1, 2, 3, \dots$. Hence for every p

$$x_{n_l} \in \bigcup_{n=0}^{\infty} A_{p+n}, \quad \text{for all } l \geq 1.$$

This implies that

$$x \in \bigcap_{l=1}^{\infty} \overline{\bigcup_{n=0}^{\infty} A_{l+n}}.$$

Conversely let us assume that, for every $p, x \in \overline{\bigcup_{n=0}^{\infty} A_{p+n}}$. Then there is a sequence $\{x_{n_l}^p\} \subset \bigcup_{n=0}^{\infty} A_{p+n}$ such that $x_{n_l}^p \rightarrow x$ as $l \rightarrow \infty$ for every natural. Let there exists $x_1 = x_{n_1}^1$ such that

$x_{n_1}^1 \in \mathcal{A}_{p_1}$ and $d_\alpha(x_{n_1}^1, x) < 1$. Similarly, let $x_2 = x_{n_2}^{l_1+1}$ such that $p_2 > p_1$ and $d_\alpha(x_{n_2}^{l_1+1}, x) < \frac{1}{2}$, $x_{n_2}^{l_1+1} \in \mathcal{A}_{p_2}$. By continuing this process, we have $x_{l+1} = x_{n_{l+1}}^{l_l+1}$ such that $d_\alpha(x_{n_{l+1}}^{l_l+1}, x) < \frac{1}{l+1}$ and $x_{n_{l+1}}^{l_l+1} \in \mathcal{A}_{p_{l+1}}$, $p_l < p_{l+1}$. Thus, we have $x_l \rightarrow x$ as $l \rightarrow \infty$ and $x_l \in \mathcal{A}_l$ for $l = 1, 2, 3, \dots$. Hence by Theorem 2.2, $x \in \overline{Lt\mathcal{A}_l}$. It completes the proof. \square

Corollary 2.2

$$\lim_{l \rightarrow \infty} \mathcal{A}_l = \overline{\overline{Lt\mathcal{A}_l}} = \overline{Lt\mathcal{A}_l}.$$

Proof Let us assume that $a \in \overline{\overline{Lt\mathcal{A}_l}}$, then there is a sequence $a_n \in \overline{Lt\mathcal{A}_l}$ for $n = 1, 2, 3, \dots$ such that $a_n \rightarrow a$ as $n \rightarrow \infty$. Consequently, there exists an integer p_{l_1} such that $a_{l_1} \in \mathcal{A}_{l_1}$ and $d_\alpha(a_{l_1}, a) < 1$. Similarly, there exists an integer $p_{l_2} > p_{l_1}$ such that $d_\alpha(a_{l_2}, a) < \frac{1}{2}$. Continuing this process, we can find an increasing sequence $\{p_{l_n}\}$ of integers with $a_{l_n} \in \mathcal{A}_{l_n}$ for $n = 1, 2, 3, \dots$ such that

$$d_\alpha(a_{l_n}, a_n) < \frac{1}{n}, \quad \text{for all } n.$$

Thus, by the triangle inequality, we get

$$d_\alpha(a_{l_n}, a) \leq \alpha(a_{l_n}, a_n)d_\alpha(a_n, a) + \alpha(a_n, a)d_\alpha(a_n, a).$$

Note that, as we take n to infinity, the distance between $\{a_{l_n}\}$ and a converges to zero, so it follows that $\{a_{l_n}\}$ converges to a . Hence, by Theorem 2.2, $a \in \overline{Lt\mathcal{A}_l}$. It follows that

$$\overline{\overline{Lt\mathcal{A}_l}} \subset \overline{Lt\mathcal{A}_l}. \tag{7}$$

Conversely, let us assume that $a \in \overline{Lt\mathcal{A}_l}$, then, in a similar way,

$$\overline{Lt\mathcal{A}_l} \subset \overline{\overline{Lt\mathcal{A}_l}}. \tag{8}$$

From Eqs. (7) and (8), we have

$$\overline{Lt\mathcal{A}_l} = \overline{\overline{Lt\mathcal{A}_l}}.$$

The remaining part of the theorem can be verified by the similar way. \square

Theorem 2.4 *If (X, d_α) be a complete controlled metric space with $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$, for all $x_n, x_m \in X$, where $\kappa \geq 1$. Then $(CLD(X), H_\alpha)$ is complete.*

Proof Let $\{\mathcal{A}_n\}$, $n = 1, 2, \dots$ be a Cauchy sequence in $CLD(X)$. Then, by the definition, for each $\varepsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that

$$H_\alpha(\mathcal{A}_n, \mathcal{A}_m) < \varepsilon, \quad \text{for all } n, m \geq N. \tag{9}$$

Let $\mathcal{A} = \overline{Lt\mathcal{A}_n}$. We will prove that $\mathcal{A} \in CLD(X)$ and $\mathcal{A}_n \rightarrow \mathcal{A}$. From Theorem 2.3, $\mathcal{A} \in CLD(X)$. Next, we will show that $\{\mathcal{A}_n\}$ converges to \mathcal{A} , i.e. there exists a positive integer

N such that $H_\alpha(\mathcal{A}_n, \mathcal{A}) < \varepsilon$ for all $n \geq N$. By the triangle inequality for all $n, m \geq N$,

$$H_\alpha(\mathcal{A}_n, \mathcal{A}) \leq \max \left\{ \sup_{a_n \in \mathcal{A}_n} \alpha(a_n, a_m), \alpha(a_m, \mathcal{A}_n) \right\} H_\alpha(\mathcal{A}_n, \mathcal{A}_m) + \max \left\{ \sup_{a_m \in \mathcal{A}_m} \alpha(a_m, a), \alpha(a, \mathcal{A}_m) \right\} H_\alpha(\mathcal{A}_m, \mathcal{A}).$$

For $n, m \geq N$, we have from (9)

$$H_\alpha(\mathcal{A}_n, \mathcal{A}) \leq \max \left\{ \sup_{a_n \in \mathcal{A}_n} \alpha(a_n, a_m), \alpha(a_m, \mathcal{A}_n) \right\} \varepsilon + \max \left\{ \sup_{a_m \in \mathcal{A}_m} \alpha(a_m, a), \alpha(a, \mathcal{A}_m) \right\} H_\alpha(\mathcal{A}_m, \mathcal{A}). \tag{10}$$

Now, we will prove that

$$H_\alpha(\mathcal{A}_m, \mathcal{A}) \leq \max \left\{ \sup_{a_m \in \mathcal{A}_m} \alpha(a_m, a_{n_r}), \alpha(a_{n_r}, \mathcal{A}_m) \right\} \varepsilon.$$

For this purpose, we will show the following inequalities:

$$d_\alpha(a_m, a^*) \leq \alpha(a_m, a_{n_r})\varepsilon, \quad \text{for all } a_m \in \mathcal{A}_m, \tag{11}$$

$$d_\alpha(a^*, \mathcal{A}_m) \leq \alpha(a_{n_r}, \mathcal{A}_m)\varepsilon. \tag{12}$$

From (9), we get

$$\mathcal{A}_n \subset \mathcal{A}_{m_\varepsilon}, \quad \text{for all } n > m \geq N.$$

Next from Corollary 2.1, we have

$$\mathcal{A} \subset \overline{\mathcal{A}_n \cup \mathcal{A}_{n+1} \cup \dots} \subset \overline{\mathcal{A}_{m_\varepsilon}},$$

hence from Lemma 2.3, we get, for $a^* \in \mathcal{A}$,

$$d_\alpha(a^*, \mathcal{A}_m) \leq \alpha(a_{n_r}, \mathcal{A}_m)\varepsilon.$$

Thus, condition (12) is fulfilled.

Now, we have to prove (11). Since $\{\mathcal{A}_n\}$ is a Cauchy sequence in $CLD(X)$, we can find a strictly increasing sequence of positive integers $\{n_r\} = \{\varepsilon l^{-r}\}$ for $r = 1, 2, 3, \dots$ such that $n_r > N$, where $N \in \mathbb{N}$ and $H_\alpha(\mathcal{A}_n, \mathcal{A}_m) < \varepsilon l^{-r}$, for all $n, m \geq n_r$. Take arbitrary $a_m \in \mathcal{A}_m$, where $a_m = a_{n_0}$. Since $H_\alpha(\mathcal{A}_n, \mathcal{A}_{n_0}) < \varepsilon$, for $n > n_0$, there exists $a_{n_1} \in \mathcal{A}_{n_1}$ such that $d_\alpha(a_{n_0}, a_{n_1}) < \varepsilon$, for $n = n_1 > n_0$. Similarly, $H_\alpha(\mathcal{A}_n, \mathcal{A}_{n_1}) < \frac{\varepsilon}{7}$, so there exists $a_{n_2} \in \mathcal{A}_{n_2}$ such that $d_\alpha(a_{n_1}, a_{n_2}) < \frac{\varepsilon}{7}$, for $n = n_2 > n_1$. By continuing this process, we can form a sequence $\{a_{n_r}\}$ with $a_{n_r} \in \mathcal{A}_{n_r}$, for $r = 0, 1, 2, \dots$ and

$$d_\alpha(a_{n_r}, a_{n_{r+1}}) < \frac{\varepsilon}{l^r}, \quad a_{n_0} = a. \tag{13}$$

Next, we will verify that $\{a_{n_r}\}$ is a Cauchy sequence, from the triangle inequality, we have

$$\begin{aligned}
 & d_\alpha(a_{n_r}, a_{n_{r+l}}) \\
 & \leq \alpha(a_{n_r}, a_{n_{r+1}})d_\alpha(a_{n_r}, a_{n_{r+1}}) + \alpha(a_{n_{r+1}}, a_{n_{r+l}})d_\alpha(a_{n_{r+1}}, a_{n_{r+l}}) \\
 & \leq \alpha(a_{n_r}, a_{n_{r+1}})d_\alpha(a_{n_r}, a_{n_{r+1}}) + \alpha(a_{n_{r+1}}, a_{n_{r+k}})\alpha(a_{n_{r+1}}, a_{n_{r+2}})d_\alpha(a_{n_{r+1}}, a_{n_{r+2}}) \\
 & \quad + \alpha(a_{n_{r+1}}, a_{n_{r+l}})\alpha(a_{n_{r+2}}, a_{n_{r+l}})d_\alpha(a_{n_{r+2}}, a_{n_{r+l}}) \\
 & \leq \alpha(a_{n_r}, a_{n_{r+1}})d_\alpha(a_{n_r}, a_{n_{r+1}}) \\
 & \quad + \alpha(a_{n_{r+1}}, a_{n_{r+l}})\alpha(a_{n_{r+2}}, a_{n_{r+l}})\alpha(a_{n_{r+2}}, a_{n_{r+3}})d_\alpha(a_{n_{r+2}}, a_{n_{r+3}}) \\
 & \quad + \alpha(a_{n_{r+1}}, a_{n_{r+l}})\alpha(a_{n_{r+2}}, a_{n_{r+l}})\alpha(a_{n_{r+3}}, a_{n_{r+l}})d_\alpha(a_{n_{r+3}}, a_{n_{r+l}}) \\
 & \leq \dots \\
 & \leq \alpha(a_{n_r}, a_{n_{r+1}})d_\alpha(a_{n_r}, a_{n_{r+1}}) + \sum_{i=r+1}^{r+l-2} \left(\prod_{j=r+1}^i \alpha(a_{n_j}, a_{n_{r+l}}) \right) \alpha(a_{n_i}, a_{n_{i+1}})d_\alpha(a_{n_i}, a_{n_{i+1}}) \\
 & \quad + \prod_{j=r+1}^{r+l-1} \alpha(a_{n_j}, a_{n_{r+l}})\alpha(a_{n_{r+l-1}}, a_{n_{r+l}})d_\alpha(a_{n_{r+l-1}}, a_{n_{r+l}}) \\
 & \leq \alpha(a_{n_r}, a_{n_{r+1}})d_\alpha(a_{n_r}, a_{n_{r+1}}) + \sum_{i=r+1}^{r+l-1} \left(\prod_{j=r+1}^i \alpha(a_{n_j}, a_{n_{r+l}}) \right) \alpha(a_{n_i}, a_{n_{i+1}})d_\alpha(a_{n_i}, a_{n_{i+1}}) \\
 & \leq \alpha(a_{n_r}, a_{n_{r+1}})d_\alpha(a_{n_r}, a_{n_{r+1}}) + \sum_{i=r+1}^{r+l-1} \left(\prod_{j=r+1}^i \alpha(a_{n_j}, a_{n_{r+l}}) \right) \alpha(a_{n_i}, a_{n_{i+1}})d_\alpha(a_{n_i}, a_{n_{i+1}}).
 \end{aligned}$$

From Eq. (13), we have

$$d_\alpha(a_{n_r}, a_{n_{r+l}}) \leq \alpha(a_{n_r}, a_{n_{r+1}}) \frac{\varepsilon}{l^r} + \sum_{i=r+1}^{r+l-1} \left(\prod_{j=r+1}^i \alpha(a_{n_j}, a_{n_{r+l}}) \right) \alpha(a_{n_i}, a_{n_{i+1}}) \frac{\varepsilon}{l^i}. \tag{14}$$

As $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m) \kappa < 1$, for all $x_n, x_m \in X$. Thus the series

$$\sum_{i=r+1}^{r+l-1} \left(\prod_{j=r+1}^i \alpha(a_{n_j}, a_{n_{r+l}}) \right) \alpha(a_{n_i}, a_{n_{i+1}}) \frac{\varepsilon}{l^i}$$

converges by the ratio test. By taking the limit $r \rightarrow \infty$ in Eq. (14), we get

$$\lim_{r \rightarrow \infty} d_\alpha(a_{n_r}, a_{n_{r+l}}) = 0.$$

Hence, we conclude that $\{a_{n_r}\}$ is a Cauchy sequence. Since (X, d_α) is complete, there exists $a_* \in X$ such that $a_{n_r} \rightarrow a_* \in X$, and clearly $a_* \in \mathcal{A}$. Again, by the triangle inequality, we have

$$\begin{aligned}
 d_\alpha(a_{n_0}, a_{n_r}) & \leq \alpha(a_{n_0}, a_{n_1})d_\alpha(a_{n_0}, a_{n_1}) + \alpha(a_{n_1}, a_{n_r})d_\alpha(a_{n_1}, a_{n_r}) \\
 & \leq \alpha(a_{n_0}, a_{n_1})d_\alpha(a_{n_0}, a_{n_1}) + \alpha(a_{n_1}, a_{n_r})\alpha(a_{n_1}, a_{n_2})d_\alpha(a_{n_1}, a_{n_2})
 \end{aligned}$$

$$\begin{aligned}
 & + \alpha(a_{n_1}, a_{n_r})\alpha(a_{n_2}, a_{n_r})d_\alpha(a_{n_2}, a_{n_r}) \\
 & \leq \dots \\
 & \leq \alpha(a_{n_0}, a_{n_1})d_\alpha(a_{n_0}, a_{n_1}) + \sum_{i=1}^{r-2} \left(\prod_{j=1}^i \alpha(a_{n_j}, a_{n_r}) \right) \alpha(a_{n_i}, a_{n_{i+1}})d_\alpha(a_{n_i}, a_{n_{i+1}}) \\
 & \quad + \prod_{j=1}^{r-1} \alpha(a_{n_j}, a_{n_r})\alpha(a_{n_{r-1}}, a_{n_r})d_\alpha(a_{n_{r-1}}, a_{n_r}) \\
 & \leq \alpha(a_{n_0}, a_{n_1})d_\alpha(a_{n_0}, a_{n_1}) + \sum_{i=1}^{r-1} \left(\prod_{j=1}^i \alpha(a_{n_j}, a_{n_r}) \right) \alpha(a_{n_i}, a_{n_{i+1}})d_\alpha(a_{n_i}, a_{n_{i+1}}) \\
 & \leq \alpha(a_{n_0}, a_{n_1})d_\alpha(a_{n_0}, a_{n_1}) + \sum_{i=1}^{r-1} \left(\prod_{j=1}^i \alpha(a_{n_j}, a_{n_r}) \right) \alpha(a_{n_i}, a_{n_{i+1}})d_\alpha(a_{n_i}, a_{n_{i+1}}).
 \end{aligned}$$

From Eq. (13), we have

$$d_\alpha(a_{n_0}, a_{n_r}) \leq \alpha(a_{n_0}, a_{n_1})\varepsilon + \sum_{i=1}^{r-1} \left(\prod_{j=1}^i \alpha(a_{n_j}, a_{n_r}) \right) \alpha(a_{n_i}, a_{n_{i+1}}) \frac{\varepsilon}{l^i}. \tag{15}$$

As $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$, for all $x_n, x_m \in X$. Thus, the series

$$\sum_{i=1}^{r-1} \left(\prod_{j=m+1}^i \alpha(a_{n_j}, a_{n_r}) \right) \alpha(a_{n_i}, a_{n_{i+1}}) \frac{\varepsilon}{l^i}$$

converges by the ratio test. By taking the limit $r \rightarrow \infty$ in Eq. (15), we get

$$\lim_{r \rightarrow \infty} d_\alpha(a_{n_0}, a_{n_r}) < \frac{1}{\kappa} \varepsilon < \varepsilon.$$

Next, from the triangle inequality, we have

$$d_\alpha(a_*, a_m) \leq \alpha(a_*, a_{n_r})d_\alpha(a_*, a_{n_r}) + \alpha(a_{n_r}, a_m)d_\alpha(a_{n_r}, a_m).$$

Hence, $d_\alpha(a_*, a_m) \leq \alpha(a_{n_r}, a_m)\varepsilon$, when $r \rightarrow \infty$. So the condition (11) is fulfilled.

Hence, from (10), we obtain

$$\begin{aligned}
 H_\alpha(\mathcal{A}_n, \mathcal{A}) & \leq \max \left\{ \sup_{a_n \in \mathcal{A}_n} \alpha(a_n, a_m), \alpha(a_m, \mathcal{A}_n) \right\} \varepsilon + \max \left\{ \sup_{a_m \in \mathcal{A}_m} \alpha(a_m, a), \alpha(a, \mathcal{A}_m) \right\} \\
 & \quad + \max \left\{ \sup_{a_m \in \mathcal{A}_m} \alpha(a_m, a_{n_r}), \alpha(a_{n_r}, \mathcal{A}_m) \right\} \varepsilon.
 \end{aligned}$$

Since $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$, for all $x_n, x_m \in X$, by taking the limit $n, m \rightarrow \infty$ in the above inequality, we get a positive real number on right side. Hence \mathcal{A}_n approaches \mathcal{A} , which completes the proof. □

Next, we will prove some fixed point results over controlled Hausdorff metric spaces.

Lemma 2.4 *Let $\mathcal{A}, \mathcal{B} \in CLD(X)$, then for all $\epsilon > 0$ and $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that*

$$d_\alpha(a, b) \leq H_\alpha(\mathcal{A}, \mathcal{B}) + \epsilon. \tag{16}$$

Proof From Definition 2.1, for $\mathcal{A}, \mathcal{B} \in CLD(X)$ and for any $b \in \mathcal{B}$, we have

$$d_\alpha(\mathcal{A}, b) \leq H_\alpha(\mathcal{A}, \mathcal{B}).$$

By definition of infimum, we may assume a sequence a_n in \mathcal{A} such that

$$d_\alpha(b, a_n) < d_\alpha(b, \mathcal{A}) + \epsilon, \quad \text{where } \epsilon > 0. \tag{17}$$

Since \mathcal{A} is closed, there exists $a \in \mathcal{A}$ such that $a_n \rightarrow a$. Therefore, by (17), we have

$$d_\alpha(a, b) < d_\alpha(\mathcal{A}, b) + \epsilon \leq H_\alpha(\mathcal{A}, \mathcal{B}) + \epsilon. \quad \square$$

Theorem 2.5 *Let $T : X \rightarrow CLD(X)$ be a mapping on a complete controlled metric space (X, d_α) . If T satisfies the inequality*

$$H_\alpha(Tx, Ty) \leq \kappa d_\alpha(x, y), \quad \text{for all } x, y \in X, \tag{18}$$

where $\kappa \in [0, 1)$ is a real constant such that $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$, for all $x_n, x_m \in X$. Then T has a fixed point.

Proof Let us consider $\kappa > 0$, $x_0 \in X$ and choose $x_1 \in Tx_0$. As $Tx_0, Tx_1 \in CLD(X)$ and $x_1 \in Tx_0$, then, by Lemma 2.4, there exists $x_2 \in Tx_1$ such that

$$d_\alpha(x_1, x_2) \leq H_\alpha(Tx_0, Tx_1) + \epsilon.$$

Now since $Tx_1, Tx_2 \in CLD(X)$ and $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that

$$d_\alpha(x_2, x_3) \leq H_\alpha(Tx_1, Tx_2) + \epsilon^2.$$

Continuing in this fashion, we obtain a sequence $\{x_n\}$ of elements of X such that $x_{n+1} \in Tx_n$, for $n = 0, 1, 2, \dots$ and

$$d_\alpha(x_n, x_{n+1}) \leq H_\alpha(Tx_{n-1}, Tx_n) + \epsilon^n, \quad \text{for all } n \geq 1.$$

From Eq. (18), we have

$$\begin{aligned} d_\alpha(x_n, x_{n+1}) &\leq \epsilon d_\alpha(x_{n-1}, x_n) + \epsilon^n \\ &\leq \epsilon(\kappa d_\alpha(x_{n-2}, x_{n-1}) + \epsilon^{n-1}) + \epsilon^n \\ &\leq \kappa^2 d_\alpha(x_{n-2}, x_{n-1}) + 2\epsilon^n. \end{aligned}$$

Continuing in this way, we have

$$d_\alpha(x_n, x_{n+1}) \leq \kappa^n d_\alpha(x_0, x_1) + n\epsilon^n, \quad \text{for all } n \geq 1. \tag{19}$$

From the triangle inequality and Eq. (19) for $m > n$, we have

$$\begin{aligned}
 d_\alpha(x_n, x_m) &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)d_\alpha(x_{n+1}, x_m) \\
 &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \alpha(x_n, x_m)\alpha(x_{n+1}, x_{n+2})d_\alpha(x_{n+1}, x_{n+2}) \\
 &\quad + \alpha(x_n, x_m)\alpha(x_{n+2}, x_m)d_\alpha(x_{n+2}, x_m) \\
 &\leq \dots \\
 &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \sum_{i=1}^{m-2} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})d_\alpha(x_i, x_{i+1}) \\
 &\quad + \prod_{j=1}^{m-1} \alpha(x_j, x_m)\alpha(x_{m-1}, x_m)d_\alpha(x_{m-1}, x_m) \\
 &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})d_\alpha(x_i, x_{i+1}) \\
 &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})d_\alpha(x_i, x_{i+1}) \\
 &\leq \alpha(x_n, x_{n+1})[\kappa^n d_\alpha(x_0, x_1) + n\kappa^n] \\
 &\quad + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})[\kappa^i d_\alpha(x_0, x_1) + i\kappa^i].
 \end{aligned}$$

This implies that

$$\begin{aligned}
 d_\alpha(x_n, x_m) &\leq d_\alpha(x_0, x_1)[\alpha(x_n, x_{n+1})\kappa^n + \alpha(x_n, x_m)n\kappa^n] \\
 &\quad + d_\alpha(x_0, x_1) \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})\kappa^i \\
 &\quad + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})i\kappa^i.
 \end{aligned}$$

Since $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$ for all $x_n, x_m \in X$, $\alpha(x_n, x_m)$ is finite and the series $\sum_{n=1}^\infty \kappa^n \prod_{i=1}^n \alpha(x_i, x_m)\alpha(x_i, x_{i+1})$ converges by the ratio test for each $m \in \mathbb{N}$. If we take $S_n = \kappa^n \prod_{i=1}^n \alpha(x_i, x_m)\alpha(x_i, x_{i+1})$ and $S_{n+1} = \kappa^{n+1} \prod_{i=1}^{n+1} \alpha(x_i, x_m)\alpha(x_i, x_{i+1})$, then $\frac{S_{n+1}}{S_n} < 1$, when $n \rightarrow \infty$. By the same procedure $\sum_{n=1}^\infty n\kappa^n \prod_{i=1}^n \alpha(x_i, x_m)\alpha(x_i, x_{i+1})$ is convergent. Let

$$S = \sum_{n=1}^\infty \kappa^n \prod_{i=1}^n \alpha(x_i, x_m)\alpha(x_i, x_{i+1}), \quad S_n = \sum_{j=1}^n \kappa^j \prod_{i=1}^j \alpha(x_i, x_m)\alpha(x_i, x_{i+1}),$$

and

$$S' = \sum_{n=1}^\infty n\kappa^n \prod_{i=1}^n \alpha(x_i, x_m)\alpha(x_i, x_{i+1}), \quad S'_n = \sum_{j=1}^n j\kappa^j \prod_{i=1}^j \alpha(x_i, x_m)\alpha(x_i, x_{i+1}).$$

Thus, for $m > n$, we have

$$d_\alpha(x_n, x_m) \leq d_\alpha(x_0, x_1) [\alpha(x_n, x_{n+1})\kappa^n + \alpha(x_n, x_m)n\kappa^n] d_\alpha(x_0, x_1) [S_{m-1} - S_n] + [S'_{m-1} - S'_n].$$

By letting $n \rightarrow \infty$, we conclude that $\{x_n\}$, for $n = 0, 1, 2, \dots$ is a Cauchy sequence. Since X is complete, there exists $x_* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x_*$. Now by the triangle inequality

$$\begin{aligned} d_\alpha(Tx_*, x_*) &\leq \alpha(Tx_*, x_n) d_\alpha(Tx_*, x_n) + \alpha(x_n, x_*) d_\alpha(x_n, x_*) \\ &\leq \alpha(Tx_*, x_n) [\kappa d_\alpha(x_*, x_{n-1})] + \alpha(x_n, x_*) d_\alpha(x_n, x_*) \\ &\leq \alpha(Tx_*, x_n) [\kappa^2 d_\alpha(x_*, x_{n-2})] + \alpha(x_n, x_*) d_\alpha(x_n, x_*) \\ &\vdots \\ &\leq \alpha(Tx_*, x_n) [\kappa^n d_\alpha(x_*, x_0)] + \alpha(x_n, x_*) d_\alpha(x_n, x_*). \end{aligned}$$

Since $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$ for all $x_n, x_m \in X$, $\alpha(x_n, x_m)$ is finite. Thus, by taking the limit $n \rightarrow \infty$ in the above inequality, we get

$$d_\alpha(Tx_*, x_*) = 0.$$

T is closed, therefore $x_* \in Tx_*$. Hence x_* is a fixed point of T . □

Definition 2.4 ([18]) A multivalued mapping $T : X \rightarrow CLD(X)$ on a controlled metric space (X, d_α) is said to be a multivalued almost F -contraction, if $F \in \mathcal{F}$ and there exist two constants $\tau > 0$ and $\gamma \geq 0$ such that

$$\tau + F(H_\alpha(Tx, Ty)) \leq F(d_\alpha(x, y)) + \gamma d_\alpha(y, Tx), \tag{20}$$

for all $x, y \in X$ with $H_\alpha(Tx, Ty) > 0$.

By putting $F(\alpha) = \ln \alpha$, then every multivalued almost contraction (1) is also a multivalued almost F -contraction.

Theorem 2.6 Let $T : X \rightarrow CLD(X)$ be a multivalued almost F -contraction on a complete controlled metric space (X, d_α) with $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$, for all $x_n, x_m \in X$, where $\kappa \geq 1$. If $F \in \mathcal{F}^*$, then T has a fixed point in X .

Proof Let $x_0 \in X$. Since Tx is nonempty for all $x \in X$, we may choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T . Therefore let us suppose that $x_1 \notin Tx_1$. Since Tx_1 is closed, $d_\alpha(x_1, Tx_1) > 0$, and also $d_\alpha(x_1, Tx_1) \leq H_\alpha(Tx_0, Tx_1)$. From axiom (F1) of Definition 1.1, we have

$$F(d_\alpha(x_1, Tx_1)) \leq F(H_\alpha(Tx_0, Tx_1)).$$

From Eq. (20), we obtain

$$\begin{aligned} F(d_\alpha(x_1, Tx_1)) &\leq F(H_\alpha(Tx_0, Tx_1)) \\ &\leq F(d_\alpha(x_0, x_1)) + \gamma d_\alpha(x_1, Tx_0) - \tau. \end{aligned}$$

As $d_\alpha(x_1, Tx_0) = d_\alpha(x_1, x_1) = 0$, from above inequality, we have

$$F(d_\alpha(x_1, Tx_1)) \leq F(d_\alpha(x_0, x_1)) - \tau. \tag{21}$$

From condition (F4), we can write

$$F(d_\alpha(x_1, Tx_1)) = \inf_{y \in Tx_1} F(d_\alpha(x_1, y)).$$

Thus, from Eq. (21), we have

$$\inf_{y \in Tx_1} F(d_\alpha(x_1, y)) \leq F(d_\alpha(x_0, x_1)) - \tau. \tag{22}$$

From Eq. (22), there exists $x_2 \in Tx_1$ such that

$$F(d_\alpha(x_1, x_2)) \leq F(d_\alpha(x_0, x_1)) - \tau.$$

If $x_2 \in Tx_2$, then the proof is complete, otherwise in the same way there exists $x_3 \in Tx_2$ such that

$$F(d_\alpha(x_2, x_3)) \leq F(d_\alpha(x_1, x_2)) - \tau.$$

By continuing the same procedure recursively, we get a sequence $\{x_n\}$ in X , for $n = 0, 1, 2, \dots$ such that $x_{n+1} \in Tx_n$ and

$$F(d_\alpha(x_n, x_{n+1})) \leq F(d_\alpha(x_{n-1}, x_n)) - \tau. \tag{23}$$

If $x_n \in Tx_n$, then x_n is a fixed point of T . Therefore, suppose that for every $n \in \mathbb{N}$ $x_n \notin Tx_n$. Denote by $\mathcal{A}_n = d_\alpha(x_n, x_{n+1})$, for $n = 0, 1, 2, \dots$. Thus, for all $n = 0, 1, 2, \dots$, $d_\alpha(x_n, x_{n+1}) > 0$. From Eq. (23), we get

$$F(\mathcal{A}_n) \leq F(\mathcal{A}_{n-1}) - \tau \leq F(\mathcal{A}_{n-2}) - 2\tau \leq \dots \leq F(\mathcal{A}_0) - n\tau. \tag{24}$$

By taking the limit $n \rightarrow \infty$ in Eq. (24), we get $\lim_{n \rightarrow \infty} F(\mathcal{A}_n) = -\infty$. Thus, from condition (F2) of Definition 1.1, we have

$$\lim_{n \rightarrow \infty} \mathcal{A}_n = 0.$$

Also from condition (F3), there exists $l \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} \mathcal{A}_n^l F(\mathcal{A}_n) = 0.$$

From Eq. (24), for all $n \in \mathbb{N}$, the following holds:

$$\lim_{n \rightarrow \infty} \mathcal{A}_n^l F(\mathcal{A}_n) - \lim_{n \rightarrow \infty} \mathcal{A}_n^l F(\mathcal{A}_0) \leq \lim_{n \rightarrow \infty} -\mathcal{A}_n^l n\tau \leq 0. \tag{25}$$

By letting $n \rightarrow \infty$ in (25), we obtain

$$\lim_{n \rightarrow \infty} n\mathcal{A}_n^l = 0. \tag{26}$$

From Eq. (26), there exists $n_1 \in \mathbb{N}$ such that $n\mathcal{A}_n^l \leq 1$ for all $n \geq n_1$. Thus, for all $n \geq n_1$, we have

$$\mathcal{A}_n \leq \frac{1}{n^{\frac{1}{l}}}. \tag{27}$$

From the triangle inequality and Eq. (27) for $m > n \geq n_1$, we have

$$\begin{aligned} d_\alpha(x_n, x_m) &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)d_\alpha(x_{n+1}, x_m) \\ &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \alpha(x_n, x_m)\alpha(x_{n+1}, x_{n+2})d_\alpha(x_{n+1}, x_{n+2}) \\ &\quad + \alpha(x_n, x_m)\alpha(x_{n+2}, x_m)d_\alpha(x_{n+2}, x_m) \\ &\leq \dots \\ &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \sum_{i=1}^{m-2} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})d_\alpha(x_i, x_{i+1}) \\ &\quad + \prod_{j=1}^{m-1} \alpha(x_j, x_m)\alpha(x_{m-1}, x_m)d_\alpha(x_{m-1}, x_m) \\ &\leq \alpha(\mathcal{A}_n, \mathcal{A}_{n+1})d_\alpha(\mathcal{A}_n, \mathcal{A}_{n+1}) + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})d_\alpha(x_i, x_{i+1}) \\ &\leq \alpha(x_n, x_{n+1})d_\alpha(x_n, x_{n+1}) + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})d_\alpha(x_i, x_{i+1}) \\ &= \alpha(x_n, x_{n+1})\mathcal{A}_n + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})\mathcal{A}_i \\ &= \alpha(x_n, x_{n+1})\frac{1}{n^{\frac{1}{l}}} + \sum_{i=1}^{m-1} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})\frac{1}{i^{\frac{1}{l}}} \\ &\leq \alpha(x_n, x_{n+1})\frac{1}{n^{\frac{1}{l}}} + \sum_{i=1}^{\infty} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1})\frac{1}{i^{\frac{1}{l}}}. \end{aligned}$$

Since $\lim_{n,m \rightarrow \infty} \alpha(x_{n+1}, x_m) < 1$ for all $x_n, x_m \in X$, the series $\sum_{i=1}^{\infty} \left(\prod_{j=1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) \frac{1}{i^{\frac{1}{l}}}$ converges by the ratio test for each $m \in \mathbb{N}$. Therefore, by taking the limit $n \rightarrow \infty$ in the above inequality, we get $d_\alpha(x_n, x_m) \rightarrow 0$. Since X is complete, there exists $x_* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x_*$. Now, we prove that x_* is a fixed point of T . From the construction of $\{x_n\}$

for $n = 0, 1, 2, \dots$, there is a subsequence $\{x_p\}$ such that

$$x_p \in Tx_{p-1}. \tag{28}$$

Since $\lim_{p \rightarrow \infty} x_p = x_*$, we have

$$\lim_{p \rightarrow \infty} d_\alpha(x_*, Tx_{p-1}) = 0. \tag{29}$$

From Lemma 2.1 and (20), we have

$$\begin{aligned} d_\alpha(x_*, Tx_*) &\leq \alpha(x_*, x_p)d_\alpha(x_*, x_p) + \alpha(x_p, Tx_*)d_\alpha(x_p, Tx_*) \\ &\leq \alpha(x_*, x_p)d_\alpha(x_*, x_p) + \alpha(x_p, Tx_*)H_\alpha(Tx_{p-1}, Tx_*) \\ &\leq \alpha(x_*, x_p)d_\alpha(x_*, x_p) + \alpha(x_p, Tx_*)[d_\alpha(x_{p-1}, x_*) + \gamma d_\alpha(x_*, Tx_{p-1})]. \end{aligned}$$

Since $\lim_{n,m \rightarrow \infty} \alpha(x_n, x_m)\kappa < 1$ for all $x_n, x_m \in X$, $\alpha(x_n, x_m)$ is finite. Thus by taking the limit $p \rightarrow \infty$ in the above inequality and from (29), we get $d_\alpha(x_*, Tx_*) = 0$. Hence $x_* \in Tx_*$, and x_* is a fixed point of T . □

Remark 2.1 Theorem 2.6 is a generalization of Theorem 1.1 and Theorem 1.3.

Example 2.1 Let $X = [0, \infty)$. Define $d_\alpha : X \times X \rightarrow [0, \infty)$ as

$$d_\alpha(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \frac{1}{x}, & \text{if } x \geq 1 \text{ and } y \in [0, 1); \\ \frac{1}{y}, & \text{if } y \geq 1 \text{ and } x \in [0, 1); \\ 1, & \text{otherwise.} \end{cases}$$

Hence (X, d_α) is a complete controlled metric space, where $\alpha : X \times X \rightarrow [1, \infty)$ is defined as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1); \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

Define a mapping $T : X \rightarrow CLD(X)$ by

$$T\mathfrak{S} = \begin{cases} [\frac{x}{3}, \frac{x}{2}], & \text{if } x, y \in [0, 1); \\ \{x\}, & \text{if } x \geq 1. \end{cases}$$

Now, consider the mapping F defined by $F(\mathcal{A}) = \ln \mathcal{A}$. Then T is multivalued almost F -contraction with $\tau = \ln 2$ and $\gamma = 10$. As $H_\alpha(Tx, Ty) > 0$ for $x \neq y$. So (20) is equivalent to the following equation:

$$H_\alpha(Tx, Ty) \leq e^{-\tau}d_\alpha(x, y) + \gamma e^{-\tau}d_\alpha(y, Tx),$$

and so

$$H_\alpha(Tx, Ty) \leq \frac{1}{2}d_\alpha(x, y) + 5d_\alpha(y, Tx). \quad (30)$$

Now, we will consider the following cases:

Case (1) If $x, y \in [0, 1)$, then

$$H_\alpha(Tx, Ty) = 1 = d_\alpha(x, y),$$

and hence (30) is satisfied.

Case (2) If $x, y \geq 1$, then

$$H_\alpha(Tx, Ty) = 1 = d_\alpha(x, y) = d_\alpha(y, Tx).$$

Clearly, (30) is satisfied.

Case (3) If $x \geq 1$ and $y \in [0, 1)$, then

$$H_\alpha(Tx, Ty) = \frac{1}{x} = d_\alpha(x, y) = d_\alpha(y, Tx).$$

Equation (30) is satisfied.

Case (4) If $y \geq 1$ and $x \in [0, 1)$, then

$$H_\alpha(Tx, Ty) = \frac{1}{y} = d_\alpha(x, y) = d_\alpha(y, Tx).$$

Hence (30) is satisfied.

3 Conclusion

In the present study, we defined the concept of a Pompeiu–Hausdorff metric on the class of nonempty closed subsets of controlled metric spaces and we showed that if (X, d_α) is complete, then $(H_\alpha, CLD(X))$ is also complete. Also, we analyzed some topological properties of such spaces. Then we established some fixed point results for multivalued mappings satisfying almost F -contractive condition on controlled metric spaces which generalize many existing results in the literature. We think that different versions of contractive conditions can be considered in such spaces by using a Pompeiu–Hausdorff metric. Also, this new working area will be a powerful tool for the existence solution of the systems of integral inclusions and fractional differential inclusions.

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