JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 78B, No. 2, April–June 1974

Fixed-Point Solution of Plant Input/Location Problems*

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(March 14, 1974)

This paper considers the following generalization of the Weber plant location problem: the plant's output level is fixed, and its levels of input from its supply points, as well as its location, are among the decision variables. Hurter and Wendell (J. Reg. Sci., 1972) showed that this problem admits a kind of separability when the plant's production function lies in a certain class including the Cobb-Douglas forms. The present paper (a) determines the extent of that function-class, (b) carries out the explicit separation for the *CES* generalization of the Cobb-Douglas functions, and (c) discusses simple fixed-point-type iterative solution algorithms, similar to that well-known for the ordinary Weber problem, for several production functions (Cobb-Douglas, *CES*, and various two-stage technologies). Local convergence of these algorithms is established; computational experience will be reported in a separate Part II.

Key words: *CES*; economics; Leontief; location theory; plant location; production functions; transportation; Weber problem, mathematical programming.

1. Introduction

The "ordinary" Weber plant-location problem, set in the real n-dimensional space R^n, can be described as requiring the selection of $x \epsilon R^n$ to minimize the function

$$\phi_W(x) = \sum_1^m t_i ||x - s_i|| q_i. \tag{1.1}$$

Here the decision variable x represents the location of a plant which requires m inputs for its operation; s_i is the source of the ith input, t_i the associated unit transportation cost, and q_i the level of the ith input. In (1.1), $||\cdot||$ denotes some appropriate norm on R^n, which will be taken as the Euclidean norm throughout. The abbreviation

$$\rho_i = ||x - s_i|| \tag{1.2}$$

will prove convenient.

This problem can be generalized by including the vector $q = (q_1, \ldots, q_m)$ of input levels among the decision variables. A plant output level q^0 is specified, as is the plant's production function $f(q)$ and the unit prices p_i of the inputs at their respective sources s_i. Now the problem is to choose $x \epsilon R^n$ and $q \epsilon R^m$ so as to achieve

$$\min_x \min_q \sum_1^m \{t_i \rho_i + p_i\} q_i \tag{1.3}$$

subject to

$$f(q) = q^0. \tag{1.4}$$

AMS Classification: Primary 90B30; Secondary 90A99, 90C30, 90C50.

*Affectionally dedicated to Professor A. W. Tucker on the occasion of his retirement from the Department of Mathematics, Princeton University.

A one-dimensional version of this problem was studied by Sakashita [1] [1] and extended to a network location problem by Wendell and Hurter [2]. The general case was formulated by Wendell and Hurter in a paper [3] which is the point of departure for the present work.

The production function f will be said to lie in "class $C(q^0)$" if there is a positive constant $K(q^0)$ such that, for all positive q satisfying (1.4),

$$\sum_1^m q_i \partial f / \partial q_i = K(q^0). \tag{1.5}$$

It is noted in [3] that for this class of production functions, the problem has a kind of separability: it can be transformed into

$$\min_x \mu(x) \tag{1.6}$$

where $\mu(x)$ is the Lagrange multiplier corresponding to constraint (1.4) for the "inner" minimization in (1.3).

If f lies in class $C(q^0)$ for all $q^0>0$, we say it lies in "class C". As noted in [3], it is a consequence of Euler's theorem that class C contains all differentiable homogeneous functions, but it contains other functions as well. In section 2, we determine the extent of class C.

Since class C contains the homogeneous production functions, it in particular includes the familiar Cobb-Douglas functions [4]

$$f(q) = \exp\left\{a + \sum_1^m a_i \log q_i\right\} \qquad (a_i > 0) \tag{1.7}$$

as well as the multi-input *CES* ("constant elasticity of substitution") functions of Arrow, Chenery, Minhas and Solow [5],

$$f(q) = \left(\sum_1^m b_i q_i^{-c}\right)^{-1/c} \qquad (b_i>0,\ c>-1,\ c\neq 0). \tag{1.8}$$

In [3] the separation (1.6) is carried out explicitly for the Cobb-Douglas case, but the corresponding problem for other homogeneous functions is noted to be "difficult". In section 3, we perform the explicit separation for the *CES* functions and for several functions representing two-stage technologies.

The ordinary Weber problem can be regarded as arising from a Leontief production function (in which specifying the output-level q^0 fixes the values q_i), and is a convex programming problem. In contrast, the mathematical programming problems (1.6) arising from the cases studied here are nonconvex, so that their numerical solution is nontrivial. In section 4, simple iterative fixed-point-type solution algorithms are presented, patterned after one well known for the Weber problem. Local convergence is established for the low dimensions of practical interest, and the analyses necessary to handle the singularities s_i are performed. A subsequent Part II will report our computational experience, to date, with these algorithms. Further work should take up the case of time-varying prices p_i, transport costs t_i, and output levels q^0. Another natural line of generalization would incorporate consideration of market location.

2. Determination of Class C

From (1.4) and (1.5), it is readily seen that C is the class of production functions f which satisfy a partial differential equation of the form

$$\sum_1^m q_i \partial f / \partial q_i = F[f(q)] \tag{2.1}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

for all positive q. Since a production function f has positive first-order derivatives, F is positive-valued; we also assume it continuous.

THEOREM: *Class* C *consists of the production functions of the form* $f(q)=M[h(q)]$, *where* h *is a production function homogeneous of degree* 1, *and* M *is an increasing differentiable function.*

Before proving this theorem, we note three consequences of it. *First*, it identifies class C with the class of "homothetic" production functions introduced by Shephard (see p. 30 of [6]), apart from questions of smoothness and other properties in a theoretical definition of "production function". *Second*, it "explains" the examples of nonhomogeneous members of C given in [3], which in fact are the logarithms of Cobb-Douglas functions. *Third*, it implies that analyses of our generalized Weber problem can be confined to production functions which are homogeneous of degree 1, since with $f=M[h]$ as above, the constraint (1.4) can be replaced by the equivalent $h(q)=M^{-1}(q^0)$.

For the proof, first assume $f=M[h]$ as in the theorem's statement. By Euler's theorem on homogeneous functions,

$$\sum_1^m q_i \partial h/\partial q_i = h(q),$$

and so by the chain rule of differentiation

$$\sum_1^m q_i \partial f/\partial q_i = M'[h(q)]h(q),$$

an instance of (2.1) with $F(v)=M'[M^{-1}(v)]M^{-1}(v)$.

Conversely, suppose $f\epsilon C$ satisfies (2.1). Define a function $M^{-1}(u)$ by

$$M^{-1}(u)=\exp\left\{\int_{u_0}^{u}[1/F(v)]dv\right\} \tag{2.2}$$

for some $u_0>0$. Since F is positive-valued and continuous, M^{-1} is increasing and differentiable, and thus has an increasing differentiable inverse function M. Define $h=M^{-1}[f]$; then by (2.1) and (2.2)

$$\sum_1^m q_i \partial h/\partial q_i = (M^{-1})'[f] \sum_1^m q_i \partial f/\partial q_i$$

$$=\{M^{-1}[f]/F[f]\}F[f]=h,$$

so that the converse of Euler's theorem implies that h is homogeneous of degree 1. Since $f=M[h]$, the proof is complete.

Before leaving this topic, we note that class C also arises in a multi-output generalization of the problem under discussion. Namely, suppose the m inputs are used jointly to produce several outputs in accordance with a vector production function having one component $f_j(q)$ per output. Suppose furthermore that the level of each output is prescribed, and that $\mu_j(x)$ denotes the Lagrange multiplier corresponding to the jth of these constraints in the inner minimization of (1.3). The first-order optimality conditions for that minimization are

$$t_i\rho_i+p_i=\sum_j \mu_j(x)\partial f_j/\partial q_i \qquad (i=1, 2, \ldots, m),$$

so that with the inner minimization accomplished for each x (and the $\partial f_j/\partial q_i$ evaluated at its solution) the objective function (1.2) becomes

$$\sum_i (t_i\rho_i+p_i)q_i=\sum_j \mu_j(x) \sum_i q_i \partial f_j/\partial q_i.$$

If now each f_j is in class C, with corresponding F_j in (2.1), then this minimand is equal to

$$\phi(x)=\sum_j \mu_j(x)F_j(f_j^0) \tag{2.3}$$

where f_j^0 is the prescribed level of the jth output. Result (2.3) is a multi-output extension of the separability expressed by (1.6). This (multiple output) line of generalization will not be pursued further in the present paper.

3. Some Cases of Explicit Separation

To take full advantage of the separability expressed in (1.6), it is necessary to find an explicit expression for $\mu(x)$, so that the resultant "pure location" problem is in explicit form. This will be possible, in particular, if f satisfies a suitable set of identities

$$q_i=F_i[f(q),\,\partial f/\partial q_i]. \tag{3.1}$$

To see why this is so, recall the first-order optimality conditions

$$t_i\rho_i+p_i=\mu(x)\partial f/\partial q_i$$

for the inner minimization in (1.3). The solution $q(x)$ of that minimization will therefore satisfy, by (3.1),

$$q_i(x)=G_i[(t_i\rho_i+p_i)/\mu(x)],$$

where $G_i(\cdot)=F_i(q^0,\cdot)$. It follows that

$$q^0=f[q(x)]=f\{G_1[(t_1\rho_1+p_1)/\mu(x)],\ .\ .\ .,\ G_m[(t_m\rho_m+p_m)/\mu(x)]\}\cdot \tag{3.2}$$

Typically (hence the adjective "suitable" above (3.1)) this equation can be solved to obtain the desired explicit form for $\mu(x)$.

For the Cobb-Douglas case (1.7), one has in (3.1)

$$F_i(u,\,v)=a_iu/v;$$

thus (3.2) yields

$$\log q^0=a+\sum_1^m a_i\log[a_iq^0\mu(x)/(t_i\rho_i+p_i)]$$

$$=[a+\sum_1^m a_i\log(a_iq^0)]+[\log\mu(x)]\sum_1^m a_i-\sum_1^m a_i\log(t_i\rho_i+p_i)\cdot$$

It follows that

$$\log\mu(x)=\left(\sum_1^m a_i\right)^{-1}\sum_1^m a_i\log(t_i\rho_i+p_i)+\text{const.}$$

or, with the abbreviation

$$\pi_i=p_i/t_i,$$

$$\log\mu(x)=\left(\sum_1^m a_i\right)^{-1}\sum_1^m a_i\log(\rho_i+\pi_i)+\text{const.} \tag{3.3}$$

Thus the pure location problem (1.6) is equivalent in the Cobb-Douglas case to minimizing

$$\phi_{CD}(x) = \sum_{1}^{m} a_i \log (\rho_i + \pi_i), \tag{3.4}$$

a result derived in [3].

For the *CES* production function (1.8), one has in (3.1)

$$F_i(u, v) = u(b_i/v)^{1/(c+1)}.$$

With $d = c/(c+1)$, (3.2) yields

$$\begin{aligned}(q^0)^{-c} &= \sum_{1}^{m} b_i[q^0(b_i\mu(x)/\{t_i\rho_i + p_i\})^{1/(c+1)}]^{-c} \\ &= (q^0)^{-c}[\mu(x)]^{-d} \sum_{1}^{m} b_i^{1-d}\{t_i\rho_i + p_i\}^d \\ &= (q^0)^{-c}[\mu(x)]^{-d} \sum_{1}^{m} c_i\{\rho_i + \pi_i\}^d\end{aligned}$$

where π_i is as above and

$$c_i = b_i^{1-d} t_i^d. \tag{3.5}$$

If $c > 0$, so that $0 < d < 1$, then minimizing $\mu(x)$ is equivalent to minimizing $[\mu(x)]^d$, and it follows that the pure location problem for the *CES* case can be expressed as demanding the minimization of

$$\phi_{CES}(x) = \sum_{1}^{m} c_i\{\rho_i + \pi_i\}^d \qquad (0 < d < 1). \tag{3.6}$$

If $d < 0$ (i.e., $-1 < c < 0$), the pure location problem involves *maximizing* the form (3.6), or equivalently minimizing its negative. Our subsequent discussion of the *CES* situation is readily adapted to this subcase, but will for simplicity be confined to the subcase $0 < d < 1$; the reader is *warned* that the later discussion does not as it stands refer to the case $c < 0$ (i.e., $d < 0$), though the revision is simple.

The three functions ϕ_W, ϕ_{CD}, ϕ_{CES} all have the form

$$\phi(x) = \sum_{1}^{m} \phi_i(\rho_i) \tag{3.7}$$

where the functions ϕ_i are defined and nonnegative on $(0, \infty)$, positive-valued and twice differentiable on $(0, \infty)$, and satisfy

$$\phi_i' > 0 \qquad \text{on } (0, \infty). \tag{3.8}$$

But while for ϕ_W, which has $\phi_i(u) = (t_i q_i)u$, each summand in (3.7) is a convex function of x, neither ϕ_{CD} nor ϕ_{CES} is convex, so that the *CD* and *CES* cases give rise to nonconvex programming problems. This nonconvexity is most easily seen in the one-dimensional case; whereas ϕ_W is linear on each (open) interval between successive points s_i, both ϕ_{CD} and ϕ_{CES} satisfy $\phi'' < 0$ (the antithesis of convexity) on those intervals.

The absence of convexity suggests the possibility of multiple local minima, and these can in fact occur. They may occur at a point s_i (in the one-dimensional case, local minima occur *only* at points s_i), which however would not be routinely identified as a critical point since s_i is a singular point of

$$\text{grad}\,\rho_i = (x - s_i)/\rho_i$$

and thus of grad ϕ. We therefore proceed to develop a special test for the existence of a local minimum at an s_i, say s_1. It will be assumed that

$$s_i = s_1 \qquad \text{for } 1 \leqslant i \leqslant r, \qquad s_i \neq s_1 \qquad \text{for } i > r.$$

Let θ be a nonnegative scalar variable, and $w \epsilon R^n$ be a variable "direction vector", i.e., $\| w \| = 1$. Set

$$g(\theta, w) = \phi(s_1 + \theta w). \tag{3.9}$$

Then a necessary condition for a local minimum at s_1 is that

$$\inf_w \partial g(0+, w)/\partial\theta \geqslant 0 \tag{3.10}$$

holds. For $x = s_1 + \theta w$, one has

$$\rho_i = \theta \qquad \text{for } i \leqslant r, \qquad \rho_i = \|\theta w - (s_i - s_1)\| \qquad \text{for } i > r,$$

so that

$$g(\theta, w) = \sum_1^r \phi_i(\theta) + \sum_{r+1}^m \phi_i(\|\theta w - (s_i - s_1)\|),$$

$$\partial g/\partial\theta = \sum_1^r \phi_i'(\theta) + \sum_{r+1}^m [\phi_i'(\rho_i)/\rho_i]\,[\theta - (w, s_i - s_1)] \tag{3.11}$$

where $(w, s_i - s_1)$ denotes the scalar product. With the notations

$$b = \sum_{r+1}^m [\phi_i'(\|s_i - s_1\|)/\|s_i - s_1\|](s_i - s_1), \tag{3.12}$$

$$A = \sum_1^r \phi_i'(0), \tag{3.13}$$

it follows from (3.11) that

$$\partial g(0+, w)/\partial\theta = A - (w, b). \tag{3.14}$$

The Cauchy-Schwarz inequality implies

$$(w, b) \leqslant \|b\|,$$

with equality for all w if $b = 0$, and for $w = b/\|b\|$ and its negative if $b \neq 0$. It follows from (3.14) that

$$\min_w \partial g(0+, w)\partial\theta = A - \|b\|,$$

and so

$$A \geqslant \|b\| \tag{3.15}$$

is a necessary condition for a local minimum at s_1.

Conversely, suppose

$$\sigma = A - \|b\| > 0. \tag{3.16}$$

Choose any positive $\delta_1 < \min_{i>r} ||s_i - s_i||$. Then the right-hand side of (3.11), call it $g^*(\theta, w)$, is continuous on the compact domain $[0, \delta_1] \times \{w: \| w \| = 1\}$, hence uniformly continuous. In particular there is a $\delta > 0$, with $\delta \leq \delta_1$, such that if $0 \leq \theta \leq \delta$ then for all w

$$g^*(\theta, w) > g^*(0, w) - \sigma.$$

Since the previous analysis implies $g^*(0, w) \geq \sigma$, we have demonstrated the existence of a $\delta > 0$ such that

$$\partial g(\theta, w)/\partial\theta = g^*(\theta, w) > 0 \qquad \text{for } 0 \leq \theta \leq \delta \text{ and all } w.$$

Thus in the δ-neighborhood of s_1, ϕ is uniquely minimized at the point s_1, so that (3.16) is a sufficient condition for a strict local minimum at s_1. It is a generalization of its specialization (given by Kuhn and Kuenne [7]) for the ordinary Weber problem.

Next we consider some cases in which the "inputs" transported to the plant from the source-points s_i are best interpreted as "factors of production", not for the process yielding the plant's final output, but rather for intermediate on-site processes producing these "final factors." Note that the production functions for these intermediate processes, as well as that for the final process, must now be specified. The levels of the final factors are (intermediate) variables of the problem; these levels will be denoted

$$Q = (Q_1, Q_2, \ldots, Q_M)$$

and the production function for the final process will be denoted $f(Q)$.

A variety of interesting problems can be posed in this context; we will briefly take up just a few of them. For notation, it will be convenient to partition the input-indexing set $\{1, 2, \ldots, m\}$ into subsets $\{I(\nu): \nu = 1, 2, \ldots, M\}$, where $i \epsilon I(\nu)$ signifies that the ith input goes into making the νth final factor. Assuming disjointness of these sets $I(\nu)$ is not really a restriction on the technology—so long as capacity constraints at the sources are omitted—since otherwise-identical inputs can be artifically distinguished according to the final factor in which they will be embodied.

Suppose first that each intermediate process follows a simple Leontief production law; that is, there are positive constants K_i such that

$$q_i = K_i Q_\nu \qquad \text{for all } i \epsilon I(\nu). \tag{3.17}$$

Then the problem can be written

$$\min_x \min_Q \sum_\nu \Big[\sum_{i \epsilon I(\nu)} \{t_i \rho_i + p_i\} K_i \Big] Q_\nu \tag{3.18}$$

subject to

$$f(Q) = q^0. \tag{3.19}$$

Reduction to a pure location problem follows the same pattern as before; if f is a Cobb-Douglas function with parameters a_ν, or a CES function with parameters b_ν and c, the result is an objective function

$$\phi^L_{CD}(x) = \sum_\nu a_\nu \log \Big(\sum_{i \epsilon I(\nu)} L_i \{\rho_i + \pi_i\} \Big), \tag{3.20}$$

with $L_i = K_i t_i$, $\pi_i = p_i / t_i$, or

$$\phi^L_{CES}(x) = \sum_\nu b_\nu^{1-d} \Big(\sum_{i \epsilon I(\nu)} K_i \{t_i \rho_i + p_i\} \Big)^d$$

$$= \sum_\nu \Big(\sum_{i \epsilon I(\nu)} L_i \{\rho_i + \pi_i\} \Big)^d, \tag{3.21}$$

with $d=c/(c+1)$, π_i as above, and $L_i=b_\nu^{1/c}K_i t_i$.

Next suppose that the final process is of Cobb-Douglas type, with parameters a_ν. If each intermediate process is also Cobb-Douglas, then the q_i's are related to the plant output by a Cobb-Douglas function, so that the material leading to (3.4) applies. Let us suppose, instead, that each intermediate process is of *CES* type; assume the νth final factor has a *CES* production function with parameters $\{b_i : i \epsilon I(\nu)\}$ and c_ν. The result is a (composite) production function, for the plant, of Uzawa-*CES* type [8],

$$\psi(q)=\exp\left\{a-\sum_\nu (a_\nu/c_\nu)\log\left(\sum_{i\epsilon I(\nu)} b_i q_i^{-c_\nu}\right)\right\}. \tag{3.22}$$

It is readily verified that

$$\partial\psi/\partial q_i = a_\nu b_i q_i^{-(c_\nu+1)}\psi\Big/\sum_{j\epsilon I(\nu)} b_j q_j^{-c_\nu}\ (i\epsilon I(\nu)) \tag{3.23}$$

Although identities of type (3.1) are lacking, the general approach can still be carried out. Let

$$\alpha_i(x)=t_i\rho_i+p_i, \tag{3.24}$$

then (3.23) and the first-order optimality conditions below (3.1) yield

$$\alpha_i=\mu a_\nu b_i q_i^{-(c_\nu+1)}\psi\Big/\sum_{j\epsilon I(\nu)} b_j q_j^{-c_\nu} \qquad (i\epsilon I(\nu)),$$

or equivalently, with $d_\nu=c_\nu/(c_\nu+1)$,

$$q_i^{-c_\nu}=(\mu\psi)^{-d_\nu}(\alpha_i/a_\nu b_i)^{-d_\nu}\left(\sum_{j\epsilon I(\nu)} b_j q_j^{-c_\nu}\right)^{d_\nu} \qquad (i\epsilon I(\nu)).$$

Multiply both sides by b_i and sum over $i\epsilon I(\nu)$; the result is

$$\sum_{i\epsilon I(\nu)} b_i q_i^{-c_\nu}=(\mu\psi a_\nu)^{-d_\nu}\left(\sum_{j\epsilon I(\nu)} b_j q_j^{-c_\nu}\right)^{d_\nu}\sum_{i\epsilon I(\nu)} b_i(\alpha_i/b_i)^{d_\nu},$$

or equivalently

$$\sum_{i\epsilon I(\nu)} b_i q_i^{-c_\nu}=(\mu\psi a_\nu)^{-c_\nu}\left[\sum_{i\epsilon I(\nu)} b_i(\alpha_i/b_i)^{d_\nu}\right]^{c_\nu+1},$$

i.e.,

$$\log\left(\sum_{i\epsilon I(\nu)} b_i q_i^{-c_\nu}\right)=-c_\nu\log\mu-c_\nu\log(\psi a_\nu)+(c_\nu+1)\log\left[\sum_{i\epsilon I(\nu)} b_i(\alpha_i/b_i)^{d_\nu}\right],$$

leading via (3.22) to

$$\log q^0=a+\left(\sum_\nu a_\nu\right)\log\mu+\sum_\nu a_\nu\log(q^0 a_\nu)-\sum_\nu (a_\nu/d_\nu)\log\left[\sum_{i\epsilon I(\nu)} b_i(\alpha_i/b_i)^{d_\nu}\right].$$

Thus, with the abbreviations

$$\beta_\nu=a_\nu/d_\nu, \qquad \pi_i=p_i/t_i,\ L_i=b_i^{1-d_\nu}t_i^{d_\nu},$$

the objective function for the pure location problem takes the form

$$\phi_{CD}^{CES}(x)=\sum_\nu \beta_\nu\log\left[\sum_{i\epsilon I(\nu)} L_i\{\rho_i+\pi_i\}^{d_\nu}\right]. \tag{3.25}$$

If the f_ν's are as above, but f is a *CES* function with parameters b_ν and c, then the composite production function is

$$\psi(q)=\left\{\sum_\nu b_\nu\left[\sum_{i\in I(\nu)} b_i q_i^{-c_\nu}\right]^{c/c_\nu}\right\}^{-1/c}. \tag{3.26}$$

Manipulations like those above lead to an objective function

$$\phi_{CES}^{CES}(x)=\sum_\nu\left[\sum_{i\in I(\nu)} L_i\{\rho_i+\pi_i\}^{d_\nu}\right]^{d/d_\nu}, \tag{3.27}$$

with $d=c/(c+1)$, $d_\nu=c_\nu/(c_\nu+1)$, $\pi_i=p_i/t_i$ and $L_i=b_\nu^{d_\nu/c}b_i^{1-d_\nu}t_i^{d_\nu}$.

Now assume f is as above, but the f_ν's are Cobb-Douglas functions with parameters a_ν and $\{a_i : i\epsilon I(\nu)\}$. Then the composite production function is

$$\psi(q)=\left\{\sum_\nu b_\nu\left[\exp\left(a_\nu+\sum_{i\in I(\nu)} a_i\log q_i\right)\right]^{-c}\right\}^{-1/c}. \tag{3.28}$$

For the explicit separation to be tractable, it appears necessary to require each f_ν to be homogeneous of the same degree, i.e.,

$$\sum_{i\in I(\nu)} a_i=\Delta \qquad (\text{all } \nu). \tag{3.29}$$

With the notations

$$\sigma=c/(1+c\Delta), \qquad A_\nu=a_\nu+\sum_{i\in I(\nu)} a_i\log a_i,$$

$$\pi_i=p_i/b_i, \qquad a_i'=\sigma a_i$$

$$a_\nu'=\sum_{i\in I(\nu)} a_i'\log t_i+(1-\sigma\Delta)\log b_\nu-\sigma A_\nu,$$

an objective function

$$\phi_{CES}^{CD}(x)=\sum_\nu \exp\left\{a_\nu'+\sum_{i\in I(\nu)} a_i'\log(\rho_i+\pi_i)\right\}, \tag{3.30}$$

a sum of Cobb-Douglas functions, is obtained for the pure location problem.

The reasonableness of the restriction (3.29) is supported by the following observation, which applies to the situations (3.25), (3.27), (3.30) above. Suppose $f\epsilon C$, with associated function F in (3.1), and that the functions f_ν are homogeneous of respective degrees Δ_ν. For the two-stage technology to admit the kind of analysis given in this paper, the composite production function ψ must lie in C. Now

$$\sum_i q_i\partial\psi/\partial q_i=\sum_\nu \partial f/\partial Q_\nu \sum_{i\in I(\nu)} q_i\partial f_\nu/\partial q_i$$

$$=\sum_\nu \Delta_\nu Q_\nu \partial f/\partial Q_\nu,$$

and only if all Δ_ν have a common value Δ can we continue to the

$$=\Delta F[\psi]$$

which shows that $\psi\epsilon C$.

We return now to the matter of testing for a local minimum at a point s_i. Since the objective functions (3.20), (3.21), (3.25), (3.27) and (3.30) are not of the form (3.7), the test (3.16) does not apply. Instead, these objective functions have the more general form

$$\phi(x) = \sum_\nu \phi_\nu(\rho_\nu) \tag{3.31}$$

where ρ_ν is a vector with components $\{\rho_i : i\epsilon I(\nu)\}$. For the point s being tested, define the index-sets

$$E(\nu) = \{i\epsilon I(\nu) : s_i = s\}, \qquad U(\nu) = I(\nu) - E(\nu),$$

and introduce a variable vector u_ν with components $\{u_i : i\epsilon I(\nu)\}$ as general argument of ϕ_ν. As in the analysis leading to (3.16), let

$$g(\theta, w) = \phi(s + \theta w)$$

with θ a nonnegative scalar variable and $w\epsilon R^n$ a direction vector. Then

$$\begin{aligned}\partial g/\partial\theta &= \sum_\nu \sum_{i\epsilon I(\nu)} [\partial\phi_\nu/\partial u_i](\{||\theta w - (s_j - s)||\}_{j\epsilon I(\nu)}) \\ &\qquad [\theta - (w, s_i - s)]/||\theta w - (s_i - s)|| \\ &= \sum_\nu \sum_{i\epsilon E(\nu)} [\partial\phi_\nu/\partial u_i][\{||\theta w - (s_j - s)||\}_{j\epsilon I(\nu)}] \\ &\quad + \sum_\nu \sum_{i\epsilon U(\nu)} [\partial\phi_\nu/\partial u_i](\{||\theta w - (s_j - s)||\}_{j\epsilon I(\nu)}] \\ &\qquad [\theta - (w, s_i - s)]/||\theta w - (s_i - s)||.\end{aligned}$$

It follows that

$$\begin{aligned}\partial g(0+, w)/\partial\theta &= \sum_\nu \sum_{i\epsilon E(\nu)} [\partial\phi_\nu/\partial u_i](\{||s_j - s||\}_{j\epsilon I(\nu)}) \\ &\quad - \sum_\nu \sum_{i\epsilon U(\nu)} [\partial\phi_\nu/\partial u_i](\{||s_j - s||\}_{j\epsilon I(\nu)})(w, s_i - s)/||s_i - s||.\end{aligned}$$

Arguing as in the derivation of (3.16), we obtain the criterion

$$A > ||b|| \tag{3.32}$$

where now

$$A = \sum_\nu \sum_{i\epsilon E(\nu)} [\partial\phi_\nu/\partial u_i](\{||s_j - s||\}_{j\epsilon I(\nu)}), \tag{3.33}$$

$$b = \sum_\nu \sum_{i\epsilon U(\nu)} (s_i - s)[\partial\phi_\nu/\partial u_i](\{||s_j - s||\}_{j\epsilon I(\nu)})/||s_i - s||. \tag{3.34}$$

Note that in (3.33) the arguments $||s_j - s|| = 0$ for $j\epsilon E(\nu)$.

One might also consider a two-stage process with the final stage of Leontief type. But then fixing q^0 fixes all Q_ν, so that the problem is equivalent to one of the single-stage multi-output type described at the end of section 2.

The final situation to be considered is that the various inputs $i\epsilon I(\nu)$ are of the νth final factor itself (without further processing), but are distinguished merely by being from different sources. That is, the νth intermediate process has as "production function" the additive

$$Q_\nu = \sum_{i\epsilon I(\nu)} q_i. \tag{3.35}$$

Here formal use of the preceding approach would lead to nonsense. The reason lies in the reliance of that approach upon the optimality condition below (3.1) to characterize the inner minimum in (1.3). In fact, that condition is guaranteed only for those q_i which are strictly positive at the minimum, a condition which indeed is satisfied in all the previous cases treated, but is violated here since each final factor would be purchased only from the least expensive of its sources.

With $\alpha_i(x)$ defined as in (3.24), let

$$\alpha_\nu^*(x) = \min \{\alpha_i(x) : i \epsilon I(\nu)\}; \tag{3.36}$$

then (1.3) for the case (3.35) can be written

$$\min_x \min_Q \sum_\nu \alpha_\nu^*(x) Q_\nu.$$

If for example f is Cobb-Douglas with parameters a_ν, the resultant pure location problem has in analogy with (3.4) the objective function

$$\phi_{CD}^*(x) = \sum_\nu a_\nu \log [\alpha_\nu^*(x)], \tag{3.37}$$

while if f is CES with parameters b_ν and c, the result is

$$\phi_{CES}^*(x) = \sum_\nu b_\nu^{1-d} [\alpha_\nu^*(x)]^d \tag{3.38}$$

where $d = c/(c+1)$, analogous with (3.6). Under the plausible assumption (in the present context) that $\{t_i : i \epsilon I(\nu)\}$ has a single member t_ν, (3.37) and (3.38) can be replaced by

$$\phi_{CD}^*(x) = \sum_\nu a_\nu \log [\min_{i \epsilon I(\nu)} \{\rho_i + \pi_i\}], \tag{3.39}$$

$$\phi_{CES}^*(x) = \sum_\nu c_\nu [\min_{i \epsilon I(\nu)} \{\rho_i + \pi_i\}]^d \tag{3.40}$$

where $c_\nu = b_\nu^{1-d} t_\nu^d$ and $\pi_i = p_i / t_i$. Note that under the further assumption that $\{p_i : i \epsilon I(\nu)\}$ has a single member p_ν,

$$\min_{i \epsilon I(\nu)} \{\rho_i + \pi_i\} = [\min_{i \epsilon I(\nu)} \rho_i] + \pi_\nu$$

with $\pi_\nu = p_\nu / t_\nu$.

4. Iterative Solution Methods

The pure location problems obtained in section 3, by working out several cases of "explicit separation", require the minimization of fairly complex nonconvex functions $\phi(x)$. Since such problems are computationally nontrivial, it seems useful to present a class of iterative solution methods which are simple in concept and simple to program. These algorithms, which are based on characterizing an optimal location as a fixed point of an associated transformation of R^n, are presented in the present section, while computational experience with them will be reported in Part II.

As noted in (3.7) and (3.8), several of these problems have an objective function of the form

$$\phi(x) = \sum_1^m \phi_i(\rho_i) \tag{4.1}$$

where the functions $\phi_i(u)$ are twice differentiable for nonnegative arguments, and satisfy

$$\phi_i' > 0 \qquad (i = 1, 2, \ldots, m). \tag{4.2}$$

At any point $\bar{x}$ not an s_i, one has

$$\text{grad } \phi = \sum_{1}^{m} \phi_i' \ (\rho_i) \ (\bar{x} - s_i)/\rho_i. \tag{4.3}$$

If $\bar{x}$ is to be a local minimum, then grad $\phi = 0$ must hold, or equivalently

$$\bar{x} = \left[\sum_{1}^{m} s_i \phi_i' \ (\rho_i)/\rho_i \right] \Big/ \left[\sum_{1} \phi_i' \ (\rho_i)/\rho_i \right], \tag{4.4}$$

a formula which displays $\bar{x}$ as a fixed point of the function on the right-hand side, and incidentally as lying in the convex hull of the points s_i. This formula suggests the iterative scheme

$$x^{(k+1)} = \left[\sum_{1}^{m} s_i \phi_i' \ (\rho_i^{(k)})/\rho_i^{(k)} \right] \Big/ \left[\sum_{1}^{m} \phi_i' \ (\rho_i^{(k)})/\rho_i^{(k)} \right]. \tag{4.5}$$

Because of the presence of (in general, uncancelled) denominators $\rho_i^{(k)}$, this form can be unsuitable for numerical work when $x^{(k)}$ is near some s_i, say s_j, and should be replaced by the algebraically equivalent form obtained by multiplying numerator and denominator through by $\rho_j^{(k)}$. (This alternative form also shows that each s_j is a fixed point of the transformation given by (4.5).) Of course, the test (3.16) for a local minimum at s_j should be applied in such cases.

For the ordinary Weber problem, with objective function ϕ_W given by (1.1), the algorithm reads

$$x^{(k+1)} = \left[\sum_{1}^{m} s_i t_i q_i/\rho_i^{(k)} \right] \Big/ \left[\sum_{1}^{m} t_i q_i/\rho_i^{(k)} \right]. \tag{4.6}$$

This iterative scheme, which has been repeatedly rediscovered (e.g. [7], [9–11]), goes back at least as far as Weiszfeld [12], who also gave a convergence proof; the rapidity of that convergence has been confirmed in a number of instances, e.g. [13].

For the Cobb-Douglas case, with objective function ϕ_{CD} given by (3.4), the algorithm reads

$$x^{(k+1)} = \left[\sum_{1}^{m} s_i a_i/(\rho_i^{(k)} + \pi_i)\rho_i^{(k)} \right] \Big/ \left[\sum_{1}^{m} a_i/(\rho_i^{(k)} + \pi_i)\rho_i^{(k)} \right], \tag{4.7}$$

while for the *CES* case, with objective function ϕ_{CES} given by (3.6), it reads

$$x^{(k+1)} = \left[\sum_{1}^{m} s_i c_i/(\rho_i^{(k)} + \pi_i)^{1-d}\rho_i^{(k)} \right] \Big/ \left[\sum_{1}^{m} c_i/(\rho_i^{(k)} + \pi_i)^{1-d}\rho_i^{(k)} \right] \tag{4.8}$$

This scheme (4.8) was considered by Cooper [14] for the case of all $\pi_i = 0$, a limiting case of the situations of interest here. Note that if some $\pi_j = 0$, and if the algorithm leads to an $x^{(k)}$ near s_j, then the numerator and denominator need to be multiplied by $[\rho_j^{(k)}]^2$ in (4.7), and $[\rho_j^{(k)}]^{2-d}$ in (4.8), not just $\rho_j^{(k)}$. Note also that if some $\pi_j = 0$, then for both ϕ_{CD} and ϕ_{CES} the test for a local minimum at s_j yields $A = \infty$ in (3.13) and thus an affirmative result for the test; for ϕ_{CES} with all $\pi_j = 0$, the fact that each s_j yields a local minimum was observed by Cooper [15].

By the *local convergence property* (*LCP*) for the pure location problem, we shall mean that each strict local minimum $\bar{x}$ of ϕ, other than the points s_i, has a neighborhood $N(\bar{x})$ such that if the iterative process enters $N(\bar{x})$ at some stage, then it subsequently converges to $\bar{x}$ (in fact, in an-least-geometric fashion). In a paper [16] dealing with the general scheme (4.5), Katz (*op. cit.*, Theorem 4) shows that the *LCP* holds if, in addition to (4.2), the functions ϕ_i satisfy

$$\phi_i''(u) \leq (3 - n)\phi_i'(u)/u. \tag{4.9}$$

For both ϕ_{CD} and ϕ_{CES} one has $\phi_i'' < 0$, so that for the low dimensions ($n \leq 3$) of greatest practical interest, (4.9) is satisfied and hence the *LCP* is assured.

The objective functions (3.20), (3.21), (3.25), (3.27) and (3.30) have the more general form (3.31),

$$\phi(x) = \sum_{\nu} \phi_\nu(\rho_\nu), \tag{4.10}$$

with ρ_ν the vector with components $\{\rho_i : i\epsilon I(\nu)\}$. The twice-differentiable positive-valued functions $\phi_\nu(u_\nu)$, where u_ν is a vector of nonnegative variables $\{u_i : i\epsilon I(\nu)\}$, satisfy in all these cases the analog

$$\partial\phi_\nu/\partial u_i > 0 \qquad (\text{all } i\epsilon I(\nu)) \tag{4.11}$$

of (4.2). The analysis by Katz [16] can be mimicked to obtain a generalization of (4.9) which, together with (4.11), is sufficient to assure the *LCP* for the generalization (given later, below) of (4.5).

The details of this imitative analysis are straightforward by reference to [16], and therefore will not be repeated here. The result is that the condition

$$\sum_{\nu} \left\{ (n-3) \sum_{i\epsilon I(\nu)} (1/\rho_i)[\partial\phi_\nu/\partial u_i](\rho_\nu) + \sum_{i,\, j\epsilon I(\nu)} [(\bar{x}-s_i,\, \bar{x}-s_j)/\rho_i\rho_j][\partial^2\phi_\nu/\partial u_i\partial u_j](\rho_\nu) \right\} \leq 0, \tag{4.12}$$

together with (4.11), sufficies for local convergence at $\bar{x}$. It follows that the conditions

$$(n-3) \sum_{i\epsilon I(\nu)} (1/\rho_i)[\partial\phi_\nu/\partial u_i](\rho_\nu) + \sum_{i,\, j\epsilon I(\nu)} [(\bar{x}-s_i,\, \bar{x}-s_j)/\rho_i\rho_j][\partial^2\phi_\nu/\partial u_i\partial u_j](\rho_\nu) \leq 0 \tag{4.13}$$

for all ν, together with (4.11), are sufficient. In particular, if for each ν the local minimum $\bar{x}$ lies outside the convex hull of the points $\{s_i : i\epsilon I(\nu)\}$, so that in (4.13) each scalar product $(\bar{x}-s_i,\, \bar{x}-s_j) > 0$, and if each ϕ_ν has all $\partial^2\phi_\nu/\partial u_i\partial u_j \leq 0$, and if $n \leq 3$, then local convergence holds at $\bar{x}$. For a more useful condition, one can employ the consequence

$$(\bar{x}-s_i,\, \bar{x}-s_j)/\rho_i\rho_j \geq (-1) \tag{4.14}$$

of the Cauchy-Schwartz inequality. If each ϕ_ν satisfies

$$\partial^2\phi_\nu/\partial u_i\partial u_j \leq 0 \qquad (i,\, j\epsilon I(\nu);\ i \neq j), \tag{4.15}$$

then it follows from (4.13) and (4.14) that

$$\sum_{i\epsilon I(\nu)} \{(n-3)\ (1/u_i)\partial\phi_\nu/\partial u_i + \partial^2\phi_\nu/\partial u_i^2\} - \sum \{\partial^2\phi_\nu/\partial u_i\partial u_j : i,\, j\epsilon I(\nu);\ i \neq j\} \leq 0, \tag{4.16}$$

together with (4.11), is sufficient to assure that *LCP*. Note that (4.16) is a generalization of (4.9).

Consider first the objective functions ϕ_{CD}^L and ϕ_{CES}^L of (3.20) and (3.21). For each of them, $\phi_\nu(u_\nu)$ has the form

$$\phi_\nu(u_\nu) = g_\nu\left(\sum_{i\epsilon I(\nu)} L_i\{u_i + \pi_i\}\right), \tag{4.17}$$

so that (4.16) takes the form

$$\sum_{i\epsilon I(\nu)} \{(n-3)\ (L_i/u_i)g_\nu' + L_i^2 g_\nu''\} - \sum \{L_i L_j g_\nu'' : i,\, j\epsilon I(\nu),\ i \neq j\} \leq 0 \tag{4.18}$$

with g_ν' and g_ν'' evaluated at the g_ν-argument of (4.17). For (3.21), with $g_\nu(v) = v^d$, this condition is

$$\sum_{i\epsilon I(\nu)} \left\{ (n-3)\ (L_i/u_i) \sum_{j\epsilon I(\nu)} L_j(u_j + \pi_j) - (1-d)L_i^2 \right\} + (1-d) \sum \{L_i L_j : i,\, j\epsilon I(\nu);\ i \neq j\} \leq 0,$$

or equivalently

$$\sum_{i\epsilon I(\nu)} L_i^2\{(n-3)\ (1/u_i)\ (u_i+\pi_i)-(1-d)\}$$

$$+\sum \{L_iL_j[(n-3)\{(u_j+\pi_j)/u_i+(u_i+\pi_i)/u_j\}+2(1-d)]:i,j\epsilon I(\nu),\ i<j\}\leqslant 0.$$

Since $d<1$, assuming $n\leqslant 3$ assures that the first sum is <0. The generic summand of the second sum, divided by L_iL_j, is (if $n\leqslant 3$)

$$\leqslant (n-3)[u_j/u_i+u_i/u_j]+2(1-d);$$

applying to the first term the inequality $z+1/z\geqslant 2$ for $z>0$, proves for $n\leqslant 3$ that the last displayed expression is

$$\leqslant 2(n-3)+2(1-d)=2(n-2-d),$$

which is negative for $n\leqslant 2$. Thus, for ϕ^L_{CES}, *LCP* holds for the planar and one-dimensional cases. The same argument, with $d=0$ in the later steps, yields the same conclusion for ϕ^L_{CD}.

For the objective functions ϕ^{CES}_{CD} and ϕ^{CES}_{CES} of (3.25) and (3.27), we have the generalization

$$\phi_\nu(u_\nu)=g_\nu\Big(\sum_{i\epsilon I(\nu)} L_i\{u_i+\pi_i\}^{d_\nu}\Big) \tag{4.19}$$

of (4.17). Thus (4.16) takes the form

$$\sum_{i\epsilon I(\nu)} L_i(u_i+\pi_i)^{d_\nu-2}[\{(n-3)(u_i+\pi_i)/u_i-(1-d_\nu)g'_\nu+L_i(u_i+\pi_i)^{d_\nu}g''_\nu d_\nu]$$

$$-d_\nu\sum\{L_iL_j(u_i+\pi_i)^{d_\nu-1}(u_j+\pi_j)^{d_\nu-1}g''_\nu:i,j\epsilon I(\nu);\ i\neq j\}\leqslant 0.$$

For (3.27), with $g_\nu(v)=v^{\delta_\nu}$ where $\delta_\nu=d/d_\nu$, this yields

$$\sum_{i\epsilon I(\nu)} L_i(u_i+\pi_i)^{d_\nu-2}[\{(n-3)(u_i\pi_i)/u_i-(1-d_\nu)\}\sum_{j\epsilon I(\nu)} L_j(u_j+\pi_j)^{d_\nu}$$

$$+L_i(u_i+\pi_i)^{d_\nu}(d-d_\nu)]-(d-d_\nu)\sum\{L_iL_j(u_i+\pi_i)^{d_\nu-1}(u_j+\pi_j)^{d_\nu-1}:i,j\epsilon I(\nu);\ i\neq j\}\leqslant 0$$

or equivalently

$$\sum_{i\epsilon I(\nu)} L_i^2(u_i+\pi_i)^{2d_\nu-2}[(n-3)\ (u_i+\pi_i)/u_i-(1-d_\nu)+(d-d_\nu)]$$

$$+\sum\{L_iL_j(u_i+\pi_i)^{d_\nu-1}\ (u_j+\pi_j)^{d_\nu-1}\ [\{(n-3)\ (u_i+\pi_i)/u_i-(1-d_\nu)\}(u_j+\pi_j)^2$$

$$+\{(n-3)\ (u_j+\pi_j)/u_j-(1-d_\nu)\}(u_i+\pi_i)^2-2(d-d_\nu)\ (u_i+\pi_i)\ (u_j+\pi_j)]\}$$

$$:i,j\epsilon I(\nu);\ i<j\}\leqslant 0.$$

Since $d<1$, the first sum is negative for $n\leqslant 3$. As for the second sum, the factor [---] in its generic summand is for $n\leqslant 3$

$$\leqslant -[\{(1-d_\nu)-(n-3)\}\{(u_i+\pi_i)^2+(u_j+\pi_j)^2\}+2(d-d_\nu)(u_i+\pi_i)(u_j+\pi_j)]$$

$$=-\{(1-d_\nu)-(n-3)\}Q(u_i+\pi_i,u_j+\pi_j),$$

where the coefficient $(1-d_\nu)-(n-3)$ is positive for $n\leqslant 3$, and the quadratic form Q is given by

$$Q(v,w)=v^2+w^2+2kvw,\qquad k=(d-d_\nu)/\{(1-d_\nu)-(n-3)\}.$$

Q is positive definite for $k^2<1$. Since $k<1$ follows from the fact that $d<1$, it suffices to have

$k > (-1)$. For $n=3$ this is true if $d+1 > 2d_\nu$, while for $n \leqslant 2$ it follows without additional restriction. For (3.25) the analysis is similar, corresponding to $d=0$.

For the objective function ϕ_{CES}^{CD} of (3.30), condition (4.16) leads to

$$\sum_{i\epsilon I(\nu)} a_i' (u_i + \pi_i)^{-2} \{ (n-3)\ (u_i + \pi_i)/u_i - 1\}$$
$$- \sum \{a_i' a_j' / (u_i + \pi_i)\ (u_j + \pi_j) : i, j \epsilon I(\nu);\ i \neq j\} \leqslant 0,$$

which holds for $n \leqslant 3$.

The generalization of (4.5) to the situation (4.7), for which the preceding convergence analyses employing (4.16) were given, is obtained using the generalization

$$\text{grad}\ \phi = \sum_{\nu} \sum_{i\epsilon I(\nu)} [\partial\phi_\nu/\partial u_i]\,(\rho_\nu)\ (\bar{x} - s_i)/\rho_i$$

of (4.3). The iterative scheme is

$$x^{(k+1)} = \left[\sum_{\nu} \sum_{i\epsilon I(\nu)} s_i [\partial\phi_\nu/\partial u_i]\,(\rho_\nu)/\rho_i \right] \div \left[\sum_{\nu} \sum_{i\epsilon I(\nu)} [\partial\phi_\nu/\partial u_i]\,(\rho_\nu)/\rho_i \right]. \tag{4.20}$$

For $x^{(k)}$ near some s_i, precautions like those noted under (4.3) are required.

It remains to consider objective functions like the ϕ_{CD}^* and ϕ_{CES}^* of (3.37) and (3.38), whose general form is

$$\phi^*(x) = \sum_{\nu} \phi_\nu [\alpha_\nu^*(x)], \qquad \alpha_\nu^*(x) = \min\ \{\alpha_i(x) : i\epsilon I(\nu)\}$$

with α_i defined by (3.24). Let us call x *exceptional* if a tie occurs in the definition of some $\alpha_\nu^*(x)$; for any nonexceptional point x, let $i(\nu, x)\epsilon I(\nu)$ be the unique index for which the minimum occurs. For nonexceptional points, the previous analyses can be carried over by replacing $\alpha_\nu^*(x)$ with $\alpha_{i(\nu, x)}(x)$. The reason is that these analyses—testing for a local minimum at a point s_i, or for local convergence to a local minimum (not an s_i) of an iterative scheme based on a zero gradient (itself a local construct)—deal only with *local* behavior of ϕ^*, and each nonexceptional point x has a neighborhood consisting entirely of nonexceptional points y for which $i(\nu, y) = i(\nu, x)$ for all ν. But the "radius of convergence" around a local minimum is reduced by the need to avoid contact with the set of exceptional points; a local minimum lying near this set may therefore be "hard to get at" for the algorithm. If the algorithm generates an $x^{(k)}$ which is an exceptional point, it is natural to proceed by breaking the tie arbitrarily, and the effect of this seems difficult to predict. (An alternative is to employ a more complex logic involving "branching" when an exceptional $x^{(k)}$ is encountered.) The ability to detect a minimizing point which is exceptional is *a priori* dubious. These problems are explored on an empirical basis in some of the computational experiments to be reported in Part II.

5. References

[1] Sakashita, N., Production function, demand function and location theory of the firm, Papers of Reg. Sci. Assoc. **20**, 109–122 (1967).

[2] Wendell, R. E., and Hurter, A. P., Optimal locations on a network, Trans. Sci. **7**, 18–33 (1973).

[3] Hurter, A. P., and Wendell, R. E., Location and production—A special case, J. Reg. Sci. **12**, 243–247 (1972).

[4] Cobb, C. W., and Douglas, P. H., A theory of production, Amer. Eco. Rev. (Papers & Proceedings) **18**, 139–205 (1926).

[5] Arrow, K., Chenery, H., Minhas, B. S., and Solow, R. M., Capital-labor substitution and economic efficiency. Rev. Eco. & Statistics **63**, 225–250 (1961).

[6] Shephard, R. W., Theory of Cost and Production Functions, Princeton Univ. Press, Princeton, N.J., 1970.

[7] Kuhn, H. W., and Kuenne, R. E., An efficient algorithm for the numerical solution of the generalized Weber problem in spatial economics, J. Reg. Sci. **4**, 21–33 (1962).
[8] Uzawa, H., Production functions with constant elasticities of substitution, Rev. Eco. Studies **30**, (1962).
[9] Miehle, W., Link-length minimization in networks, Oper. Res. **6**, 232–243 (1958).
[10] Palermo, F. P., A network minimization problem, IBM J. Res. & Dev. **5**, 335–337 (1961).
[11] Cooper, L., Location-allocation problems, Oper. Res. **11**, 331–343 (1963).
[12] Weiszfeld, E., Sur le point pour lequel la somme des distances de n points donnes est minimum, Tohuku Math. J. **43**, 355–386 (1936).
[13] Jordan, R. H., et al., Systems Analysis of Inland Consolidation Centers for Marine Cargo, Nat. Bur. Stand. (U.S.), Tech. Note 530, 162 pages (Nov. 1970).
[14] Cooper, L., Solution of generalized locational equilibrium problems, J. Reg. Sci. **7**, 1–18 (1967).
[15] Cooper, L., An extension of the generalized Weber problem, J. Reg. Sci. **8**, 181–197 (1968).
[16] Katz, I. N., On the convergence of a numerical scheme for solving some locational equilibrium problems, SIAM J. Appl. Math. **17**, 1224–1231 (1969).

(Paper 78B2–403)